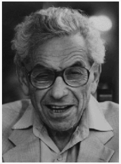
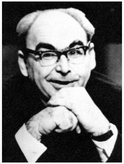
Extremal combinatorics in random discrete structures





Mathias Schacht

Fachbereich Mathematik
Universität Hamburg

December 2014

Extremal graph theory

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- property $\mathcal{P}$

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Today

$\mathcal{P}$: F-free

Turán's problem

For a graph F and $n \in \mathbb{N}$ set

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Theorem (Mantel, Turán, Erdős, Stone, Simonovits)

For every F

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- only very few results for hypergraphs are known

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Turán's theorem: extremal K_t-free graph is complete $(t-1)$-partite

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Theorem (Erdős '67, Simonovits '68)

If H is an n-vertex, F-free graph and $e(H) \geq ex(n,F) - o(n^2)$,

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Theorem (Erdős '67, Simonovits '68)

If H is an n-vertex, F-free graph and $e(H) \geq ex(n, F) - o(n^2)$, then one can remove of $o(n^2)$ edges to obtain a $(\chi(F)-1)$-partite graph.

Typical K_t-free graphs

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Theorem (Kolaitis, Prömel, Rothschild '85)

Almost every K_t-free graph on n vertices is $(t-1)$-partite.

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Theorem (Kolaitis, Prömel, Rothschild '85)

Almost every K_t-free graph on n vertices is $(t-1)$-partite.

- $t = 3$ was proved by Kleitman, Rothschild and Erdős '73

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First results

- $G(n, Cn^{-1/2})$ for $F = K_3$ — Frankl and Rödl '86

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- $G(n, 1/2)$ for $F = K_t$ — Babai, Simonovits, Spencer '90

1st Poznań Seminar on Random Graphs 1983

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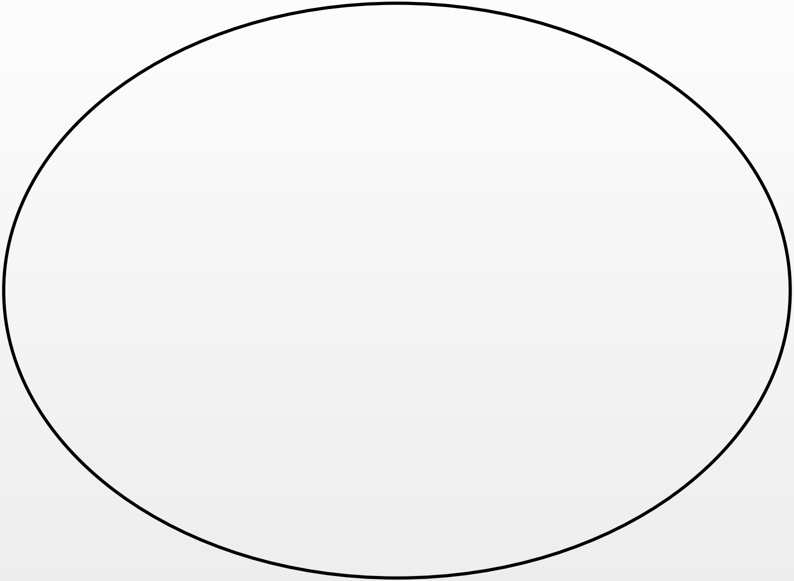

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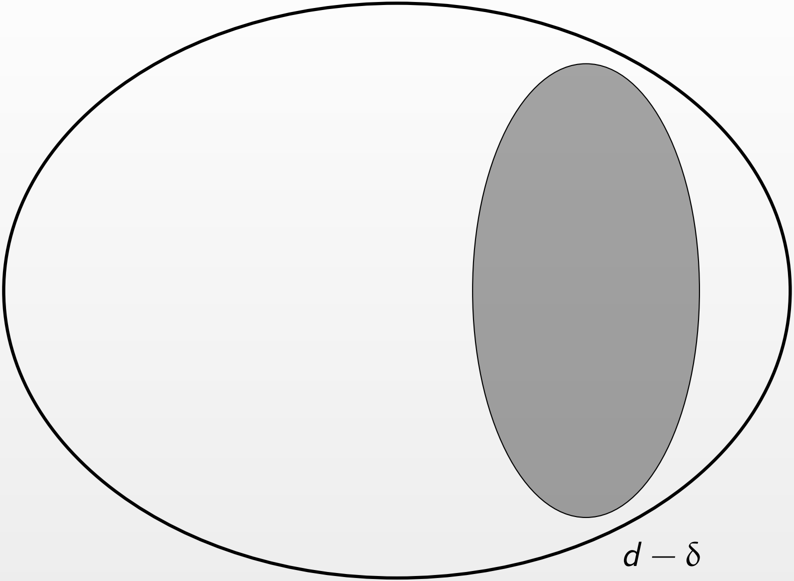

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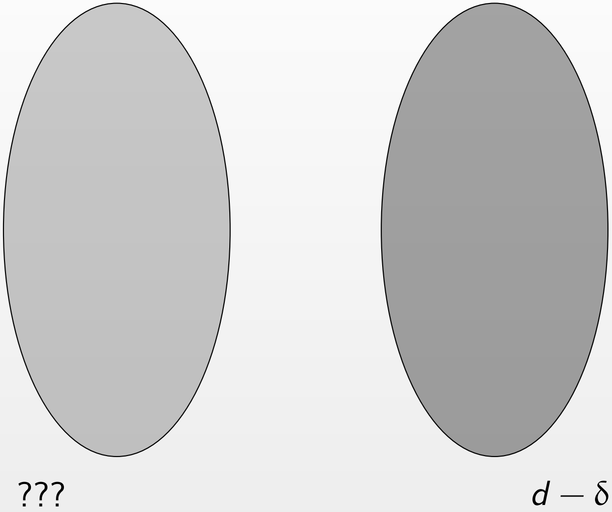

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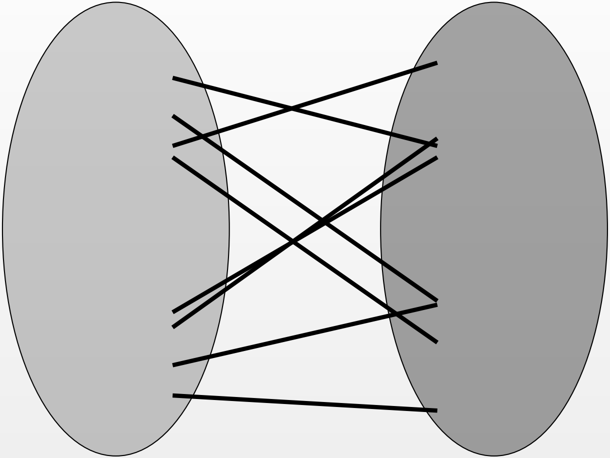

???

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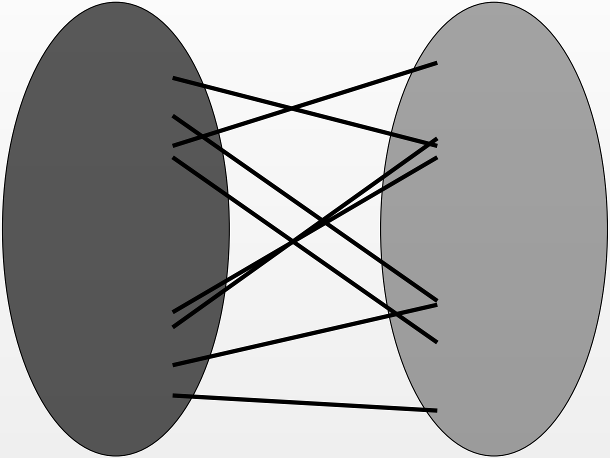

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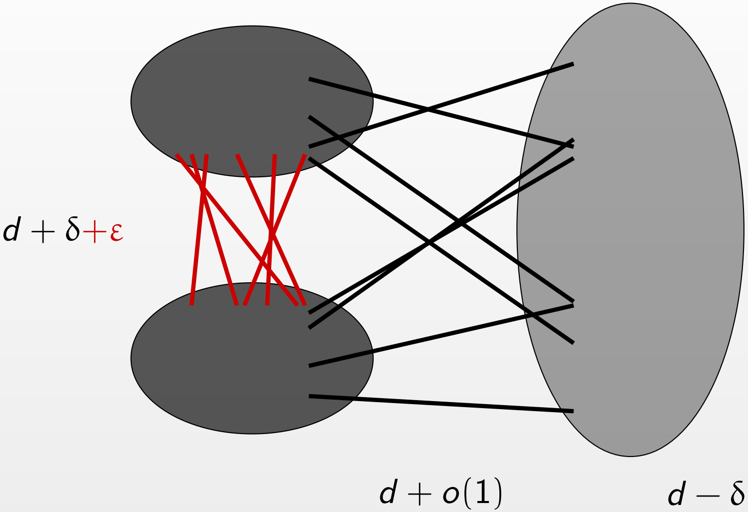

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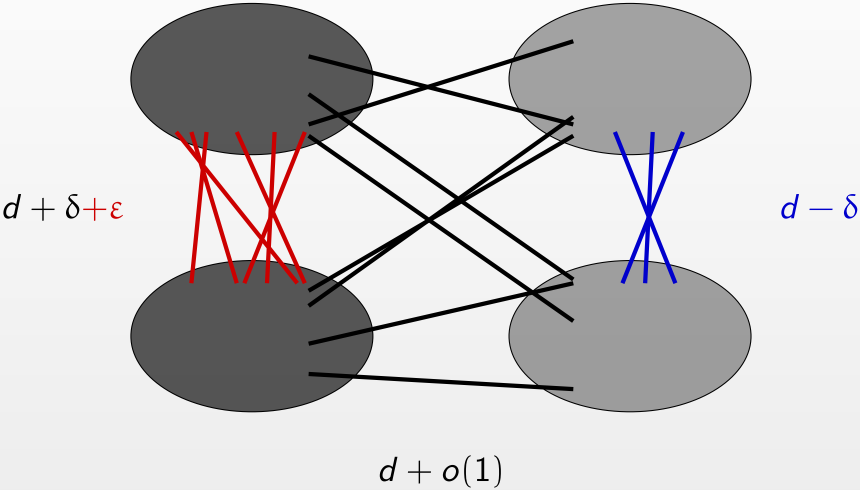

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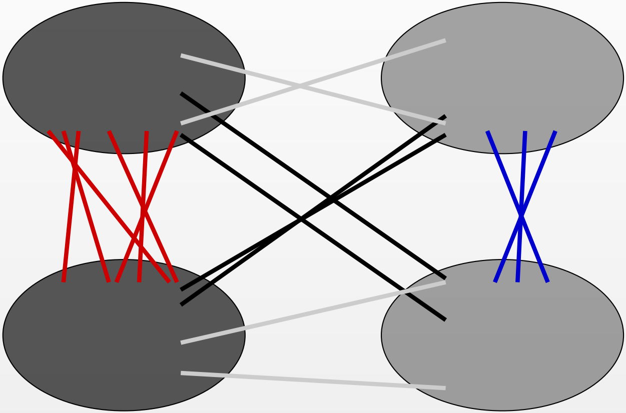

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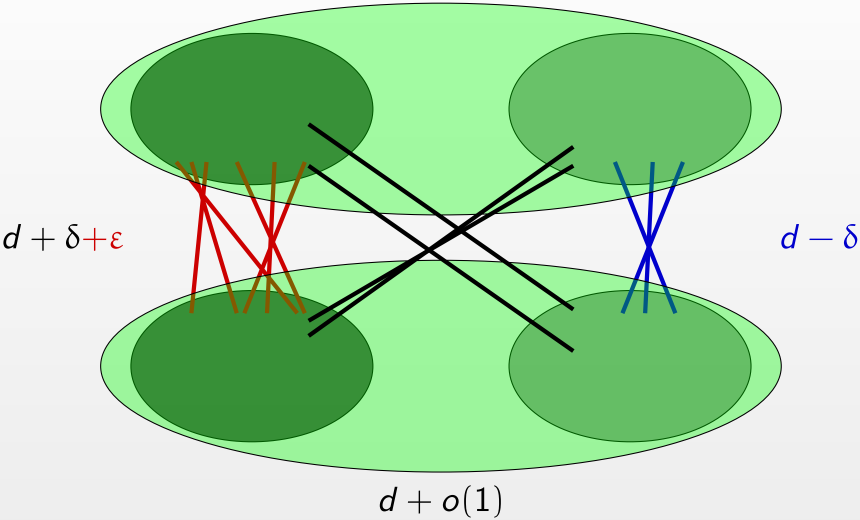

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Definition

$$m_F = \max\{d_{F'} \colon \emptyset \neq F' \subseteq F\}$$

where

$$d_{F'} = \begin{cases} 1/2, & \text{if } e(F') = 1 \\ \frac{e(F')-1}{v(F')-2}, & \text{otherwise.} \end{cases}$$

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Conjecture (Kohayakawa, Łuczak & Rödl)

Threshold for $\mathrm{ex}(G(n,p),F) = (\pi_F + o(1))p\binom{n}{2}$ is $p = n^{-1/m_F}$.

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- for $K_3, \ldots, K_6$ Frankl, Rödl; KŁR; Gerke, Schickinger, Steger; Gerke
- for cycles Haxell, Kohayakawa, Łuczak
- for all graphs Conlon and Gowers (balanced), Sch.

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Remark:

- implies solution of the general Erdős-Nešetřil problem since $m_{K_{t+1}} > m_{K_t}$

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Results:

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- for $p \geq n^{-\varepsilon_t}$ — Brightwell, Panagiotou, Steger
- optimal: $p \geq Cn^{-\frac{1}{m_{K_t}}}\mathrm{poly}(\log n)$ — DeMarco, Kahn
  $\longrightarrow \#\{K_t \subseteq G(n,p)\} \geq pn^2 \log n$

Remark:

- for approximate $(t-1)$-partiteness $\mathrm{poly}(\log n)$ is not required

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- K_t Balogh, Morris, Samotij, Warnke

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- Several approaches:
  - Conlon, Gowers
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- several other results can/could be transferred to subgraphs of $G(n, p)$

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Definition (α-dense)

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$\forall\delta > 0$, $\exists\xi > 0$ and n_0 such that $\forall n \geq n_0$ we have If $U \subseteq V_n$ and

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What are the asymtotics of the smallest sequence $(p_n)_{n\in\mathbb{N}}$ of probabilities such that α-density from $(H_n)_{n\in\mathbb{N}}$ can be transferred to $(H_n[V_{n,p_n}])_{n\in\mathbb{N}}$?

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Examples:

- $p^k n^2 \gg pn$ — Szemerédi's theorem
- $p^{e(F')} n^{v(F')} \gg pn^k \quad \forall F' \subseteq F$ — Turán

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Corollary (probabilistic version of Szemerédi's theorem)

$\forall k \geq 3$, $\forall \delta > 0$, $\exists 0 < c < C$, such that $\forall (q_n)_{n\in\mathbb{N}}$

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- Main result yields probabilistic versions of many extremal results
  - multidimensional and polynomial variants of Szemerédi's theorem
  - maximal sum-free subsets
  - theorems of Turán and of Erdős and Stone for $G(n,p)$ and $G^{(k)}(n,p)$

Motivation for the Erdős-Nešetřil question

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For every integer t does there exist a K_{t+1}-free graph H s.t.

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Question (Folkman function $f_r(t)$)

How large is the smallest such H?

New bounds for the Folkman number

$f_r(t) := \min\{v(H)\colon\ \omega(H) = t \text{ and } H \longrightarrow (K_t)_r\}$

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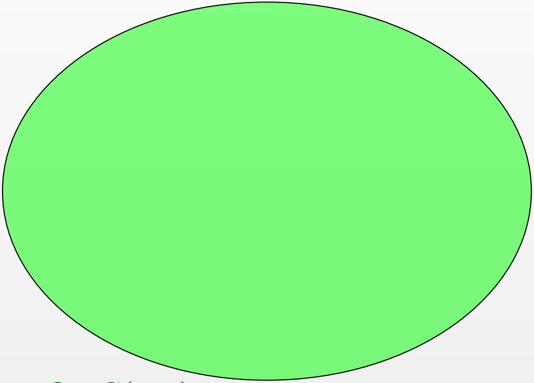
Question

Can t^4 be improved to $o(t^2)$ for $r = 2$?

Sketch of Proof



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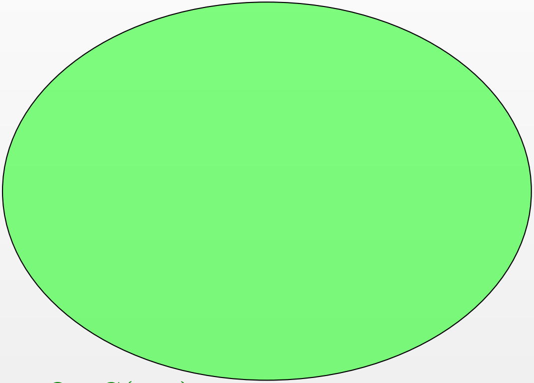

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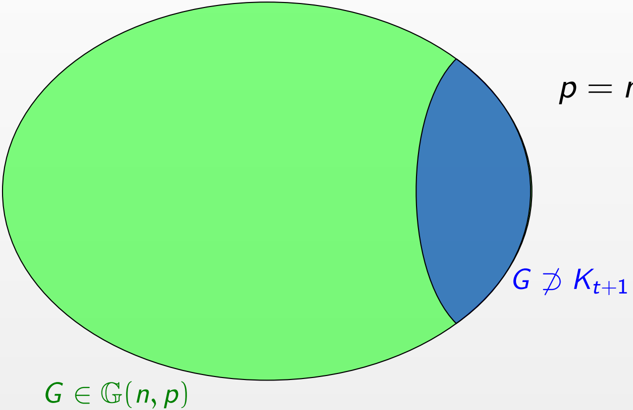

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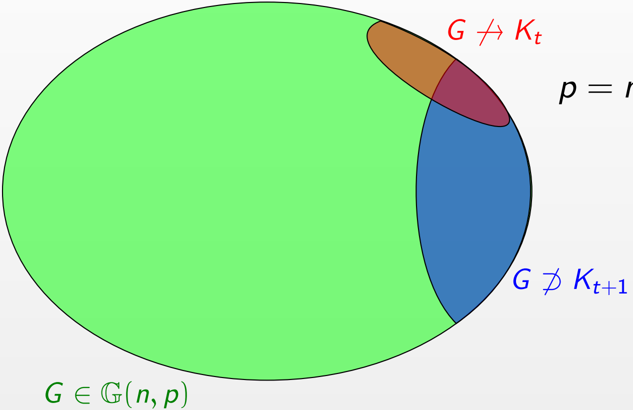

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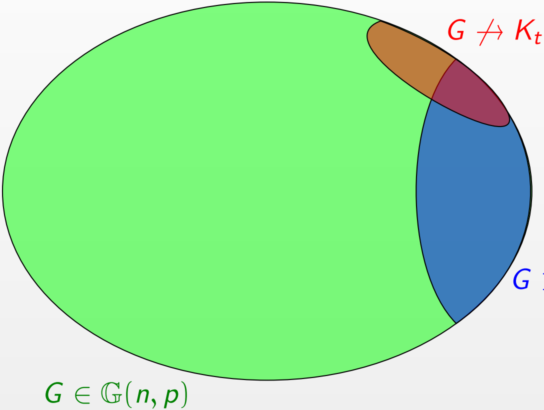

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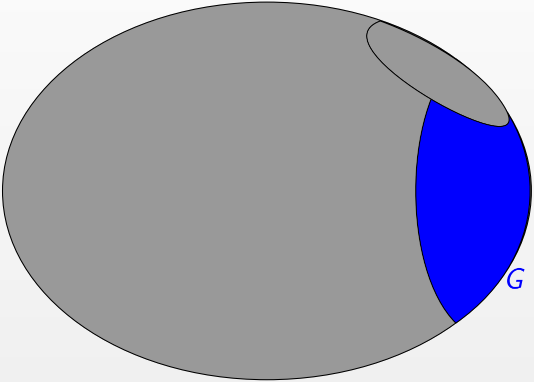

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