List of Problems

This list was assembled during the *Workshop on Combinatorics, Algorithms, and Applications* (`http://www.ime.usp.br/ yoshi/pronex/Workshop/`), held in Ubatuba, 1–5 September, 2003.

1. **(Cornuéjols)** A $0, \pm 1$ matrix is *totally unimodular* if all its square submatrices have determinant equal to 0, +1 or −1.

   A $0, \pm 1$ matrix A is *minimally non-totally unimodular* if it is not totally unimodular, but all its proper submatrices are. (So A must be square.)

   A $0, \pm 1$ matrix H is an *umbalanced hole matrix* if it is minimally non-totally unimodular and it has exactly two non-zero entries per row and per column.

   Prove or disprove that, for every minimally non-totally unimodular matriz A, there exists a totally unimodular matrix U such that UA is an unbalanced hole matrix.

2. **(Spinrad)** [Forbiden Subgraph Recognition] The goal is to show that, for every graph F which is not a subset of P_4, F-free induced subgraph recognition is as hard as triangle-free recognition.

   The rule is that you must find a reduction from triangle-free graph recognition on G to F-free recognition on G' which runs in $O(n^2)$ time, and such that G' has $O(n)$ vertices.

   This has been proved for many graphs F, e.g. all graphs on 5 vertices. The two open problems are:

   (a) Show that C_4-free recognition is as hard as triangle-free recognition. (You will also get credit if you have a C_4-free recognition algorithm which is faster than matrix multiplication.)

   (b) Show that, for every graph F with at least 6 vertices, F-free recognition is as hard as triangle-free recognition.

   All subgraphs referred to are induced subgraphs.

3. **(de Figueiredo, Meidanis and Mello)** [Edge-colouring of chordal graphs] A graph $G = (V, E)$ is *overful* if $|E| > \Delta \lfloor \frac{|V|}{2} \rfloor$, where Δ is the maximum vertex degree.

   A graph G is *subgraph overful* if it contains a sugraph H such that $\Delta(H) = \Delta(G)$ and H is overful.

   A graph G is *neighbourhood overful* if it contains a vertex v with degree $d(v) = \Delta$ such that $N[v]$ is overful. Clearly neighbourhood overful graphs are Class 2 (need $\Delta + 1$ colours to be edge-coloured).

   Conjecture 1 *For chordal graphs, Class 2 = neighbourhood overful.*

Conjecture 2 *Every chordal graph with Δ odd is Class 1.*

(Note that Conjecture 1 implies Conjecture 2.)

4. **(Reed)** Prove or disprove:

   If G is a cubic connected graph with n vertices then G has a dominating set D with $|D| \leq \lceil \frac{n}{3} \rceil$.

5. **(Konstadinidis)** Let n be a non-negative integer. A *proper colouring* of R^n is a colouring of each point of R^n in which no two points at distance one have the same color.

   The *chromatic number of* R^n is the number

   $$\tilde{\chi}(n) \;=\; \min\{k : \text{there is a proper colouring of } R^n \text{ with } k \text{ colors}\}.$$

   Clearly $\tilde{\chi}(n)$ is a non-decreasing function.

   Conjecture 3 $\tilde{\chi}(n)$ *is a scrictly increasing function.*

   In other words, if you increase the dimension by one, then you will need to use at least one more colour to get the job done.

6. **(Mandel)** Find a good characterization/algorithm for the following problem:

   Graph formulation: Given a bipartite graph $G = (A \cup B, E)$, when can one order the elements on each side, $A = \{a_1, a_2, \ldots, a_n\}$ and $B = \{b_1, b_2, \ldots, b_m\}$, so that all edges "go to the right": $(a_i, b_j) \in E$ implies $i \leq j$?

   Matrix formulation: Given a (binary) matrix A, do there exist permutation matrices P and Q such that PAQ is 0 below the main diagonal? ($PAQ_{ij} \neq 0$ implies $i \leq j$)

7. **(Duffus)** [Partition into union-free classes; H. Abbott and D. Hanson $\sim$ 1970] Let $f(n)$ be the minimum k such that there exists a partition $\mathcal{P}([n]) = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \cdots \cup \mathcal{C}_k$ ($\mathcal{P}([n])$ is the power set of $[n]$) where each $\mathcal{C}_i$ is (non-trivial) union-free, that is, if $A, B \in \mathcal{C}_i$, $A \cup B \notin \mathcal{C}_i$ (unless $A \subseteq B$ or $B \subseteq A$).

   Upper bound:

   (Abbott and Hanson): $f(n) \leq \lfloor \frac{n+1}{2} \rfloor$.

   Partition $\mathcal{P}([n])$ according to cardinality:

   - $\mathcal{C}_0$: 0-, 1-, 3-, 7-, ... element subsets;
   - $\mathcal{C}_1$: 2-, 5-, 11-, 23-, ... element subsets;
   - $\mathcal{C}_2$: 4-, 9-, 19-, 39-, ... element subsets;
   - ...

   Lower bounds:

   (Erdős and Shelah): $\frac{n}{4} \leq f(n)$.

Just by considering the $n^2/4$ sets of the form $\{i, i+1, \ldots, j-1, j\}$, where $i \leq n/2 < j$, they showed $n/4$ classes are needed.

(Aigner, Duffus, Kleitman $\sim$ 1990): $\frac{\ln 2}{2} n \approx .35n \leq f(n)$.

Problem: Improve the bounds on $f(n)$.

Conjecture 4 $f(n) \approx \frac{n}{2}$.

8. **(Lins)** Consider a rectangle of size $A \times B$ and another of size $a \times b$, $a \leq A$ and $b \leq B$, a, b, A, B positive numbers.

   What is the maximum number of $a \times b$-rectangles that can be packed into the $A \times B$-rectangle?

   Obs: Consider only "orthogonal" packings, that is, those where all the $a \times b$-rectangle have their sizes parallel to the ones of the big $A \times B$-rectangle.

   Can anyone prove or disprove that this problem is NP-hard?

9. **(Kohayakawa and Moreira)** Given integers a and b and a finite set F, an (a, b)*-regular family of subsets of* F is a family $\mathcal{C}$ such that $|X| = a$, for all $X \in \mathcal{C}$, and $deg(x) = \{X \in \mathcal{C} : x \in X\} = b$, for all $x \in F$.

   We define $\alpha(F, \mathcal{C}) = \min\{r \in N : \exists X_i \in \mathcal{C}, 1 \leq i \leq r, \text{ such that } \cup_{i=1}^{r} X_i = F\}$.

   Given a positive integer a, and an integer $n \geq a$, we define $\alpha(a, n) = \max\{\alpha(F, \mathcal{C}) : |F| = n$ and $\mathcal{C}$ is an (a, b)-regular family for some $b\}$, and $f(a) = \limsup_{n \to \infty} \frac{a}{n} \alpha(a, n)$.

   We know that $f(2) = 4/3$ and that, for any a,

$$\ln a - \ln \ln a \;\leq\; f(a) \;\leq\; H_a \;=\; \sum_{k=1}^{a} \frac{1}{k} \;=\; \ln a + \gamma + O(\frac{1}{a}).$$

   Problems:

   (a) Compute $f(3)$ and $f(4)$ (and ideally $f(a)$ for any a).

   (b) Decide whether $H_a - f(a)$ is bounded.

10. **(do Lago)** (LCS-query problem) Let A be an alphabet, that is, a set of *letters*. Any finite sequence $w = w_1 w_2 \cdots w_k$ of letters is called a *word*. A word u is called a *factor* of w if there are $0 < i \leq j \leq k$ such that $u = w_i w_{i+1} \cdots w_j$.

    The number $k = |w|$ is called the *length* of the word $w = w_1 w_2 \cdots w_k$. A word u is called a *subsequence* of w (or a subword of w) if there are $0 \leq l \leq k$ and $1 \leq i_1 < i_2 < \cdots < i_l \leq k$ such that $w_{i_1} w_{i_2} \cdots w_{i_l}$.

    Let LCS(u, v) be the length of a longest common subsequence of u and v, that is, the largest l such that there is a word w of length l and w is a subsequence of u and of v.

    The number LCS(u, v) can be computed in $O(n^2)$ for $n = \max(|u|, |v|)$ by standard dynamic programming.

    There are $O(n^2)$ factors of u.

    Problem: Given two words u and v of length up to n, we can preprocess them in such a way that queries to compute LCS(x, y), for x a factor of u and y a factor of v, can be answered in $O(1)$. The preprocessing can be done in $O(n^4)$ time. Can we do it in $O(n^2)$ time?