

On an Anti-Ramsey Property of Ramanujan Graphs

P.E. Haxell¹ and Y. Kohayakawa^{1,2}

¹Department of Pure Mathematics and Mathematical Statistics,
University of Cambridge, 16 Mill Lane, Cambridge CB2 1SB, England

²Instituto de Matemática e Estatística, Universidade de São Paulo,
Caixa Postal 20570, 01452-990 São Paulo, SP, Brazil

Abstract. If G and H are graphs, we write $G \rightarrow H$ (respectively, $G \rightarrow TH$) if for any proper edge-colouring γ of G there is a subgraph $H' \subset G$ of G isomorphic to H (respectively, isomorphic to a subdivision of H) such that γ is injective on $E(H')$. Let us write C^ℓ for the cycle of length ℓ . Spencer (cf. Erdős [10]) asked whether for any $g \geq 3$ there is a graph $G = G_g$ such that (i) G has girth $g(G)$ at least g , and (ii) $G \rightarrow TC^3$. Recently, Rödl and Tuza [22] answered this question in the affirmative by proving, using non-constructive methods, a result that implies that for any $t \geq 1$ there is a graph $G = G_t$ of girth $t + 2$ such that $G \rightarrow C^{2t+2}$. In particular, condition (ii) may be strengthened to (iii) $G \rightarrow C^\ell$ for some $\ell = \ell(G)$. For $G = G_t$ above $\ell = \ell(G) = 2t + 2 = 2g(G) - 2$. Here, we show that suitable Ramanujan graphs constructed by Lubotzky, Phillips, and Sarnak [18] are explicit examples of graphs $G = G_g$ satisfying (i) and (iii) above. For such graphs $\ell = \ell(G)$ in (iii) may be taken to be roughly equal to $(3/2)g(G)$, thus considerably improving the value $2g(G) - 2$ given in the result of Rödl and Tuza. It is not known whether there are graphs G of arbitrarily large girth for which (iii) holds with $\ell = \ell(G) = g(G)$.

1. Introduction

In this note we shall consider proper edge-colourings of graphs, and we shall be concerned with the existence of subgraphs whose edges are all coloured differently. To be more precise, let a graph G be given, and suppose $\gamma : E(G) \rightarrow \mathbb{N}$ is a proper edge-colouring of G . We

shall say that a subgraph $H \subset G$ of G is *multicoloured* with respect to the colouring γ if γ is injective on $E(H)$. A question raised by Spencer (see [10], p. 29) is as follows: are there graphs G of arbitrarily large girth such that every proper edge-colouring of G admits a multicoloured cycle?

Let G and H be graphs. We write $G \xrightarrow[p]{\text{mc}} H$ if for any proper edge-colouring γ of G there is a subgraph $H' \subset G$ of G isomorphic to H such that γ is injective on $E(H')$. Recently, Rödl and Tuza [22] settled the above problem of Spencer in the affirmative, by proving the following stronger statement. As customary, we denote the girth of a graph G by $g(G)$, and write C^ℓ for a cycle of length $\ell \geq 3$.

Theorem 1. *For every positive integer t , every real δ such that $0 < \delta < 1/(2t + 1)$, and every n sufficiently large with respect to t and δ , there is a graph $G = G^n$ of order n such that (i) $g(G) = t + 2$, and (ii) $G \xrightarrow[p]{\text{mc}} C^\ell$ for all $2t + 2 \leq \ell \leq n^\delta$. \square*

Erdős [10] remarked that random methods should be suitable for tackling Spencer's problem, and indeed in [22] the result above is proved by probabilistic means. In fact, it is shown that if $p = n^{\varepsilon-1}$ and $\varepsilon = \varepsilon(t, \delta) > 0$ is appropriately chosen, then a graph G obtained by suitably altering a random graph $G_p \in \mathcal{G}(n, p)$ almost surely satisfies (i) and (ii). No explicit construction for a graph G as in Theorem 1 is given in [22].

Following [22], for a graph H , let us denote by $t(H)$ the smallest possible girth for a graph G satisfying $G \xrightarrow[p]{\text{mc}} H$. Formally, let $t(H) = \inf\{g(G) : G \xrightarrow[p]{\text{mc}} H\}$. In the spirit of the well-known works of Nešetřil and Rödl [20], in [22], Rödl and Tuza raise the following very attractive problem.

Problem 2. *Does $t(H) = g(H)$ hold for every graph H ?*

As pointed out in [22], we do clearly have $t(H) = g(H)$ if H is a forest or if H contains a triangle. The complete bipartite graph $K^{4,4}$ shows that $t(C^4) = 4$. By considering the incidence graph of a projective plane, one may check that $t(C^6) = 6$, and a random construction involving projective planes gives that $t(C^5) = 5$. Thus $t(C^\ell) = \ell$ for $3 \leq \ell \leq 6$. (See Section 5 for details concerning these small cases of ℓ .) More generally, Theorem 1 gives that $t(C^\ell) \geq \lfloor \ell/2 \rfloor + 1$ for all $\ell \geq 4$. Our main results here significantly improve this estimate on $t(C^\ell)$ for large even values of ℓ , and give explicit graphs for Spencer's original problem. Indeed, Theorems 7 and 11 below show that suit-

able Ramanujan graphs X of Lubotzky, Phillips, and Sarnak [18] are as required. Roughly speaking, we take a large enough bipartite d -regular X and show that $X \xrightarrow[p]{\text{mc}} C^{2s}$ for all s varying from about $(3/4)g(X)$ up to $\Omega(d)$. (See Theorem 7.) Then the asymptotic lower bound for $t(C^\ell)$ (ℓ even) that we obtain is $t(C^{2s}) \geq (4/3 + o(1))s$. When deducing this bound, we shall arrive at a simple number-theoretic problem concerning the existence of certain primes. Although this question can be dealt with in a standard manner, for completeness we include a separate section, Section 4, dedicated to it.

The methods used in the proof of Theorem 7 are different from the methods used by Rödl and Tuza; they are based on the eigenvalue method of Alon and Milman (see, *e.g.* [4], Chapter 9, Section 2) and on the techniques used by Beck [6] to prove that the size-Ramsey number of an ℓ -path is $O(\ell)$. The fact that Ramanujan graphs are ‘explicit examples’ of extremal or nearly extremal graphs with respect to many problems is well known, and our result gives another example of such a problem. This note was inspired by the results of Alon and Chung [2], and of Friedman and Pippenger [12]. Let us also mention that an extremal problem concerning the relation $\xrightarrow[p]{\text{mc}}$ is studied in Babai [5], and that the corresponding problem for uniform hypergraphs is considered in Lefmann [17]. Problem 2 is also related to canonical versions of Ramsey’s theorem, in the sense of Erdős and Rado [11].

In the next section we state a few properties of the Ramanujan graphs constructed by Lubotzky, Phillips, and Sarnak that we shall use, and we give some lemmas needed in the proof of Theorem 7. Section 3 is devoted to the proof of Theorem 7, and in Section 4 we prove Theorem 11. In Section 5, which is included for completeness, we prove that $t(C^\ell) = \ell$ for $3 \leq \ell \leq 6$. In the last section we give some generalisations of our results, and we mention a related result.

Finally, we remark that appropriate modifications of our results here can be shown to hold for the Ramanujan graphs $Y_m = Y_{m,p}$ of Margulis [19]. However we have decided to restrict ourselves to the graphs $X^{p,q}$ because a more detailed analysis of these graphs is available in the literature. Moreover, unfortunately, the use of the Y_m does not seem to give significantly better results.

2. Auxiliary lemmas

Let p and q be two distinct primes such that $p, q \equiv 1 \pmod{4}$, $p \leq q^2$, and p is not a square in $\mathbb{Z}/q\mathbb{Z}$. Throughout this note, except for Section 6, we let $G = G_{p,q} = X^{p,q}$ be the Ramanujan graph of Lubotzky, Phillips, and Sarnak [18]. Thus G is a bipartite graph, with bipartition $V = V(G) = V_0 \cup V_1$, say. The order $|G|$ of G is $2n = q(q^2 - 1)$, and G is a d -regular graph where $d = p + 1$. We now recall three important results concerning the graph G . Let us collect three important properties of G in the following lemma.

Lemma 3. *Let $G = G_{p,q} = X^{p,q}$ be as above. Then the following hold.*

- (i) *The girth $g(G)$ of G satisfies $4 \log_p(q/\sqrt{2}) \leq g(G) < 4 \log_p(q\sqrt{2p})$. In particular, setting $t = \lceil (2/3) \log_p(n/\sqrt{2}) \rceil$, we have $d > p \geq 2^{-1/3t} n^{2/3t}$ and $g(G) \geq 2t$.*
- (ii) *The eigenvalues of the adjacency matrix of G are $\pm \lambda_0, \dots, \pm \lambda_{n-1}$, where $0 \leq \lambda_{n-1} \leq \dots \leq \lambda_0 = d$, and $|\lambda_i| \leq 2p^{1/2} = 2(d-1)^{1/2}$ for all $i \neq 0$.*
- (iii) *For any $U \subset V_0$ and any $W \subset V_1$, the number of edges $e(U, W)$ between U and W satisfies $|e(U, W) - (d/n)|U||W|| \leq 2\{(d-1)|U||W|\}^{1/2}$. □*

Property (ii), often called the *Ramanujan property*, is proved in [18], together with the lower bound for $g(G)$ in (i). The upper bound in (i) is a result of Biggs and Boshier [7]. Finally, we remark that the technique given in Alon and Spencer [4], Chapter 9, Section 2, is enough to prove (iii).

For the rest of this note, we let $\varepsilon = 2 \times 10^{-6}$, and set $\ell = \lceil 3t/2 \rceil$ where t is as in Lemma 3(i). Our aim is to show that if p and q are larger than a certain absolute constant, $t \geq 2$, and $d \geq 16\varepsilon^{-2\ell}$, then $G = G_{p,q} \xrightarrow[p]{\text{mc}} C^{2s}$ for all $\ell + 1 \leq s \leq (\varepsilon^\ell/6)n^{1/\ell} + 1$.

In what follows, several of the inequalities are claimed to hold for large enough values of p and q only. We also assume that p and q are so that $t \geq 2$. (For instance, we may require that $q \geq 3p/2$ to guarantee that $t \geq 2$.) Note that, in particular, we have $d \geq d_0 = 2^{-1/6} n^{2/3t}$. Below, our indices will be taken modulo 2 so that, e.g. $V_0 = V_2 = \dots$, etc.

For convenience, we introduce the following simple definition. Let J be a bipartite graph with a fixed bipartition, say $V(J) = X \cup Y$, and let b and f be positive real numbers. Then we shall say that J is a (b, f) -*expander* if for all $U \subset X$ and $U \subset Y$ such that $|U| \leq b$ we have $|\Gamma_J(U)| \geq f|U|$. Note that for a (b, f) -expander J if $V(J) \neq \emptyset$ and $f > 1$, then $|X|, |Y| > b$.

If H is a graph and $U, W \subset V(H)$ are disjoint sets of vertices, we write $H[U, W]$ for

the bipartite subgraph of H with vertex set $U \cup W$ and edge set $E_H(U, W) = \{uw \in E(H) : u \in U, w \in W\}$. Also, if $A, B \subset V(H)$, then $e(A, B) = e_H(A, B)$ denotes the number of edges in H that have one endvertex in A and the other in B . As usual, we denote the minimal degree $\min_{x \in H} d_H(x)$ of the graph H by $\delta(H)$. Recall $G = G_{p,q} = X^{p,q}$ has order $2n = q(q^2 - 1)$. Throughout this note, we let $r = n/2$. Note that clearly $n = q(q^2 - 1)/2$ is even, and hence r is an integer. We are now ready to state and prove our first lemma.

Lemma 4. *Suppose $d \geq 360^2$. Let $S_\sigma \subset V_\sigma$ with $|S_\sigma| = r = n/2$ ($\sigma = 0, 1$) be given. Let $H \subset G' = G[S_0, S_1]$ be a spanning subgraph of G' such that $e(H) \geq (1/3)e(G')$. Then there are sets $\bar{S}_\sigma \subset S_\sigma$ ($\sigma = 0, 1$) such that if $J = H[\bar{S}_0, \bar{S}_1]$, then (i) $\delta(J) \geq d/18$, and (ii) J is an $(r/10f, f)$ -expander for any $0 < f \leq 360^{-2}d$. In particular, we have $|\bar{S}_\sigma| \geq n/20$ ($\sigma = 0, 1$).*

Proof. Let $d' = e(G')/r$ be the average degree of $G' = G[S_0, S_1]$. Note that as $d \geq 144$, we have $d' \geq d/3$. Let $Z \subset S_0 \cup S_1$ be a minimal non-empty set satisfying $e(H[Z]) \geq (1/6)d'|Z|$. Let $\bar{S}_\sigma = Z \cap S_\sigma$ ($\sigma = 0, 1$), and let $J = H[\bar{S}_0, \bar{S}_1]$. Then, for any $X \subset Z$, we have $e_J(X, Z) \geq (1/6)d'|X|$. Indeed, we have $e_J(X, Z) = e(H[Z]) - e(H[Z \setminus X]) \geq (d'/6)|Z| - (d'/6)|Z \setminus X| = (d'/6)|X|$. Noting that for $x \in Z = V(J)$ we have $d_J(x) = e_J(\{x\}, Z)$, we see that $\delta(J) = \min_{x \in J} d_J(x) \geq d'/6 \geq d/18$, which proves (i).

Now assume $0 < f \leq 360^{-2}d$. We shall show that J is an $(r/10f, f)$ -expander. Suppose $U \subset \bar{S}_\sigma$, where $\sigma = 0$ or 1 , and $u = |U| \leq r/10f$. Let $W = \Gamma_J(U) \subset \bar{S}_{\sigma+1}$, and set $w = |W|$. We claim that $w \geq fu$. Suppose for a contradiction that $w < fu$. Then $du/18 \leq d'u/6 \leq e_J(U, Z) = e_J(U, W) \leq e_G(U, W) \leq (d/n)uw + 2(duw)^{1/2} < du/20 + 2(duw)^{1/2}$, and hence $w \geq 360^{-2}du \geq fu$, which is a contradiction.

To see that $|\bar{S}_\sigma| \geq n/20$ ($\sigma = 0, 1$), it suffices to note that $f_0 = 360^{-2}d > 1$. Indeed, as J is an $(r/10f, f)$ -expander for all $0 < f \leq f_0$, we have $|\bar{S}_\sigma| > r/10f$ ($\sigma = 0, 1$) for all $1 < f \leq f_0$. Letting f tend to 1, we conclude that $|\bar{S}_\sigma| \geq r/10$ ($\sigma = 0, 1$). \square

We now let $\gamma : E(G) \rightarrow \mathbb{N}$ be an arbitrary fixed proper edge-colouring of $G = G_{p,q}$. Let $S_\sigma \subset V_\sigma$ such that $|S_\sigma| = r = n/2$ ($\sigma = 0, 1$) be given. Also, let $H \subset G' = G[S_0, S_1]$ be a spanning subgraph of G' such that $e(H) \geq (1/3)e(G')$. Suppose $\bar{S}_\sigma \subset S_\sigma$ ($\sigma = 0, 1$) are as in Lemma 4, and again let $J = H[\bar{S}_0, \bar{S}_1]$.

For tidiness, if $P \subset G$ is a path, we write $\gamma(P)$ for $\gamma(E(P))$, and say that γ is injective on P if it is injective on $E(P)$. We may now state and prove the key lemma needed in the proof of our main result.

Lemma 5. *Suppose $\ell \leq \ell' \leq \varepsilon n^{1/\ell}$. Let $c_0 \in \mathbb{N}$, and $x \in \bar{S}_\sigma$ ($\sigma = 0$ or 1) be given. Then there is a set $T \subset \bar{S}_{\sigma+\ell'}$ such that (i) $|T| \geq \varepsilon^\ell n$, and (ii) for all $y \in T$ there is an x - y path P_y of length ℓ' in J with (a) $c_0 \notin \gamma(P_y)$, and (b) γ injective on P_y .*

Proof. First recall that the minimal degree $\delta(J)$ of J satisfies $\delta(J) \geq d/18 \geq 3\ell'$. Thus clearly there is a path $P_0 \subset J$ in J , starting at x , of length $\ell' - \ell$, such that (a) $c_0 \notin \gamma(P_0)$, and (b) γ is injective on P_0 . Suppose P_0 is an x - x' path, and note that $x' \in S_{\sigma'}$, where $\sigma' = \sigma + \ell' - \ell$. We now let $d'_0 = d_0$ if t is even, and let $d'_0 = 2^{-1/6} n^{1/\ell}$ if t is odd. Note that clearly $d'_0 \leq d_0 \leq d$.

Let $N_0 = \{x'\}$. Let $1 \leq i \leq \ell$, and suppose that pairwise disjoint sets $N_j \subset \bar{S}_0 \cup \bar{S}_1$ ($0 \leq j < i$) have been defined so that, for every $0 \leq j < i$, we have

- (i) $N_j \subset S_{\sigma'+j}$,
- (ii) $|N_j| = \left\lceil (d'_0/3 \times 360^2)^j \right\rceil$,
- (iii) for all $z \in N_j$, there is an x' - z path P_z in J such that (a) $V(P_0) \cap V(P_z) = \{x'\}$,
 (b) $|V(P_z) \cap N_k| = 1$ ($0 \leq k \leq j$), and (c) $P'_z = P_0 P_z$ is an x - z path in J with $c_0 \notin \gamma(P'_z)$ and γ injective on P'_z .

We shall now extend the sequence N_j ($0 \leq j < i$) by defining $N_i \subset \bar{S}_0 \cup \bar{S}_1$ suitably. Let $U = N_{i-1}$, $W = \Gamma_J(U)$, and $f = d'_0/360^2$. Note that $|U| = \left\lceil (d'_0/3 \times 360^2)^{i-1} \right\rceil \leq r/10f$, and hence $|W| \geq f|U|$. Let $W' = W \cap \left(V(P_0) \cup \bigcup_{j=0}^{i-1} N_j \right)$. Then we have

$$|W'| \leq \left\lceil \frac{1}{2} (\ell' - \ell + 1) \right\rceil + \sum_{j=0}^{i-2} |N_j| \leq \ell' - 1 + 2|N_{i-2}|.$$

For each $z \in N_{i-1}$, fix an x' - z path $P_z \subset J$ as in (iii) above, and let $P'_z = P_0 P_z$. Now let

$$W'' = \bigcup_{z \in N_{i-1}} \{w \in \Gamma_J(z) : \gamma(zw) \in \{c_0\} \cup \gamma(P'_z)\}.$$

Then $|W''| \leq \sum_{z \in N_{i-1}} (1 + |\gamma(P'_z)|) \leq \ell' |N_{i-1}|$. Let $N'_i = W \setminus (W' \cup W'')$. Then

$$\begin{aligned} |N'_i| &\geq |W| - (|W'| + |W''|) \geq |W| - (\ell' - 1 + 2|N_{i-2}| + \ell' |N_{i-1}|) \\ &\geq f|N_{i-1}| - 2\ell' |N_{i-1}| \geq \frac{d'_0}{3 \times 360^2} |N_{i-1}| \geq \left(\frac{d'_0}{3 \times 360^2} \right)^i. \end{aligned}$$

Let $N_i \subset N'_i$ be such that $|N_i| = \lceil (d'_0/3 \times 360^2)^i \rceil$. Note that then (i), (ii), and (iii) hold for $j = i$. Thus by induction there are sets N_j ($0 \leq j \leq \ell$) satisfying (i), (ii), and (iii) for all $0 \leq j \leq \ell$. To finish the proof, let $T = N_\ell$. \square

3. The main result

As in the previous section, we let a proper edge-colouring $\gamma : E(G) \rightarrow \mathbb{N}$ be fixed. Let the partition of $E = E(G)$ naturally induced by γ be $E = E_1 \cup \dots \cup E_k$, where, say, $E_i = \gamma^{-1}(\{i\}) \neq \emptyset$ ($1 \leq i \leq k$). For the rest of this section, let $V_\sigma = S_\sigma^{(0)} \cup S_\sigma^{(1)}$, with $|S_\sigma^{(0)}| = |S_\sigma^{(1)}| = r = n/2$, be a fixed partition of V_σ ($\sigma = 0, 1$). Let $G^{(i)} = G[S_0^{(i)}, S_1^{(i)}]$ ($i = 0, 1$).

We shall use the following slightly non-standard notation. If H is a graph and $F \subset E(H)$, we write $H[F]$ for the *spanning* subgraph of H with edge set F . The following result is a simple technical lemma.

Lemma 6. *Suppose $d \geq 108 \log 3$. Then there exists a partition $[k] = C_0 \cup C_1$ of $[k] = \{1, \dots, k\}$ such that if we let $H^{(i)} = G^{(i)} \left[E(G^{(i)}) \cap \bigcup_{j \in C_i} E_j \right]$ ($i = 0, 1$), then $e(H^{(i)}) \geq (1/3)e(G^{(i)})$ ($i = 0, 1$).*

Proof. Set $a_i^{(j)} = |E_i \cap E(G^{(j)})|$ for $1 \leq i \leq k$ and $j = 0, 1$. Note that $0 \leq a_i^{(j)} \leq r$. Put $i \in [k]$ in C_0 with probability $1/2$, independently from all other $j \in [k]$. Let $X_i = 1$ if $i \in C_0$ and $X_i = 0$ if $i \in C_1$. Let $H^{(i)}$ ($i = 0, 1$) be as in the statement of our lemma. Then clearly $e(H^{(0)}) = \sum_{1 \leq i \leq k} a_i^{(0)} X_i$. Let $Y_i = a_i^{(0)} X_i / r$ ($1 \leq i \leq k$), and set $Y = \sum_{1 \leq i \leq k} Y_i$. Then, for any $0 \leq \delta \leq 1$, by Hoeffding's inequality we have that $\mathbb{P}\{Y \leq (1 - \delta)\mathbb{E}(Y)\} \leq \exp\{-\delta^2 \mathbb{E}(Y)/2\}$. Note that $\mathbb{E}(Y) = e(G^{(0)})/2r \geq d/6 \geq 18 \log 3$, and therefore we have that $\mathbb{P}\{e(H^{(0)}) \leq (1/3)e(G^{(0)})\} \leq 1/3$. Similarly $\mathbb{P}\{e(H^{(1)}) \leq (1/3)e(G^{(1)})\} \leq 1/3$, and thus a partition as required does exist. \square

We are now ready to prove our main result.

Theorem 7. *Let p and q be distinct primes such that $p, q \equiv 1 \pmod{4}$, $q^2 \geq p$, and p is not a square in $\mathbb{Z}/q\mathbb{Z}$. Let $n = q(q^2 - 1)/2$, $t = \lceil (2/3) \log_p(n/\sqrt{2}) \rceil$, $\ell = \lceil 3t/2 \rceil$, and $\varepsilon = 2 \times 10^{-6}$. Suppose $t \geq 2$, $d = p + 1 \geq 16\varepsilon^{-2\ell}$, and let $G = G^{2n} = X^{p,q}$. Then,*

if $\ell + 1 \leq s \leq (\varepsilon^\ell/6)n^{1/\ell} + 1$, we have $G \xrightarrow[p]{\text{mc}} C^{2s}$.

Proof. Recall that a partition $V_\sigma = S_\sigma^{(0)} \cup S_\sigma^{(1)}$ with $|S_\sigma^{(0)}| = |S_\sigma^{(1)}| = r = n/2$ of V_σ ($\sigma = 0, 1$) has been fixed. Let us now fix C_i and $H^{(i)}$ ($i = 0, 1$) as in Lemma 6. For both $i = 0, 1$, we apply Lemma 4 to obtain $\bar{S}_\sigma^{(i)} \subset S_\sigma^{(i)}$ with $|\bar{S}_\sigma^{(i)}| \geq n/20$ ($\sigma = 0, 1$) such that $J^{(i)} = H^{(i)}[\bar{S}_0^{(i)}, \bar{S}_1^{(i)}]$ is an $(r/10f, f)$ -expander for all $0 < f \leq d/360^2$. Since $|\bar{S}_0^{(0)}|, |\bar{S}_1^{(1)}| \geq n/20$, we have $e_G(\bar{S}_0^{(0)}, \bar{S}_1^{(1)}) > 0$. Let $x_0x_1 \in E(G)$ be such that $x_0 \in \bar{S}_0^{(0)}$ and $x_1 \in \bar{S}_1^{(1)}$. Let $c_0 = \gamma(x_0x_1)$, and suppose $\ell \leq \ell' \leq (\varepsilon^\ell/6)n^{1/\ell} \leq \varepsilon n^{1/\ell}$. We shall now proceed to find a multicoloured $2s$ -cycle $C^{2s} \subset G$ where $s = \ell' + 1$.

We first apply Lemma 5 to $J^{(i)}$ ($i = 0, 1$) and obtain $T^{(i)} \subset \bar{S}_{i+\ell'}$ such that (i) $|T^{(i)}| = a = \lceil \varepsilon^\ell n \rceil$, and (ii) for all $y_i \in T^{(i)}$, there is an x_i - y_i path $P_{y_i} \subset J^{(i)}$ in $J^{(i)}$ of length ℓ' with γ injective on P_{y_i} and $c_0 \notin \gamma(P_{y_i})$.

For each $y_i \in T^{(i)}$ ($i = 0, 1$), fix an x_i - y_i path $P_{y_i} \subset J^{(i)}$ as given in (ii) above. We now notice that, since $d \geq 16\varepsilon^{-2\ell}$, we have $e_G(T^{(0)}, T^{(1)}) \geq (d/n)a^2 - 2d^{1/2}a \geq (d/2n)a^2$. For $i = 0, 1$, let

$$F_i = \bigcup_{y_i \in T^{(i)}} \{y_iz \in E(G) : z \in T^{(i+1)}, \text{ and } \gamma(y_iz) \in \{c_0\} \cup \gamma(P_{y_i})\},$$

and set $F = F_0 \cup F_1$. Then $|F| \leq (\ell' + 1)|T^{(0)} \cup T^{(1)}| \leq 2(\ell' + 1)a$. Note that since $\ell \geq 3$ and $d \geq 16\varepsilon^{-2\ell}$, we have $\varepsilon^\ell d \geq 2 \times 10^{18}$. Thus $\ell' + 1 \leq (\varepsilon^\ell/6)n^{1/\ell} + 1 \leq (\varepsilon^\ell/6)2^{1/6}d + 1 < (\varepsilon^\ell/4)d$, and therefore $|E_G(T^{(0)}, T^{(1)}) \setminus F| \geq (d/2n)a^2 - 2(\ell' + 1)a = a(\varepsilon^\ell d/2 - 2(\ell' + 1)) > 0$. Finally, suppose $y_0y_1 \in E_G(T^{(0)}, T^{(1)}) \setminus F$. Then, if $P_{y_0} = x_0z_1z_2 \dots y_0$, $P_{y_1} = x_1z'_1z'_2 \dots y_1$, we have that $C = x_0z_1z_1 \dots y_0y_1 \dots z'_2z'_1x_1x_0$ is a $2s$ -cycle in G and γ is injective on $E(C)$. This finishes the proof of the theorem. \square

Recall that for G as in Theorem 7 we have $g(G) \geq 2t$, and in fact $g(G)$ is roughly equal to $2t$ (cf. Lemma 3(i)). Thus in the above result we have $G \xrightarrow[p]{\text{mc}} C^{2s}$ for $2s$ about $(3/2)g(G)$. Notice, in fact, that this means that the *short* multicoloured cycles guaranteed by Theorem 7 are necessarily induced. Moreover, note that if ℓ is not too large, then the upper bound for s in Theorem 7 is nearly of the same order as $d = p + 1$. Thus, since the chromatic index of G is d , the restriction that s should not exceed $(\varepsilon^\ell/6)n^{1/\ell} + 1$ cannot be essentially weakened.

4. The existence of suitable primes

Theorem 7 states that $X = X^{p,q} \xrightarrow[p]{\text{mc}} C^{2s}$ provided p , q , and s satisfy certain conditions. For brevity, let us say that a pair (p, q) of primes is *good* if $p, q \equiv 1 \pmod{4}$, $q \geq 2p$, and p is a quadratic non-residue modulo q . Note that *a priori* it is not clear that there is any good pair (p, q) of primes that admits an integer s satisfying the conditions in Theorem 7. Our first aim in this section is to present a simple result, Lemma 9, that implies that for any even $\ell \geq 18$ there is a good pair (p, q) such that $X = X^{p,q} \xrightarrow[p]{\text{mc}} C^\ell$ (see Theorem 11). Moreover, we shall see that we may further require the girth $g(X)$ of X to be roughly as large as $(2/3)\ell$.

As usual, for an integer $n \geq 1$, let $\varphi(n)$ denote the number of integers $1 \leq m \leq n$ coprime to n . Also, let $\Lambda(n) = \log p$ if n is a positive integer power of a prime p , and let $\Lambda(n) = 0$ otherwise. Let a real $x > 0$ and integers $h \geq 2$, $a \geq 1$ be given. We put

$$\psi(x; h, a) = \sum \{\Lambda(n) : 2 \leq n \leq x, n \equiv a \pmod{h}\}.$$

A well-known number-theoretic result states that $\psi(x; h, a) = (1 + o(1))x/\varphi(h)$ as $x \rightarrow \infty$, provided h and a are fixed and $(a, h) = 1$. In fact, the following refinement of Dirichlet's celebrated theorem on primes in arithmetic progressions is an easy consequence of the above result. For fixed h and a with $(a, h) = 1$, we have $\pi(x; h, a) = (1 + o(1))x/\varphi(h) \log x$ ($x \rightarrow \infty$), where $\pi(x; h, a)$ is the number of primes $2 \leq p \leq x$ with $p \equiv a \pmod{h}$. We refer the reader to Davenport [9] for this and other related results from analytic number theory.

Below, we shall be interested in the error term in the above asymptotic estimate for $\psi(x; h, a)$. Let us introduce some notation. We put $E(x; h, a) = |\psi(x; h, a) - x/\varphi(h)|$, $E(x, h) = \max\{E(x; h, a) : (a, h) = 1\}$, and $E^*(x, h) = \max\{E(y, h) : 0 < y \leq x\}$. The following beautiful and far-reaching result was proved by Bombieri [8] (see also Davenport [9], §28).

Theorem 8. *Let $A > 0$ be fixed. Then there is a constant $C = C(A)$ such that, for $x^{1/2}(\log x)^{-A} \leq Q \leq x^{1/2}$, we have $\sum_{2 \leq h \leq Q} E^*(x, h) \leq Cx^{1/2}Q(\log x)^5$. \square*

In particular, the above result implies that $E^*(x, h)$ is a great deal smaller than $x/\varphi(h)$ for most $h \leq x^{1/2}(\log x)^{-A}$. Furthermore, clearly, Theorem 8 states that under the given

conditions the average error $\text{Ave}_{2 \leq h \leq Q} E^*(x, h)$ is $O\{x^{1/2}(\log x)^5\}$. We remark in passing that assuming the generalised Riemann hypothesis, one may prove that $E(x, h) = O\{x^{1/2}(\log x)^2\}$ for $x \geq h$ (cf. §20 in [9]). Such a bound for $E(x, h)$ would allow us to give a more direct proof of the existence of appropriate good pairs of primes.

The main lemma of this section is Lemma 9. For its proof we shall need the following simple result. For $x > 0$ put $\psi_2(x) = \sum \Lambda(n)$, where the sum is extended over all $2 \leq n = p^k \leq x$ with p a prime and $k \geq 2$ an integer. Then, by the prime number theorem, we have that $\psi_2(x) \leq (2 + o(1))x^{1/2}$ as $x \rightarrow \infty$. Let us now turn to Lemma 9. Suppose a set $H = \{h_i \in \mathbb{N} : i \geq 1\}$ of integers is fixed, and assume $2 \leq h_1 < h_2 < \dots$. Moreover, let integers $1 \leq a_i < h_i$ ($i \geq 1$) be given.

Lemma 9. *Suppose that for a constant $\rho > 0$ we have $|H \cap [n]| = (\rho + o(1))n/\log n$ as $n \rightarrow \infty$. Let $0 < \eta < 1/2$ and $b > 0$ be fixed. Then there is $h_i \in H$ and a prime q such that $h_i \geq b$, $q \equiv a_i \pmod{h_i}$, and $(h_i/4)^{1/\eta}/4 \leq q \leq (h_i/4)^{1/\eta}$.*

Proof. Let $A = 7$ in Theorem 8, and set $C = C(7)$. Pick $Q \geq 2$, and let $x_0 = x_0(Q)$ be such that $Q = x_0^{1/2}(\log x_0)^{-7}$. Assume that Q is sufficiently large so that (i) $4x_0^\eta \geq b$, (ii) $\log x_0 \geq (80C/\rho)2^\eta$, (iii) $x_0^{1/2-\eta} \geq 60 \times 2^\eta$, and (iv) $\psi_2(x_0) \leq 3x_0^{1/2}$. Now note that, as $x \rightarrow \infty$,

$$\begin{aligned} N(x) &= |\{h \in H : 4x^\eta < h \leq 4(2x)^\eta\}| \\ &= (4\rho + o(1)) \frac{(2x)^\eta}{\eta \log(2x)} - (4\rho + o(1)) \frac{x^\eta}{\eta \log x}, \end{aligned}$$

which is, for large enough x , at least

$$\frac{4\rho x^\eta}{\eta \log x} \left(2^{3\eta/4} - 2^{\eta/4}\right) \geq 2\rho(\log 2)x^\eta(\log x)^{-1} \geq \rho x^\eta(\log x)^{-1}.$$

Thus we may further assume that Q is sufficiently large to guarantee that $N(x_0) \geq \rho x_0^\eta(\log x_0)^{-1}$. Now, by Theorem 8, we clearly have that $E^*(x_0, h) \geq (2C/\rho)x_0^{1-\eta}(\log x_0)^{-1}$ holds for at most $(\rho/2)x_0^\eta(\log x_0)^{-1}$ values of $2 \leq h \leq Q$. Thus there is $h_i \in H$ such that $b \leq 4x_0^\eta < h_i \leq 4(2x_0)^\eta$, and $E^*(x_0, h_i) \leq (2C/\rho)x_0^{1-\eta}(\log x_0)^{-1}$. Hence we have $E(x_0; h_i, a_i)$, $E(x_0/2; h_i, a_i) \leq (2C/\rho)x_0^{1-\eta}(\log x_0)^{-1}$. Therefore

$$\begin{aligned} \psi(x_0; h_i, a_i) - \psi(x_0/2; h_i, a_i) &\geq \frac{x_0}{2\varphi(h_i)} - \frac{4C}{\rho} \cdot \frac{x_0^{1-\eta}}{\log x_0} \\ &\geq \frac{x_0}{10\varphi(h_i)} + \psi_2(x_0) > \psi_2(x_0), \end{aligned}$$

where for the second inequality we use (ii) and (iii) above. Thus there is a prime q such that $x_0/2 < q \leq x_0$ and $q \equiv a_i \pmod{h_i}$. Finally, note that $(h_i/4)^{1/\eta}/4 \leq x_0/2 < q \leq x_0 \leq (h_i/4)^{1/\eta}$. \square

We now apply the above lemma to our particular set-up. For $a, b \geq 2$, let $\ell(a, b) = \lceil (3/2) \lceil (2/3) \log_a(n(b)/\sqrt{2}) \rceil \rceil$, where $n(b) = b(b^2 - 1)/2$. Note that in Theorem 7, we have $\ell = \ell(p, q)$.

Lemma 10. *Let $P \geq 2$ and $\ell_0 \geq 8$ be given. Then (i) if $\ell_0 \equiv 0$ or $2 \pmod{3}$, there is a good pair (p, q) of primes with $p \geq P$ and $\ell(p, q) = \ell_0$, and (ii) if $\ell_0 \equiv 1 \pmod{3}$, then there is a good pair (p, q) of primes with $p \geq P$ and $\ell(p, q) = \ell_0 + 1$.*

Proof. (i) We may clearly assume that $P \geq 53$. Let p_i ($i \geq 1$) be the i th prime congruent to 1 modulo 4. Let $h_i = 4p_i$, and set $H = \{h_i : i \geq 1\}$. Note that $H \cap [n] = \{h_i = 4p_i : p_i \leq n/4\}$, and so $|H \cap [n]| = \pi(n/4; 4, 1)$. Thus $|H \cap [n]| = (1/8 + o(1))n/\log n$ as $n \rightarrow \infty$. Now let $1 \leq s_i < p_i$ ($i \geq 1$) be such that $4s_i \equiv 1 \pmod{p_i}$, and let $1 \leq r_i < p_i$ ($i \geq 1$) be a quadratic non-residue modulo p_i . Let $1 \leq a_i < h_i$ ($i \geq 1$) be such that $a_i \equiv 4s_i r_i + p_i \pmod{h_i}$. We now define $0 < \eta < 1/2$ as follows. If $\ell_0 \equiv 0 \pmod{3}$, we let $\eta = 3/\ell_0$, and if $\ell_0 \equiv 2 \pmod{3}$, we let $\eta = 6/(2\ell_0 - 1)$. Set $b = 4P$. Lemma 9 then gives that, for some $i \geq 1$ and some prime q , we have $p_i \geq P$, $q \equiv 4s_i r_i + p_i \pmod{4p_i}$, and $(1/4)p_i^{1/\eta} \leq q \leq p_i^{1/\eta}$. But then $q \equiv 1 \pmod{4}$ and q is a quadratic non-residue modulo p_i . By the law of quadratic reciprocity, we have that p_i is a quadratic non-residue modulo q . Since $\eta < 1/2$, we may deduce from $q \geq (1/4)p_i^{1/\eta}$ that $q \geq 2p_i$. Thus (p_i, q) is good. Moreover, a simple but a little tedious calculation shows that indeed $\ell(p_i, q) = \ell_0$ (the assumption $P \geq 53$ is used here). This finishes the proof of (i). Assertion (ii) follows immediately from (i). \square

With Lemma 10 in hand, we may easily show that for any even $L \geq 18$ there is a good pair (p, q) such that $X^{p,q} \xrightarrow[p]{\text{mc}} C^L$.

Theorem 11. *Suppose $s \geq 9$ and $L = 2s$. Then there is a good pair (p, q) of primes such that (i) $X = X^{p,q} \xrightarrow[p]{\text{mc}} C^L$, and (ii) $g(X) \geq (2/3)(L - 4)$ if $s \equiv 0$ or $1 \pmod{3}$ and $g(X) \geq (2/3)(L - 2)$ otherwise. \square*

5. Very short multicoloured cycles

In the introduction we remarked that we have $t(C^\ell) = \ell$ for all $3 \leq \ell \leq 6$. For $\ell \in \{3, 4\}$ this statement can be verified by considering the complete graph K^{14} and the complete bipartite graph $K^{4,4}$. Now, for $\ell = 6$ we take a projective plane \mathcal{P} of large order, and consider its incidence bipartite graph, namely, the bipartite graph $I = I(\mathcal{P})$ with vertex classes V_0, V_1 , where $V_0 = V_0(\mathcal{P})$ is the set of points of \mathcal{P} and $V_1 = V_1(\mathcal{P})$ is the set of lines of \mathcal{P} , and with $xy \in E(I)$ ($x \in V_0, y \in V_1$) if and only if x and y are incident in \mathcal{P} . By estimating the maximal possible number of non-multicoloured 6-cycles in a properly edge-coloured $I = I(\mathcal{P})$, we may verify that $I \xrightarrow[\mathcal{P}]{\text{mc}} C^6$ for a large enough \mathcal{P} . Since $g(I) = 6$, we conclude that $t(C^6) = 6$.

For completeness, we check here that $t(C^5) = 5$: using probabilistic methods we shall ‘construct’ a suitable graph G to verify this fact. Our method of proof might be of some use to prove upper bounds for $t(C^\ell)$ for large odd ℓ . Our random construction of G is based again on large projective planes \mathcal{P} .

The construction of G . Let \mathcal{P} be a projective plane of order q , and let $n = q^2 + q + 1$. Thus I is a $(q+1)$ -regular bipartite graph and we have $|V_0| = |V_0(\mathcal{P})| = |V_1| = |V_1(\mathcal{P})| = n$. By considering random partitions, we may assume that $V_0 = A_0 \cup A_1 \cup A_2, V_1 = B_1 \cup B_2$ are partitions of V_0 and V_1 such that, for all $i \in \{0, 1, 2\}$ and $j \in \{1, 2\}$, we have (i) $n/6 \leq |A_i| \leq n/2, n/3 \leq |B_j| \leq 2n/3$, and (ii) $|\Gamma_I(x) \cap B_j| \geq (q+1)/3$ for all $x \in V_0$, and $|\Gamma_I(y) \cap A_i| \geq (q+1)/4$ for all $y \in V_1$.

We next define $H \subset I$ by deleting the edges in $E_I(A_1, B_2) \cup E_I(A_2, B_1)$ from I . Now let $p = \omega/n$, where $\omega = \omega(n) = 4 \log n$. For all pairs $(a_1, a_2) \in A_1 \times A_2$, add the edge $a_1 a_2$ to H independently with probability p , and let J be the resulting random graph. Our aim now is to verify that, almost surely, we may delete some edges from $E_J(A_1, B_1) \cup E_J(A_2, B_2) \cup E_J(A_1, A_2)$ from J in such a way that the resulting graph G is such that (iii) $|\Gamma_G(y) \cap A_i| \geq (q+1)/5$ for all $y \in V_1, i \in \{1, 2\}$, (iv) for any $U \subset A_1, W \subset A_2$ with $|U|, |W| \geq n/15$, we have $e_G(U, W) \geq p|U||W|/2$, and (v) all vertices $z \in A_1 \cup A_2$ have degree $d_{G[A_1, A_2]}(z) = O(\omega)$ in $G[A_1, A_2]$.

Let us choose which edges from $E_J(A_1, B_1)$ we shall delete. To this end, we first concentrate our attention on a vertex $y \in B_1$. Choose a maximum collection of edge-disjoint quadrilaterals $C^4 \subset J[A_1, A_2 \cup \{y\}]$ that contain y , and let E_y be the edges incident to y present in this collection. The edges we delete from $E_J(A_1, B_1)$ are the ones

in $E_1 = \bigcup_{y \in B_1} E_y$. In exactly the same way, we define a set $E_2 \subset E_J(A_2, B_2)$ of edges of J to be deleted from J .

Finally, let $E_{12} \subset E_J(A_1, A_2)$ be the set of all the edges present in 4-cycles $C^4 \subset J[A_1, A_2]$. Define now G to be the subgraph $G = J - (E_1 \cup E_2 \cup E_{12})$ of J . It is quite easy to check that (iii), (iv), and (v) above hold in G almost surely. Let us fix G satisfying such conditions. This completes the ‘construction’ of G .

Claim. We have $g(G) = 5$ and $G \xrightarrow[\text{p}]{\text{mc}} C^5$.

Proof of the Claim. It is immediate that $g(G) \geq 5$. To verify the second assertion in our claim, we use the counting argument of Rödl and Tuza [22], which applies very easily here owing to our definition of G . It is easily seen that any vertex $a_0 \in A_0$ belongs to $\Omega(\omega n)$ cycles $C^5 \subset G$, and hence the total number of copies of C^5 in G is $\Omega(\omega n^2)$. Let now $\gamma : E(G) \rightarrow \mathbb{N}$ be a fixed proper edge-colouring of G , and let us count the number of non-multicoloured C^5 contained in G . Let us write $a_0 b_1 a_1 a_2 b_2$ for a 5-cycle of G , where naturally $a_i \in A_i$ and $b_j \in B_j$ ($i \in \{0, 1, 2\}$, $j \in \{1, 2\}$). How many such C^5 have the edges $a_0 b_1$ and $a_2 b_2$ of the same colour? For such cycles, the number of possibilities for $a_0 b_1$ is $O(n^{3/2})$ and for $a_0 b_2$, given $a_0 b_1$, is $O(n^{1/2})$. Also, given $a_0 b_1$ and $a_0 b_2$, there is at most one possibility for $a_2 b_2$ as this edge must be incident to b_2 and must have the same colour as $a_0 b_1$. Finally, there is at most one vertex $a_1 \in A_1$ with $a_1 a_2, a_1 b_1 \in E(G)$. Thus the total number of 5-cycles $a_0 b_1 a_1 a_2 b_2$ in G with the edges $a_0 b_1, a_2 b_2$ of the same colour is $O(n^2)$. Analysing a few more cases, one checks that the number of non-multicoloured 5-cycles in G is $O(n^2)$. Since the total number of 5-cycles in G is $\Omega(\omega n^2)$, we are done. \square

In view of our results for $t(C^\ell)$ for ℓ even, the method above might apply to give improved lower estimates for $t(C^\ell)$ for large odd ℓ . This would be an interesting problem to consider.

6. Some generalisations

The technique used in [22] to prove Theorem 1 gives two results of a more general nature. Roughly speaking, these results assert the existence of graphs G of large girth satisfying $G \xrightarrow[\text{p}]{\text{mc}} H$ for any very sparse graph H . Given a graph H , let $m(H) = \max\{e(H')/|H'|\}$, where the maximum is taken over all subgraphs $H' \subset H$ of positive order. Using a simple

‘decomposition’ result given in [22] (cf. the proof of Theorem 2 in [22]), and the methods of Sections 2 and 3 above applied to the *non-bipartite* Ramanujan graphs $X^{p,q}$ of Lubotzky, Phillips, and Sarnak [18] (so here p is a quadratic residue modulo q), one may prove the following result.

Theorem 12. *Let $t, h \geq 3$ be given. Then there exist primes p and q , a real $\mu > 0$, and an integer $g \geq 3$ such that if $G = G_{t,h} = X^{p,q}$, then (i) $g(G) \geq t$ and (ii) $G \xrightarrow[p]{\text{mc}} H$ for every graph H with $m(H) \leq 1 + \mu$, $g(H) \geq g$, and $|H| \leq h$.*

Theorem 12 above has as an immediate corollary Theorem 13 below. For two graphs G and H , let us write $G \xrightarrow[p]{\text{mc}} TH$ if for any proper edge-colouring γ of G , there is a subgraph $H' \subset G$ of G that is a subdivision of H and such that γ is injective on $E(H')$.

Theorem 13. *Let $t, k \geq 3$ be given. Then there exist primes p and q such that if $G = G_{t,k} = X^{p,q}$, then (i) $g(G) \geq t$ and (ii) $G \xrightarrow[p]{\text{mc}} TK^k$. \square*

Note that this result settles a generalised version of Spencer’s original question. Indeed, when $k = 3$ this theorem asserts the existence of graphs of arbitrarily large girth that have the property that however they are properly edge-coloured, we may always find a multicoloured cycle in them. Moreover, Theorem 13 tells us that suitable Ramanujan graphs have this property. The existence of the graphs $G_{t,h}$ and $G_{t,k}$ as above was proved in [22] by non-constructive means, and no explicit examples of such graphs were given. Our aim in this section is to give a sketch of proof for Theorem 12.

We start with some preparation concerning the decomposition result given in [22]. Let an integer $s \geq 1$ be given. A sequence of graphs $(F_k)_0^m$ is an *s-decomposition series* for a graph H if $K^1 = F_0 \subset \dots \subset F_m = H$ and, for all $1 \leq k \leq m$, the graph F_k is obtained from F_{k-1} either by the addition of an isolated vertex or of a vertex of degree one, or else by the addition of an $x'-y'$ path of order at least s , by joining x' (respectively y') to a vertex $x \in F_{k-1}$ (respectively $y \in F_{k-1}$), where x and y are not necessarily distinct. Thus, more precisely, we either have $|F_k| = |F_{k-1}| + 1$, $V(F_k) = V(F_{k-1}) \cup \{x\}$, and $d_{F_k}(x) \leq 1$, or else we have $|F_k| \geq |F_{k-1}| + s$, for all $w \in W = V(F_k) \setminus V(F_{k-1})$ we have $d_{F_k}(w) = 2$, and W induces a path in F_k .

For sparse graphs of large girth, as shown by Rödl and Tuza [22], we have the following simple result on the existence of decomposition series.

Lemma 14. *If a graph H is such that $m(H) < 1 + 1/(3s - 1)$ and $g(H) \geq s + 1$, then H admits an s -decomposition series. \square*

To prove Theorem 12, in addition to Lemma 14 we need a somewhat cumbersome technical lemma, namely Lemma 15 below. Before we can state that result, we need to introduce a few definitions and some notation. Let us fix a non-bipartite Ramanujan graph $G = G^n = X^{p,q}$, and recall that (†) the inequality in Lemma 3(iii) holds for any two disjoint sets $U, W \subset V(G)$ (cf. Alon and Spencer [4], Chapter 9, Section 2). Suppose $\gamma : E(G) \rightarrow \mathbb{N}$ is a fixed proper edge-colouring of G . Let $m \geq 1$ be fixed, and let $V = V(G) = V_0 \cup \dots \cup V_m$ be a partition of V such that $n' = |V_1| = \dots = |V_m|$ is even and $\lfloor n/2m \rfloor \leq n' \leq \lfloor n/2m \rfloor + 1$. Also, for $1 \leq k \leq m$, let $V_k = S_k \cup T_k$ be a partition of V_k with $|S_k| = |T_k|$, and let $G'_k = G[S_k, T_k]$ be the bipartite graph induced by the vertex classes S_k and T_k in G . Now let $\mathbb{N} = C_1 \cup \dots \cup C_m$ be a partition of \mathbb{N} , and let $G_k = G'_k[\gamma^{-1}(C_k)]$ ($1 \leq k \leq m$). Now, the argument used in the proof of Lemma 6 gives that there is a partition $(C_k)_1^m$ of \mathbb{N} such that $e(G_k) \geq (1/2m)e(G'_k)$ ($1 \leq k \leq m$), provided $d = p + 1$ is at least as large as a constant $d_0 = d_0(m)$ depending only on m . In the sequel, we assume that our partition $(C_k)_1^m$ is as above. Moreover, from now on we tacitly assume that $d = p + 1$ is large enough with respect to m .

Lemma 15. *Let $m \geq 1$ be fixed. Then there are constants $c_i > 0$ ($0 \leq i \leq 3$) depending only on m satisfying the following property. For each $i \in \{0, 1\}$ and $1 \leq k \leq m$, there are sets $S_k^{(i)} \subset S_k$, $T_k^{(i)} \subset T_k$, and a vertex $x_0 \in V_0$ such that $S_k^{(0)} \cap S_k^{(1)} = \emptyset$ and $T_k^{(0)} \cap T_k^{(1)} = \emptyset$ ($1 \leq k \leq m$) and such that the following holds. Let $J_k^{(i)} = G_k[S_k^{(i)}, T_k^{(i)}]$ ($i \in \{0, 1\}$, $1 \leq k \leq m$). Then, for all $1 \leq k \leq m$ and $i \in \{0, 1\}$, we have*

- (i) $|S_k^{(i)}|, |T_k^{(i)}| \geq r_k = c_1^{m-k}r$, where $r \geq c_0n$,
- (ii) $|\Gamma_G(x_0) \cap S_k^{(i)}| \geq d|S_k^{(i)}|/2n$ and $|\Gamma_G(x_0) \cap T_k^{(i)}| \geq d|T_k^{(i)}|/2n$,
- (iii) for all $k < \ell \leq m$, and $i_k, i_\ell \in \{0, 1\}$, if $x \in J_k^{(i_k)}$ then $|\Gamma_G(x) \cap S_\ell^{(i_\ell)}| \geq d|S_\ell^{(i_\ell)}|/2n$ and $|\Gamma_G(x) \cap T_\ell^{(i_\ell)}| \geq d|T_\ell^{(i_\ell)}|/2n$,
- (iv) $J_k^{(i)}$ is $(c_2r_k/f, f)$ -expanding for all $0 < f \leq c_3d$. \square

We only briefly describe a proof of Lemma 15. We start by applying the ‘bipartite version’ of Szemerédi’s lemma for subgraphs of pseudo-random graphs (see [16]) to the graphs $G_k \subset G'_k$, with an appropriately small $\varepsilon = \varepsilon(m) > 0$. This gives us sets $S'_k \subset S_k$,

$T'_k \subset T_k$ ($1 \leq k \leq m$) with $|S'_k| = |T'_k| = \Omega(n)$ for all k and such that $G_k[S'_k, T'_k]$ is (ε, G_k, G'_k) -regular and $e(G_k[S'_k, T'_k]) \geq (1/3m)e(G'_k[S'_k, T'_k])$. (For the definition of these terms, the reader is referred to [16].) We now define the sets $S_k^{(i)}, T_k^{(i)}$ as required in the lemma by reverse induction on k , and we define the vertex x_0 last.

Let us first split S'_m and T'_m into pairs of sets of cardinality as equal as possible, say $S'_m = S_{m,0} \cup S_{m,1}$ and $T'_m = T_{m,0} \cup T_{m,1}$. Applying the argument we used in the proof of Lemma 4, we may find suitable sets $S_m^{(i)} \subset S_{m,i}, T_m^{(i)} \subset T_{m,i}$ ($i \in \{0,1\}$) as required. Now we alter the sets S'_k, T'_k ($1 \leq k < m$) to guarantee (iii) in our lemma: we delete from $S'_k \cup T'_k$ any vertex x for which $|\Gamma_G(x) \cap S_m^{(i)}| < d|S_m^{(i)}|/2n$ or $|\Gamma_G(x) \cap T_m^{(i)}| < d|T_m^{(i)}|/2n$ for some $i \in \{0,1\}$. Let $S''_k \subset S'_k, T''_k \subset T'_k$ be the sets we obtain in this way, and then note that by (\dagger) we have $|S''_k| \geq |S'_k|/2, |T''_k| \geq |T'_k|/2$ ($1 \leq k < m$). Moreover, by the (ε, G_k, G'_k) -regularity of $G_k[S'_k, T'_k] \subset G'_k[S'_k, T'_k]$, we have $e(G_k[S''_k, T''_k]) \geq (1/6m)e(G'_k[S''_k, T''_k])$ ($1 \leq k < m$).

We now consider S''_k, T''_k ($1 \leq k < m$), and by induction find sets $S_k^{(i)}, T_k^{(i)}$ ($i \in \{0,1\}, 1 \leq i < k$) satisfying (i), (iii), and (iv) in Lemma 15. This completes the definition of the sets $S_k^{(i)}, T_k^{(i)}$. The existence of a suitable vertex $x_0 \in V_0$ satisfying Lemma 15(ii) follows from (\dagger) . Finally, we note that Theorem 12 follows from Lemmas 14 and 15.

Proof of Theorem 12. Let $t, h \geq 3$ be fixed, and note that we may clearly assume that $t \geq 4$. Let $g = 4 + 3t$ and suppose $\mu < 1/(9t + 8)$ is a fixed real. Moreover, let p, q be primes congruent to 1 modulo 4 with p a quadratic residue modulo q . Arguing in the same way as in Section 4, we see that there are arbitrarily large primes p and q as above satisfying the additional property that $p^{t/2} \leq q \leq 2p^{t/2}$. Let us fix such a pair (p, q) with p large enough with respect to h . From the properties of p and q above, we see that the Ramanujan graph $G = G^n = X^{p,q}$ of Lubotzky, Philips, and Sarnak [18] is non-bipartite, has order $n = q(q^2 - 1)/2$, has girth $g(G) \geq t$, and is d -regular with $d = p + 1 \geq 2^{-4/9}n^{2/3t}$. We claim that G will do in Theorem 12 for the given values of t and h .

Let H be a graph of order h with $m(H) \leq 1 + \mu$ and girth $g(H) \geq g$. Let $\gamma : E(G) \rightarrow \mathbb{N}$ be a proper edge-colouring of G . We shall show that G admits a multicoloured copy of H with respect to γ . To do so, let $(F_k)_0^m$ be an s -decomposition series for H with $s = 3t + 3$. Also, fix partitions $(V_k)_0^m$ and $(C_k)_1^m$ as in the paragraph before Lemma 15, and apply that lemma. Now, the argument used in the proof of Lemma 5 and Theorem 7, coupled with a simple induction, successively gives that multicoloured copies of the graphs $K^1 = F_0 \subset F_1 \subset \dots$ are present in G , and hence a multicoloured copy of H does appear in G ,

as required. We omit the details. \square

We close with a Ramsey type result. As customary, for graphs G and H and an integer $r \geq 2$, write $G \rightarrow (H)_r$ if in any (not necessarily proper) edge-colouring of G with r -colours we may find a monochromatic copy of H . By making use of Szemerédi's regularity lemma for subgraphs of pseudo-random graphs [16], one may prove the following result (cf. [15]).

Theorem 16. *Let $r \geq 2$ be given. Then there are constants $c_0 = c_0(r) > 0$ and $p_0 = p_0(r)$ such that the following holds. Let p and q be two unequal primes such that $p, q \equiv 1 \pmod{4}$, $p \leq q^2$, and p is a square in $\mathbb{Z}/q\mathbb{Z}$. Let $t \geq 3$ be given. If $p \geq p_0$ and $q \geq q_0 = q_0(r, p, t)$, where $q_0 = q_0(r, p, t)$ depends only on r, p , and t , then $G = G^n = X^{p, q}$ is such that (i) $e(G) = (p + 1)n/2$, (ii) $g(G) \geq t$, and (iii) $G \rightarrow (C^s)_r$ for any integer s with $4 \log_2 n + 1 \leq s \leq c_0 n$. \square*

This result has a simple consequence concerning multicoloured cycles in edge-coloured graphs. Let $r \geq 1$ be given. Let us say that a (not necessarily proper) edge-colouring $\gamma : E(G) \rightarrow \mathbb{N}$ of a graph G is r -bounded if $|\gamma^{-1}(\{i\})| \leq r$ for all $i \in \mathbb{N}$. Let us write $G \xrightarrow[r]{\text{mc}} H$ if, for any r -bounded edge-colouring of G , there is a multicoloured copy of H in G . Clearly, if for two graphs G and H we have $G \rightarrow (H)_r$, then $G \xrightarrow[r]{\text{mc}} H$. Thus, Theorem 16 implies that in r -bounded edge-colourings of Ramanujan graphs, we can always find very long multicoloured cycles. This is of some interest in view of the remark in the last paragraph of Section 3. For a related result, the reader is referred to Frieze and Reed [13], where it is proved that, for some large constant A , any r -bounded edge-colouring of the complete graph K^n admits a multicoloured Hamilton cycle, where $r = r(n) = n/A \log n$. Hahn and Thomassen [14] conjecture that the statement above holds for some $r = r(n) = \Omega(n)$.

Acknowledgements. This work is part of the PhD thesis of the first author, who worked under the supervision of Dr B. Bollobás at the University of Cambridge, and was supported by an NSERC Canada 1967 Science and Engineering Scholarship. The second author was supported by Churchill College, Cambridge, where he was a Junior Research Fellow while this work was being done. The authors are indebted to Drs B. Bollobás and L. Babai for their insightful comments on previous versions of this note.

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