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Adjacency of Vertices on the Clique Partitioning Polytope

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Dedicated to Professor Roberto W. Frucht in honor of his 80th birthday

Abstract. Given a graph $G = [V, E]$, a subset $A \subseteq E$ is called a *clique partitioning* of G if there is a partition of V into sets $V_1, V_2, \dots, V_k$ such that the subgraph induced by each V_i is a clique and $A = \bigcup_{i=1}^{k}\{uv : u, v \in V_i, u \neq v\}$. The *clique partitioning polytope* $\mathcal{P}_n$ is defined as the convex hull of the incidence vectors of all clique partitionings of the complete graph of order n. In this paper we present a condition characterizing the adjacency of vertices on the polytope $\mathcal{P}_n$. We also prove that the diameter of $\mathcal{P}_n$ is equal to three for every $n \geq 4$.

1. Introduction

The *clique partitioning polytope* $\mathcal{P}_n$ is defined as the convex hull of the incidence vectors of all clique partitionings of the complete graph of order n. It is the natural 0/1–polytope associated with the problem of finding a minimum weight clique partitioning in a weighted complete graph of order n. A description of a cutting plane algorithm for this $\mathcal{NP}$–hard problem and computational results on the applications of this algorithm to solve some clustering problems in Data Analysis are presented in Grötschel and Wakabayashi [**3**, 1987b]. Other theoretical results on the facial structure of the polytope $\mathcal{P}_n$ are mentioned in Grötschel and Wakabayashi [**2**, 1987a].

In this paper we give a characterization of adjacency of vertices on the polytope $\mathcal{P}_n$. It turns out that the adjacency test of two vertices on $\mathcal{P}_n$ can be performed by an algorithm of time complexity $\mathcal{O}(n^2)$. We also show that for every $n \geq 4$ the polytope $\mathcal{P}_n$ has diameter equal to three.

2. Definitions and Notation

We assume that the reader is familiar with the basic concepts of graph theory. The definitions not given here can be found in Bondy and Murty [**1**, 1976] or Harary [**4**, 1969]. We consider only simple graphs. A graph G with node set V and edge set E is denoted by $G = [V, E]$. The complete graph on n nodes is denoted by $K_n = [V_n, E_n]$. If A is an edge set of $G = [V, E]$, then the subgraph of G with edge set A and node set consisting of the end nodes of the edges in A is denoted by $[A]$ and is said to be *induced* by A. For notational convenience, we denote an edge $\{u, v\}$ simply by uv.

If S and T are disjoint node sets of a graph $G = [V, E]$, then

$$(S : T) := \{uv : u \in S, v \in T\}.$$

A *clique* of a graph G is a complete nonempty subgraph of G. Note that the complete subgraph need not be maximal, as it is sometimes assumed in the literature. If Q is a clique, then we denote by VQ and EQ its node set and edge set, respectively. If $EQ = \emptyset$, then Q is called a *degenerate clique*; otherwise it is called a *nondegenerate clique*.

We say that $\Gamma = \{V_1, \dots, V_k\}$ $(k \geq 1)$ is a *partition* of V if $V_i \cap V_j = \emptyset$ for $1 \leq i < j \leq k$ and $V = V_1 \cup \dots \cup V_k$. A set of edges A in a graph $G = [V, E]$ is called a *clique partitioning* of G if there is a partition $\Gamma = \{V_1, \dots, V_k\}$ of V such that the subgraph induced by each V_i, $1 \leq i \leq k$, is a clique in G, and $A = \bigcup_{i=1}^{k} \{uv : u, v \in V_i, u \neq v\}$. In this case, if for $1 \leq i \leq k$ the clique induced by V_i is denoted by Q_i, then we say that $\mathcal{C}(A) := \{Q_1, \dots, Q_k\}$ is the *clique set defined by* A.

The *distance* $d(u,v)$ between two nodes u and v in a graph is the length of a shortest path between u and v, if it exists; otherwise $d(u,v) = \infty$.

We introduce now the basic concepts of polyhedral theory that we need in the sequel. If E is a finite set and X any set, then we denote by X^E the set of (column) vectors with $|E|$ components indexed by E and with entries in X; that is, if $E = \{e_1, \dots, e_m\}$ then $X^E = \left\{x \;:\; x^T = (x_{e_1}, \dots, x_{e_m}),\; x_{e_i} \in X, 1 \leq i \leq m\right\}$. Here it is always assumed that the elements of E are ordered and that the indexing corresponds to the given ordering.

If E is a finite set and $A \subseteq E$ then we denote by χ^A the *incidence vector* of A, $\chi^A = (\chi_e^A)_{e \in E}$, defined as $\chi_e^A = 1$ if $e \in A$, and $\chi_e^A = 0$ if $e \notin A$. If $A \subseteq E$ and $w \in X^E$, then we use the following notation :

$$w(A) := \sum_{e \in A} w_e .$$

If $S \subseteq \mathbf{R}^m$, we denote by $\operatorname{conv}(S)$ the *convex hull* of the elements of S, defined as the set of all vectors in $\mathbf{R}^m$ which are convex combinations of finitely many vectors of S.

A set $P \subseteq \mathbf{R}^m$ is called a *polyhedron* if P is the solution of a system of linear inequalities; that is,

$$P = \{x \in \mathbf{R}^m : Ax \leq b\},$$

for some (n,m)-matrix A and vector $b \in \mathbf{R}^n$. If P is bounded, then P is called a *polytope*.

Let $P \subseteq \mathbf{R}^m$ be a polyhedron. An inequality $a^T x \leq a_0$ is called *valid* for P, if $P \subseteq \left\{x \in \mathbf{R}^m \;:\; a^T x \leq a_0\right\}$. A set $F \subseteq P$ is called a *face* of P, if there exists an inequality $a^T x \leq a_0$ which is valid for P such that

$$F = P \cap \left\{x \;:\; a^T x = a_0\right\}.$$

A face F of P is called *proper* if $F \neq P$, and *nontrivial* if $\emptyset \neq F \neq P$. A nontrivial face of P is called a *facet* if F is not contained in any other proper face of P. A face of P consisting of exactly one point is called a *vertex*. Two vertices x^A and x^B of P are called *adjacent* (with respect to P) if there is a 1–dimensional face of P, called an edge, containing both of them. Other equivalent formulations of the adjacency concept are presented in the next theorem (cf. Murty [**5**, 1971]).

Theorem 2.1. *Let $P \subset \mathbf{R}^m$ be a polytope and Q be the set of its vertices. Let x^A and x^B be distinct vertices of P. Then the following statements are equivalent :*

(i) *x^A and x^B are adjacent (with respect to P).*
(ii) *No point y on the line segment connecting x^A and x^B, $y \neq x^A$, $y \neq x^B$, can be represented as a convex combination of vertices in $Q \setminus \{x^A, x^B\}$.*
(iii) *Any point y on the line segment connecting x^A and x^B, $y \neq x^A$, $y \neq x^B$, can be represented uniquely as a convex combination of vertices in Q.*
(iv) *There is a vector $c \in \mathbf{R}^m$ such that x^A and x^B are the only vertices of P which maximize the function $c^T x$ over P.*

We denote by $A \triangle B$ the *symmetric difference* of two sets A and B, i.e.,

$$A \triangle B := \{x \;:\; x \in A \setminus B \text{ or } x \in B \setminus A\}.$$

Throughout this paper we assume that $\mathcal{P}_n$ denotes the *clique partitioning polytope* (associated with $K_n = [V_n, E_n]$), defined as

$$\mathcal{P}_n := \operatorname{conv}\left\{\chi^A \;:\; A \text{ clique partitioning of } K_n\right\}.$$

3. Characterization of Adjacency of Vertices on $\mathcal{P}_n$

Theorem 3.1. *Let $\mathcal{P}_n$ be the clique partitioning polytope associated with the complete graph $K_n = [V_n, E_n]$, $n \geq 2$, and let A and B be distinct clique partitionings of K_n. Then the vertices χ^A and χ^B of $\mathcal{P}_n$ are adjacent if and only if the following condition holds:*

$$\begin{array}{ll} & \textit{The graph induced by } A\triangle B \textit{ is connected, and} \\ (3.2) & \textit{if } A \subset B\,, \textit{ then } \mathcal{C}(B) \textit{ has no clique containing} \\ & \textit{more than 2 cliques of } \mathcal{C}(A)\,. \end{array}$$

Proof: Let $H = [A \triangle B]$ be the graph induced by $A \triangle B$. Assume that A and B define the clique sets

$$\mathcal{C}(A) := \{A_1, A_2, \dots, A_r\}, \quad r \geq 1, \quad \text{and}$$
$$\mathcal{C}(B) := \{B_1, B_2, \dots, B_s\}, \quad s \geq 1,$$

respectively. Let VA_i (resp. VB_j) be the node set and EA_i (resp. EB_j) be the edge set of the clique A_i (resp. B_j), $1 \leq i \leq r$ (resp. $1 \leq j \leq s$).

(a) We prove first the necessity of condition (3.2). For that, let us assume that χ^A and χ^B are adjacent and $H = [A \triangle B]$ is not connected. Then H has at least two components, say $H_1 = [V_1, E_1]$ and $H_2 = [V_2, E_2]$. For $i = 1, 2$ let $H_i^* = [V_i, E_i^*]$ be the subgraph of $[A \cup B]$ induced by the nodes in V_i, and set

$$A' := (A \setminus \{EA_i \;:\; EA_i \subseteq E_1^*,\; 1 \leq i \leq r\}) \cup \{EB_j \;:\; EB_j \subseteq E_1^*,\; 1 \leq j \leq s\}$$

and

$$B' := (B \setminus \{EB_j \;:\; EB_j \subseteq E_1^*,\; 1 \leq j \leq s\}) \cup \{EA_i \;:\; EA_i \subseteq E_1^*,\; 1 \leq i \leq r\}.$$

Note that A' (resp. B') is a clique partitioning of K_n with $\mathcal{C}(A')$ (resp. $\mathcal{C}(B')$) consisting of all cliques from $\mathcal{C}(A)$ (resp. $\mathcal{C}(B')$) not contained in H_1^* and all cliques from $\mathcal{C}(B)$ (resp.

$\mathcal{C}(A'))$ contained in H_1^*. Hence $A' \neq A$, $A' \neq B$, $B' \neq B$ and $B' \neq A$. Furthermore, $A' \neq B'$,

$$A \cup B = A' \cup B' \quad \text{and} \quad A \cap B = A' \cap B'.$$

Thus

$$\frac{1}{2}\chi^A + \frac{1}{2}\chi^B = \frac{1}{2}\chi^{A'} + \frac{1}{2}\chi^{B'},$$

which implies that χ^A and χ^B are not adjacenct, contradicting our assumption.

Let us assume now that H is connected and $A \subset B$. Assume also that B_1 is the clique in $\mathcal{C}(B)$ which contains at least 2 cliques of $\mathcal{C}(A)$, say $A_1, A_2, \dots, A_k$ with $k \geq 2$. Note that this clique is unique since $[A \triangle B]$ is connected.

Suppose by contradiction that $k \geq 3$. Since χ^A and χ^B are adjacent there is a vector $c = (c_e : e \in E_n)$ such that χ^A and χ^B are the only two vertices of $\mathcal{P}_n$ which maximize $c^T x$ over $\mathcal{P}_n$. For all pairs i, j with $1 \leq i < j \leq k$, let

$$A_{ij} := A \cup (VA_i : VA_j).$$

Then A_{ij} is a clique partitioning of K_n with $\mathcal{C}(A_{ij})$ consisting of all cliques from $\mathcal{C}(A)$ different from A_i and A_j and a new clique obtained by combining A_i and A_j into a unique clique.

Clearly $A_{ij} \neq A$ and furthermore, since $k \geq 3$, $A_{ij} \neq B$. Thus, $c^T\chi^{A_{ij}} < c^T\chi^A$ and therefore $c(VA_i : VA_j) < 0$ for all pairs i, j with $1 \leq i < j \leq k$.

On the other hand, since $B \backslash A = \{e \in (VA_i : VA_j) : 1 \leq i < j \leq k\}$, then $c(B \backslash A) < 0$ and hence $c(B) = c(A) + c(B \setminus A) < c(A)$, i.e., $c^T\chi^B < c^T\chi^A$, a contradiction. Thus $k = 2$, and this completes the proof of part (a).

(b) To prove the sufficiency of condition (3.2) we first introduce the terminology to be used. For $A_i \in \mathcal{C}(A)$ and $B_j \in \mathcal{C}(B)$ the set $\mathcal{I} := VA_i \cap VB_j \neq \emptyset$ is called a *binding intersection set* if $\mathcal{I} \neq VA_i$ and $\mathcal{I} \neq VB_j$. In this case, we say that A_i and B_j are *binding cliques* and that (A_i, B_j) is a *pair of binding cliques* (see Figure 3.1).

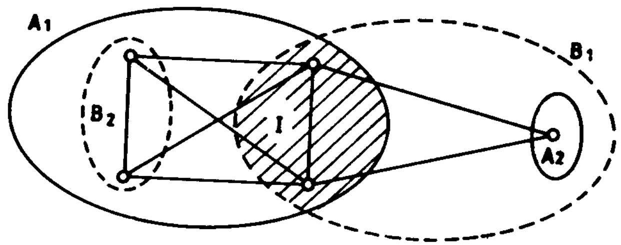

Figure 3.1.

Suppose now that condition (3.2) holds and χ^A, χ^B are not adjacent with respect to $\mathcal{P}_n$. Thus, by Theorem (2.1), there is a point on the line segment between χ^A and χ^B

which has another different convex representation, i.e., there exist clique partitionings $D_1, \dots, D_l$ and scalars λ's such that

$$(1) \qquad \begin{cases} \lambda_A \chi^A + \lambda_B \chi^B = \sum_{i=1}^{l} \lambda_i \chi^{D_i}, \text{ and} \\ \lambda_A + \lambda_B = 1, \quad \lambda_A, \lambda_B > 0, \\ \sum_{i=1}^{l} \lambda_i = 1, \quad \lambda_i \geq 0, \\ D_i \notin \{A, B\}, \quad i = 1, \dots, l. \end{cases}$$

For simplicity we set $L := \{1, 2, \dots, l\}$ and assume without loss of generality that $\lambda_i > 0$ for all $i \in L$.

By (1) it follows that if $e \in A \cap B$ then

$$1 = \sum_{i \in L} \lambda_i \chi_e^{D_i},$$

and if $e \in E_n \setminus A \cup B$ then

$$0 = \sum_{i \in L} \lambda_i \chi_e^{D_i}.$$

Since $\sum_{i=1}^{l} \lambda_i = 1$ and $\lambda_i > 0$ for all $i \in L$, it follows that

$$\chi_e^{D_i} = \begin{cases} 1, & \text{if } e \in A \cap B, \\ 0, & \text{if } e \in E_n \setminus A \cup B, \end{cases}$$

and thus

$$(2) \qquad A \cap B \subseteq D_i \subseteq A \cup B \quad \text{for all } i \in L.$$

Our aim is to prove that there exists $k \in L$ such that $D_k = A$ or $D_k = B$, obtaining this way a contradiction to (1).

Case 1. $A \subset B$ or $B \subset A$. Assume without loss of generality that $A \subset B$ and let $e \in B \setminus A$. Then by our assumptions in (1) there must exist $k \in L$ such that $\chi_e^{D_k} = 1$. We may assume that $e \in B_1$. In this case, B_1 must contain precisely two cliques of $\mathcal{C}(A)$, say A_1 and A_2, and therefore all edges in $EB_1 \setminus A$ are incident to a node in $VA_1 \cup VA_2$. Since $EA_1 \cup EA_2 \subseteq A \cap B \subseteq D_k$ and $e \in D_k$ then $EB_1 \subseteq D_k$. Now using the fact that $\mathcal{C}(B) \setminus \{B_1\} = \mathcal{C}(A) \setminus \{A_1, A_2\}$ and (2) holds we obtain that $D_k = B$.

Case 2. $A \not\subset B$ and $B \not\subset A$. In this case we establish first two claims.

Claim 1. Let $k \in L$, $1 \leq i \leq r$ and $1 \leq j \leq s$. If (A_i, B_j) is a pair of binding cliques and there exists $e \in EA_i \setminus B$, e incident to $\mathcal{I} := VA_i \cap VB_j$ such that $e \in D_k$, then $A_i \in \mathcal{C}(D_k)$.

[Proof of Claim 1]. Assume that $e := uv$ with $u \in \mathcal{I}$. It is immediate that no edge in $E_n \setminus A_i$ incident to e is in D_k, otherwise we would have a contradiction to the fact that $D_k \subseteq A \cup B$. Thus if $EA_i = \{e\}$, then clearly $A_i \in \mathcal{C}(D_k)$. Now assume that there exists an edge $f := uy$ in $EA_i \setminus B$, $f \notin D_k$. Then there must exist an index $p \in L \setminus \{k\}$ such that $f \in D_p$, otherwise we would get a contradiction to our assumptions in (1). Let

$$\begin{aligned} L_1 &:= \{p \in L \,:\, e \in D_p \,,\, f \notin D_p\} \,, \\ L_2 &:= \{p \in L \,:\, f \in D_p\} \quad \text{and} \\ L_3 &:= \{p \in L \,:\, e \notin D_p \,,\, f \notin D_p\} \,. \end{aligned} \tag{3}$$

Clearly, these sets are all pairwise disjoint and $L = L_1 \cup L_2 \cup L_3$. Thus

$$1 = \sum_{p \in L} \lambda_p = \sum_{p \in L_1} \lambda_p + \sum_{p \in L_2} \lambda_p + \sum_{p \in L_3} \lambda_p \,. \tag{4}$$

Since $e, f \in A \setminus B$, then by (1) and (3) it follows that

$$\sum_{p \in L_1} \lambda_p \leq \sum_{p \in L_2} \lambda_p = \lambda_A \,. \tag{5}$$

On the other hand, since there exists an edge $h := uw$, $h \in EB_j \setminus A$, then $\chi_h^{D_p} = 0$ for all $p \in L_1 \cup L_2$ and therefore

$$\sum_{p \in L_3} \lambda_p \geq \lambda_B \,. \tag{6}$$

Now combining (4), (5), (6) and the fact that $\lambda_A + \lambda_B = 1$, we get $\sum_{p \in L_1} \lambda_p = 0$, a contradiction.

Thus we have proved that all edges in $EA_i \setminus B$ incident to u are also in D_k. Now using the fact that D_k is a clique partitioning and $A \cap B \subseteq D_k \subseteq A \cup B$ we obtain immediately that $A_i \in \mathcal{C}(D_k)$ and complete this way the proof of Claim 1.

By making use of Claim 1 we can now prove the following

Claim 2. Let $k \in L$ and i, j be such that $1 \leq i \leq r$ and $1 \leq j \leq s$. If (A_i, B_j) is a pair of binding cliques then $A_i \in \mathcal{C}(D_k)$ if and only if $B_j \notin \mathcal{C}(D_k)$.

[Proof of Claim 2] If $A_i \in \mathcal{C}(D_k)$, then it is immediate that $B_j \notin \mathcal{C}(D_k)$. Now assume that $B_j \notin \mathcal{C}(D_k)$. To prove that $A_i \in \mathcal{C}(D_k)$ we assume the contrary and set

$$\begin{aligned} L_A &:= \{p \in L \,:\, A_i \in \mathcal{C}(D_p)\} \,, \quad \text{and} \\ L_B &:= \{p \in L \,:\, B_j \in \mathcal{C}(D_p)\} \,. \end{aligned}$$

So, by our assumption $k \in L \setminus L_A \cup L_B$, and by the previous result $L_A \cap L_B = \emptyset$. Observing that there is an edge $e \in EA_i \setminus B$, and an edge $f \in EB_j \setminus A$, both incident to $\mathcal{I} := VA_i \cap VB_j$, using Claim 1 and (1) we obtain

$$\sum_{p \in L_A} \lambda_p = \lambda_A \quad \text{and} \quad \sum_{p \in L_B} \lambda_p = \lambda_B \,.$$

Since $\lambda_k > 0$, then $1 = \sum_{p \in L} \lambda_p > \sum_{p \in L_A} \lambda_p + \sum_{p \in L_B} \lambda_p = \lambda_A + \lambda_B = 1$, a contradiction. This proves that $A_i \in \mathcal{C}(D_k)$ and concludes the proof of Claim 2.

Finally, using the two claims we can complete the proof of Case 2. For that, consider a pair (A_i, B_j) of binding cliques and let e be an edge such that $e \in A \setminus B$ and e is incident to $VA_i \cap VB_j$. By (1), there exists an index $k \in L$ such that $e \in D_k$, and therefore by Claim 1, $A_i \in \mathcal{C}(D_k)$. Now using Claim 2 and the hypothesis that $[A \triangle B]$ is connected, it follows that all binding cliques in $\mathcal{C}(A)$ are in $\mathcal{C}(D_k)$. Recalling that $A \cap B \subseteq D_k \subseteq A \cup B$ we conclude that $D_k = A$. Thus, the sufficiency of (3.2) is settled and the proof of the theorem is complete. ∎

An immediate consequence of Theorem (3.1) is the following

Corollary 3.3. *Adjacency of vertices on the polytope $\mathcal{P}_n$ can be tested by an algorithm of time complexity $\mathcal{O}(n^2)$.*

4. The Diameter of $\mathcal{P}_n$

The adjacency structure of a polytope P can be better understood by studying the properties of the so-called *adjacency graph* or *skeleton* of P, $G(P)$, defined as follows: $G(P)$ has a node for each vertex of P, and two nodes of it are joined by an edge if and only if the corresponding vertices are adjacent (with respect to P). One interesting information about $G(P)$ which gives good insight into the adjacency structure of the polytope P concerns the *diameter* of $G(P)$ — also called the *diameter* of P and denoted by $\operatorname{diam}(P)$. It is defined as the maximum length of a shortest path between any pair of nodes of $G(P)$. Thus if d is the diameter of a polytope P, this means that starting from any vertex one can reach any other by "walking" along at most d edges of P. Note that an edge of P is a face of dimension 1 defined by $\operatorname{conv}\left(\{x^1, x^2\}\right)$, where x^1, x^2 are adjacent vertices of P.

To investigate the diameter of the polytope $\mathcal{P}_n$, we assume for ease of notation that if A and B are clique partitionings of K_n, then $d(A, B)$ denotes the distance between the vertices χ^A and and χ^B in $G(\mathcal{P}_n)$.

Theorem 4.1. *Let $\mathcal{P}_n$ be the clique partitioning polytope. Then $\operatorname{diam}(\mathcal{P}_2) = 1$, $\operatorname{diam}(\mathcal{P}_3) = 2$ and $\operatorname{diam}(\mathcal{P}_n) = 3$ for all $n \geq 4$.*

Proof: It is immediate that $\operatorname{diam}(\mathcal{P}_2) = 1$ and $\operatorname{diam}(\mathcal{P}_3) = 2$. So assume that $n \geq 4$ and let A and B be distinct clique partitionings of K_n. We shall prove that $d(A, B) \leq 3$, and that equality holds in some cases.

Case 1. $A = \emptyset$.

(1.1) Suppose $\mathcal{C}(B)$ has only one nondegenerate clique, say B_1.

a) Assume first that $B_1 \not\cong K_n$. By Theorem (3.1), if $B_1 \cong K_2$ then $d(A, B) = 1$, otherwise $d(A, B) \geq 2$. In the latter case, take $u \in VB_1$ and $v \notin VB_1$ and define a clique partitioning D consisting only of the edge uv. Then by Theorem (3.1), $d(A, D) = 1 = d(D, B)$, and therefore $d(A, B) = 2$.

b) Assume that $B_1 \cong K_n$. Let $\mathcal{C}_1$ be the set of all clique partitionings of K_n consisting of one edge, and let $\mathcal{C}_2$ be the set of all clique partitionings B' such that $|\mathcal{C}(B')| = 2$. Thus, by Theorem (3.1), $d(A, A') = 1$ if and only if $A' \in \mathcal{C}_1$ and $d(B, B') = 1$ if and only if $B' \in \mathcal{C}_2$. Since $n \geq 4$, then $\mathcal{C}_1 \cap \mathcal{C}_2 = \emptyset$ and therefore $d(A, B) \geq 3$. Now in order to prove that $d(A, B) = 3$ note that there exist $A' \in \mathcal{C}_1$ and $B' \in \mathcal{C}_2$ such that

$d(A', B') = 1$. In fact, if we take any two nodes u, v in K_n and set $A' := \{uv\}$ and $B' := E_n \setminus \{e \in E_n \ : \ e \text{ incident to } u\}$, then, by Theorem (3.1), $d(A', B') = 1$.

(1.2) Suppose $\mathcal{C}(B)$ has more than one nondegenerate clique. In this case, $d(A, B) \geq 2$. If $\mathcal{C}(B)$ has exactly 2 nondegenerate cliques, say B_1 and B_2, then take $u \in VB_1$ and $v \in VB_2$ and define the clique partitioning $D := \{uv\}$. Clearly, $d(A, D) = 1 = d(B, D)$ and therefore $d(A, B) = 2$. Now suppose $\mathcal{C}(B)$ has more than 2 nondegenerate cliques. Let $\mathcal{C}_1$ be as defined in the case 1.1.b. Since for every $A' \in \mathcal{C}_1$ the graph $[A' \Delta B]$ is not connected, then $d(A', B) \geq 2$ and therefore $d(A, B) \geq 3$. Now let u be a node in a nondegenerate clique of $\mathcal{C}(B)$ and set $D := E_n \setminus \{e \in E_n \ : \ e \text{ incident to } u\}$. Clearly, $d(B, D) = 1$. On the other hand, by the case 1.1.a already analysed, $d(A, D) = 2$. Thus $d(A, B) = 3$.

Case 2. $A \neq \emptyset$ and $B \neq \emptyset$.

(2.1) $A \subset B$. Let A_1 be a nondegenerate clique of $\mathcal{C}(A)$. Take a node $u \in VA_1$ and set $D := E_n \setminus \{e \in E_n \ : \ e \text{ incident to } u\}$. Then $d(A, D) = 1 = d(B, D)$ and therefore $d(A, B) \leq 2$.

(2.2) $A \not\subset B$ and $B \not\subset A$. If $[A \Delta B]$ is connected then $d(A, B) = 1$. So let us assume that $[A \Delta B]$ is not connected. Then $d(A, B) \geq 2$. Suppose $\mathcal{C}(A) = \{A_1, \ldots, A_r\}$ and $\mathcal{C}(B) = \{B_1, \ldots, B_s\}$. If there exists a node u with the property that there are nodes v, w such that $uv \in A_1$ and $uw \in B_j$ (eventually with $v = w$), then set $D := E_n \setminus \{e \in E_n \ : \ e \text{ incident to } u\}$. Clearly, $D \neq A$, $D \neq B$ and furthermore, $d(A, D) = d(B, D) = 1$. Thus, $d(A, B) = 2$. In case such a node u does not exist, then every nondegenerate clique of $\mathcal{C}(A)$ (resp. $\mathcal{C}(B)$) contains only degenerate cliques of $\mathcal{C}(B)$ (resp. $\mathcal{C}(A)$). Let $\mathcal{C}'$ be the set of those cliques in $\mathcal{C}(A) \cup \mathcal{C}(B)$ with this property. Then by our assumptions, $\mathcal{C}'$ must contain at least a clique of $\mathcal{C}(A)$ and a clique of $\mathcal{C}(B)$, and furthermore all cliques in $\mathcal{C}'$ are pairwise disjoint. Now choose a node in each clique of $\mathcal{C}'$, call W the set of these nodes and set $D := \{e \in E_n \ : \ e \text{ has both ends in } W\}$. Since $[A \Delta D]$ and $[B \Delta D]$ are connected, $A \not\subset D$, $D \not\subset A$ and $B \not\subset D$, $D \not\subset B$, then $d(A, D) = 1 = d(B, D)$ and thus $d(A, B) = 2$. ∎

An analysis of the proof of Theorem (4.1) gives immediately the following two results:

Corollary 4.2. *Let $\mathcal{P}_n$ be the clique partitioning polytope and let A and B be distinct clique partitionings of K_n, $n \geq 4$. Then $d(A, B) \leq 3$ and equality holds if and only if $A = \emptyset$ and either $B \cong K_n$ or $\mathcal{C}(B)$ has more than 2 nondegenerate cliques.*

Corollary 4.3. *Let K_n be the complete graph on n nodes, $n \geq 4$, and let*

$$\mathcal{P}'_n := \operatorname{conv}\{\chi^A \ : \ A \neq \emptyset, A \text{ clique partitioning of } K_n\}.$$

Then $\operatorname{diam}(\mathcal{P}'_n) = 2$.

Remark 4.4. A polyhedron P with dimension m and k facets has the ***Hirsch property*** if $\operatorname{diam}(P) \leq k - m$. Using the results presented in our former paper (cf. Grötschel and Wakabayashi [**2**, 1987a]) on the facets of $\mathcal{P}_n$, we can conclude that the polytope $\mathcal{P}_n$ has the Hirsch property for all $n \geq 2$. This fact is however, implied by a more general result proved recently (personal communication) by D. Naddef (Grenoble, France) which states that ***all polytopes with 0/1-vertices have the Hirsch property.***

5. Concluding Remarks

The fact that the diameter of $\mathcal{P}_n$ is very small is not so surprising, since examples of polytopes associated with $\mathcal{NP}$-hard problems which have small diameter are well-known in the literature. This is what occurs for instance with the travelling salesman polytope and the linear ordering polytope which are known to have diameter 2 (Padberg and Rao [**6**, 1974] and Young [**8**, 1978]). We should remark, however, that whereas in some cases the (non-)adjacency test of two vertices can be easily performed, as on the polytope $\mathcal{P}_n$ or the linear ordering polytope, it might happen — as in the case of the travelling salesman polytope — that deciding whether two vertices are non-adjacent is an $\mathcal{NP}$-complete problem. (Papadimitriou [**7**, 1978]).

Note Added in Proof. The main results of this paper — Theorem (3.1) and Theorem (4.1) — were obtained first by Simon Régnier and published in technical reports of "Centre de Calcul de la Maison des Sciences de l'Homme", Paris, in the period from 1971 to 1975. A reprint of Régnier's papers can be found in "Mathématiques et Sciences Humaines, 21^e année, No. 82, 1983 — a special issue dedicated to his contributions to mathematical taxonomy. This work was done independently and uses an entirely different proof technique.

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