Algorithms and complexity of graph convexity problems

Vinícius F. dos Santos

Centro Federal de Educação Tecnológica de Minas Gerais (CEFET-MG)
Belo Horizonte, Brazil

Joint Work with:
Igor da Fonseca Ramos
Jayme L. Szwarcfiter

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Convexity

Definition: **convexity**

Given a finite ground set V and a family $\mathcal{C}$ of subsets of V, $\mathcal{C}$ is a **convexity** if the following conditions are satisfied:

(a) The sets $\emptyset$ and V belong to $\mathcal{C}$; and

(b) $\mathcal{C}$ is closed for intersections.

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Definition: convex set

A set is **convex** if it belongs to $\mathcal{C}$.

Graph convexity

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Path convexity

- We can use a family of paths $\mathcal{F}_G$ to define $\mathcal{C}$:
- $S \subseteq V(G)$ is convex if, for every pair $u, v \in S$, every $w \in V(G)$ in a uv-path in $\mathcal{F}_G$ is also in S.
- Interval function: $I(S)$ is the set of vertices that belong to at least one uv-path of $\mathcal{F}_G$, $u, v \in S$

Convexity

Some well studied graph convexities

- Geodetic Convexity
- Monophonic Convexity
- P_3 Convexity
- P_3^* Convexity

Geodetic convexity

- Family of paths $\mathcal{F}_G$ consists of all the shortest paths of G.

Convex sets

$S \subseteq V(G)$ is convex if and only if for every pair $u, v \in S$, every vertex in a uv-shortest path in G is also in S.

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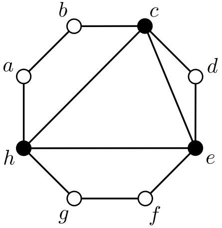

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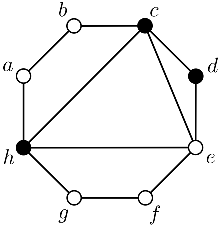

Figure: Non-convex set.

Monophonic convexity

- Family of paths $\mathcal{F}_G$ consists of the induced paths of G.

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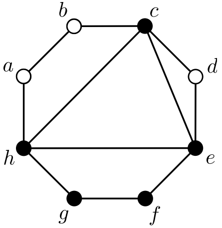

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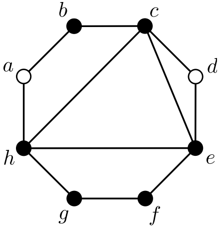

Figure: Non-convex set.

P_3 convexity

- Family of paths $\mathcal{F}_G$ contain the paths of order 3 of G.

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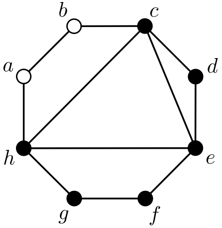

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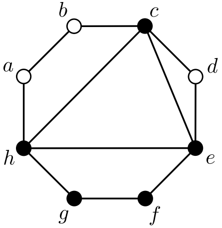

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Convex Hull in graphs

- Convex hull of S: smallest convex superset containing S.
- Denoted by $H(S)$.
- If $v \in H(S) \setminus S$, we say that v is **generated** by S.

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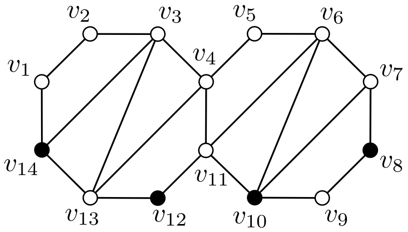

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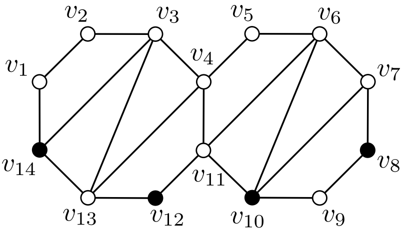

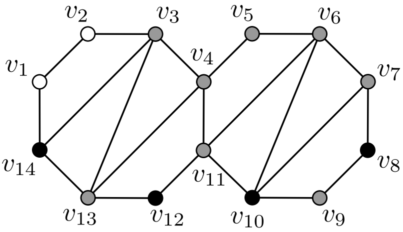

Algorithm to determine the convex hull

- Greedy algorithm.
- Start with $H(S) = S$.
- Repeat:
  - Let $R = I(H(S))$;
  - $H(S) \leftarrow H(S) \cup R$.

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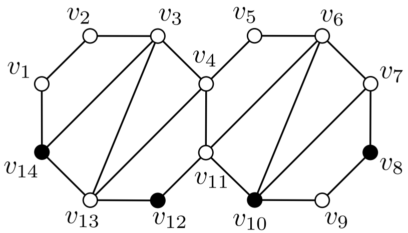

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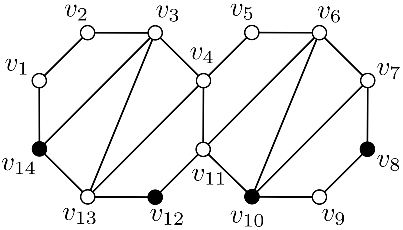

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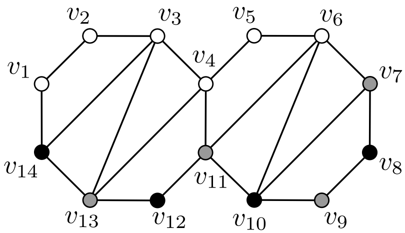

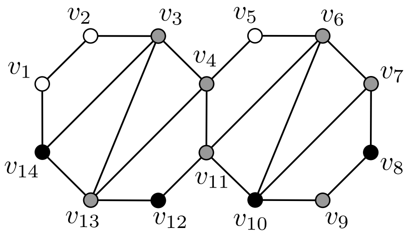

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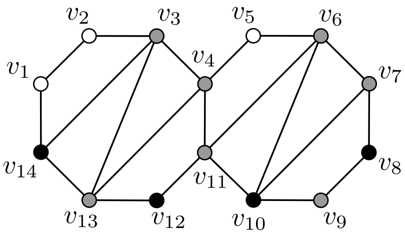

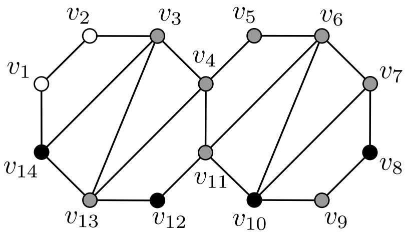

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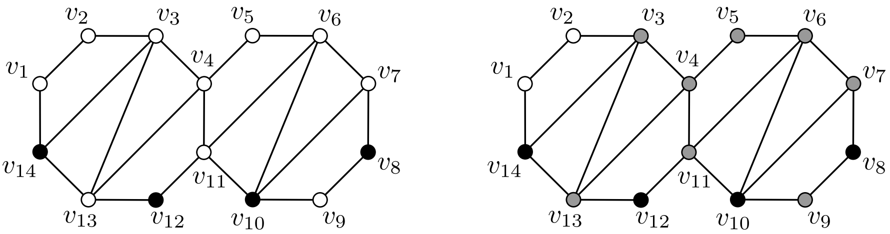

Hull set

Definition: **hull set**

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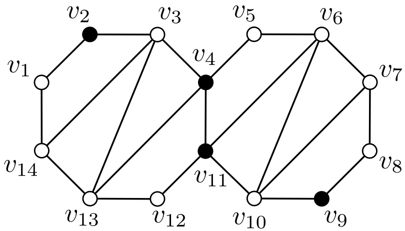

Figure: A hull set.

Parameters of a graph convexity

Definition: **hull number**

The hull number of a graph is the cardinality of its smallest hull set.

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Definition: **interval number**

The minimum size of a set S such that $I(S) = V(G)$.

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The maximum size of a convex set S such that $S \neq V(G)$.

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Definition: partition number

The maximum integer k such that $V(G)$ can be partitioned into p convex sets for any $p \in \{1, \ldots, k\}$.

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Definition: **percolation time**

The maximum number of iteractions of the greed algorithm to generate the whole graph.

Radon number

Definition: Radon partition

Given a set R, a partition $R = R_1 \cup R_2$ such that $H(R_1) \cap H(R_2) \neq \emptyset$ is a Radon partition.

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Definition: Radon-independent set

If a set does not admit a Radon partition we say it is a Radon-independent set.

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The Radon number $r(G)$ is given by $\max\{|R|, R \text{ is a Radon-independent set}\} + 1$.

Carathéodory number

Definition: **Carathéodory-independent set**

A set S such that

$$H(S) \setminus \bigcup_{v \in S} H(S \setminus \{v\}) \neq \emptyset.$$

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Definition: **Carathéodory number**

Largest cardinality of a Carathéodory-independent set.

Smallest integer k such that for every $U \subseteq V$ and every $u \in H(U)$, there is a set $F \subseteq U$ with $|F| \leq k$ and $u \in H(F)$.

Helly number

Definition: Helly-independent set

A set S such that

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Helly number

Definition: **Helly-independent set**

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Definition: **Helly number**

The cardinality of a maximum Helly-independent set.

Convexly independent sets

Definition: **convexly independent set**

$S \subseteq V(G)$ is a convexly independent set if, for all $v \in S$, $v \notin H(S - v)$. Equivalently, no vertex of S is generated by the others.

If a vertex of $S \subseteq V(G)$ is generated by the others, S is **convexly dependent**.

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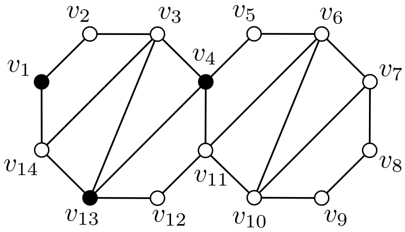

Figure: Convexly dependent set.

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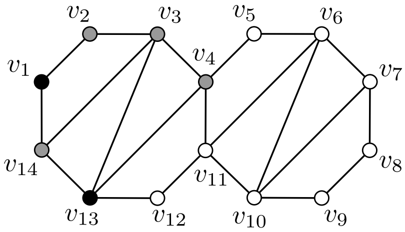

Figure: $v_4 \in H(\{v_1, v_{13}\})$.

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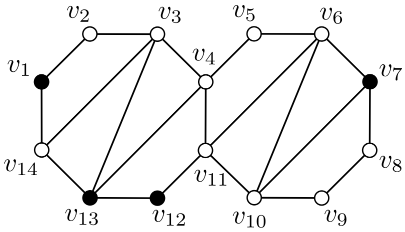

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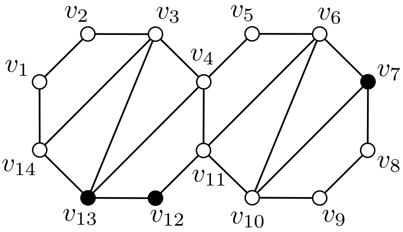

Figure: $v_1 \notin H(\{v_7, v_{12}, v_{13}\})$.

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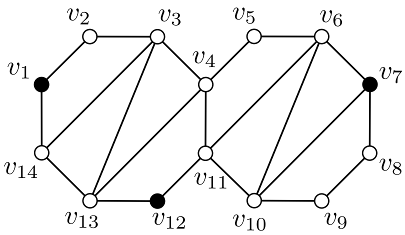

Figure: $v_{13} \notin H(\{v_1, v_7, v_{12}\})$.

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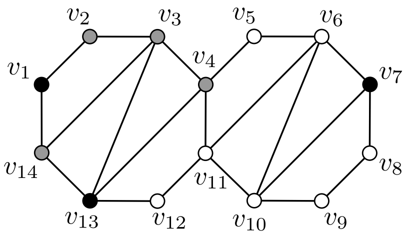

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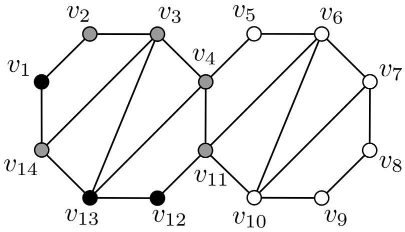

Figure: $v_7 \notin H(\{v_1, v_{12}, v_{13}\})$.

The rank of a graph

Definition: **rank**

The rank of a graph, denoted by $rk(G)$, is the cardinality of a maximum convexly independent set of G.

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CONVEXLY INDEPENDENT SET

INPUT: A graph G and an integer k.

QUESTION: $rk(G) \geq k$?

State of the art

Parameter	Geodetic	Monophonic	P_3	P_3^*
Interval of a set	Polynomial	NP-complete	Polynomial	Polynomial
Hull	NP-complete	Polynomial	NP-complete	NP-complete
Convexity	NP-complete	NP-complete	NP-complete	NP-complete
Radon	NP-complete	NP-complete	NP-hard	NP-hard
Carathéodory	NP-complete	Polynomial	NP-complete	NP-complete
Helly	coNP-complete	Polynomial	Open	Open
Rank	NP-complete	NP-complete	NP-complete	NP-complete
Partition	NP-complete	Open	NP-complete	Open
Percolation time	NP-complete	NP-complete	NP-complete	NP-complete

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The open packing number

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Open Packing Number
Input: A graph G and an integer k.
Question: $\rho_o(G) \geq k$?

$\rho_o(G)$ and $rk(G)$

- If S is an open packing, $H(S) = S$. Hence, every open packing is convexly independent.
- Not every convexly independent set is an open packing.

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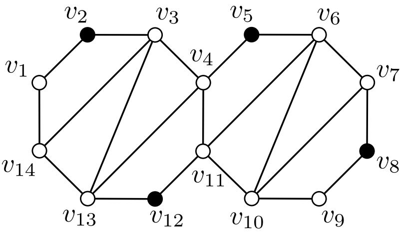

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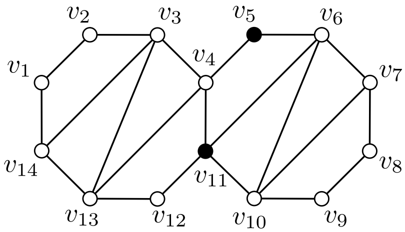

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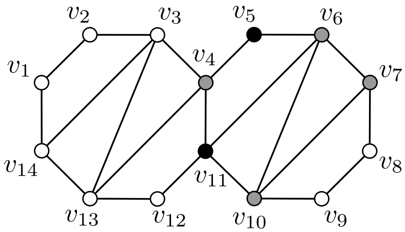

Split graphs on the P_3 convexity

Theorem

The CONVEXLY INDEPENDENT SET *problem is NP-complete on the P_3 convexity, even for split graphs with $\delta(G) \geq 2$.*

Proof sketch

Reduction from SET PACKING.

SET PACKING
INPUT: A family $\mathcal{S}$ of non-empty subsets $S_i \in \mathcal{S}$ of a ground set and an integer k.
QUESTION: $\mathcal{S}$ has at least k pairwise disjoint sets?

Proof sketch

$S_1 = \{1, 6\}, S_2 = \{1, 2\}, S_3 = \{2, 3\}, S_4 = \{3, 4\}, S_5 = \{4, 5\}, S_6 = \{5, 6\}.$

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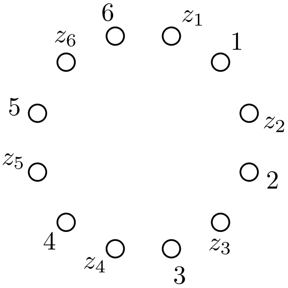
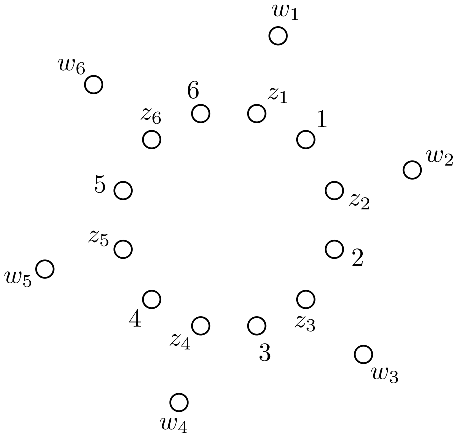


Proof sketch

$S_1 = \{1,6\}, S_2 = \{1,2\}, S_3 = \{2,3\}, S_4 = \{3,4\}, S_5 = \{4,5\}, S_6 = \{5,6\}.$



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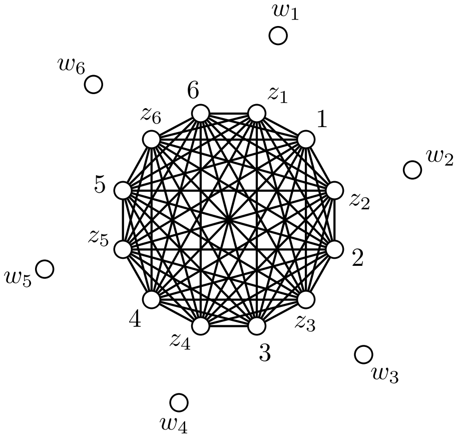

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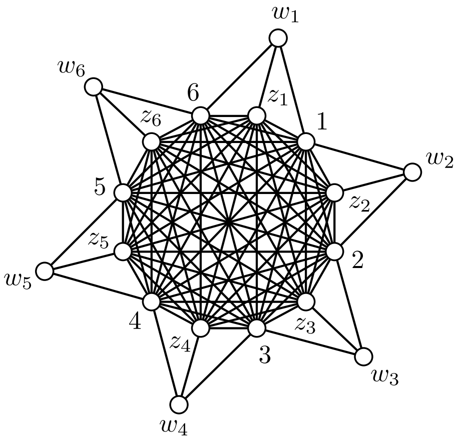

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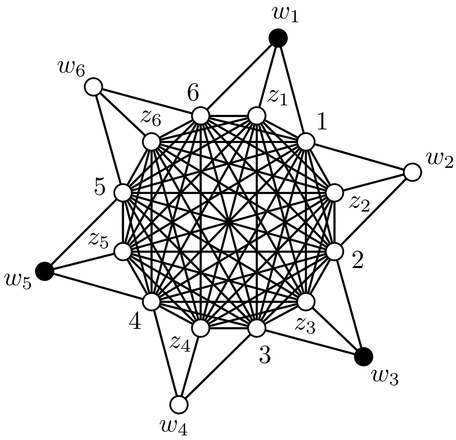

Open packing number

Corollary

The OPEN PACKING NUMBER *is NP-complete for split graphs with* $\delta(G) \geq 2$.

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It was known that the problem was NP-hard for chordal graphs [Henning and Slater, 1999].

Bipartite graphs on the P_3 convexity

Theorem

The Convexly Independent Set *problem is NP-complete on the P_3 convexity, even for bipartite graphs with diameter at most 3.*

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Reduction from Convexly Independent Set for split graphs with $\delta(G) \geq 2$.

Threshold graphs on the P_3 convexity

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The Convexly Independent Set *problem can be solved in linear time for threshold graphs on the P_3 convexity.*

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If G is a connected threshold graph with $|V(G)| \geq 3$ and $D \subseteq V(G)$ is a set with all vertices of minimum degree G, then:

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If G is a connected threshold graph with $|V(G)| \geq 3$ and $D \subseteq V(G)$ is a set with all vertices of minimum degree G, then:

(i) *if G is a star, then $rk(G) = |V(G)| - 1$;*

(ii) *otherwise, if $d(v) = 1$ for all $v \in D$, then $rk(G) = |D| + 1$;*

(iii) *otherwise, $rk(G) = 2$.*

Trees on the P_3 convexity

Theorem

The CONVEXLY INDEPENDENT SET *problem can be solved in time $O(n \log \Delta(T))$ for trees on the P_3 convexity.*

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The CONVEXLY INDEPENDENT SET *problem can be solved in time $O(n \log \Delta(T))$ for trees on the P_3 convexity.*

Dynamic Programming.

Algorithm idea

Given a tree T.

- Select a root $r \in V(T)$.

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- $P_v(i, j, k)$ is the **contribuition** of v: maximum number of vertices of the subtree rooted on v that can be on the maximum convexly independent set under the condition given by i, j e k:
  - $i = 1$: the parent of v sends charge to v.
  - $j = 1$: v is in the convexly independent set being considered.
  - k: number of children sending charge to v.

Algorithm idea

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  - $i = 1$: the parent of v sends charge to v.
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  - k: number of children sending charge to v.
- Define:
  - $f(v, i) = \max\{P_v(i, 0, 0), P_v(i, 0, 1)\}$.
  - $h(v, i) = \max\{\max_{2 \leq k < d(v)}\{P(i, 0, k)\}, \max_{0 \leq k \leq d(v)} P_v(i, 1, k)\}$.
  - $g(v, i_1, i_2) = h(v, i_1) - f(v, i_2)$.

Recurrence relation I

$$P_v(0,0,0) = \sum_{u \in N'(v)} f(u,0); \tag{1}$$

$$P_v(0,0,1) = \begin{cases} -\infty, & \text{if } v \text{ have no children,} \\ \sum_{u \in N'(v)} f(u,0) + \max_{u \in N'(v)} g(u,0,0), & \text{otherwise;} \end{cases} \tag{2}$$

$$P_v(0,0,2) = \begin{cases} -\infty, & \text{if } v \text{ has at most 1 child,} \\ \sum_{u \in N'(v)} f(u,1) + \max_{\substack{\forall X \subseteq N'(v) \\ |X|=2}} \sum_{u \in X} g(u,0,1), & \text{otherwise;} \end{cases} \tag{3}$$

$$\underset{k \geq 3}{P_v(0,0,k)} = \begin{cases} -\infty, & \text{if } v \text{ less than } k \text{ children,} \\ \sum_{u \in N'(v)} f(u,1) + \max_{\substack{\forall X \subseteq N'(v) \\ |X|=k}} \sum_{u \in X} g(u,1,1), & \text{otherwise;} \end{cases} \tag{4}$$

Recurrence relation II

$$P_v(0,1,0) = \sum_{u \in N'(v)} f(u,1) + 1; \tag{5}$$

$$P_v(0,1,1) = \begin{cases} -\infty, & \text{if } v \text{ have no children,} \\ \displaystyle\sum_{u \in N'(v)} f(u,1) + \max_{u \in N'(v)} g(u,1,1) + 1, & \text{otherwise;} \end{cases} \tag{6}$$

$$\underset{k \geq 2}{P_v(0,1,k)} = -\infty; \tag{7}$$

$$P_v(1,0,0) = \begin{cases} -\infty, & \text{if } v = r, \\ \displaystyle\sum_{u \in N'(v)} f(u,0), & \text{otherwise;} \end{cases} \tag{8}$$

Recurrence relation III

$$P_v(1,0,1) = \begin{cases} -\infty, & \text{if } v \text{ have no children ou } v = r, \\ \displaystyle\sum_{u \in N'(v)} f(u,1) + \max_{u \in N'(v)} g(u,0,1), & \text{otherwise;} \end{cases} \tag{9}$$

$$\underset{k \geq 2}{P_v(1,0,k)} = \begin{cases} -\infty, & \text{if } v \text{ has less than } k \text{ children or } v = r, \\ \displaystyle\sum_{u \in N'(v)} f(u,1) + \max_{\substack{\forall S \subseteq N'(v) \\ |S|=k}} \sum_{u \in S} g(u,1,1), & \text{otherwise;} \end{cases} \tag{10}$$

$$P_v(1,1,0) = \begin{cases} -\infty, & \text{if } v = r, \\ \displaystyle\sum_{u \in N'(v)} f(u,1) + 1, & \text{otherwise;} \end{cases} \tag{11}$$

$$\underset{k \geq 1}{P_v(1,1,k)} = -\infty. \tag{12}$$

Graphs without separating cliques on the monophonic convexity

Theorem

The CONVEXLY INDEPENDENT SET *problem is NP-complete on the monophonic convexity, even for graphs without separating clique.*

Graphs without separating cliques on the monophonic convexity

Theorem [Dourado, Protti, Szwarcfiter, 2010]

If G is a graph with no separating clique, but is not a complete graph, then every pair of non-adjacent vertices is a hull set of G on the monophonic convexity.

Graphs without separating cliques on the monophonic convexity

Theorem [Dourado, Protti, Szwarcfiter, 2010]

If G is a graph with no separating clique, but is not a complete graph, then every pair of non-adjacent vertices is a hull set of G on the monophonic convexity.

Lemma

The CLIQUE *problem is NP-complete, even for graphs without separating clique.*

Open problems

Parameter	**Geodetic**	**Monophonic**	P_3	P_3^*
Interval of a set	Polynomial	NP-complete	Polynomial	Polynomial
Hull	NP-complete	Polynomial	NP-complete	NP-complete
Convexity	NP-complete	NP-complete	NP-complete	NP-complete
Radon	NP-complete	NP-complete	NP-hard	NP-hard
Carathéodory	NP-complete	Polynomial	NP-complete	NP-complete
Helly	coNP-complete	Polynomial	Open	Open
Rank	NP-complete	NP-complete	NP-complete	NP-complete
Partition	NP-complete	Open	NP-complete	Open
Percolation time	NP-complete	NP-complete	NP-complete	NP-complete

Open problems

- Fill the gaps on the table.

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  - Generating Degree.
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Open problems

- Fill the gaps on the table.
- New parameters:
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  - Exchange number.
- New convexities.
- Refine results:
  - Finding tractable cases.
  - Exact (FPT?) algorithms.
  - Strengthening hardness results.

Thanks

Thank you for your attention!

¡Gracias por su atención!