Kneser Transversals

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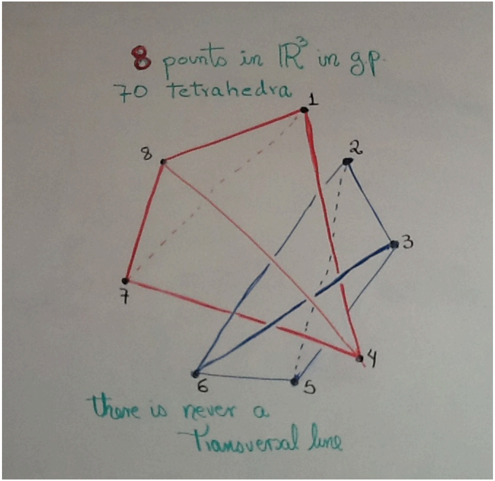
GRAPHS AND COMBINATORICS WORSHOP
FOCM MONTEVIDEO 2014

Overview

8 points in R^3 in general position



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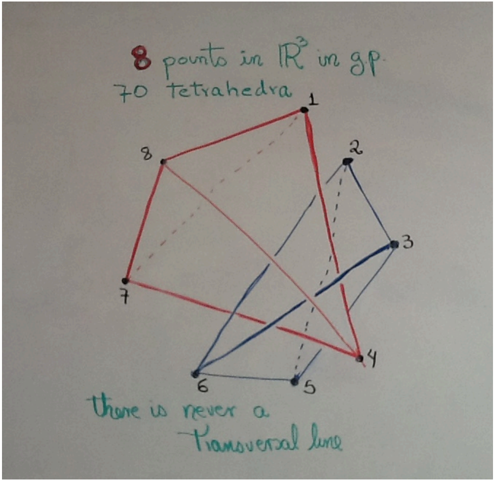
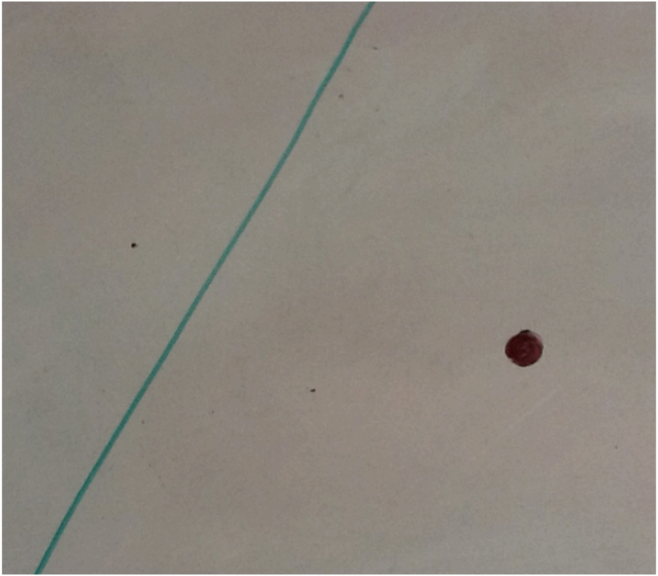


Is there a transversal line to the convex hull of all tethahedra ?

8 points in R^3 in general position NEVER



- *8 POINTS, NEVER*
- *6 POINTS, ALWAYS*
- *7 POINTS, SOMETIMES YES, SOMETIMES NOT*

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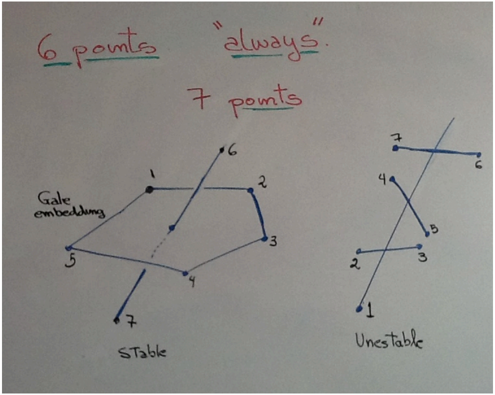

Basic Definitions

always

$m(k, d, \lambda) =$ the maximum positive integer n such that every set of n points in R^d has the property that the convex hull of all k-sets have a transversal $(d - \lambda)$-plane

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$M(k, d, \lambda) =$ the minimum positive integer n such that for every set of n points in $\mathbb{R}^d$ in general position, the convex hull of the k-sets do not have a transversal $(d - \lambda)$-plane.

$$m(k, d, \lambda) < M(k, d, \lambda)$$

$M(k, d, \lambda)$ NEVER

$$M(k, d, \lambda) = d + 2(k - \lambda) + 1$$

- $M(4, 3, 2) = 8$
- *The inequality:* $M(k, d, \lambda) \leq d + 2(k - \lambda) + 1$ *is a Combinatorial argument*
- *Gale Emmbeddings give rice to the inequality* $M(k, d, \lambda) \geq d + 2(k - \lambda) + 1$

$m(k, d, \lambda)$ ALWAYS

$$d - \lambda + k + \left\lceil \frac{k}{\lambda} \right\rceil - 1 \leq m(k, d, \lambda).$$

- $m(4, 3, 2) = 6$
- *The proof of this inequality uses Schubert calculus in the cohomology ring of Grassmannian manifolds.*
- *This inequality has strong connetions with the chromatic number of the Kneser hypergraphs*

Kneser Graphs

- Let $[n]$ denote the set $\{1, \ldots, n\}$
- $\binom{[n]}{k}$ the collection of k-subsets of $[n]$
- The well known Kneser graph has vertex $\binom{[n]}{k}$ and two k-subsets are connected by an edge if they are disjoint.

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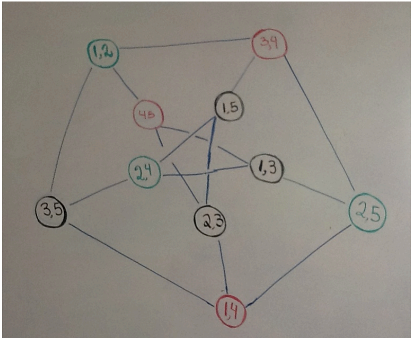
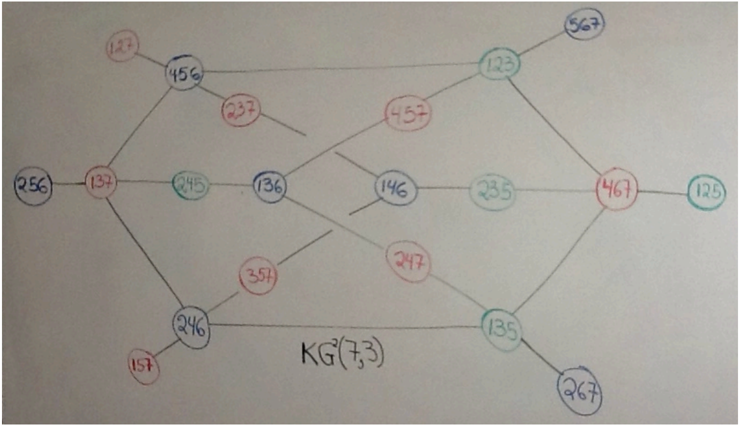
$$KG^2(5,2)$$



Kneser Hypergraph $KG^2(73)$



Chromatic Number of Kneser Hypergraphs

The Kneser Hypergraph $KG^{\lambda}(n, k)$ has vertex $\binom{[n]}{k}$ and λ k-subsets $\{S_1, ..., S\lambda\}$ give rise to an hyperedge if $S_1 \cap ... \cap S_{\lambda} = \emptyset$.

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The chromatic number $\chi(KG^{\lambda+1}(n,k))$ of the Kneser hypergraph is the smallest number m such that a proper coloring of $KG^{\lambda}(n,k)$ with m colors exist.

Connection between $m(k, d, \lambda)$ and the Chromatic Number of Kneser Hypergraphs

The connections between the chromatic number and $m(k, d, \lambda)$ are of the following sort:

PROPER COLORATIONS

$\downarrow$

KNESER TRANSVERSAL

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PROPER COLORATIONS

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KNESER TRANSVERSAL

If the triangles of a set X of 8 points in R^4 can be colored with 3 colors in such a way that the triangles of the same color has the 2-Helly property (3 by 3 have a common point), then there is a plane transversal to all triangles of X.

Upper bound for $m(k, d, \lambda)$

$$\downarrow$$

Lower bound for $\chi(KG^{\lambda+1}(n, k))$

Upper bound for $m(k, d, \lambda)$

$$\downarrow$$

Lower bound for $\chi(KG^{\lambda+1}(n, k))$

If $\quad m(k, d, \lambda) < n, \quad$ then $\quad d - \lambda + 1 < \chi(KG^{\lambda+1}(n, k)).$

Upper bound for $m(k, d, \lambda)$

$$\downarrow$$

Lower bound for $\chi(KG^{\lambda+1}(n, k))$

If $m(k, d, \lambda) < n$, then $d - \lambda + 1 < \chi(KG^{\lambda+1}(n, k))$.

As consequence, of the lower bound
$m(k, d, \lambda) < d - 2(k - \lambda) + 1 = M(k, d, \lambda)$
we obtain:

Upper bound for $m(k, d, \lambda)$

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Lower bound for $\chi(KG^{\lambda+1}(n, k))$

If $\quad m(k, d, \lambda) < n, \quad$ then $\quad d - \lambda + 1 < \chi(KG^{\lambda+1}(n, k)).$

As consequence, of the lower bound
$m(k, d, \lambda) < d - 2(k - \lambda) + 1 = M(k, d, \lambda)$
we obtain:

$$n - 2k + \lambda \quad < \quad \chi(KG^{\lambda+1}(n, k))$$

.

As a Corollary we obtain a proof of the Kneser Conjecture first proved by L. Lovasz.

Lovasz

$$\chi(KG^2(n,k)) = n - 2k + 2$$

.

Rado's Central Point Theorem

Another interesting connection concerns the

Rado's Theorem

Let X be a finite set of n points in $\mathbb{R}^d$. Then there exist a point $x \in \mathbb{R}^d$ such that any closed half-space H through x contains at least $\left\lceil \frac{n}{d+1} \right\rceil$ points of X.

The inequality $d - \lambda + k + \lceil \frac{k}{\lambda} \rceil - 1 \leq m(k, d, \lambda)$ is equivalent to the following theorem which generalizes Rados Theorem.

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Theorem

Let X be a finite set of n points in $\mathbb{R}^d$. Then there exist a $(d - \lambda)$-plane L such that any closed half-space H through L contains at least $\left\lfloor \frac{n-d+2\lambda)}{\lambda+1} \right\rfloor + (d - \lambda)$ points of X.

Conjecture

$$d - \lambda + k + \left\lceil \frac{k}{\lambda} \right\rceil - 1 \quad = \quad m(k, d, \lambda)$$

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The inequality $m(k, d, \lambda) < M(k, d, \lambda)$ shows that the conjecture is true if either $\lambda = 1$, or $k \leq \lambda + 1$ or $k = 2, 3$.

Our next purpose is to improve the inequality

$$d - \lambda + k + \left\lceil \frac{k}{\lambda} \right\rceil - 1 \leq m(k, d, \lambda) \quad < \quad d + 2(k - \lambda) + 1$$

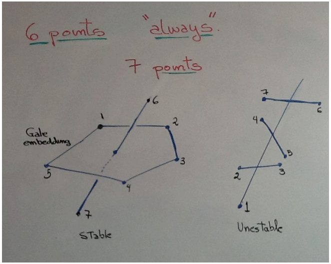
Next Purpose

Find an embedding of $d + 2(k - \lambda)$ *points in* R^d
WITHOUT
$(d - \lambda)$*-transversal line to the convex hull of their* k*-subsets*

Proving that $m(k, d, \lambda) < d + (k - \lambda)$.

In this range we have two classes of $(d-\lambda)$ Kneser Transversals

- Unestable
- Stable $=$ contain $(d - lambda + 1)$ of our points



We shall prove later that the cyclic politope with does not admit an stable Kneser Transversal

Special Kneser Transversals

Given a collection X of points in R^d a $(d-\lambda)$-plane L es called a special Kneser Transversal if and only if

- L contains $(d - lambda + 1)$ points of X
- L intersect the convex hull of all k-subsets of X

Oriented Matroids

order type $\rightarrow$ collection of of points

Kneser Transversals $\rightarrow$ Special Kneser Transversals

$$m(k, d, \lambda) \quad \rightarrow \quad m^*(k, d, \lambda)$$

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Geometry $\rightarrow$ Oriented Matroids

$$M^*(k, d, \lambda) = M(k, d, \lambda)$$

$$M^*(k,d,\lambda) = M(k,d,\lambda)$$

If $2\lambda \geq d+1$, then

$$m^*(k,d,\lambda) = d - \lambda + k$$

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$$k+(d-\lambda)+1 \leq m^*(k,d,\lambda) \leq \left\lfloor \left(\frac{2\lceil \frac{d}{2} \rceil - \lambda + 1}{\lceil \frac{d}{2} \rceil}\right)(k-1) \right\rfloor + (d-\lambda)+1.$$

Last inequality was obtained by proving that the cyclic polytope in $\mathbb{R}^d$ with more than

$$\left\lfloor \left(\frac{2\lceil \frac{d}{2} \rceil - \lambda + 1}{\lceil \frac{d}{2} \rceil}\right)(k-1) \right\rfloor + (d-\lambda) + 1.$$

vertices DOES NOT admit an special transversal $(d-\lambda)$-plane to the convex hulls of all their k-subset.

If $k-1 > \lceil \frac{d}{2} \rceil$, then

$$\left\lfloor \left(\frac{2\lceil \frac{d}{2} \rceil - \lambda + 1}{\lceil \frac{d}{2} \rceil}\right)(k-1) \right\rfloor + (d-\lambda) + 1 < 2d + (k-\lambda)$$

If $k-1 > \left\lceil \frac{d}{2} \right\rceil$, then

$$\left\lfloor \left(\frac{2\left\lceil \frac{d}{2} \right\rceil - \lambda + 1}{\left\lceil \frac{d}{2} \right\rceil}\right)(k-1) \right\rfloor + (d-\lambda) + 1 < 2d + (k-\lambda)$$

If $k-1 > \left\lceil \frac{d}{2} \right\rceil$, then

$$m(d,k,\lambda) < 2d + (k-\lambda)$$

Coauthors

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- Jorge Ramirez Alfonsin
- Martin Tancer

THANK YOU

FOR YOUR

KIND ATTENTION