DECOMPOSITIONS OF GRAPHS

Abstract. The topic of research comprises problems on graph decompositions, in a very broad sense. We mention some problems on this topic that we are investigating or would like to investigate.

Given a graph G, a family of subgraphs $\{H_1, \ldots, H_k\}$ of G is a **decomposition** of G if $E(H_i) \cap E(H_j) = \emptyset$ for every $i \neq j$, and $\bigcup_{i=1}^{k} E(H_i) = E(G)$. Given a family $\mathcal{H}$ of graphs, an **$\mathcal{H}$-decomposition** of G is a decomposition $\mathcal{D}$ of G such that each element of $\mathcal{D}$ is isomorphic to an element of $\mathcal{H}$. If there exists an $\mathcal{H}$-decomposition of G, we say that G **admits** an $\mathcal{H}$-decomposition. If $\mathcal{H} = \{H\}$ we say that an $\mathcal{H}$-decomposition is an H-decomposition.

We say that a path decomposition $\mathcal{D}$ of a graph G is **minimum** if for every path decomposition $\mathcal{D}'$ of G we have $|\mathcal{D}'| \geq |\mathcal{D}|$. The **path number** of a graph G, denoted by $\mathbf{pn(G)}$, is the size of a minimum path decomposition. According to Lovász [5], in 1966 Erdős asked about this parameter, and Gallai stated the following conjecture.

Conjecture [Gallai, 1966] *If G is an n-vertex connected graph, then* $\text{pn}(G) \leq (n+1)/2$.

This parameter is not known for most of the graph classes. Lovász (1968) found an upper bound for a similar parameter. He proved that *every n-vertex connected graph G can be decomposed into at most $n/2$ paths and cycles.* From this result, it follows that Gallai's conjecture holds on connected graphs with at most one even-degree vertex. Some upper bounds for the path number of a connected graph in terms of its number of odd-degree and even-degree vertices have been obtained by Lovász (1968), and improved by Donald (1980) and Dean and Kouider (2000). Finding good upper bounds for $\text{pn}(G)$ is a challenging problem; even for special classes of graphs not much is known.

Another related problem concerns H-decompositions of a graph, where H is some fixed graph. This is also a topic that has raised much interest of the researchers, and a number of interesting results have been found. An interesting variant concerns the case in which H is a tree. In this respect, there is a conjecture posed by Barát and Thomassen [1], which states that, *for each tree T, there exists a natural number k_T such that, if G is a k_T-edge-connected graph and $|E(T)|$ divides $|E(G)|$, then G admits a T-decomposition.* In a series of papers, Thomassen has proved this conjecture for stars, some bistars, paths of length 3, and paths whose length is a power of 2. Recently, Botler, Mota, Oshiro and Wakabayashi [2] have verified this conjecture for paths of length 5.

The topic is very broad, so an interested student may choose one of the variants to focus on, or may write a survey collecting the known results. Some PhD students and post-doc fellows of our Combinatorics group at IME-USP have been working (or have worked) on this topic. Some other results on decompositions of regular graphs into paths [3, 4] or bistars have also been proved by members of this group. We would be happy to welcome students interested to work with us.

References

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5. László Lovász, *On covering of graphs*, Theory of Graphs (Proc. Colloq., Tihany, 1966), Academic Press, New York, 1968, pp. 231–236. MR 0233723 (38 #2044)