HOMOMORPHISMS
STRUCTURAL RAMSEY THEORY
LIMITS

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FoCM MONTEVIDEO
DEC 13, 2014

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MODERN DICHOTOMIES

SPARSE	VS	DENSE
STRUCTURE	VS	RANDOMNES

MODEST GOAL

THREE RESULTS

&

ONE CONJECTURE(S)
TWO

ALL MATHEMATICAL CONTEXT
OF COMBINATORICS

$\boxed{\text{I}}$ COUNTING HOMOMORPHISM (SUBGRAPHS)

$\boxed{\text{DEF.}}$ A HOMOMORPHISM IS A MAP WHICH PRESERVES ALL INFORMATION (DEFINING STRUCTURE)

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FOR GRAPHS: $f: V \longrightarrow V'$ IS HOMOMORPHISM OF $G=(V,E) \longrightarrow G'=(V',E')$ IF $xy \in E \Longrightarrow f(x)f(y) \in E'$.

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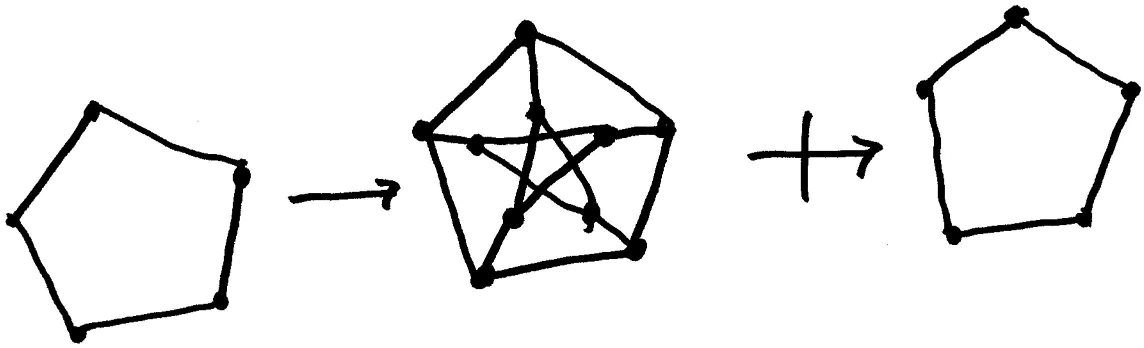

$$hom(F,G) = \left| \left\{ f ; f : F \xrightarrow[\text{HOMO}]{} G \right\} \right|$$

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$\Big(\sim$ G-COLOURING OF F $\Big)$

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$\Big($ $\sim$ G-COLOURING OF F $\Big)$

$\sim$ PARTITION FUNCTION

$$\hom(F, G) = \left| \{ f \, ; \, f : F \xrightarrow[\text{HOMO}]{} G \} \right|$$

(~ G-COLOURING OF F)

~ PARTITION FUNCTION

~ CONSTRAINT SATISFACTION

$$\hom(F, G) = \left| \{ f ; f \underset{\text{HOMO}}{:} F \to G \} \right|$$

(~ G-COLOURING OF F)

~ PARTITION FUNCTION

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~ IMPORTANT INVARIANT

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(~ G-COLOURING OF F)

~ PARTITION FUNCTION

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~ IMPORTANT INVARIANT

WHAT CAN WE SAY ABOUT

$$\frac{\log \hom(F,G)}{\log |G|} \quad ?$$

(i.e. TRYING TO EXPRESS $\hom(F,G)$ AS $|G|^{\alpha}$)

$\boxed{\text{THM}}$ (N., P. OSSONA DE MENDEZ)

FOR ANY CLASS $\mathcal{C}$ OF GRAPHS

$$\sup_{G} \frac{\log \hom(F,G)}{\log |G|} \longrightarrow \{-\infty, 0, 1, \ldots, |F|\}$$

FOR ANY F.

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COUNTER EXAMPLE :

$$\mathcal{C} = \{ G ;\ G \not\ni \square \}$$

SUP NOT INTEGER $\left(3/2 \right)$

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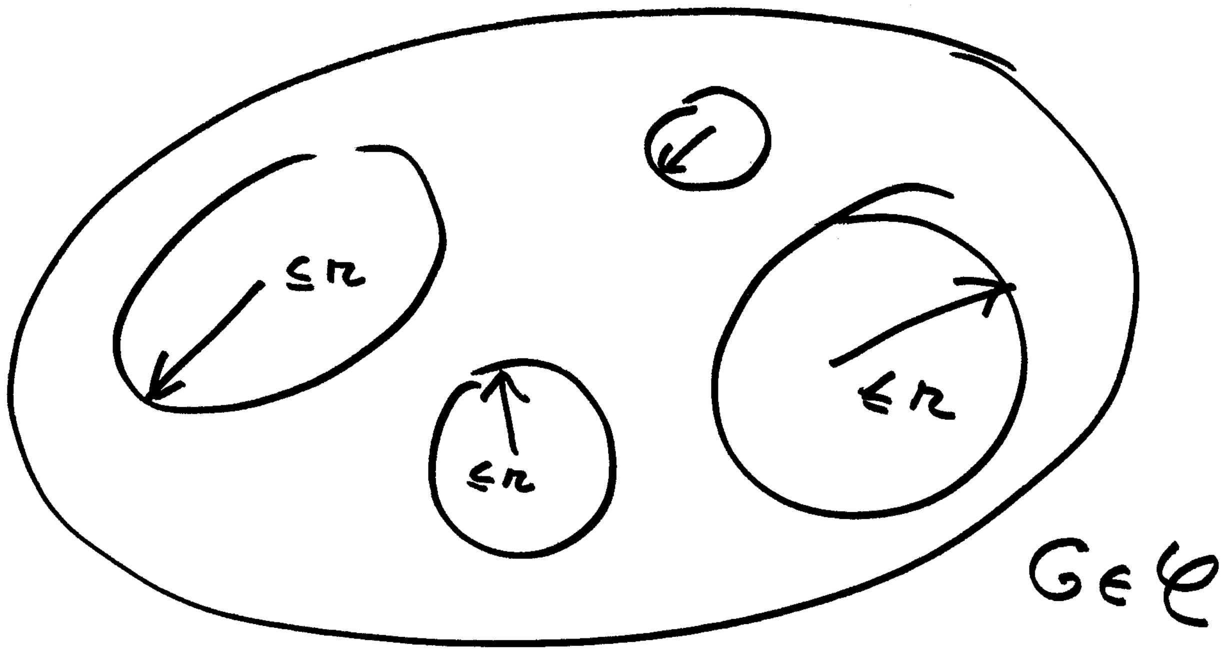
CENTRAL QUESTION OF EXTREMAL THEORY

HIERARCHY OF CLASSES

$$\mathcal{C} \subseteq \mathcal{C}\nabla 0 \subseteq \mathcal{C}\nabla 1 \subseteq \ldots.$$

$\mathcal{C}\nabla i$ = ALL GRAPHS WHICH CAN BE OBTAINED FROM GRAPHS IN $\mathcal{C}$ BY CONTRACTING SUBGRAPHS WITH RADIUS $\leq i$.



$\boxed{\text{THM I}}$

FOR ANY CLASS $\mathcal{C}$ AND ANY F:

$$\lim_{i\to\infty} \sup_{G\in\mathcal{C}\nabla i} \frac{\log \hom(F,G)}{\log |G|} \in \{-\infty, 0, 1, \ldots, \underbrace{|F|}_{\alpha(F)}\}$$

&

$$\lim\sup < |F| \iff \mathcal{C}\nabla i \neq \text{ALL GRAPHS FOR ALL } i$$

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&

$$\limsup < |F| \iff \mathcal{C}\nabla i \neq \text{ALL GRAPHS} \text{ FOR ALL } i$$

(NOWHERE DENSE CLASSES)

VS

SOMEWHERE DENSE CLASSES

(DVOŘÁK, NORINE, KREUTZER, GROHE, KRÁL, THOMAS, ...

Algorithms and Combinatorics 28

Jaroslav Nešetřil
Patrice Ossona de Mendez

Sparsity

Graphs, Structures, and Algorithms

Springer

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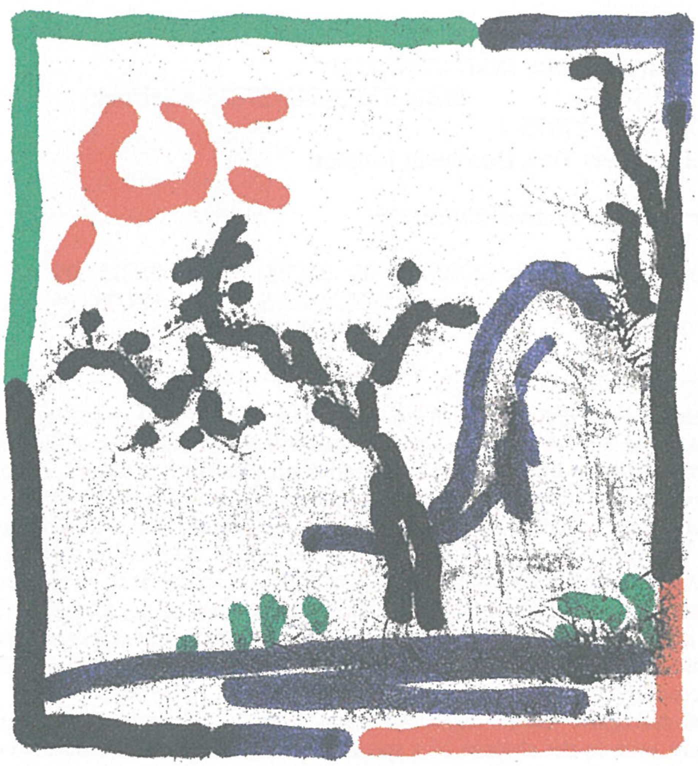


Springer

COROLLARY

$\mathcal{C}$ NOWHERE DENSE

$\Updownarrow$

$$\lim_{i\to\infty} \limsup_{G\in\mathcal{C}\nabla i} \frac{\log(\# F\subseteq G)}{\log|G|}$$

$\in$

$$\{-\infty, 0, 1, \dots, \alpha(F)\}$$

FOR EVERY F. WITH AT LEAST ONE EDGE SO THAT $\alpha(F) < |F|$.

$\boxed{\text{PROOF}}$

IT SUFFICES TO PROVE $\Downarrow$

PROOF COMBINES ALL CHARACTERIZATION

- TREE DEPTH td
- LOW TREE DEPTH DECOMPOSITIONS χ_p
- STEPPING UP LEMMA FOR "(k, F) - SUNFLOWERS"
- COUNTING OF COLORED TREES

PROOF II

FIRST CASE: $F = K_2$

THM (TRICHOTOMY THM)

FOR EVERY INFINITE CLASS $\mathcal{C}$

THE LIMIT

$$\lim_{i \to \infty} \limsup_{G \in \mathcal{C} \triangledown i} \frac{\log \|G\|}{\log |G|}$$

$$\in$$

$$\{-\infty, 0, 1, 2\}$$

≤ 1 IFF $\mathcal{C}$ IS NOWHERE DENSE

2 IFF $\mathcal{C}$ IS SOMEWHERE DENSE

$\boxed{\text{II}}$ CHARACTERISATION OF RAMSEY CLASSES

FINITE RAMSEY THEOREM

$$\forall p, k, n \quad \exists N = N(p,k,n) : N \longrightarrow (n)^p_k$$

IF $|X| = N$, $\binom{X}{p} = \{A \subseteq X ; |A| = p\}$

$\binom{X}{p} = \mathcal{A}_1 \cup \dots \cup \mathcal{A}_k$ (ARBITRARY PARTITION)

THEN

$\exists Y, Y \subseteq X, |Y| = n \quad \exists i_0$

$$\binom{Y}{p} \subseteq \mathcal{A}_{i_0}$$

$\boxed{\text{DEF}}$ RAMSEY CLASS $\mathcal{K}$

(OF STRUCTURES)

$A, B, C, \ldots \in \mathcal{K}$

$\binom{B}{A}$ ALL SUBSTRUCTURES OF B ISOMORPHIC TO A

$\mathcal{K}$ IS RAMSEY CLASS IFF (LEEB, N., RODL)

$$\forall A \, \forall B \, \forall k \;\; \exists C : \quad C \longrightarrow (B)^A_k$$

IN $\mathcal{K}$

(ERDŐS-RADO) PARTITION ARROW :

FOR EVERY $\binom{C}{A} = \mathcal{A}_1 \cup \ldots \cup \mathcal{A}_k$ $\exists \, B' \in \binom{C}{B}$

SUCH THAT $\binom{B'}{A} \subseteq \mathcal{A}_{i_0}$

RAMSEY CLASSES $\equiv$ TOP OF THE LINE OF "RAMSEY PROPERTIES"

EXAMPLES

ALL COMPLETE GRAPHS (FRT)

ALL FINITE LINEAR ORDERS (FRT)

ALL FINITE VECTOR SPACES

ALL COMBINATORIAL CUBES

NOT

GRAPHS, ...

NECESSARY CONDITIONS FOR RAMSEY CLASSES

1 RIGIDITY

(ORDERING, ORDERING LEMMA)

N. RODL

ANGEL, KECHRIS, LYONS

NECESSARY CONDITIONS FOR RAMSEY CLASSES

① RIGIDITY

(ORDERING, ORDERING LEMMA)

N. RODL

ANGEL, KECHRIS, LYONS

② GLOBAL SYMMETRY

(AGE OF ULTRAHOMOGENEOUS STRUCTURE)

N.

KECHRIS, PESTOV, TODORCEVIC

CHARACTERIZATION PROGRAM

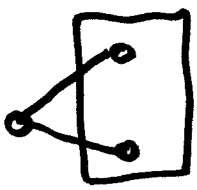
HOMOMORPHISM DEFINE | CLASSES

$$\mathcal{F} = \{F_1, F_2, \dots, F_t\}$$

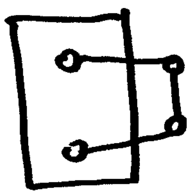
$$\mathrm{FORB}(\mathcal{F}) = \{G ;\ F_i \not\to G \ \ i = 1, \dots, t\}$$

FORB(⬠) = ALL GRAPHS NOT CONTAINING $\triangle$ OR ⬠

$$\boxed{\text{FORB(⬠) NOT RAMSEY}}$$



BLUE



RED

BUT

FORB () + BLUE-RED

LIFT OF FORB()

(EXPANSION)

IS RAMSEY

BUT

FORB() + BLUE-RED

LIFT OF FORB()
(EXPANSION)

IS RAMSEY

$\boxed{\text{THM II}}$ (N. 13)

FOR EVERY FINITE SET OF CONNECTED GRAPHS $\mathcal{F}$ THE CLASS FORB($\mathcal{F}$) HAS A FINITE LIFT WHICH FORM RAMSEY CLASS.

$\boxed{\text{THM 2}}$

POSETS FORM RAMSEY CLASS
(N. RODL)

FINITE METRIC SPACES
(N.)

UNIVERSAL OBJECTS FOR FORB($\mathcal{F}$)
(CSS)

$\boxed{\text{THM 2}}$

POSETS FORM RAMSEY CLASS (N. RODL)

FINITE METRIC SPACES (N.)

UNIVERSAL OBJECTS FOR FORB($\mathcal{F}$) (CSS)

$\boxed{\text{PROBLEM}}$ EVERY ULTRAHOMOGENEOUS STRUCTURE HAS A FINITARY LIFT WHICH FORM RAMSEY CLASS.

$\boxed{\text{THM 2}}$

POSETS FORM RAMSEY CLASS (N. RODL)

FINITE METRIC SPACES (N.)

UNIVERSAL OBJECTS FOR FORB($\mathcal{F}$) (CSS)

$\boxed{\text{PROBLEM}}$ EVERY ULTRAHOMOGENEOUS STRUCTURE HAS A FINITARY LIFT WHICH FORM RAMSEY CLASS.

BOWTIE-FREE GRAPHS (KOMJATH) HAVE RAMSEY LIFT (HUBIČKA, N.)

$\boxed{\text{COROLLARY}}$

EVERY HOMOMORPHISM CLOSED FIRST ORDER DEFINABLE CLASS OF STRUCTURES HAS A RAMSEY LIFT.

(PF: THEOREM II
+
ROSSMAN)

III. LIMITS BY SATISFACTION

$$\exists\, f : F \longrightarrow G$$

$$\Updownarrow$$

$$G \models \varphi_f$$

$$\left(\varphi_f = \bigwedge_{ij \in E(F)} x_i x_j \right)$$

φ FIRST ORDER FORMULA

$\varphi(x_1, \dots, x_t)$ WITH t FREE VARIABLES

$$\varphi(G) = \{ (v_1, \dots, v_t) ; \; G \models \varphi(v_1, \dots, v_t) \}$$

$$\langle \varphi, G \rangle = \frac{|\varphi(G)|}{|G|^t}$$

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$$\langle \varphi, G \rangle = \frac{|\varphi(G)|}{|G|^t}$$

"STONE BRACKET"

PROBABILITY THAT t VERTICES CAN BE EXTENDED TO SATISFACTION OF φ

EXAMPLE

$$\varphi(x_1, x_2) = \exists x \left(xx_1 \in E \,\&\, xx_2 \in E \;\&\; x_1 \neq x_2 \right)$$

$$|||$$

BETWEEN x_1 AND x_2 THERE IS A PATH OF LENGTH 2.

$\langle \varphi, G \rangle$ = PROB. THAT 2 VERTICES OF G ARE JOINED BY A PATH OF LENGTH 2.

$G_1, G_2, \dots, G_n, \dots$ IS $\boxed{\text{FO-CONVERGENT}}$

IFF

FOR EVERY φ THE NUMBERS

$$\langle \varphi, G_i \rangle$$

CONVERGE.

N. P. OSSONA DE MENDEZ

$G_1, G_2, \dots, G_n, \dots$ IS $\boxed{\text{FO-CONVERGENT}}$

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CONVERGE.

N. P. OSSONA DE MENDEZ

X – CONVERGENCE

⋮

FOR EVERY $\varphi \in X$ ....

- - - - -

$FO(\mathcal{L})$ BOOLEAN ALGEBRA

$\cup\!|$

X BOOLEAN SUBALGEBRA

$S(X)$ STONE SPACE OF X

(ULTRAFILTERS OR
HOMOMORPHISMS $f : X \to \{0,1\}$)

+ TOPOLOGY BY

$$K_X(\varphi) = \{ U \mid \varphi \in U \}$$
$$= \{ f \mid f(\varphi) = 1 \}$$

STONE DUALITY

$\boxed{\text{THM}}$

X SUBALGEBRA OF $FO(\mathcal{L})$.
FOR EVERY X-CONVERGENT SEQUENCE $(G_n)_{n\in\mathbb{N}}$ THERE EXISTS UNIQUE PROBABILITY MEASURE μ ON $S(X)$ SUCH THAT FOR EVERY FORMULA $\varphi(x_1,\dots,x_p)\in X$ HOLDS:

$$\int_{S(X)} \mathbb{1}_{K(\varphi)}(x)\,d\mu(x) =$$

$$= \lim_{n\to\infty} \langle \varphi, G_n \rangle$$

EXTREME CASE

$\mathbb{G}$ MODELLING IS STANDARD BOREL SPACE

SUCH THAT EVERY FIRST ORDER DEFINABLE SET IS MEASURABLE

EXTREME CASE

$\mathcal{G}$ MODELLING IS STANDARD BOREL SPACE

SUCH THAT EVERY FIRST ORDER DEFINABLE SET IS MEASURABLE

$\boxed{\text{THM 3}}$

EVERY FO-CONVERGENT SEQUENCE (G_n) CONVERGES TO MODELLING

$\Downarrow$

$\mathcal{C}$ IS NOWHERE DENSE

CONVERSE TRUE FOR

- GRAPHS WITH BOUNDE TREE DEPTH
- GRAPHS WITH BOUNDED DEGREES
- TREES

? $\Uparrow$?

THANK YOU FOR YOUR ATTENTION