# An Historical Note: Euler's Königsberg Letters

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## ABSTRACT

In this paper we discuss three little known letters on the Königsberg bridges problem. These letters indicate more clearly Euler's attitude to the problem and to his solution of it.

## 1. INTRODUCTION

On August 26, 1735, Leonhard Euler presented a paper on the Königsberg bridges problem to the Academy of Sciences in St. Petersburg (now Leningrad). In the following year he wrote up his solution in his celebrated paper *Solutio problematis ad geometriam situs pertinentis* (The solution of a problem relating to the geometry of position) [2]. In this paper Euler formulated necessary and sufficient conditions under which, given any arrangement of islands and bridges, one can find a connected trail that crosses each bridge exactly once. Euler discussed, but did not prove, the sufficiency of his conditions; a valid proof of sufficiency was not published until 1873, by Carl Hierholzer [4]. For further information about the history of the Königsberg bridges problem, see [1] or [8].

We have recently tried to find out how Euler became aware of the Königsberg problem and why it intrigued him. Although we have been unable to discover the full story, we have found some letters from Euler's correspondence that shed some light on his involvement with it.

A persistent theme running through the correspondence is Euler's preoccupation with the geometry of position. In 1679 Leibniz [5, pp. 18–19] had declared

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I am not content with algebra, in that it yields neither the shortest proofs nor the most beautiful constructions of geometry. In view of this, I consider that we need yet another kind of analysis, geometric or linear, which deals directly with position, as algebra deals with magnitude . . .

Although Leibniz was probably anticipating vector analysis when he wrote this, it was widely interpreted as referring to topics we now consider "topological". In particular, Euler and others regarded the Königsberg problem as a problem in *geometria situs*. Further information on the various interpretations that have been ascribed to this term can be found in [6] or [7].

## 2. EHLER'S LETTER TO EULER

Carl Leonhard Gottlieb Ehler was mayor of Danzig, a friend of Euler, and a lover of mathematics. From 1735 to 1742 he corresponded with Euler in St. Petersburg [3, pp. 282–387], acting as an intermediary between Euler and Heinrich Kühn (1690–1769), professor of mathematics at the academic gymnasium in Danzig. Via Ehler, Kühn communicated with Euler about the Königsberg problem. In a letter dated March 9, 1736, Ehler wrote to Euler (see Figs. 1 and 2):

You would render to me and our friend Kühn a most valuable service, putting us greatly in your debt, most learned Sir, if you would send us the solution, which you know well, to the problem of the seven Königsberg bridges, together with a proof. It would prove to be an outstanding example of the calculus of position [Calculi Situs], worthy of your great genius. I have added a sketch of the said bridges . . . .

It emerges from this letter that Ehler and Euler had already exchanged ideas on the Königsberg problem, but we have been unable to locate any earlier references.

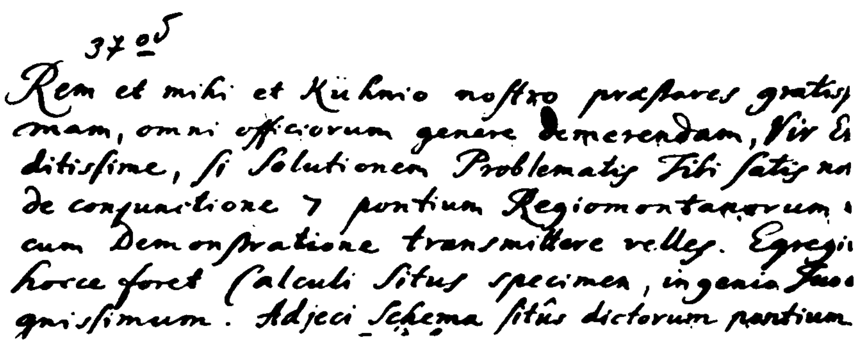
Rem et mihi et Kühnio nostro praestares gratis-
simam, omni officiorum genere demerendam, Vir Eru-
ditissime, si Solutionem Problematis Tibi satis noti
de conjunctione 7 pontium Regiomontanorum,
cum Demonstratione transmittere velles. Egregii
hocce foret Calculi Situs specimen, ingenio Tuo
dignissimum. Adjeci Schema situs dictorum pontium

FIGURE 1

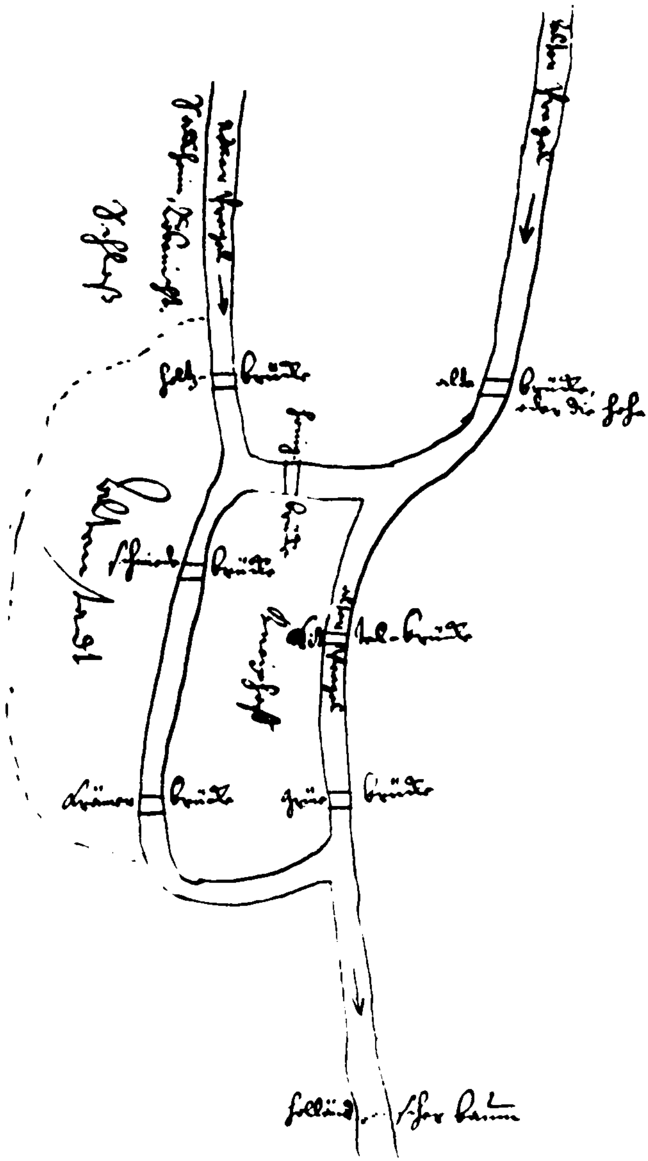

FIGURE 2

## 3. EULER'S LETTER TO MARINONI

Giovanni Jacobo Marinoni (1670–1755) was an Italian mathematician and engineer who lived in Vienna from 1730 and received from Kaiser Leopold I the title of Court Astronomer. On March 13, 1736, Euler wrote to Marinoni, describing his solution, but demonstrating (as in his paper) only the necessity of the conditions and not their sufficiency. He introduced his ideas on the problem as follows (see Fig. 3):

A problem was proposed to me about an island in the city of Königsberg, surrounded by a river spanned by seven bridges, and I was asked whether someone could traverse the separate bridges in a connected walk in such a way that each bridge is crossed only once. I was informed that hitherto no-one had demonstrated the possibility of doing this, or shown that it is impossible. This question is so banal, but seemed to me worthy of attention in that neither geometry, nor algebra, nor even the art of counting [ars combinatoria] was sufficient to solve it. In view of this, it occurred to me to wonder whether it belonged to the geometry of position [geometria situs], which Leibniz had once so much longed for. And so, after some deliberation, I obtained a simple, yet completely established, rule with whose help one can immediately decide for all examples of this kind, with any number of bridges in any arrangement, whether such a round trip is possible, or not . . . .

## 4. EULER'S LETTER TO EHLER

On April 3, 1736, Euler replied to Ehler's letter of March 9. The following extract is interesting in that it reveals much more clearly his attitude toward the Königsberg problem (see Fig. 4):

Thus you see, most noble Sir, how this type of solution bears little relationship to mathematics, and I do not understand why you expect a mathematician to produce it, rather than anyone else, for the solution is based on reason alone, and its discovery does not depend on any mathematical principle. Because of this, I do not know why even questions which bear so little relationship to mathematics are solved more quickly by mathematicians than by others. In the meantime, most noble Sir, you have assigned this question to the geometry of position, but I am ignorant as to what this new discipline involves, and as to which types of problem Leibniz and Wolff expected to see expressed in this way . . . .

## 5. EULER'S 1736 PAPER

In spite of the above remarks, Euler considered the problem important enough to write a paper on it. This was his celebrated 1736 paper, mentioned in the introduction. In this work he explicitly ascribed the Königsberg problem to the geometry of position, as follows:

1. In addition to that branch of geometry which is concerned with distances, and which has always received the greatest attention, there is another branch, hitherto almost unknown, which Leibniz first mentioned, calling it the *geometry of position*. This branch is concerned only with the determination of position and its properties; it does not involve distances, nor calculations made with them. It has not yet been satisfactorily determined what kinds of problem are relevant to this geometry of position, or what methods should be used in solving them. Hence, when a problem was recently mentioned which seemed geometrical but was so constructed that it did not require the measurement of distances, nor did calculation help at all, I had no doubt that it was concerned

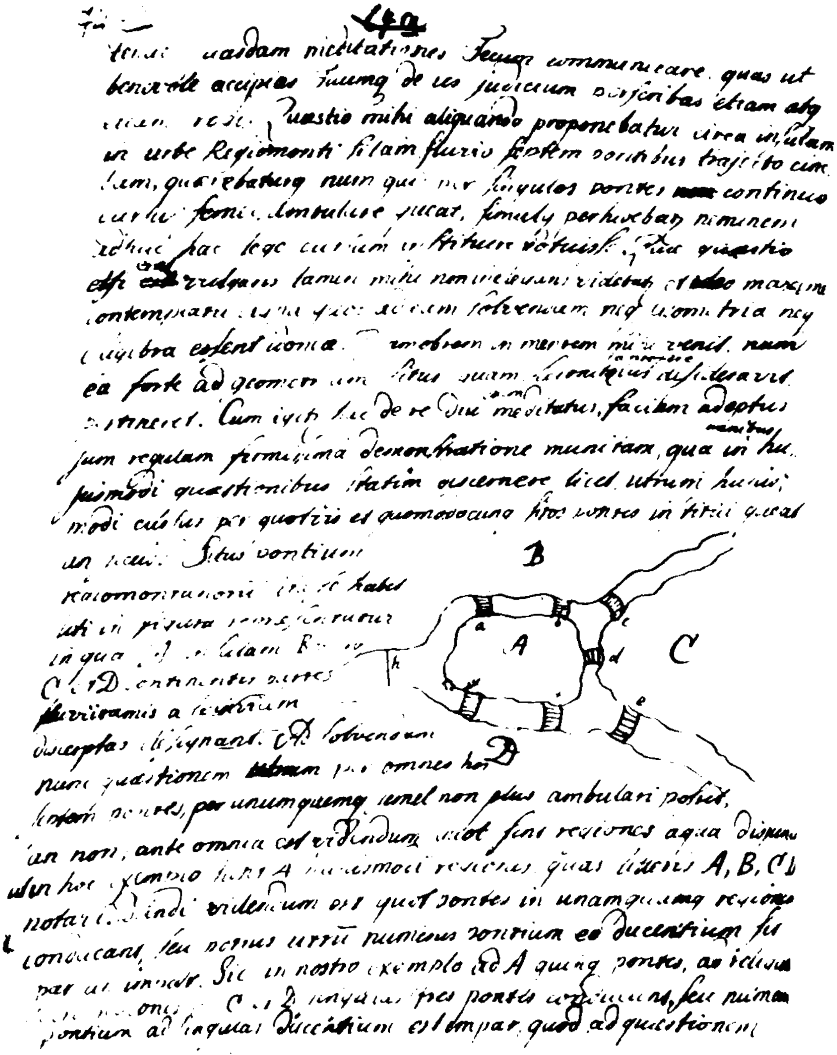


FIGURE 3

with the geometry of position — especially as its solution involved only position, and no calculation was of any use. I have therefore decided to give here the method which I have found for solving this kind of problem, as an example of the geometry of position. 2. The problem, which I am told is widely known, is as follows: in Königsberg . . . . *(A full English translation of this paper appears in [1].)*

It emerges from this quotation that by the time Euler wrote this paper he had become convinced that the geometry of position, in the sense in which he un-

Vides ergo Vir Amplissime solutionem hanc ita esse comparatam ut vix ad mathesin pertinere videatur. neque ego comprehendo cur ea potius a Mathematico sit expectanda quam a quovis alio homine, sola enim ratione nititur ista solutio nec ullis mathesi propriis principiis ad eam inveniendam opus fuit. Nescio igitur quomodo fit ut quaestiones etiam

ad mathesin minime spectantes citius a mathematicis solvantur quam ab aliis. Caeterum interim huic quaestioni concedis Vir Amplissime in Geometria Situs, de qua autem nova disciplina fateor me ignorare cujusmodi problemata ad eam referenda velint Leibnitius et Wolffius. Rogo igitur Te si me idoneum judicas in hac nova disciplina quicquam praestandi ut mihi aliquot definita problemata eo spectantia proponere velis, quo distinctius perspicere queam quid praecise desideretur.

FIGURE 4

derstood it, represented a significant mathematical discipline. Against this, it is worth noting that when, in 1750, he discovered his famous *polyhedral formula*

$$\text{(vertices)} + \text{(faces)} = \text{(edges)} + 2\,,$$

he seems not to have ascribed it to the geometry of position, even though later authors were to do so.

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