THE INDUCED SIZE-RAMSEY NUMBER OF CYCLES

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Abstract. For a graph H and an integer $r \geq 2$, the *induced r-size-Ramsey number* of H is defined to be the smallest integer m for which there exists a graph G with m edges with the following property: however one colours the edges of G with r colours, there always exists a monochromatic induced subgraph H' of G that is isomorphic to H. This is a concept closely related to the classical r-size-Ramsey number of Erdős, Faudree, Rousseau, and Schelp, and to the r-induced Ramsey number, a natural concept that appears in problems and conjectures due to, among others, Graham and Rödl and Trotter. Here, we prove a result that implies that the r-size-Ramsey number of the cycle C^ℓ ($\ell \geq 3$) is at most $c_r\ell$ for some constant c_r that depends only on r. Thus we settle, in a rather strong sense, a conjecture of Graham and Rödl, which states that the above holds for the path P^ℓ of order ℓ, and also generalise a result of Bollobás, Burr, and MG that states that the r-size-Ramsey number of the cycle C^ℓ is linear in ℓ. Our method of proof is heavily based on random graphs and on a variant of the well-known regularity lemma of Szemerédi.

§0. Introduction

In this article we are concerned with a basic problem in Ramsey theory: we shall show that there are very sparse graphs that have the Ramsey property with respect to long induced cycles. Before we make this precise, we give some background and terminology.

Let G and H be graphs and r a positive integer. Let us put $[r] = \{1, \dots, r\}$. We write $G \to (H)_r$ if, for any r-colouring $\chi : E(G) \to [r]$ of the edges of G, there is a *monochromatic* copy of H in G, that is, for some subgraph $H' \subset G$ of G isomorphic to H, we have that χ is constant on $E(H')$. If we are further guaranteed to find an *induced* monochromatic copy of H in G, we write $G \overset{\text{ind}}{\longrightarrow} (H)_r$. The well-known theorem of Ramsey implies that for any given graph H and any $r \geq 2$, we have $G \to (H)_r$ if G is a sufficiently large complete graph. On the other hand, a classical result proved independently by Deuber **[9]**, Erdős, Hajnal, and Pósa **[11]**, and Rödl **[19]** states that, for any graph H and any $r \geq 1$, there is a graph G such that $G \overset{\text{ind}}{\longrightarrow} (H)_r$.

For a graph G, we write $|G|$ for its *order*, that is, the number of vertices in G, and we write $e(G)$ for its *size*, the number of edges in G. Let K^n be

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the complete graph of order n. Now, for a graph H and a positive integer r, let $r(H,r) = \min\{|G| : G \to (H)_r\}$, and let $r_{\mathrm{e}}(H,r) = \min\{e(G) : G \to (H)_r\}$. Thus Ramsey's theorem guarantees that $r(H,r)$, $r_{\mathrm{e}}(H,r) < \infty$ for any H and any r. Similarly, replacing the $\to$ relation by the $\xrightarrow{\text{ind}}$ relation in the definitions above, we obtain $r^{\text{ind}}(H,r)$ and $r_{\mathrm{e}}^{\text{ind}}(H,r)$. The result of Deuber, Erdős, Hajnal, Pósa, and Rödl gives that $r^{\text{ind}}(H,r)$, $r_{\mathrm{e}}^{\text{ind}}(H,r) < \infty$ for any H and any $r \geq 1$. We call $r(H,r)$ the *r-Ramsey number* of H, and $r_{\mathrm{e}}(H,r)$ the *r-size-Ramsey* number of H (cf. **[10]**). We refer to the induced analogues of these parameters as the *induced* r-Ramsey number and as the *induced* r-size-Ramsey number of H. For simplicity, in the basic case in which the number r of colours is 2, we omit this parameter from our notation.

Many problems in Ramsey theory involve the functions r, r^{ind}, r_{e} and $r_{\mathrm{e}}^{\text{ind}}$. Undoubtedly, the best-known of these problems concerns the order of growth of the standard Ramsey number $R(n,n) = r(K^n)$. A celebrated problem of Erdős asks whether $\lim_{n\to\infty} R(n,n)^{1/n}$ exists, and, if it does exist, what the value is. (See for instance [1, Appendix B].)

We now turn to the specific problems that we shall deal with here. Let P^ℓ be the path of order ℓ. Settling a problem of Erdős, Beck **[2]** proved the rather striking result that, for any fixed positive integer r, there is a constant c_r such that $r_{\mathrm{e}}(P^\ell, r) \leq c_r\ell$ for any $\ell \geq 1$. This result suggests the following two problems. Graham and Rödl **[12]** raised the natural question whether such a linear upper bound also holds for $r^{\text{ind}}(P^\ell, r)$ for any fixed r. Moreover, writing C^ℓ for the cycle of order ℓ, the result of Beck naturally suggests investigating whether $r_{\mathrm{e}}(C^\ell, r)$ is also linear in ℓ. Our main result here settles these two questions in the affirmative. We prove that $r_{\mathrm{e}}^{\text{ind}}(C^\ell, r) \leq c_r\ell$ for some constant c_r that depends only on r.

We in fact prove more. Theorem 10 below states that, for any fixed $r \geq 2$ and any $n \geq 1$, there is a graph $G = G_r = G_r^n$ of order n and size $e(G) = O(n)$ satisfying the following property: for any r-edge-colouring of G, there is a colour c such that, for any ℓ with $B \log n \leq \ell \leq bn$, there is a monochromatic induced ℓ-cycle C^ℓ in G of colour c. Here, $B = B(r) > 0$ and $b = b(r) > 0$ are two real constants that depend only on r. In particular, for any ℓ as above, we have $G \xrightarrow{\text{ind}} (C^\ell)_r$.

Note that Theorem 10 is an intrinsically Ramsey-theoretical result. Let us introduce some notation to make this precise. Suppose G and H are graphs and γ is a real number with $0 \leq \gamma \leq 1$. We write $G \to_\gamma H$ if any subgraph $J \subset G$ with $e(J) \geq \gamma e(G)$ contains a copy of H. Moreover, let us write $G \xrightarrow[\gamma]{\text{ind}} H$ if J above necessarily contains an *induced* copy of H. It is easy to see that, owing to the odd cycles, the immediate analogues of Theorem 10 for the relations $\to_\gamma$ and $\xrightarrow[\gamma]{\text{ind}}$ cannot hold for $\gamma \leq 1/2$. If we restrict our attention to even cycles we may however prove a result analogous to Theorem 10. This 'density', rather than 'partition', type result is given in §4 (see Theorem 24). To guarantee all cycles in a range as in Theorem 10, we need to have $\gamma > 1/2$. This result is given in Theorem 25.

Our method is based on random graphs, and on a variant of the powerful lemma of Szemerédi concerning regular partitions of graphs. Part of the method was developed in **[17]** to deal with induced cycles in random graphs. Our variant of Szemerédi's lemma, given in Section 2.1 below, asserts the existence of regular partitions for subgraphs of pseudo-random graphs. We remark that this lemma was independently observed in **[16]** and by Rödl **[20]**. An overview of the proof of Theorem 10 is given in §1 below.

We close by mentioning a few related problems and results. Theorem 10 immediately implies that $r_{\mathrm{e}}(C^\ell, r) = O(\ell)$ for any fixed r, a result proved by Bollobás, Burr, and an MG[1] **[6]**. Our proof of Theorem 10 may be considerably simplified to give a direct proof of this result. In **[2]** and **[3]**, Beck investigated $r_{\mathrm{e}}(H, r)$ for the case in which H is a tree. Further results on the size-Ramsey number of trees may be found in **[13]** and in **[15]**. Beck **[3]** has also studied the *induced* size-Ramsey number of trees. It is proved in **[3]** that $r_{\mathrm{e}}^{\mathrm{ind}}(T, r) = O\{|T|^3(\log|T|)^4\}$ for any tree T and any fixed r, and it is also observed that there is a tree T_0 with $r_{\mathrm{e}}^{\mathrm{ind}}(T_0, 2) = \Omega(|T_0|^2)$. Finally, we remark that the estimation of $r^{\mathrm{ind}}(H)$ presents very interesting and challenging problems.

A problem of Graham and Rödl **[12]** asks whether $r^{\mathrm{ind}}(H) \le \exp\{c|H|\}$ for any graph H and some absolute constant $c > 0$. The best results so far are due to Rödl, who has proved that this is indeed the case for bipartite graphs H, and that for general graphs one at least has that $r^{\mathrm{ind}}(H) \le \exp\{\exp\{|H|^{1+o(1)}\}\}$ as $|H| \to \infty$ (cf. **[12]**). On the other hand, Trotter has asked whether if we consider graphs H of bounded maximal degree, then $r^{\mathrm{ind}}(H)$ is of polynomial order in $|H|$ (see **[12]**). It is worth noting that, for such graphs H, the Ramsey number $r(H)$ is indeed *linear* in $|H|$, as proved by Chvátal, Rödl, Szemerédi, and Trotter **[8]**. (See also **[4]** and **[7]**.)

§1. Sketch of the Method of Proof

In this section we outline the proof of the following result: for any fixed $r \ge 2$ and any integer $n \ge 1$, there is a graph $G = G_r = G_r^n$ of order n and size $e(G) = O(n)$ with the property that $G \xrightarrow{\mathrm{ind}} (C^\ell)_r$ for any $B\log n \le \ell \le bn$, where $B = B(r) > 0$ and $b = b(r) > 0$ are constants that depend only on r.

Here and in the sequel, G^n will always denote a graph of order n. Throughout this section we let an integer $r \ge 2$ be fixed. Let an integer $n \ge 1$ be given. It is enough to prove the existence of $G = G_r^n$ as above for large enough n. Hence we may and shall assume in the sequel that n is greater than a suitably large constant $n_0 = n_0(r)$ that depends only on r. We consider a binomial random graph $G' = G_p \in \mathcal{G}(N, p)$ where $p = p(N) = D/N$ and D is a constant that is very large with respect to r. Thus, $G' = G_p$ has vertex set $\{1, \dots, N\}$, say, and an edge ij $(1 \le i < j \le N)$ is present in G' with probability p, independently of all other edges. We *fix* a typical element $G' \in \mathcal{G}(N, p)$, and delete from G' all vertices that have 'large' degree and all edges that belong to 'short' cycles. Let G be the resulting graph. (Here we choose N a little larger than n so that we may further require G to have order n.) We claim that this deterministic graph G will do.

To prove this claim, fix an arbitrary r-edge-colouring of G. Now, by invoking the variant of Szemerédi's regularity lemma mentioned in the introduction, we may choose a colour i for which the argument below works. The conditions that we require on this colour i and the general set-up on which the rest of the proof is

[1] Mysterious gentleman

based are hard to describe briefly. Thus, let us assume for now that this colour has somehow been chosen. Let J be the spanning subgraph of G whose edges are the edges of G coloured i.

We first need to show that J contains a short induced cycle C that is also induced in G. It turns out that we may guarantee the existence of such a cycle C of length logarithmic in n. We now assume that we are given an induced ℓ-cycle C_0 in J that is also induced in G, where ℓ is neither too small nor too large. In the rest of the proof we show that we may 'enlarge' this cycle C_0 to an induced $(\ell+1)$-cycle C' of J that is also induced in G. Provided we succeed in showing that this process may be carried out for reasonably short and reasonably long cycles C_0, we are done. More precisely, it suffices to show that this is indeed possible for ℓ in the range $c\log n \le \ell \le n/c$ for some constant $c = c_r > 0$.

The 'enlarging' procedure is as follows: we simply find in C_0 a segment of length L that may be replaced by a path of length $L+1$ to give the required cycle C'. Here $L = \Theta(\log n)$ is to be chosen suitably. Not all segments of C_0 admit a path as above. Thus we need to try several possibilities before we find a good segment.

We first fix two vertices $x_0^{(1)}$, $x_0^{(2)}$ that determine a segment of length L in C_0. In the main iterative part of our 'enlarging' algorithm (cf. Section 3.2), we look for suitable sets of vertices $X_0^{(\sigma)}, \dots, X_k^{(\sigma)}$ ($\sigma \in \{1,2\}$). These $X_i^{(\sigma)}$ grow geometrically with i, $X_0^{(\sigma)} = \{x_0^{(\sigma)}\}$, and $|X_k^{(\sigma)}| = \Omega(n)$ ($\sigma \in \{1,2\}$). Moreover, they have the following further property: for any $x^{(1)} \in X_k^{(1)}$, $x^{(2)} \in X_k^{(2)}$, we may find vertices $x_i^{(\sigma)} \in X_i^{(\sigma)}$ ($1 \le i \le k$, $\sigma \in \{1,2\}$) for which $x_k^{(1)} = x^{(1)}$, $x_k^{(2)} = x^{(2)}$, the paths $P^{(\sigma)} = x_0^{(\sigma)} \dots x_k^{(\sigma)}$ ($\sigma \in \{1,2\}$) are induced paths of J, and (*) the only edges induced in G by $V(C_0) \cup \{x_i^{(\sigma)} : 1 \le i \le k,\ \sigma \in \{1,2\}\}$ are the $\ell + 2k$ edges in $E(C_0) \cup E(P^{(1)}) \cup E(P^{(2)})$.

However, for a given pair $(x_0^{(1)}, x_0^{(2)})$, we may fail to find such sets $X_i^{(\sigma)}$. In such a case, we find a set $Q \subset V(G)$ of vertices of G that together with $V(C_0)$ (and some other vertices) induce a 'dense' subgraph in G, *i.e.* a subgraph with many edges. We successively search for the sets $X_i^{(\sigma)}$ starting with many distinct pairs $(x_0^{(1)}, x_0^{(2)})$, and show that we eventually succeed in finding the $X_i^{(\sigma)}$ for some pair $(x_0^{(1)}, x_0^{(2)})$ (see Lemma 17). Roughly speaking, we use the fact that, if we were to fail starting from many such pairs $(x_0^{(1)}, x_0^{(2)})$, we would be able to find in G a subgraph that is far too dense for a subgraph of a random graph.

The argument above guarantees $|X_k^{(\sigma)}| = \Omega(n)$ ($\sigma \in \{1,2\}$), but the constant in the Ω-notation is quite small. In the second part of the 'enlarging' algorithm, we extend the sequences $X_0^{(\sigma)}, \dots, X_k^{(\sigma)}$ ($\sigma \in \{1,2\}$). We define $X_{k+1}^{(\sigma)}, \dots, X_{k_1}^{(\sigma)}$ with the $X_i^{(\sigma)}$ again growing geometrically with i, and with $|X_{k_1}^{(\sigma)}| \ge cn$, where $c > 0$ is to be chosen suitably. To define such sets $X_i^{(\sigma)}$ ($k < i \le k_1$, $\sigma \in \{1,2\}$), roughly speaking, we weaken property (*). Namely, we require that for any $x^{(1)} \in X_{k_1}^{(1)}$, $x^{(2)} \in X_{k_2}^{(2)}$, we may find vertices $x_i^{(\sigma)} \in X_i^{(\sigma)}$ ($1 \le i \le k$, $\sigma \in \{1,2\}$) for which $x_{k_1}^{(1)} = x^{(1)}$, $x_{k_1}^{(2)} = x^{(2)}$, $P^{(\sigma)} = x_0^{(\sigma)} \dots x_{k_1}^{(\sigma)}$ is a path in J ($\sigma \in \{1,2\}$), and (†) the only edges induced in G by $V(C_0) \cup \{x_i^{(\sigma)} : 1 \le i \le k_1,\ \sigma \in \{1,2\}\}$ are the ones in $E(C_0) \cup E(P^{(1)}) \cup E(P^{(2)})$, except for possibly some edges e incident to vertices in $\bigcup X_{k'}^{(\sigma)}$, where the union ranges over $\sigma \in \{1,2\}$, $k_0 < k' \leq k_1$, and k_0 satisfies $k_1 - k_0 = O(1)$.

Picking c large enough, we may guarantee that there is an $X_{k_1}^{(1)}$–$X_{k_1}^{(2)}$ edge $x^{(1)}x^{(2)}$ in J. This edge together with $P^{(1)}$ and $P^{(2)}$ give the path of length $L+1$ that we use to construct C'. The fact that C' is an induced cycle in G follows from (†) and the fact that G has large girth. (The edges e mentioned in (†) do not occur in C' as they would give short cycles in G.)

Let us give an outline of the contents of the following sections. The method used to choose the colour i in the argument above is given in Section 2.1. In Section 2.2 we compile the results concerning random graphs that we shall need. In Section 2.3 we give the construction of the graph $G = G_r^n$ sketched above. There we also state our first main result, Theorem 10. In §3 we give the proof of Theorem 10. In §4 we state two related results.

§2. Preliminary Results

2.1. Szemerédi's Lemma. Let a graph $G = G^n$ of order $|G| = n$ be fixed. For U, $W \subset V = V(G)$ with $U \cap W = \emptyset$, we write $E(U,W) = E_G(U,W)$ for the set of edges of G that have one endvertex in U and the other in W. We set $e(U,W) = e_G(U,W) = |E(U,W)|$. The following notion will be needed in the sequel. Suppose $0 \leq \eta \leq 1$ and $0 \leq p \leq 1$. We say that G is *η-uniform* with *density p* if, for all U, $W \subset V$ with $U \cap W = \emptyset$ and $|U|$, $|W| \geq \eta n$, we have

$$\big|e_G(U,W) - p|U||W|\big| \leq \eta p|U||W|.$$

Now let $H \subset G$ be a spanning subgraph of G. For U, $W \subset V$ with $U \cap W = \emptyset$, let

$$d_{H,G}(U,W) = \begin{cases} e_H(U,W)/e_G(U,W) & \text{if } e_G(U,W) > 0 \\ 0 & \text{if } e_G(U,W) = 0. \end{cases}$$

Suppose $\varepsilon > 0$, U, $W \subset V$, and $U \cap W = \emptyset$. We say that the pair (U,W) is *(ε, H, G)-regular*, or simply *ε-regular*, if for all $U' \subset U$, $W' \subset W$ with $|U'| \geq \varepsilon|U|$ and $|W'| \geq \varepsilon|W|$, we have

$$|d_{H,G}(U',W') - d_{H,G}(U,W)| \leq \varepsilon.$$

Now let $r \geq 1$ spanning subgraphs $H_1, \ldots, H_r \subset G$ of G be given. The pair (U,W) is said to be *$(\varepsilon, H_1, \ldots, H_r, G)$-regular* if it is (ε, H_i, G)-regular for all $1 \leq i \leq r$.

We say that a partition $P = (V_i)_0^k$ of $V = V(G)$ is *(ε,k)-equitable* if $|V_0| \leq \varepsilon n$, and $|V_1| = \ldots = |V_k|$. Also, we say that V_0 is the *exceptional* class of P. When the value of ε is not relevant, we refer to an (ε,k)-equitable partition as a *k-equitable* partition. Similarly, P is an *equitable* partition of V if it is a k-equitable partition for some k. Finally, we say that an (ε,k)-equitable partition $P = (V_i)_0^k$ of V is *$(\varepsilon, H_1, \ldots, H_r, G)$-regular*, or simply *$\varepsilon$-regular*, if at most $\varepsilon\binom{k}{2}$ pairs (V_i, V_j) with $1 \leq i < j \leq k$ are not $(\varepsilon, H_1, \ldots, H_r, G)$-regular. We can now state the extension of Szemerédi's lemma to subgraphs of η-uniform graphs.

LEMMA 1. *For given $\varepsilon > 0$ and k_0, $r \geq 1$, there are constants $\eta = \eta(\varepsilon, k_0, r) > 0$ and $K_0 = K_0(\varepsilon, k_0, r) \geq k_0$ that depend only on ε, k_0, and r for which the following holds. If G is an η-uniform graph and $H_1, \dots, H_r \subset G$ are r spanning subgraphs of G, then there is an $(\varepsilon, H_1, \dots, H_r, G)$-regular (ε, k)-equitable partition of $V = V(G)$ with $k_0 \leq k \leq K_0$.* $\square$

Lemma 1 has been observed independently by Rödl **[20]** and Kohayakawa **[16]**. We shall not give a proof of this result here since, once the set-up is clear, natural modifications to the proof in **[21]** of Szemerédi's original lemma give Lemma 1.

Now suppose $H \subset G$ is a spanning subgraph of a graph G. Let V_1, V_2, $V_3 \subset V(G)$ be three pairwise disjoint sets of vertices, and let $\varepsilon > 0$ be given. In the sequel, we say that the triple (V_1, V_2, V_3) is an (ε, H, G)-*regular triple* if all pairs (V_i, V_j) with $1 \leq i < j \leq 3$ are (ε, H, G)-regular.

LEMMA 2. *Let $0 < \varepsilon \leq 1/5$ and $0 < \rho \leq 1$ be given, and suppose $0 < \eta \leq \varepsilon/8$. Let $G = G^n$ be an η-uniform graph, and let $J \subset G$ be a spanning subgraph of G. Suppose (V_1, V_2, V_3) is an (ε, J, G)-regular triple with $d_{i,j} = d_{J,G}(V_i, V_j) \geq \rho$ for all $1 \leq i < j \leq 3$. Moreover, assume $|V_i| \geq (\eta/\varepsilon)n$ for $i \in \{1, 2, 3\}$. Then there are sets $\bar{V}_i \subset V_i$ $(i \in \{1, 2, 3\})$ such that*

$$\left|\Gamma_J(x) \cap \bar{V}_j\right| \geq (1 - 5\varepsilon/\rho) d_{i,j} e_G(V_i, V_j)/|V_i| \tag{1}$$

for all $x \in \bar{V}_i$ and $j \neq i$ $(i, j \in \{1, 2, 3\})$. Moreover, $|\bar{V}_i| \geq (1 - 2\varepsilon)|V_i|$ for all $i \in \{1, 2, 3\}$ and, in particular, $(\bar{V}_1, \bar{V}_2, \bar{V}_3)$ is $(2\varepsilon, J, G)$-regular.

Proof. We first define three sequences of sets $V_i = V_i^{(0)} \supset V_i^{(1)} \supset \cdots$ $(i \in \{1, 2, 3\})$ and a sequence of vertices $x_0, x_1, \dots$ by induction as follows. Put $V_i^{(0)} = V_i$ for $i \in \{1, 2, 3\}$. Now let $s \geq 1$ and assume $V_i^{(0)} \supset \cdots \supset V_i^{(s-1)}$ $(i \in \{1, 2, 3\})$ and $x_0, \dots, x_{s-2}$ already defined. If $|V_i^{(s-1)}| \leq (1-2\varepsilon)|V_i|$ for some $i \in \{1, 2, 3\}$, terminate the sequences. Otherwise, if putting $(\bar{V}_1, \bar{V}_2, \bar{V}_3) = (V_1^{(s-1)}, V_2^{(s-1)}, V_3^{(s-1)})$ condition (1) holds for all $x \in \bar{V}_i$ and all $j \neq i$ $(i, j \in \{1, 2, 3\})$, then terminate the sequences. However, if putting $(\bar{V}_1, \bar{V}_2, \bar{V}_3) = (V_1^{(s-1)}, V_2^{(s-1)}, V_3^{(s-1)})$ condition (1) fails for some $x_{s-1} \in V_{i_{s-1}}^{(s-1)}$ and $j_{s-1} \neq i_{s-1}$ $(i_{s-1}, j_{s-1} \in \{1, 2, 3\})$, then put $V_{i_{s-1}}^{(s)} = V_{i_{s-1}}^{(s-1)} \setminus \{x_{s-1}\}$ and $V_i^{(s)} = V_i^{(s-1)}$ for $i \neq i_{s-1}$. This completes the definition of the $V_i^{(s)}$ and of the x_s. Suppose we have obtained the sequences $\big(V_i^{(s)}\big)_{i=1}^t$ $(i \in \{1, 2, 3\})$. We claim that $|V_i^{(t)}| \geq (1 - 2\varepsilon)|V_i|$ for $i \in \{1, 2, 3\}$.

Assume the contrary, and suppose without loss of generality that $|V_1^{(t)}| \leq (1 - 2\varepsilon)|V_1|$ and $|V_i^{(t)}| > (1 - 2\varepsilon)|V_i|$ for $i = 2$, 3. Then, for any $x \in V_1 \setminus V_1^{(t)}$, there are $s = s_x$ and $j = j_x \in \{2, 3\}$ such that $x = x_{s-1}$ and

$$\left|\Gamma_J(x) \cap V_j^{(t)}\right| \leq \left|\Gamma_J(x) \cap V_j^{(s-1)}\right| < (1 - 5\varepsilon/\rho) d_{1,j} e_G(V_1, V_j)/|V_1|.$$

We may assume that there is a set $U \subset V_1 \setminus V_1^{(t)}$ with $|U| \geq |V_1 \setminus V_1^{(t)}|/2 \geq \varepsilon|V_1|$ such that, for all $x \in U$, we have $|\Gamma_J(x) \cap V_2^{(t)}| \leq (1 - 5\varepsilon/\rho) d_{1,2} e_G(V_1, V_2)/|V_1|$. Then

$$e_J(U, V_2^{(t)}) = \sum_{x \in U} \left|\Gamma_J(x) \cap V_2^{(t)}\right| \leq (1 - 5\varepsilon/\rho) d_{1,2} |U| e_G(V_1, V_2)/|V_1|,$$

and hence

$$d_{J,G}(U,V_2^{(t)}) \le (1-5\varepsilon/\rho)d_{1,2}\frac{|U|}{|V_1|}\cdot\frac{e_G(V_1,V_2)}{e_G(U,V_2^{(t)})}.$$

By the η-uniformity of G, we have

$$e_G(U,V_2^{(t)}) \ge p|U||V_2^{(t)}| - \eta p|U||V_2^{(t)}| \ge p(1-\eta)(1-2\varepsilon)|U||V_2|,$$

and

$$e_G(V_1,V_2) \le p(1+\eta)|V_1||V_2|.$$

Therefore

$$d_{J,G}(U,V_2^{(t)}) \le \left\{1-\frac{5\varepsilon}{\rho}\right\}\frac{1+\eta}{1-\eta}\cdot\frac{d_{1,2}}{1-2\varepsilon} < \left\{1-\frac{5\varepsilon}{\rho}+4\varepsilon\right\}d_{1,2} \le d_{1,2}-\varepsilon,$$

which is a contradiction, as $|U| \ge \varepsilon|V_1|$, $|V_2^{(t)}| \ge (1-2\varepsilon)|V_2| \ge \varepsilon|V_2|$, and (V_1,V_2) is (ε,J,G)-regular. Thus our claim holds. It follows that, putting $(\bar V_1,\bar V_2,\bar V_3) = (V_1^{(t)},V_2^{(t)},V_3^{(t)})$, condition (1) holds for all $x \in \bar V_i$ and all $j \ne i$ (i, $j \in \{1,2,3\}$). Moreover, since $|\bar V_i| \ge (1-2\varepsilon)|V_i| \ge |V_i|/2$ ($i \in \{1,2,3\}$), as a simple argument shows, $(\bar V_1,\bar V_2,\bar V_3)$ is a $(2\varepsilon,J,G)$-regular triple. □

The following is an easy consequence of Turán's theorem **[22]**.

LEMMA 3. *Let an integer $\alpha \ge 1$ be given, and suppose $H = H^k$ is a graph of order $k > (\alpha-1)^2$ such that $e(H) \le \alpha^{-1}\binom{k}{2}$. Then $\alpha(H) \ge \alpha$.*

Proof. By Turán's theorem, if $\alpha(H) < \alpha$, then $e(H) \ge e(K^{k_1} \cup \cdots \cup K^{k_{\alpha-1}})$, where $k_i = \lfloor (k+i-1)/(\alpha-1)\rfloor$ $(1 \le i < \alpha)$ and $K^{k_1} \cup \cdots \cup K^{k_{\alpha-1}}$ is the disjoint union of the K^{k_i} $(1 \le i < \alpha)$. Therefore, if $\alpha(H) < \alpha$, we have

$$e(H) \ge \sum_{1\le i<\alpha}\binom{k_i}{2} \ge (\alpha-1)\binom{k/(\alpha-1)}{2} = \frac{k^2}{2(\alpha-1)}\left(1-\frac{\alpha-1}{k}\right) > \frac{1}{\alpha}\binom{k}{2},$$

where the last inequality follows from $k > (\alpha-1)^2$. □

We are now able to state and prove the main lemma of this section, Lemma 4. This result tells us how to 'choose' the colour i in the argument sketched in §1. Recall that G^n always denotes a graph of order n.

LEMMA 4. *Let $r \ge 2$ and $0 < \varepsilon < 1$ be given. Then there are constants $\eta = \eta(r,\varepsilon) > 0$ and $\mu = \mu(r,\varepsilon) > 0$ for which the following holds. Suppose $G = G^n$ is an η-uniform graph with density $p = d/n$, and let $E(G) = E_1 \cup \cdots \cup E_r$ be an r-edge-colouring χ of G. Let $G_i \subset G$ be the spanning subgraph of G with edge set E_i $(1 \le i \le r)$. Then, for some $1 \le i = i(G,\chi) \le r$ and $\mu \le \bar\mu = \bar\mu(G,\chi) \le 1$, there are pairwise disjoint sets $\bar V_1$, $\bar V_2$, $\bar V_3 \subset V(G)$ such that $(\bar V_1,\bar V_2,\bar V_3)$ is an (ε,G_i,G)-regular triple and, for all $a \ne b$ (a, $b \in \{1,2,3\}$), we have $d_{G_i,G}(\bar V_a,\bar V_b) \ge 1/r$ and $\left|\Gamma_{G_i}(x) \cap \bar V_b\right| \ge \bar\mu d/2r$ for all $x \in \bar V_a$. Moreover, $\bar\mu n/2 \le |\bar V_i| \le \bar\mu n$ for all $a \in \{1,2,3\}$.*

Proof. We start by invoking Ramsey's theorem. Let $k_1 = R_r(3)$ be the least integer R such that any r-edge-coloured complete graph of order R contains a monochromatic triangle. Put $k_0 = R_r(3)^2$. Let $\varepsilon' = \min\{R_r(3)^{-1}, 1/15r, \varepsilon/2\}$. Let $\eta = \eta(\varepsilon', k_0, r) > 0$ and $K_0 = K_0(\varepsilon', k_0, r) \geq k_0$ be as given by Lemma 1. We may clearly assume that $\eta < \varepsilon'/2K_0$. Put $\mu = 1/2K_0$. We now check that η and μ above will do for our lemma. Thus let $G = G^n$ be an η-uniform graph with density $p = d/n$, and suppose $E(G) = E_1 \cup \cdots \cup E_r$. Let $G_i \subset G$ be the spanning subgraph of G with $E(G_i) = E_i$ $(1 \leq i \leq r)$. We now apply Lemma 1 to $G_1, \ldots, G_r \subset G$ to obtain an $(\varepsilon', G_1, \ldots, G_r, G)$-regular (ε', k)-equitable partition $P = (V_i)_1^k$ of $V = V(G)$ with $k_0 \leq k \leq K_0$. Let $\bar{\mu} = |V_j|/n$ $(1 \leq j \leq k)$. By Lemma 3, for some $1 \leq j_1 < \cdots < j_{k_1} \leq k$, all pairs (V_{j_a}, V_{j_b}) with $1 \leq a < b \leq k_1$ are $(\varepsilon', G_1, \ldots, G_r, G)$-regular. By the choice of $k_1 = R_r(3)$, without loss of generality we may assume that, for some $1 \leq i \leq r$, the pairs (V_{j_a}, V_{j_b}) with $1 \leq a < b \leq 3$ are such that $d_{G_i,G}(V_{j_a}, V_{j_b}) \geq 1/r$. We now apply Lemma 2 to the (ε', G_i, G)-regular triple $(V_{j_1}, V_{j_2}, V_{j_3})$. Then we obtain $\bar{V}_a \subset V_{j_a}$ $(a \in \{1, 2, 3\})$ such that $\bar{\mu}n = |V_{j_a}| \geq |\bar{V}_a| \geq (1 - 2\varepsilon')|V_{j_a}| \geq |V_{j_a}|/2 = \bar{\mu}n/2$ and, moreover, such that for all $x \in \bar{V}_a$ and $b \neq a$ $(a,\, b \in \{1, 2, 3\})$, we have

$$\left|\Gamma_{G_i}(x) \cap \bar{V}_b\right| \geq (1 - 5\varepsilon' r)\frac{e_G(V_a, V_b)}{r|V_a|} \geq (1 - 5\varepsilon' r)(1 - \eta)p|V_b|/r \geq \bar{\mu}d/2r.$$

Finally, since $\varepsilon' \leq \varepsilon/2$, the triple $(\bar{V}_1, \bar{V}_2, \bar{V}_3)$ is (ε, G_i, G)-regular. □

For convenience, we introduce the following simple definition. Let J be a bipartite graph with a fixed bipartition, say $V(J) = X \cup Y$. Then, we shall say that J is a *(b, f)-expander*, and that it is *(b, f)-expanding*, if for all $U \subset X$ and $U \subset Y$ such that $|U| \leq b$ we have $|\Gamma_J(U)| \geq f|U|$. Also, if G is a graph and U, $W \subset V(G)$ are such that $U \cap W = \emptyset$, then we write $G[U, W]$ for the bipartite subgraph of G with vertex classes U, W and with edge set $E(U, W) = E_G(U, W)$. Lemma 5 below tells us that the colour i and the triple $(\bar{V}_1, \bar{V}_2, \bar{V}_3)$ given by Lemma 4 determine three expanding bipartite graphs $G_i[\bar{V}_a, \bar{V}_b]$ $(1 \leq a < b \leq 3)$. To prove this, however, we need to introduce another uniformity condition for graphs.

Let $G = G^n$ be a graph of order n, and suppose $A > 0$ and $0 \leq p \leq 1$. Let $d = pn$. We say that G is *(p, A)-upper-uniform* if, for all sets U, $W \subset V(G)$ with $U \cap W = \emptyset$ and $1 \leq |U| \leq |W| \leq d|U|$, we have

$$e_G(U, W) \leq p|U||W| + A\{d|U||W|\}^{1/2}. \tag{2}$$

Moreover, if for all such U, $W \subset V(G)$ we have

$$\left|e_G(U, W) - p|U||W|\right| \leq A\{d|U||W|\}^{1/2},$$

we say that G is *(p, A)-uniform*.

LEMMA 5. *Let $r \geq 2$ and $0 < \varepsilon < 1$ be given. Let $\eta = \eta(r, \varepsilon) > 0$ and $\mu = \mu(r, \varepsilon) > 0$ be as in Lemma 4, and suppose that $G = G^n$ is an η-uniform graph with density $p = d/n$ and that $E(G) = E_1 \cup \cdots \cup E_r$ is an r-edge-colouring of G. Let $1 \leq i \leq r$, $\bar{\mu} \geq \mu$, and $(\bar{V}_1, \bar{V}_2, \bar{V}_3)$ be as in Lemma 4, and let J be the spanning subgraph of G with $E(G_i) = E_i$. Put $J_{a,b} = J[\bar{V}_a, \bar{V}_b]$ for all $1 \leq a < b \leq 3$. Then,* *if G is (p, A)-upper-uniform for some $A \geq 1$, every $J_{a,b}$ is $(\bar{\mu}n/3rf, f)$-expanding for any $0 < f \leq (\bar{\mu}/6Ar)^2 d$.*

Proof. Fix $a \neq b$ (a, $b \in \{1, 2, 3\}$), and put $\widetilde{J} = J_{a,b}$. Suppose $\sigma \in \{a, b\}$ and $0 < f \leq (\bar{\mu}/6Ar)^2 d$. Let $U \subset V_\sigma$ with $u = |U| \leq \bar{\mu}n/3rf$ be fixed. Now let $W = \Gamma_{\widetilde{J}}(U)$, and suppose for a contradiction that $w = |W| < f|U|$. Then

$$\frac{\bar{\mu}}{2r} du \leq e_{\widetilde{J}}(U, W) \leq e_G(U, W) \leq puw + A(duw)^{1/2} \leq \frac{\bar{\mu}}{3r} du + A(duw)^{1/2},$$

and hence $\bar{\mu}du/6r \leq A(duw)^{1/2}$. Therefore $|W| = w \geq (\bar{\mu}/6Ar)^2 du \geq f|U|$, which is a contradiction. □

2.2. Random Graphs. Given α, $\delta > 0$, we say that a graph $G = G^n$ is (α, δ)-*locally sparse* if, for all $U \subset V(G)$ with $|U| \leq \alpha n$, we have $e(G[U]) \leq (1 + \delta)|U|$. The following lemma may be found in Łuczak **[17]**.

Lemma 6. *Let $d > 1$ be fixed, and consider the random graph $G_p = G_{n,p} \in \mathcal{G}(n, p)$ where $p = p(n) = d/n$. Then for any fixed $\delta > 0$ there is a constant $\alpha = \alpha(d, \delta) > 0$ such that almost every G_p is (α, δ)-locally sparse.* □

The following is immediate from standard estimates for tails of the binomial distribution.

Lemma 7. *Let $0 < \eta < 1$ be given, and consider the random graph $G_p = G_{n,p} \in \mathcal{G}(n, p)$ with $0 < p = p(n) < 1$. Put $d = d(n) = np(n)$. Then, there is a constant $d_0 = d_0(\eta)$ such that, if $d \geq d_0$, almost every G_p is η-uniform with density p.* □

We now verify that random graphs satisfy the rather strong uniformity condition defined just before Lemma 5.

Lemma 8. *Let $d = d(n) > 0$ be given, and put $p = p(n) = d/n$. Then a.e. $G_p = G_{n,p} \in \mathcal{G}(n, p)$ is $(p, \mathrm{e}^2\sqrt{6})$-uniform.*

Proof. We may clearly assume that $d \geq 1$. Let $\mathcal{F} = \{(U, W) : U, W \subset V = V(G_p), 1 \leq |U| \leq |W| \leq d|U|, U \cap W = \emptyset\}$. In what follows, we shall always have $(U, W) \in \mathcal{F}$, and $|U| = u$, $|W| = w$. We set

$$P_{u,w} = P(U, W) = \mathbb{P}\left[\left|e_{G_p}(U, W) - p|U||W|\right| > A\{d|U||W|\}^{1/2}\right].$$

Our aim is to show that $E = \sum_{(U,W) \in \mathcal{F}} P(U, W) = o(1)$ as $n \to \infty$. Let us put $\mu = \mu(U, W) = puw$, $b = b(U, W) = A\{duw\}^{1/2}$, and $\eta = \eta(U, W) = b/\mu = An(duw)^{-1/2}$. Let $\mathcal{F}_1 = \{(U, W) \in \mathcal{F} : \eta \leq \mathrm{e}^2\}$, $\mathcal{F}_2 = \mathcal{F} \setminus \mathcal{F}_1$, and set $E_i = \sum_{(U,W) \in \mathcal{F}_i} P(U, W)$ ($i = 1, 2$). We now claim that $E_i = o(1)$ for both $i = 1$ and 2.

(1) *We have $E_1 = o(1)$.*

Suppose $(U, W) \in \mathcal{F}_1$. We claim that $P(U, W) \leq 2\exp\{-(A^2/3\mathrm{e}^4)n\}$. To check this claim, let us first assume that $\eta = \eta(U, W) \leq 1$. Then $\eta^2\mu = A^2 n$, and hence $P(U, W) \leq 2\exp\{-(A^2/3)n\}$ by Hoeffding's inequality **[14]** (see also McDiarmid **[18]**), and our claimed estimate for $P(U, W)$ follows in this case. Suppose now that $1 \le \eta \le \mathrm{e}^2$. Then, again by Hoeffding's inequality, we have $P(U,W) \le \mathbb{P}(e(U,W) > 2\mu) \le \exp\{-\mu/3\}$. Note that $b/\mu = \eta \le \mathrm{e}^2$ gives that $\mu \ge b/\mathrm{e}^2 = (A/\mathrm{e}^2)(duw)^{1/2}$. Also, we have $An(duw)^{-1/2} = \eta \le \mathrm{e}^2$, and therefore $(duw)^{1/2} \ge (A/\mathrm{e}^2)n$. Thus $\mu \ge (A^2/\mathrm{e}^4)n$, and $P(U,W) \le \exp\{-(A^2/3\mathrm{e}^4)n\}$, as required. The upper bound for E_1 now follows easily. Indeed,

$$E_1 = \sum_{(U,W)\in\mathcal{F}_1} P(U,W) \le 2 \times 4^n \exp\left\{-\frac{A^2}{3\mathrm{e}^4}n\right\} = o(1).$$

(2) *We have $E_2 = o(1)$.*
Suppose $(U,W) \in \mathcal{F}_2$. Then $P(U,W) = \mathbb{P}(e(U,W) > \mu + b) \le \mathbb{P}(e(U,W) > b)$. Let v be such that $b = \mathrm{e}v\mu/\log v$. Then $\mathrm{e}v/\log v = b/\mu = \eta \ge \mathrm{e}^2$, and hence we may suppose $v \ge \mathrm{e}$. Also, we have $\mathrm{e}v\mu/\log v = b \ge 1 \ge \mathrm{e}/v$, and so $v^2\mu \ge \log v$. Thus $P_u \le \mathbb{P}(e(U,W) \ge b) \le \exp\{-v\mu\}$ (see Theorem 7(*ii*) in Chapter I of **[5]**). Now, we have $v\mu = (b/\mathrm{e})\log v \ge (A/\mathrm{e})(duw)^{1/2}(\log v) \ge (A/\mathrm{e})(duw)^{1/2}$. Thus $v \ge (A/\mathrm{e})n(duw)^{-1/2}$, and so $v\mu \ge (A/\mathrm{e})(duw)^{1/2}\log\{(A/\mathrm{e})n(duw)^{-1/2}\}$. Therefore

$$P(U,W) \le \exp\{-v\mu\} \le \left(\frac{\mathrm{e}(duw)^{1/2}}{An}\right)^{(A/\mathrm{e})(duw)^{1/2}}. \tag{3}$$

Recall that $u \le w \le du$, and so, setting $r = u+w$, we have $(duw)^{1/2} \ge dr/(d+1) \ge r/2$. Now let $x = \mathrm{e}/\eta = \mathrm{e}(duw)^{1/2}/An$, and note that then $0 < x \le 1/\mathrm{e}$, and that (3) states that $P(U,W) \le x^{Bx}$, where $B = (A/\mathrm{e})^2 n$. Since x^x is decreasing for $0 < x \le 1/\mathrm{e}$, we have from (3) that $P(U,W) \le \{(\mathrm{e}/2A)(r/n)\}^{(A/2\mathrm{e})r}$. Thus

$$\binom{n}{r}P(U,W) \le \left(\frac{\mathrm{e}n}{r}\right)^r\left(\frac{\mathrm{e}r}{2An}\right)^{(A/2\mathrm{e})r} = \left(\frac{\mathrm{e}n}{r}\left(\frac{\mathrm{e}r}{2An}\right)^{A/2\mathrm{e}}\right)^r \le \left(\frac{r}{4n}\right)^r. \tag{4}$$

For $1 \le s \le n$ and $1 \le t \le n-s$, let $P_{s,t} = P(S,T)$, where S, $T \subset V$ are such that $S \cap T = \emptyset$, and $|S| = s$, $|T| = t$. Then we have $E_2 = \sum_{(U,W)\in\mathcal{F}_2} P(U,W) = \sum^* \binom{n}{r}\binom{r}{u}P_{u,r-u}$, where $\sum^*$ denotes sum over all $2 \le r \le n$ and $1 \le u \le r/2$ such that $w = r - u \le du$ and $\eta = \eta(U,W) \ge \mathrm{e}^2$. Thus, by (4), we have that E_2 is at most

$$\sum\nolimits^* \binom{r}{u}\left(\frac{r}{4n}\right)^r \le \sum_{2\le r\le n}\left(\frac{r}{2n}\right)^r \le 2n^{-2} = o(1),$$

as required.

Thus $E = E_1 + E_2 = o(1)$, and the proof is complete. □

2.3. Definition of $G = G_r$. In this section we define our graph G that has the Ramsey property for long induced cycles. Throughout this section, an integer $r \ge 2$ is fixed. We now define some numerical constants that depend solely on r. Put $\varepsilon = 1/48r$, and let $\eta = \eta(r) = \eta(r,\varepsilon) > 0$ and $\mu = \mu(r) = \mu(r,\varepsilon) > 0$ be as given in Lemma 4. We may assume that $\mu \le \varepsilon$. Put $\delta = 1/140$. Fix $D = D(r) \ge 8 \times 10^5(r/\mu)^2$ such that $G_p = G_{N,p} \in \mathcal{G}(N,p)$ is $(\eta/2)$-uniform with density $p = D/N$ with probability $1 - o(1)$ as $N \to \infty$ (cf. Lemma 7). Let $\alpha = \alpha(D,\delta) > 0$ be such that G_p is (α,δ)-locally sparse with probability $1 - o(1)$ as $N \to \infty$ (cf. Lemma 6). Clearly, we may assume that $\alpha \le \mu$. Let $f_0 = 16$ and $f = 2$. Let $\gamma = \alpha/24\mu$, and $\beta = \gamma/D$. Also, put $b = \beta\mu/6$ and $\gamma_0 = \beta\mu/20$. Finally, set $g = 2\lfloor\log_f(\varepsilon/\gamma_0)\rfloor + 1$, and $B = 2/\log f$.

Lemma 9. *Let an integer $r \geq 2$ be fixed, and let $\eta = \eta(r) > 0$, $\delta = 1/140$, $D = D(r)$, $\alpha = \alpha(D, \delta) > 0$, and $g = g(r)$ be as defined above. Put $d = D/2$. Then, for any sufficiently large $n \geq 1$, there is a graph $G = G_r = G_r^n$ such that (i) the maximal degree $\Delta(G)$ of G is at most $8d$, (ii) G is η-uniform with density $p = d/n$, (iii) G is $(p, \mathrm{e}^2 2\sqrt{}3)$-upper uniform, (iv) G is (α, δ)-locally sparse, (v) G has girth $g(G) > g$.*

Proof. Put $N = 2n$, and note that $p = d/n = D/N$. We consider $G_p = G_{N,p} \in \mathcal{G}(N, p)$ and show that, with probability $1 - o(1)$ as $n \to \infty$, a suitable subgraph $G \subset G_p$ will do. By Markov's inequality, the degree $d(x)$ of a fixed vertex $x \in G_p$ is such that $\mathbb{P}(d(x) > 4D) < 1/4$. Thus, the expectation $\mathbb{E}(X)$ of the number $X = X(G_p)$ of vertices $x \in G_p$ with $d(x) > 4D$ is less than $N/4$. Again by Markov's inequality, we have $\mathbb{P}(X \geq N/2) < 1/2$. Now let $Z_j = Z_j(G_p)$ be the number of cycles of length j in G_p ($j \geq 3$), and let $Z = \sum_{3 \leq j \leq g} Z_j$. Then

$$\mathbb{E}(Z) = \sum_{3\leq j\leq g} \mathbb{E}(Z_j) = \sum_{3\leq j\leq g} \binom{N}{j} \frac{(j-1)!}{2} p^j \leq \sum_{3\leq j\leq g} \frac{D^j}{2j} \leq \frac{2D^g}{3g}.$$

Thus $\mathbb{P}(Z \geq 2D^g/g) \leq 1/3$. Recall that by the choice of D and α, we have with probability $1 - o(1)$ as $N \to \infty$ that G_p is $(\eta/2)$-uniform, $(p, \mathrm{e}^2\sqrt{}6)$-upper-uniform, and (α, δ)-locally-sparse. Let $N = 2n$ be large enough and fix a G_p satisfying these three properties, and such that $X = X(G_p) \leq N/2$ and $Z = Z(G_p) \leq 2D^g/g$. We now let $G' \subset G_p$ be an n-vertex induced subgraph of G_p such that $\Delta(G') \leq 8d$, and omit at most $2D^g/g$ edges from G' to obtain a graph G of girth $g(G) > g$. We claim that G will do. Clearly (i), (iv), and (v) of our lemma hold. Property (iii) holds as well, since the error term in (2) for G_p is $\mathrm{e}^2\sqrt{}6\{D|U||W|\}^{1/2} = \mathrm{e}^2 2\sqrt{}3\{d|U||W|\}^{1/2}$. Finally, to check (ii), it suffices to recall that at most $2D^g/g = O(1)$ edges have been omitted from G'. □

The graph $G = G_r$ whose existence is guaranteed by Lemma 9 has the Ramsey property for long induced cycles, as asserts our first main result below.

Theorem 10. *Let an integer $r \geq 2$ be fixed. The graph $G = G_r = G_r^n$ in Lemma 9 has the property that, for any r-edge-colouring of G, there is a colour c such that G contains a monochromatic induced cycle C^ℓ of colour c for all $B \log n \leq \ell \leq bn$, where $B = B(r) > 0$ and $b = b(r) > 0$ is as defined above. In particular, $G \overset{\mathrm{ind}}{\longrightarrow} (C^\ell)_r$ for all such ℓ.*

An immediate consequence of Theorem 10 is the following.

Corollary 11. *For any fixed $r \geq 2$, the induced size-Ramsey number $r_{\mathrm{e}}^{\mathrm{ind}}(C^\ell)$ of the ℓ-cycle C^ℓ is at most $c\ell$, where $c = c_r > 0$ is a constant that depends only on r.* □

§3. Proof of Theorem 10

3.1. Preparations for the proof. Let an integer $r \geq 2$ be fixed, and assume the constants ε, η, μ, δ, α, f_0, f, γ, γ_0, β, b, B, and g are as defined in Section 2.3. Let a graph $G = G^n$ satisfying (i)–(v) of Lemma 9 be given, and assume $E(G) = E_1 \cup \cdots \cup E_r$ is an r-edge-colouring of G. Let i, $\bar{\mu}$, and $\bar{V}_1$, $\bar{V}_2$, $\bar{V}_3 \subset V(G)$ be as in Lemma 4. Let G_i be the spanning subgraph of G with edge set E_i. Let $\bar{m} = \bar{\mu}n$ and $m = \lceil \mu n \rceil$. Recall that $\bar{m}/2 \le |\bar{V}_i| \le \bar{m}$ ($i \in \{1,2,3\}$). We now concentrate on the 3-partite graph J defined by G_i and the three vertex classes $\bar{V}_1$, $\bar{V}_2$, $\bar{V}_3$. Formally, $J = G_i[\bar{V}_1, \bar{V}_2] \cup G_i[\bar{V}_2, \bar{V}_3] \cup G_i[\bar{V}_1, \bar{V}_3]$.

LEMMA 12. *Let $1 \le i < j \le 3$, and put $\widetilde{J} = J[\bar{V}_i, \bar{V}_j]$. Then the graph $\widetilde{J}$ is $(\varepsilon\bar{m}, f_0)$-expanding.*

Proof. We use Lemma 5. Recall that G is (p, A)-upper-uniform for $A = 2\mathrm{e}^2\sqrt{3}$. Also, by the definition of D, we have that $f_0 = 16 \le (\mu/12r\mathrm{e}^2\sqrt{3})^2 D/2 \le (\bar{\mu}/6Ar)^2 d$. Moreover, as $\varepsilon = 1/48r$, we have $\varepsilon\bar{m} = \bar{\mu}n/3rf_0$. Thus Lemma 5 gives that $\widetilde{J}$ is $(\varepsilon\bar{m}, f_0)$-expanding, as required. □

Define a function $\varphi = \varphi_f : \mathbb{Z}_+ \to \mathbb{Z}$ by putting $\varphi(0) = 1$ and $\varphi(n) = \lceil f\varphi(n-1) \rceil$ for all $n \ge 1$. Let $L = 2\min\{k : \varphi(k-1) \ge \varepsilon\bar{m}\}$. The starting point of the proof of Theorem 10 is the following lemma.

LEMMA 13. *The graph J contains an induced $(L+1)$-cycle that is also induced in G.*

We shall not give a full proof for Lemma 13, although we shall make detailed comments in Section 3.5 on how one may alter a few steps in the arguments below to prove this lemma. We now fix an induced cycle C_0 of J that is also induced in G, and assume that C_0 has length $L + 1 \le \ell < bn$. Our aim is to construct an induced cycle C' of J that is also induced in G, and whose length is $\ell + 1$. If we show that this is possible, we shall have proved Theorem 10.

Suppose the vertices of C_0 are, in cyclic order, $x_1, \dots, x_\ell$. In the sequel, if $x \in \bar{V}_i$ for some $i \in \{1,2,3\}$, we let $\Gamma^+_{J'}(x) = \Gamma_{J'}(x) \cap \bar{V}_{i+1}$ and $\Gamma^-_{J'}(x) = \Gamma_{J'}(x) \cap \bar{V}_{i-1}$, where the indices of the $\bar{V}_i$ are considered reduced modulo 3.

LEMMA 14. *There is a set $X \subset V(C_0)$ with $|X| \ge (1 - 14\delta)|C_0|$ satisfying the following. We may choose $y_i^+ \in \Gamma^+_{J'}(x_i)$ and $y_i^- \in \Gamma^-_{J'}(x_i)$ for all $x_i \in X$ in such a way that, if $Y = \{y_i^+, y_i^- : x_i \in X\}$, all vertices in Y have degree 1 in $G[V(C_0) \cup Y]$.*

Proof. For each $1 \le i \le \ell$, let us pick $y_i^+ \in \Gamma^+_{J'}(x_i)$ arbitrarily, and let us consider the graph $G[V(C_0) \cup Y_1^+]$, where $Y_1^+ = \{y_i^+ : 1 \le i \le \ell\}$. Let

$$Y_{\mathrm{b}}^+ = \{y_i^+ : \text{ the degree of } y_i^+ \text{ in } G[V(C_0) \cup Y_1^+] \text{ is at least } 2\},$$

$X_{\mathrm{b}}^+ = V(C_0) \cap \Gamma_G(Y_{\mathrm{b}}^+)$, and $X_{\mathrm{g}}^+ = V(C_0) \setminus \Gamma_G(Y_{\mathrm{b}}^+)$. Note that all vertices y_i^+ with $x_i \in X_{\mathrm{g}}^+$ have degree 1 in $G[V(C_0) \cup Y_1^+]$. We show that X_{g}^+ is nearly all of $V(C_0)$. We have $e(G[V(C_0) \cup Y_{\mathrm{b}}^+]) \ge (3/2)|X_{\mathrm{b}}^+| + |X_{\mathrm{g}}^+| + |Y_{\mathrm{b}}^+|$. Since $|V(C_0) \cup Y_{\mathrm{b}}^+| \le 2bn \le \alpha n$, we conclude that

$$2\delta|C_0| \ge \delta(|C_0| + |Y_{\mathrm{b}}^+|) \ge e(G[V(C_0) \cup Y_{\mathrm{b}}^+]) - |V(C_0) \cup Y_{\mathrm{b}}^+| \ge \frac{1}{2}|X_{\mathrm{b}}^+|.$$

Thus $|X_{\mathrm{g}}^+| \ge (1 - 4\delta)|C_0|$. We now pick $y_i^- \in \Gamma^-_{J'}(x_i)$ for all $1 \le i \le \ell$ arbitrarily, and repeat the argument above. In this way we obtain $X_{\mathrm{g}}^- \subset V(C_0)$ with $|X_{\mathrm{g}}^-| \ge (1 - 4\delta)|C_0|$ such that all vertices y_i^- with $x_i \in X_{\mathrm{g}}^-$ have degree 1 in $G[V(C_0) \cup Y_1^-]$, where $Y_1^- = \{y_i^- : 1 \le i \le \ell\}$.

We now consider $X_{\rm g} = X_{\rm g}^+ \cap X_{\rm g}^-$, $Y_{\rm g}^+ = \{y_i^+ : x_i \in X_{\rm g}\}$, and $Y_{\rm g}^- = \{y_i^- : x_i \in X_{\rm g}\}$. Clearly $|Y_{\rm g}^+| = |Y_{\rm g}^-| = |X_{\rm g}| \geq (1-8\delta)|C_0|$. Moreover, since $|V(C_0) \cup Y_{\rm g}^+ \cup Y_{\rm g}^-| \leq 3bn \leq \alpha n$, we have that $e_G(Y_{\rm g}^+, Y_{\rm g}^-) \leq \delta|V(C_0) \cup Y_{\rm g}^+ \cup Y_{\rm g}^-| \leq 3\delta|C_0|$. Therefore, disregarding all vertices in $Y_{\rm g}^+ \cup Y_{\rm g}^-$ that are incident to $Y_{\rm g}^+$–$Y_{\rm g}^-$ edges of G, we see that there is a set $X \subset V(C_0)$ with $|X| \geq (1-14\delta)|C_0|$ satisfying the following: if $Y^+ = \{y_i^+ : x_i \in X\}$ and $Y^- = \{y_i^- : x_i \in X\}$, then in $G[V(C_0) \cup Y^+ \cup Y^-]$ every vertex in $Y^+ \cup Y^-$ has degree 1. $\square$

In the sequel, we fix a set $X \subset V(C_0)$ together with vertices y_i^+, y_i^- for all $x_i \in X$, as given by Lemma 14. Also, we let $Y^+ = \{y_i^+ : x_i \in X\}$, $Y^- = \{y_i^- : x_i \in X\}$, and $Y = Y^+ \cup Y^-$, and put $Q_0 = V(C_0) \cup Y$. For simplicity, below we consider the indices of the vertices x_i, y_i^+, and y_i^- reduced modulo $\ell = |C_0|$.

Recall $L = 2\min\{k : \varphi(k-1) \geq \varepsilon\bar{m}\}$. Suppose we are given a set $Q \subset V(J)$ with $Q \cap Q_0 = \emptyset$ and $G[Q_0 \cup Q]$ connected. We say that $x_i \in X$ is a Q-*good* vertex, or simply a *good* vertex, if both y_i^+ and y_i^- have degree 1 in $G[Q_0 \cup Q]$. Moreover, we say that $(x_i, x_{i+L}) \in X \times X$ is a Q-*good pair*, or a *good pair*, if both x_i and x_{i+L} are good.

Now suppose $(x_0^{(1)}, x_0^{(2)}) = (x_i, x_{i+L})$ is a good pair. Fix $x_1^{(1)} \in \{y_i^-, y_i^+\}$ and $x_1^{(2)} \in \{y_{i+L}^-, y_{i+L}^+\}$ in such a way that $x_1^{(1)} \in \bar{V}_{\rho(1)}$, $x_1^{(2)} \in \bar{V}_{\rho(2)}$, and $\rho(1) \neq \rho(2)$. Thus, given Q as above and a Q-good pair $(x_0^{(1)}, x_0^{(2)})$, we have associated to this pair a certain pair $(x_1^{(1)}, x_1^{(2)})$ of vertices that belong to different vertex classes of J. For brevity, we put $\Phi_Q(x_0^{(1)}, x_0^{(2)}) = (x_1^{(1)}, x_1^{(2)})$.

3.2. Main Iterative Part of the Enlarging Algorithm. To run this part of the 'enlarging' algorithm (cf. §1), we assume that we have the following set-up. First of all, we suppose that we have a set $Q \subset V(J')$ disjoint from $Q_0 = V(C_0) \cup Y$ and with $G[Q_0 \cup Q]$ connected. We remark in passing that, when this part of our algorithm is first run, we have $Q = \emptyset$. Typically, however, we run this part of the algorithm many times, and every time we do so we update the set Q to a slightly larger set. Now, continuing with the description of our set-up, besides Q, we assume that we have a Q-good pair $(x_0^{(1)}, x_0^{(2)})$, and $(x_1^{(1)}, x_1^{(2)}) = \Phi_Q(x_0^{(1)}, x_0^{(2)})$. Let $\rho(1)$, $\rho(2)$ be such that $x_1^{(\sigma)} \in \bar{V}_{\rho(\sigma)}$ $(\sigma \in \{1,2\})$. Put $\widetilde{J} = J'[\bar{V}_{\rho(1)}, \bar{V}_{\rho(2)}]$, and let $X_0^{(\sigma)} = \{x_0^{(\sigma)}\}$ and $X_1^{(\sigma)} = \{x_1^{(\sigma)}\}$ $(\sigma \in \{1,2\})$.

Given the set-up above, we aim at defining sets $X_2^{(1)}, \dots, X_k^{(1)}$, $X_2^{(2)}, \dots, X_k^{(2)} \subset V(G)$ satisfying the following properties:

(*i*) $\Gamma_{\widetilde{J}}(X_{i-1}^{(\sigma)}) \supset X_i^{(\sigma)}$ for all $2 \leq i \leq k$ and $\sigma \in \{1,2\}$.

(*ii*) $|X_i^{(\sigma)}| = \lceil f|X_{i-1}^{(\sigma)}|\rceil$ for all $2 \leq i \leq k$ and $\sigma \in \{1,2\}$.

(*iii*) The $X_i^{(\sigma)}$ $(2 \leq i \leq k,\ \sigma \in \{1,2\})$ are pairwise disjoint and disjoint from $Q_0 \cup Q$.

(*iv*) The only edges induced by $Q_0 \cup Q \cup \bigcup X_i^{(\sigma)}$ in G, where the union ranges over $\sigma \in \{1,2\}$ and $2 \leq i \leq k$, are the ones in

$$E(G[Q_0 \cup Q]) \cup \bigcup_{\sigma \in \{1,2\},\ 2 \leq i \leq k} \left\{ E(G[X_i^{(\sigma)}]) \cup E_G(X_{i-1}^{(\sigma)}, X_i^{(\sigma)}) \right\}.$$

(*v*) $|X_{k'}^{(\sigma)}| < \gamma m$ for $0 \leq k' < k$ and $|X_k^{(\sigma)}| \geq \gamma m$ for $\sigma \in \{1,2\}$.

$S^{(1)}_{k'-1} := \left(X^{(1)}_2 \cup \cdots \cup X^{(1)}_{k'-1}\right) \cup \left(X^{(2)}_2 \cup \cdots \cup X^{(2)}_{k'-1}\right);$
$A := \Gamma_{\widetilde{J}}(X^{(1)}_{k'-1}) \setminus X^{(1)}_{k'-2};\quad B := \Gamma_G \left\{(Q_0 \cup Q \cup S^{(1)}_{k'-1}) \setminus X^{(1)}_{k'-1}\right\};$
if $|A \setminus B| \geq f|X^{(1)}_{k'-1}|$
then return $X^{(1)}_{k'} \subset A \setminus B$ with $|X^{(1)}_{k'}| = \lceil f|X^{(1)}_{k'-1}|\rceil$
else return $\widetilde{Q}^{(1)} := S^{(1)}_{k'-1}$ and $\widetilde{Q}^{(2)} \subset A \cap B$
with $|\widetilde{Q}^{(2)}| = \lceil f|X^{(1)}_{k'-1}|\rceil$ and report *failure*

FIGURE 1. Algorithm I

However, starting with a particular Q-good pair $(x_0^{(1)}, x_0^{(2)})$, we may fail to find such sets $X_i^{(\sigma)}$ $(2 \leq i \leq k,\ \sigma \in \{1,2\})$. In this case, we shall find a pair of sets $\widetilde{Q}^{(1)}$, $\widetilde{Q}^{(2)}$ satisfying the following properties:

(i) $\widetilde{Q}^{(1)} \cap \widetilde{Q}^{(2)} = \emptyset$, and $\widetilde{Q} = \widetilde{Q}^{(1)} \cup \widetilde{Q}^{(2)}$ is disjoint from $Q_0 \cup Q$.
(ii) $|\widetilde{Q}^{(2)}| \geq \{(f-1)/2f\}|\widetilde{Q}|$.
(iii) $G[Q_0 \cup Q \cup \widetilde{Q}]$ is connected and $e(G[Q_0 \cup Q \cup \widetilde{Q}]) \geq e(G[Q_0 \cup Q]) + |\widetilde{Q}| + |\widetilde{Q}^{(2)}|$.

We now describe the algorithms that we shall use to generate the sets $X_i^{(\sigma)}$ or, failing that, the sets $\widetilde{Q}^{(1)}$, $\widetilde{Q}^{(2)}$. Assume first that $k' \geq 2$, and that we have already found the sets $X_i^{(\sigma)}$ $(2 \leq i \leq k'-1,\ \sigma \in \{1,2\})$ satisfying (i)–(iv) with k replaced by $k'-1$. The algorithm that we use to generate $X^{(1)}_{k'}$ is given in Figure 1.

Assume Algorithm I has returned $X^{(1)}_{k'}$. Then $\Gamma_{\widetilde{J}}(X^{(1)}_{k'-1}) \supset X^{(1)}_{k'}$, $|X^{(1)}_{k'}| = \lceil f|X^{(1)}_{k'-1}|\rceil$, $X^{(1)}_{k'}$ is disjoint from $X_i^{(\sigma)}$ $(2 \leq i \leq k'-1,\ \sigma \in \{1,2\})$ and disjoint from $Q_0 \cup Q$, the only edges of G that are both incident to a vertex in $X^{(1)}_{k'}$ and to a vertex in $Q_0 \cup Q \cup \bigcup X_i^{(\sigma)}$, where the union ranges over $2 \leq i \leq k'-1$ and $\sigma \in \{1,2\}$, are the ones in $E_G(X^{(1)}_{k'-1}, X^{(1)}_{k'})$. Now we may run Algorithm II in Figure 2 to generate $X^{(2)}_{k'}$, or else to find appropriate $\widetilde{Q}^{(1)}$, $\widetilde{Q}^{(2)}$.

$S^{(2)}_{k'-1} := S^{(1)}_{k'-1} \cup X^{(1)}_{k'};$
$A := \Gamma_{\widetilde{J}}(X^{(2)}_{k'-1}) \setminus X^{(2)}_{k'-2};\quad B := \Gamma_G \left\{(Q_0 \cup Q \cup S^{(2)}_{k'-1}) \setminus X^{(2)}_{k'-1}\right\};$
if $|A \setminus B| \geq f|X^{(2)}_{k'-1}|$
then return $X^{(2)}_{k'} \subset A \setminus B$ with $|X^{(2)}_{k'}| = \lceil f|X^{(2)}_{k'-1}|\rceil$
else return $\widetilde{Q}^{(1)} := S^{(2)}_{k'-1}$ and $\widetilde{Q}^{(2)} \subset A \cap B$
with $|\widetilde{Q}^{(2)}| = \lceil f|X^{(2)}_{k'-1}|\rceil$ and report *failure*

FIGURE 2. Algorithm II

A quick inspection gives the following.

LEMMA 15. *Suppose the sets* $X^{(\sigma)}_{k''}$ $(1 \leq k'' < k',\ \sigma \in \{1,2\})$ *satisfy conditions* (i)–(iv) *above with* k *replaced by* $k'-1$*, and that running Algorithm I and II we* *obtain* $X_{k'}^{(\sigma)}$ $(\sigma \in \{1,2\})$. *Then conditions* (i)–(iv) *hold with* k *replaced by* k'. $\square$

Now assume that either Algorithm I or Algorithm II has failed to generate $X_{k'}^{(1)}$ or $X_{k'}^{(2)}$, respectively. Thus we have defined a certain pair of sets $\widetilde{Q}^{(1)}$, $\widetilde{Q}^{(2)}$.

LEMMA 16. *Statements* (i)–(iii) *above hold for* $\widetilde{Q}^{(1)}$, $\widetilde{Q}^{(2)}$.

Proof. Let us assume that $\widetilde{Q}^{(1)}$, $\widetilde{Q}^{(2)}$ have been generated by Algorithm II. The case in which they are generated by Algorithm I is similar, and therefore omitted. Start by noticing that $\widetilde{Q}^{(1)} \cap \widetilde{Q}^{(2)} = \emptyset$, since $\widetilde{Q}^{(1)} = S_{k'-1}^{(2)}$, $\widetilde{Q}^{(2)} \subset A = \Gamma_{\widetilde{J}}(X_{k'-1}^{(2)}) \setminus X_{k'-2}^{(2)}$, and the only edges of $\widetilde{J}$ incident to $X_{k'-1}^{(2)}$ have their other endvertex either in $X_{k'-2}^{(2)}$ or outside $Q_0 \cup Q \cup S_{k'-1}^{(2)}$. This last fact also implies that $A \cap (Q_0 \cup Q) = \emptyset$ and, since $\widetilde{Q}^{(1)} = S_{k'-1}^{(2)}$ is also disjoint from $Q_0 \cup Q$, we have that $\widetilde{Q} \cap (Q_0 \cup Q) = \emptyset$. Thus (i) follows. We now check (ii). Here we use that the sets $X_i^{(\sigma)}$ grow geometrically with i, and that, by Lemma 12, the set $A = \Gamma_{\widetilde{J}}(X_{k'-1}^{(2)}) \setminus X_{k'-2}^{(2)}$ has cardinality $|A| \geq 2f|X_{k'-1}^{(2)}|$. Note that the latter fact implies that $|A \cap B| > f|X_{k'-1}^{(2)}|$, since we have $|A \setminus B| < f|X_{k'-1}^{(2)}|$. Thus we may indeed find $\widetilde{Q}^{(2)} \subset A \cap B$ with $|\widetilde{Q}^{(2)}| = \lceil f|X_{k'-1}^{(2)}|\rceil$, as required in Algorithm II. Now, we have $|\widetilde{Q}^{(1)}| = |S_{k'-1}^{(2)}| \leq (2/(1-1/f)-1)|X_{k'}^{(1)}|$. Thus $\widetilde{Q} = \widetilde{Q}^{(1)} \cup \widetilde{Q}^{(2)}$ has cardinality $|\widetilde{Q}| \leq \{(f+1)/(f-1)\}|X_{k'}^{(1)}| + |\widetilde{Q}^{(2)}| = \{(f+1)/(f-1)+1\}|\widetilde{Q}^{(2)}| = \{2f/(f-1)\}|\widetilde{Q}^{(2)}|$, as required.

We now turn to (iii). First recall that $G[Q_0 \cup Q]$ is connected, and note that, clearly, $G[Q_0 \cup Q \cup \widetilde{Q}^{(1)}]$ is also connected. It now suffices to notice that each vertex of $\widetilde{Q}^{(2)}$ sends at least two edges into $Q_0 \cup Q \cup \widetilde{Q}^{(1)}$: if $w \in \widetilde{Q}^{(2)}$, then w sends an edge into $X_{k'-1}^{(2)}$ since $w \in \Gamma_{\widetilde{J}}(X_{k'-1}^{(2)})$, and w sends an edge into $(Q_0 \cup Q \cup \widetilde{Q}^{(1)}) \setminus X_{k'-1}^{(2)}$ since $w \in B = \Gamma_G \left\{(Q_0 \cup Q \cup S_{k'-1}^{(2)}) \setminus X_{k'-1}^{(2)}\right\}$. Thus (iii) follows. $\square$

We shall run Algorithms I and II alternately until we either have obtained the sets $X_i^{(\sigma)}$ $(1 \leq i \leq k,\ \sigma \in \{1,2\})$ satisfying (i)–(v) above, or else we have found the sets $\widetilde{Q}^{(1)}$, $\widetilde{Q}^{(2)}$ satisfying (i)–(iii). Algorithm III given in Figure 3 makes this precise. Recall C_0 and Y are fixed. Let $Q \subset V(G)$ be such that $Q \cap Q_0 = \emptyset$ and $G[Q_0 \cup Q]$ is connected. Now let $(x_0^{(1)}, x_0^{(2)})$ be a Q-good pair. We may now run Algorithm III with this input.

3.3. Part I of the Enlarging Algorithm. We now consider the two possible outcomes of Algorithm III, namely, either we have generated sets $X_i^{(\sigma)}$ $(1 \leq i \leq k$, $\sigma \in \{1,2\})$ or else we have generated the sets $\widetilde{Q}^{(1)}$, $\widetilde{Q}^{(2)}$. If the latter happens, we let $\bar{Q} = Q \cup \widetilde{Q}$, we then look for a $\bar{Q}$-good pair of vertices $(\bar{x}_0^{(1)}, \bar{x}_0^{(2)})$ in C_0, and we run Algorithm III with $\bar{Q}$ and $(\bar{x}_0^{(1)}, \bar{x}_0^{(2)})$. If, on the other hand, Algorithm III succeeds in finding the sets $X_i^{(\sigma)}$, we then run Algorithm IV given below. Thus, roughly speaking, we run Algorithm III several times until it returns the sets $X_i^{(\sigma)}$. We make this precise in Algorithm IV, given in Figure 4. To run this algorithm, we only assume that C_0 has been fixed at the beginning. The main lemma in this section is the following.

$(x_1^{(1)}, x_1^{(2)}) := \Phi_Q(x_0^{(1)}, x_0^{(2)})$;
$X_0^{(1)} := \{x_0^{(1)}\}$; $X_0^{(2)} := \{x_0^{(2)}\}$; $X_1^{(1)} := \{x_1^{(1)}\}$; $X_1^{(2)} := \{x_1^{(2)}\}$;
$k := 1$;
repeat $k := k + 1$;
    run Algorithm I to generate $X_k^{(1)}$;
    if succeeded in generating $X_k^{(1)}$
      then run Algorithm II to generate $X_k^{(2)}$
until have $|X_k^{(1)}|$, $|X_k^{(2)}| \geq \gamma m$
    or else Algorithm I or Algorithm II reported *failure*;
if obtained $|X_k^{(1)}|$, $|X_k^{(2)}| \geq \gamma m$
  then return $X_0^{(1)}, \dots, X_k^{(1)}$, $X_0^{(2)}, \dots, X_k^{(2)}$
  else return $\widetilde{Q}^{(1)}$, $\widetilde{Q}^{(2)}$ given by the algorithm that failed

FIGURE 3. Algorithm III

let X, Y as in Lemma 14 be fixed;
$Q_0 := V(C_0) \cup Y$; $Q := \emptyset$;
repeat
(*) find a Q-good pair $(x_0^{(1)}, x_0^{(2)})$;
  run Algorithm III to obtain the $X_i^{(\sigma)}$ or $\widetilde{Q}^{(1)}$, $\widetilde{Q}^{(2)}$;
  if Algorithm III has failed **then** $Q := Q \cup \widetilde{Q}^{(1)} \cup \widetilde{Q}^{(2)}$
until Algorithm III returns $X_i^{(\sigma)}$ $(0 \leq i \leq k,\ \sigma \in \{1,2\})$;
return the $X_i^{(\sigma)}$

FIGURE 4. Algorithm IV

LEMMA 17. *Algorithm IV finds $Q \subset V(J')$, a Q-good pair of vertices $(x_0^{(1)}, x_0^{(2)})$, $(x_1^{(1)}, x_1^{(2)}) = \Phi_Q(x_0^{(1)}, x_0^{(2)})$, and sets $X_i^{(\sigma)}$ $(0 \leq i \leq k,\ \sigma \in \{1,2\})$ with $X_0^{(\sigma)} = \{x_0^{(\sigma)}\}$, $X_1^{(\sigma)} = \{x_1^{(\sigma)}\}$ $(\sigma \in \{1,2\})$ and such that (i)–(v) hold.*

To prove Lemma 17, it suffices to show that, in any iteration of the **repeat** loop in Algorithm IV, a Q-good pair of vertices $(x_0^{(1)}, x_0^{(2)})$ as required in (*) may always be found. We now proceed to prove this. Thus, let us assume that this loop has been iterated $j \geq 0$ times and that, in these j calls of Algorithm III, we have failed to generate the $X_i^{(\sigma)}$. Let $Q_{j'}^{(1)}$, $Q_{j'}^{(2)}$ be the sets returned by Algorithm III in the j'th call $(1 \leq j' \leq j)$. Put $Q_{j'} = Q_{j'}^{(1)} \cup Q_{j'}^{(2)}$ for all $1 \leq j' \leq j$, and let $Q = \bigcup_{1 \leq j' \leq j} Q_{j'}$. Also, let $Q^{(\sigma)} = \bigcup_{1 \leq j' \leq j} Q_{j'}^{(\sigma)}$ $(\sigma \in \{1,2\})$. We now prove two auxiliary lemmas, Lemmas 18 and 19.

LEMMA 18. *For all $1 \leq j' \leq j$, we have $|Q_{j'}| < (\alpha/2)n$.*

Proof. Fix $1 \leq j' \leq j$, and suppose that, for some $k' \geq 2$, in the j'th call of Algorithm III we generated $X_0^{(\sigma)}, \dots, X_{k'-1}^{(\sigma)}$ $(\sigma \in \{1,2\})$ and $X_{k'}^{(1)}$, but failed to generate $X_{k'}^{(2)}$. Then, for $\sigma \in \{1,2\}$,

$$|Q_{j'}| = \left| \bigcup_{\sigma' \in \{1,2\},\, 2 \le k'' < k'} X_{k''}^{(\sigma')} \right| + |X_{k'}^{(1)}| + |Q_{j'}^{(2)}| \le \left(\frac{2}{1-1/f} + 2(f+1) \right) |X_{k'-1}^{(\sigma)}|.$$

Therefore, if $|Q_{j'}| \ge (\alpha/2)n$, we have

$$|X_{k'-1}^{(\sigma)}| \ge \frac{f-1}{2(f^2+f-1)} |Q_{j'}| \ge \frac{\alpha(f-1)n}{4(f^2+f-1)} \ge \gamma m,$$

which is a contradiction. The case in which Algorithm III generates $X_0^{(\sigma)}, \dots, X_{k'-1}^{(\sigma)}$ ($\sigma \in \{1,2\}$) but fails to generate $X_{k'}^{(1)}$ for some $k' \ge 2$ can be dealt with similarly. $\square$

Lemma 19. *Let $1 \le j' \le j$ and set $Q' = Q_1 \cup \dots \cup Q_{j'}$. Then $G[Q_0 \cup Q']$ is connected and $e(G[Q_0 \cup Q']) \ge |Q_0 \cup Q'| + \big| \bigcup_{1 \le j'' \le j'} Q_{j''}^{(2)} \big|$.*

Proof. This follows from property (iii) satisfied by the $\widetilde{Q}^{(1)}$, $\widetilde{Q}^{(2)}$ generated by Algorithms I and II, and a simple induction on j'. $\square$

Recall we are assuming that the **repeat** loop of Algorithm IV has been run j times, and that we have so far failed to generate the $X_i^{(\sigma)}$. Moreover, $Q = \bigcup_{1 \le j' \le j} Q_{j'}$ is the union of the $\widetilde{Q}^{(1)}$, $\widetilde{Q}^{(2)}$ obtained by Algorithm IV in those j calls of Algorithm III. We now consider $G[Q_0 \cup Q]$. We let $X_j \subset X$ be the set of Q-good vertices of C_0. We aim at showing that there is a Q-good pair $(x_0^{(1)}, x_0^{(2)}) \in X_j \times X_j$. An easy counting argument shows that such a pair does indeed exist if $|X_j| > |C_0|/2$.

Lemma 20. *We have $|X_j| > |C_0|/2$.*

Proof. Let $x_i \in C_0$ be an $\emptyset$-good vertex that is not Q-good. Then by definition either y_i^+ or y_i^- has degree at least 2 in $G[Q_0 \cup Q]$. Note that this may be the case either because (a) x_i belonged to a good pair $(x_0^{(1)}, x_0^{(2)})$ in one of the j iterations of the **repeat** loop of Algorithm IV, or else because (b) although it never belonged to such a pair, a vertex in $Q^{(2)} = \bigcup_{1 \le j' \le j} Q_{j'}^{(2)}$ is adjacent to y_i^+ or y_i^- in G. Let us say that $x_i \in X \setminus X_j$ has been *used* if (a) above holds, and that x_i has been *spoilt* if (b) above holds.

Clearly, the number of used vertices x_i in $X \setminus X_j$ is $2j$. Let s be the number of vertices $x_i \in X \setminus X_j$ that have been spoilt. Then $|X| = 2j + s + |X_j|$. We now analyse two cases.

Case 1. We have $|Q_0 \cup Q| > \alpha n$.

In this case, let $j_0 = \max\{j' : |Q_0 \cup \dots \cup Q_{j'}| \le \alpha n\}$, and let $Q' = Q_1 \cup \dots \cup Q_{j_0}$. If $|Q'| \ge 3bn$, then $|Q'| \ge |Q_0|$, since $|Q_0| \le 3|C_0| < 3bn$. Therefore, by Lemma 19 and property (ii) of $Q_{j'}^{(1)}$, $Q_{j'}^{(2)}$, we have

$$\begin{aligned} e(G[Q_0 \cup Q']) - |Q_0 \cup Q'| &\ge \left| \bigcup_{1 \le j' \le j_0} Q_{j'}^{(2)} \right| \ge \frac{f-1}{2f} |Q'| \\ &= \frac{f-1}{2f} \frac{|Q'|}{|Q_0 \cup Q'|} |Q_0 \cup Q'| \ge \frac{f-1}{4f} |Q_0 \cup Q'| > \delta |Q_0 \cup Q'|. \end{aligned}$$

Since $|Q_0 \cup Q'| \le \alpha n$, this contradicts the fact that G is (α, δ)-locally sparse. Therefore we have $|Q'| < 3bn$. But $|Q_0 \cup Q' \cup Q_{j_0+1}| > \alpha n$, and hence $|Q_{j_0+1}| \ge (\alpha - 6b)n \ge (\alpha/2)n$. This, however, contradicts Lemma 18, and we conclude that Case 1 cannot occur.

Case 2. We have $|Q_0 \cup Q| \le \alpha n$.

Suppose first that $|Q| \ge |Q_0|$. Then, as in Case 1, we have that

$$e(G[Q_0 \cup Q]) - |Q_0 \cup Q| \ge \left| \bigcup_{1 \le j' \le j} Q_{j'}^{(2)} \right| \ge \frac{f-1}{4f} |Q_0 \cup Q| > \delta |Q_0 \cup Q|,$$

which is a contradiction. Thus $|Q| < |Q_0|$. Now suppose $j \ge |C_0|/10$. Noting that $|Q| \ge j$, we have

$$\begin{aligned} e(G[Q_0 \cup Q]) - |Q_0 \cup Q| &\ge \frac{f-1}{2f} \frac{|Q|}{|Q_0 \cup Q|} |Q_0 \cup Q| \\ &\ge \frac{f-1}{2f} \frac{j}{3|C_0| + j} |Q_0 \cup Q| \ge \frac{f-1}{62f} |Q_0 \cup Q| > \delta |Q_0 \cup Q|, \end{aligned}$$

which is again a contradiction. Therefore $j < |C_0|/10$. Finally, to complete the proof of our lemma, suppose that $|X_j| \le |C_0|/2$. Then $|C_0|(1 - 14\delta) \le |X| = 2j + s + |X_j| \le s + (7/10)|C_0|$, and hence $s \ge (3/10)|C_0| - 14\delta|C_0| \ge (1/5)|C_0|$. We remark in passing that, if $j = 0$, this is already a contradiction since in this case trivially $s = 0$. In general, we have $e(G[Q_0 \cup Q]) - |Q_0 \cup Q| \ge 1 + s \ge |C_0|/5 \ge (1/30)|Q_0 \cup Q| > \delta|Q_0 \cup Q|$, which is a contradiction. Hence $|X_j| > |C_0|/2$ and the lemma is proved. □

Now Lemma 17 follows easily.

Proof of Lemma 17. As observed above, Lemma 20 implies that we may always execute step (*) in the **repeat** loop of Algorithm IV. Thus we eventually succeed in finding the required $X_i^{(\sigma)}$, and hence Lemma 17 follows. □

3.4. Part II of the Enlarging Algorithm. Now assume that Algorithm IV has terminated with $X_0^{(\sigma)}, \ldots, X_k^{(\sigma)}$ $(\sigma \in \{1, 2\})$. Recall that $|X_k^{(\sigma)}| \ge \gamma m$ $(\sigma \in \{1, 2\})$. Let $2 \le k_0 \le k$ be such that if $S = Q_0 \cup \bigcup X_{k'}^{(\sigma)}$, where the union ranges over $2 \le k' \le k_0$ and $\sigma \in \{1, 2\}$, then $|S| \le \beta m$. Note that $|Q_0| \le 3bn \le \beta m$, and hence k_0 is well-defined. To find the induced cycle C' that we are after, we now run another algorithm, Algorithm V. This algorithm defines further sets $X_{k+1}^{(\sigma)}, X_{k+2}^{(\sigma)}, \ldots$ $(\sigma \in \{1, 2\})$, and it uses them to find C'. As usual, let $X_0^{(\sigma)} = \{x_0^{(\sigma)}\}$, $X_1^{(\sigma)} = \{x_1^{(\sigma)}\}$ $(\sigma \in \{1, 2\})$, let $x_1^{(\sigma)} \in \bar{V}_{\rho(\sigma)}$ $(\sigma \in \{1, 2\})$, and put $\widetilde{J} = J[\bar{V}_{\rho(1)}, \bar{V}_{\rho(2)}]$. Algorithm V is given in Figure 5.

For the sets $X_k^{(\sigma)}$ $(\sigma \in \{1, 2\})$ to be well-defined in the **repeat** loop in Algorithm V, we need to verify the following lemma.

LEMMA 21. *Suppose the* **repeat** *loop of Algorithm V has been run for* $k < k' < k''$. *Then* $|\Gamma_{\widetilde{J}}(X_{k''-1}^{(\sigma)}) \setminus (T_{k''-1} \cup \Gamma_G(S))| \ge f|X_{k''-1}^{(\sigma)}|$ *for* $\sigma \in \{1, 2\}$.

$T_k := Q_0 \cup \bigcup_{2\le k'\le k,\, \sigma\in\{1,2\}} X_{k'}^{(\sigma)}$; $\quad S := Q_0 \cup \bigcup_{2\le k'\le k_0,\, \sigma\in\{1,2\}} X_{k'}^{(\sigma)}$;
$k' := k$;
repeat $k' := k'+1$;
  pick $X_{k'}^{(1)} \subset \Gamma_{\widetilde{J}}(X_{k'-1}^{(1)}) \setminus (T_{k'-1} \cup \Gamma_G(S))$ with $|X_{k'}^{(1)}| = \lceil f|X_{k'-1}^{(1)}|\rceil$;
  pick $X_{k'}^{(2)} \subset \Gamma_{\widetilde{J}}(X_{k'-1}^{(2)}) \setminus (T_{k'-1} \cup \Gamma_G(S))$ with $|X_{k'}^{(2)}| = \lceil f|X_{k'-1}^{(2)}|\rceil$;
  $T_{k'} := T_{k'-1} \cup X_{k'}^{(1)} \cup X_{k'}^{(2)}$
until $|X_{k'}^{(\sigma)}| \ge \varepsilon\bar{m}$ ($\sigma\in\{1,2\}$);
$k_1 := k'$, and let $x^{(1)} \in X_{k_1}^{(1)}$, $x^{(2)} \in X_{k_1}^{(2)}$ be such that $x^{(1)}x^{(2)} \in E(\widetilde{J})$;
for $\sigma\in\{1,2\}$, let $P^{(\sigma)}$ be the $x_0^{(\sigma)}$–$x^{(\sigma)}$ path
  in $\widetilde{J}$ naturally given by the $X_i^{(\sigma)}$;
(†) let C' be the cycle in J' obtained from C_0 by replacing the
  $x_0^{(1)}$–$x_0^{(2)}$ path of length L in C_0 by $P^{(1)}x^{(1)}x^{(2)}P^{(2)}$;
return C'

FIGURE 5. Algorithm V

Proof. Fix $\sigma\in\{1,2\}$. We have that

$$|T_{k''-1}| \le \frac{2}{1-1/f}|X_{k''-1}^{(\sigma)}| + |Q_0| \le \frac{2f}{f-1}|X_{k''-1}^{(\sigma)}| + 3bn,$$

and, since $|X_{k''-1}^{(\sigma)}| \ge \gamma m$, that $|\Gamma_G(S)| \le 8d|S| \le 8d\beta m \le (8d\beta/\gamma)|X_{k''-1}^{(\sigma)}|$. Therefore, by Lemma 12, we have

$$|\Gamma_{\widetilde{J}}(X_{k''-1}^{(\sigma)}) \setminus (T_{k''-1} \cup \Gamma_G(S))| \ge \left\{f_0 - \frac{2f}{f-1} - \frac{3b}{\gamma\mu} - \frac{8d\beta}{\gamma}\right\}|X_{k''-1}^{(\sigma)}| \ge f|X_{k''-1}^{(\sigma)}|,$$

as required. □

Recall that k_0 was the largest integer with $2\le k_0\le k$ and $|S|\le\beta m$, where $S = Q_0 \cup \bigcup X_{k'}^{(\sigma)}$ and the union is over $2\le k'\le k_0$ and $\sigma\in\{1,2\}$. Since the sets $X_{k'}^{(\sigma)}$ ($1\le k'\le k_1$) grow geometrically, this choice of k_0 implies that $|X_{k_0}^{(\sigma)}| = \Omega(n)$ ($\sigma\in\{1,2\}$), and hence that $k_1 - k_0 = O(1)$.

LEMMA 22. *The* **repeat** *loop of Algorithm V generates* $X_{k+1}^{(\sigma)},\dots,X_{k_1}^{(\sigma)}$ *with* $|X_{k_1}^{(\sigma)}| \ge \varepsilon\bar{m}$ ($\sigma\in\{1,2\}$). *Moreover, we have that* $k_1 - k_0 \le \log_f(\varepsilon/\gamma_0)$.

Proof. The first statement follows from Lemma 21. For the second statement, let us first show that indeed $|X_{k_0}^{(\sigma)}| = \Omega(n)$ ($\sigma\in\{1,2\}$). If $k_0 = k$, then $|X_{k_0}^{(\sigma)}| \ge \gamma m \ge \gamma_0 n$. Assume that $k_0 < k$. Then

$$\begin{aligned}\beta m < \left|Q_0 \cup \bigcup_{2\le k'\le k_0+1,\,\sigma\in\{1,2\}} X_{k'}^{(\sigma)}\right| &\le 3bn + \frac{2}{1-1/f}|X_{k_0}^{(1)}| + 2\lceil f|X_{k_0}^{(1)}|\rceil \\ &\le 3bn + \left\{\frac{2f}{f-1} + 2(f+1)\right\}|X_{k_0}^{(1)}|.\end{aligned}$$

Hence $|X^{(1)}_{k_0}| = |X^{(2)}_{k_0}| \geq \big((f-1)/2(f^2+f-1)\big)(\beta\mu - 3b)n \geq \gamma_0 n$. Since $|X^{(\sigma)}_{k'}| = \lceil f|X^{(\sigma)}_{k'-1}|\rceil$ for all $2 \leq k' \leq k_1$ and $\sigma \in \{1,2\}$, we have $k_1 - k_0 \leq \log_f(\varepsilon/\gamma_0)$. $\square$

We have now completed the proof that the 'enlarging' algorithm succeeds.

LEMMA 23. *Algorithm V returns an induced $(\ell+1)$-cycle $C' \subset \widetilde{J}$ of $\widetilde{J}$ that is also induced in G.* $\square$

3.5. Proof of the Theorem. The methods used in the proof of Lemma 23 may be used to prove Lemma 13.

Sketch of the Proof of Lemma 13. Pick $x_0 \in J'$, and choose $y^+ \in \Gamma^+_{J'}(x_0)$, $y^- \in \Gamma^-_{J'}(x_0)$ arbitrarily. Put $Y = \{y^+, y^-\}$, $X^{(1)}_0 = X^{(2)}_0 = \{x_0\}$, $Q_0 = \{x_0\} \cup Y$, $Q = \emptyset$, $x^{(1)}_1 = y^+$, $x^{(2)}_1 = y^-$, and let $x^{(\sigma)}_1 \in \bar{V}_{\rho(\sigma)}$ $(\sigma \in \{1,2\})$. As usual, let $\widetilde{J} = J[\bar{V}_{\rho(1)}, \bar{V}_{\rho(2)}]$. Put $k=1$, and iterate the **repeat** loop in Algorithm III. By using that G is (α,δ)-locally sparse, one may prove the following claim

Claim. The **repeat** *loop in Algorithm III, run with the set-up above, succeeds in finding sets $X^{(\sigma)}_0, \ldots, X^{(\sigma)}_k$ with $|X^{(\sigma)}_k| \geq \gamma m$ $(\sigma \in \{1,2\})$.*

We omit the proof of the claim. Now replace step (†) in Algorithm V by

let C' be the $(L+1)$-cycle in J' naturally given by
the cycle C_0, the paths $P^{(1)}$, $P^{(2)}$, and the edge $x^{(1)}x^{(2)}$

Running this modified Algorithm V on the sets $X^{(\sigma)}_0, \ldots, X^{(\sigma)}_k$ $(\sigma \in \{1,2\})$ given by the above claim, we obtain a cycle C' of J' that has length $L+1$ and that is also induced in G, as required. $\square$

Proof of Theorem 10. Lemmas 4, 13 and 23 give that a graph G as in Lemma 9 is as required. $\square$

§4. DENSITY TYPE RESULTS

Theorem 10 is clearly a Ramsey type result. If, instead of colouring the edges of a graph G with r colours, we simply picked $e(G)/r$ edges of G, we could not even guarantee that we would obtain an odd cycle. However the following 'density' type results may be proved.

THEOREM 24. *For any $0 < \gamma \leq 1$, there are constants $c_1 = c_1(\gamma) > 0$, $b_1 = b_1(\gamma) > 0$, $B_1 = B_1(\gamma) > 0$, and a graph $G = G^n_\gamma$ of size $e(G) \leq c_1 n$ such that $G \xrightarrow[\gamma]{\mathrm{ind}} C^{2\ell}$ for any $B_1 \log n \leq \ell \leq b_1 n$.* $\square$

Roughly speaking, the result above says that a density type result may be proved for long even induced cycles, for any fixed positive density. If we wish to guarantee long odd induced cycles as well, we need to assume that the density is strictly above 1/2.

THEOREM 25. *For any $1/2 < \gamma \leq 1$, there are constants $c_2 = c_2(\gamma) > 0$, $b_2 = b_2(\gamma) > 0$, $B_2 = B_2(\gamma) > 0$, and a graph $G = G^n_\gamma$ of size $e(G) \leq c_2 n$ such that $G \xrightarrow[\gamma]{\mathrm{ind}} C^\ell$ for any $B_2 \log n \leq \ell \leq b_2 n$.* $\square$

To prove Theorems 24 and 25, we again consider a random graph $G_p \in \mathcal{G}(N,p)$ with $p = D/N$ and D a constant very large with respect to γ. We then take G

to be a subgraph of G_p with small maximal degree and large girth. We omit the details.

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