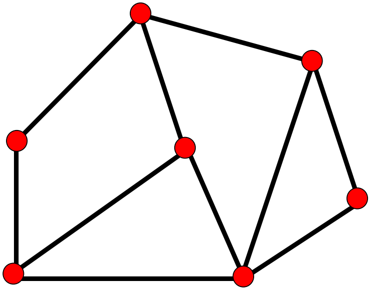
Directed cycle double covers and cut-obstacles

Andrea Jiménez

University of São Paulo

Joint work with Martin Loebl

Consider an undirected graph G



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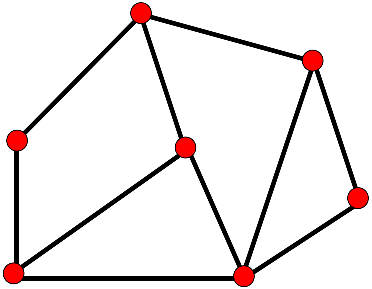
Cycle double cover (CDC): set of cycles of G covering edges exactly twice



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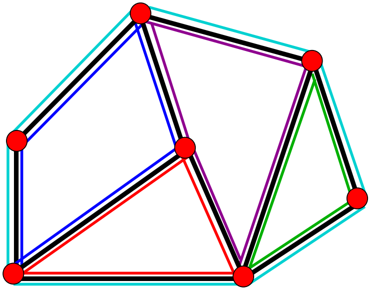

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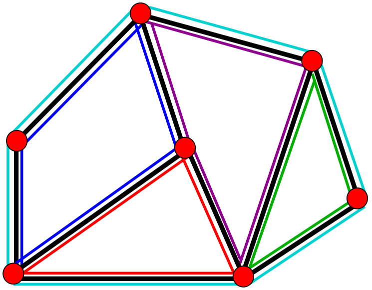

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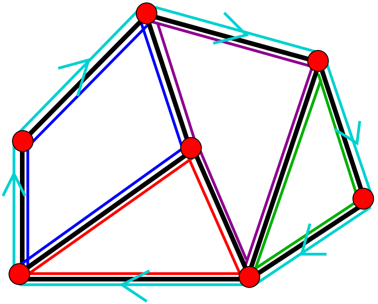

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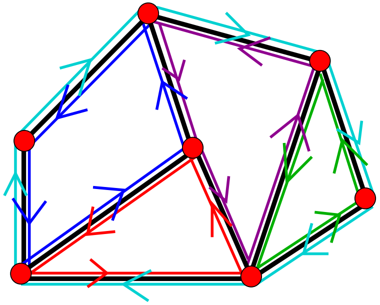

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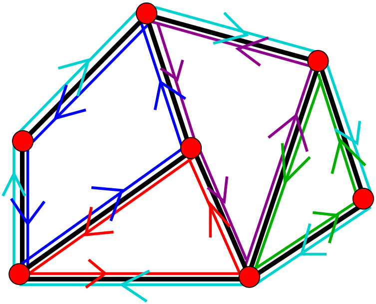

DCDC Conjecture [Jaeger – 1985]

Every bridgeless graph has a DCDC.

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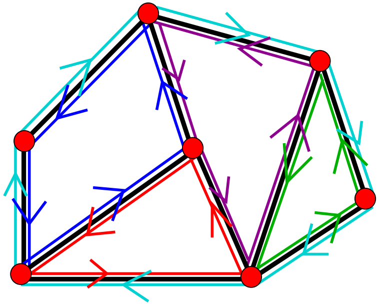

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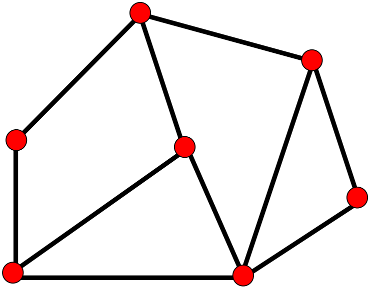

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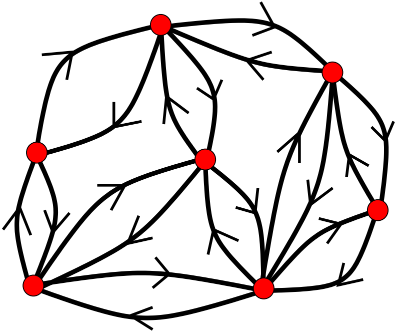

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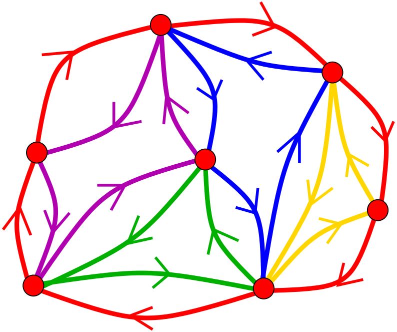

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- minimum counterexample to the DCDC conjecture: cubic, cyclically 4-edge-connected, non 3-edge-colorable.
- DCDC conjecture holds in:
  - ⋆ bridgeless planar graphs,
  - ⋆ graphs with a nowhere-zero 4-flow (Jaeger, Tutte, $\leq$ 1979),
  - ⋆ 2-connected projective-planar graphs (Ellingham & Zha, 2011),
  - ⋆ lean fork graphs (J. & Loebl, 2013+).
- Topological approach to the DCDC conjecture (Jaeger, 1979).

A trigraph H is a cubic graph with ear decomposition

$$H_0, L_1 \dots, L_n$$

s.t. H_0 is a cycle, L_i is a k-ear with $k \in \{1, 2, 3\}$ or a star.

- ear := star or path
- k-ear := path with k internal vertices

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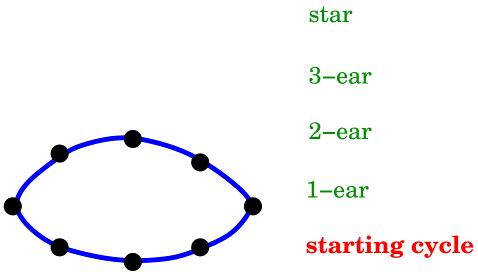

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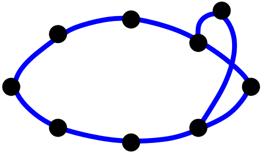

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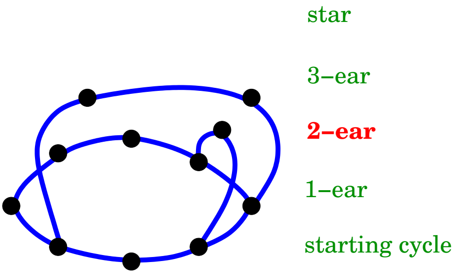

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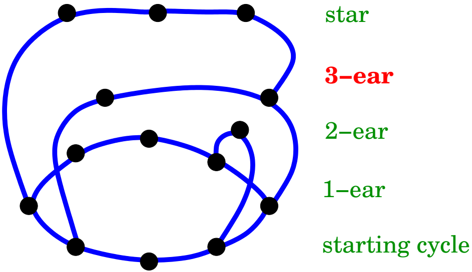

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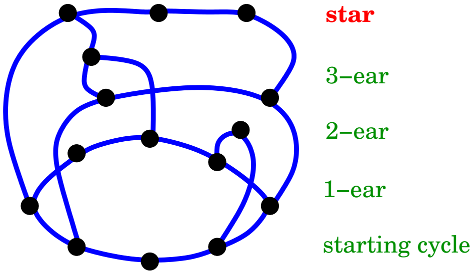

Trigraph conjecture

Every robust trigraph admits a reduction process avoiding cut-obstacles.

Theorem [J. & Loebl, 2014+]

Our “trigraph conjecture” implies general DCDC conjecture.

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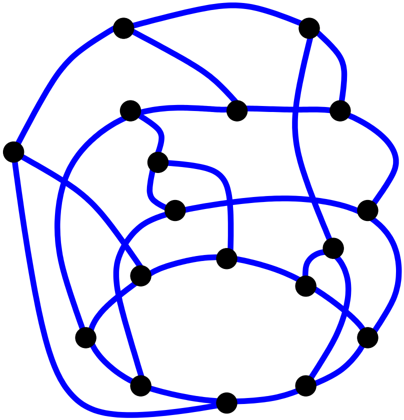
Reduction process and cut-obstacles:



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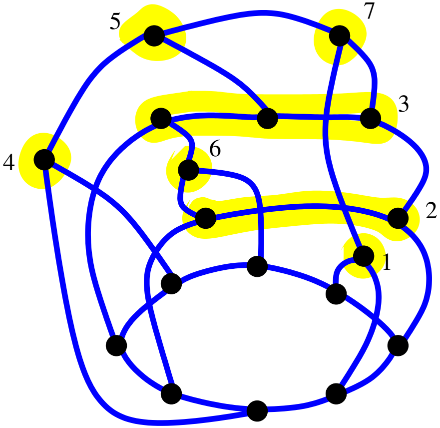

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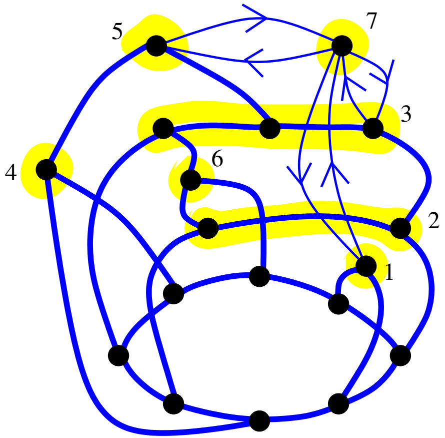

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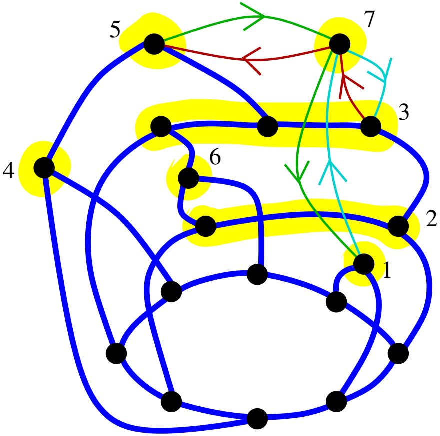

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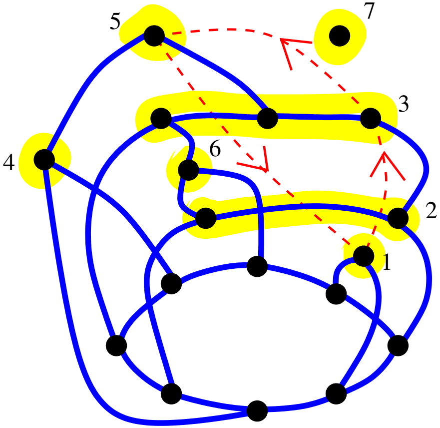

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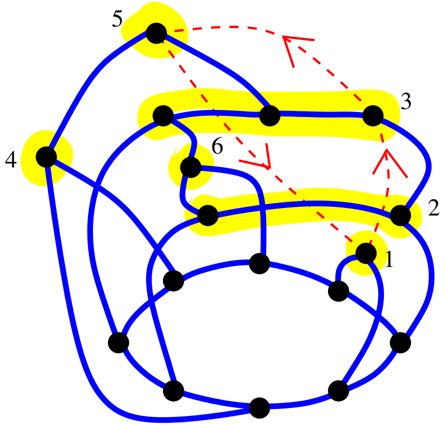

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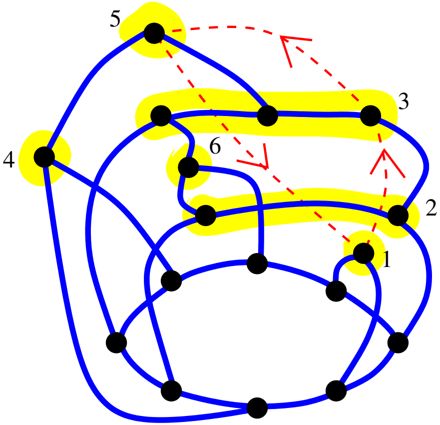

Mixed graph (V, E, A, R) R:=forbidden pairs of arcs in a DCDC

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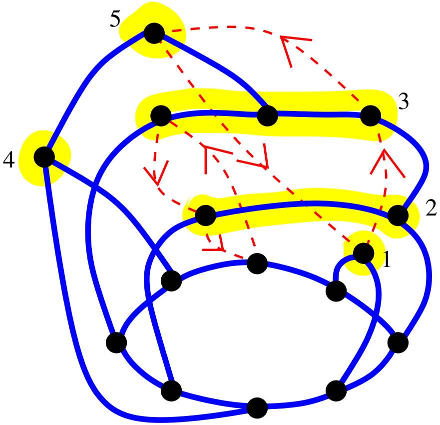

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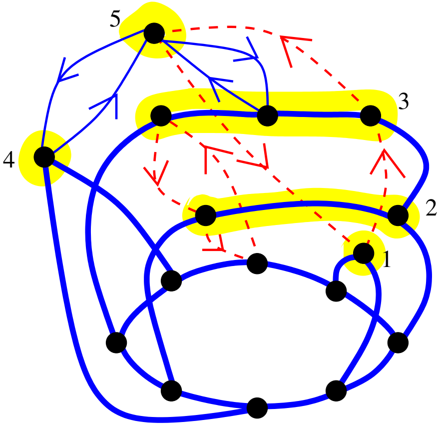

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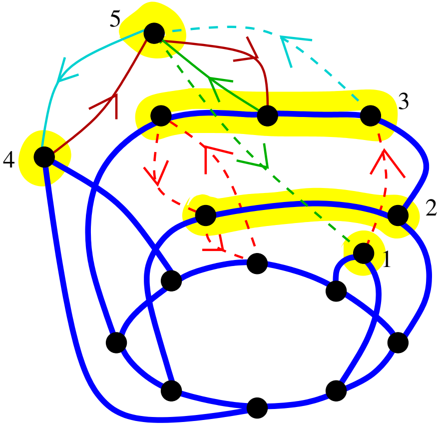

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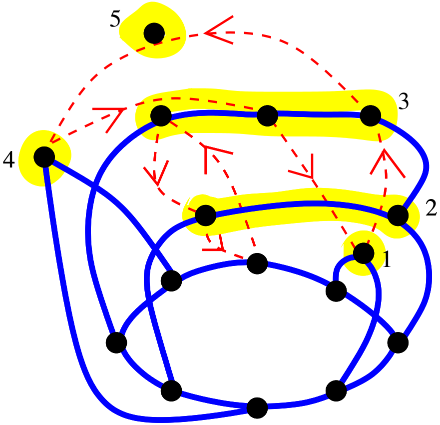

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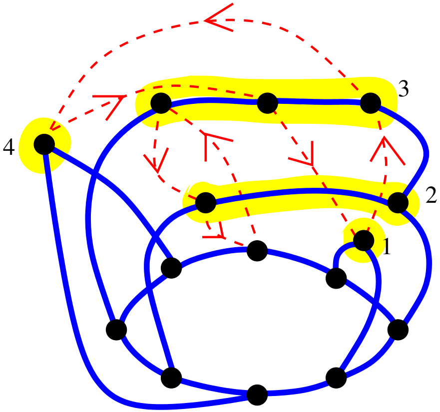

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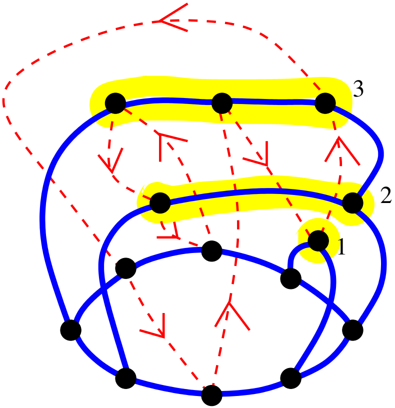

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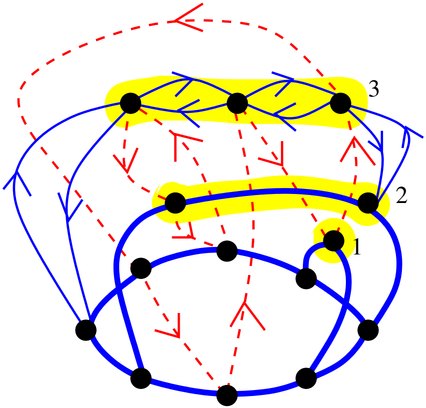

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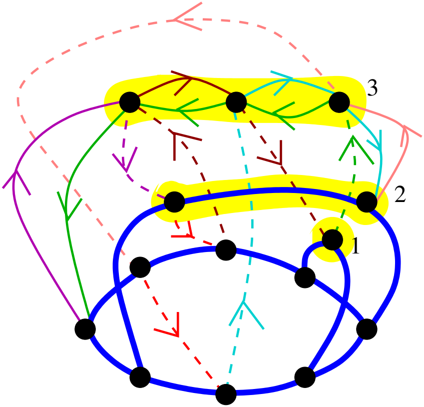

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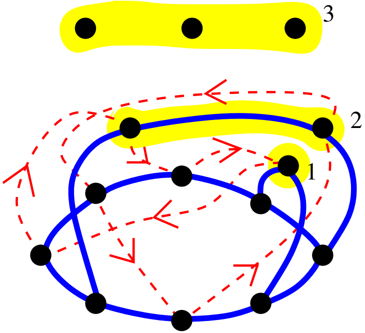

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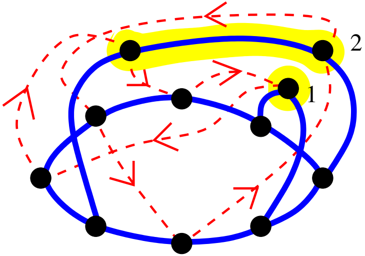

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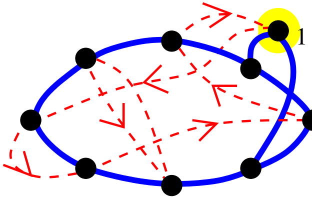

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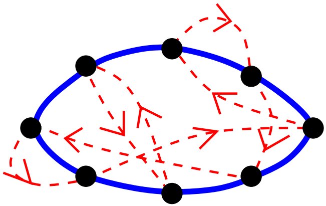

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Proposition

In general, reductions of stars, 1-ears and 2-ears always exist!!
In robust trigraphs (because of definition of robust), reductions of 3-ears always exist!!

Weak trigraph conjecture

In the closure of each robust trigraph, there exists a trigraph that admits a reduction process avoiding cut-obstacles.

Theorem [J. & Loebl, 2014+]

The “weak trigraph conjecture” implies general DCDC conjecture.

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Closure of a robust trigraph

Given trigraph $H, \mathcal{H}$. A trigraph $H', \mathcal{H}'$ is in the closure of $H, \mathcal{H}$ if $H', \mathcal{H}'$ is obtained from $H, \mathcal{H}$ by a sequence of the following two operations:

- Modification of $\mathcal{H}$ up to starting cycle and a fixed subset S of ears.
- A local exchange of edges (H' is not necessarily isomorphic to H).

Weak trigraph conjecture

In the closure of each robust trigraph, there exists a trigraph that admits a reduction process avoiding cut-obstacles.

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Proposition

Our conjecture holds in the class of planar trigraphs and for trigraphs that admit special embeddings into orientable surfaces.

"Trigraph conjecture implies general DCDC conjecture".

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Consider G bridgeless cubic graph & an ear decomposition of G.

Proposition

Complete reduction process of G constructs a DCDC of G.

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Main Lemma (Construction)

There exists a trigraph $H(G)$ such that:

- Reduction process of $H(G)$ without cut-obstacles encodes complete reduction process of G.
- If G is 3-connected, then $H(G)$ is a robust trigraph.

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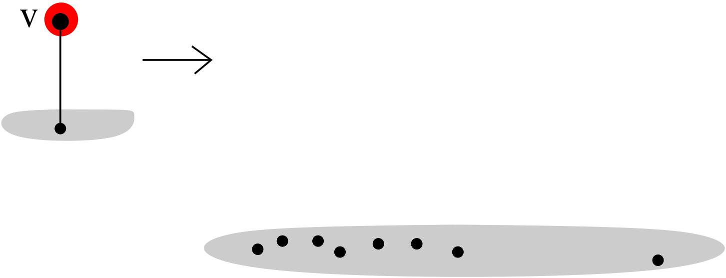
Key ingredient in the construction of $H(G)$ are: basic gadgets

$G \longrightarrow H(G)$



BASIC GADGET

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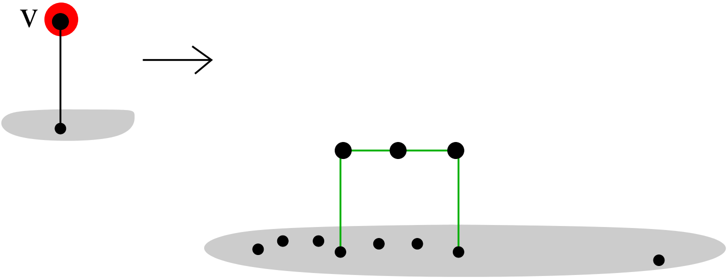
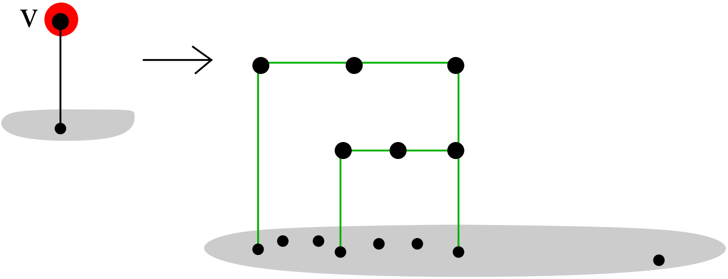


BASIC GADGET

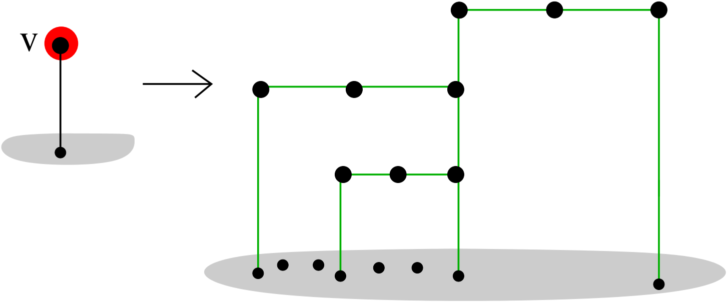
Sketch of Proof

$$G \longrightarrow H(G)$$



BASIC GADGET

G ⟶ H(G)



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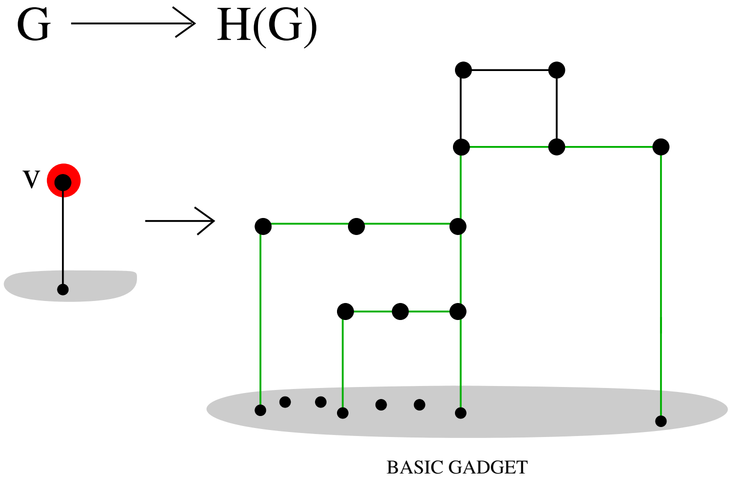
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V
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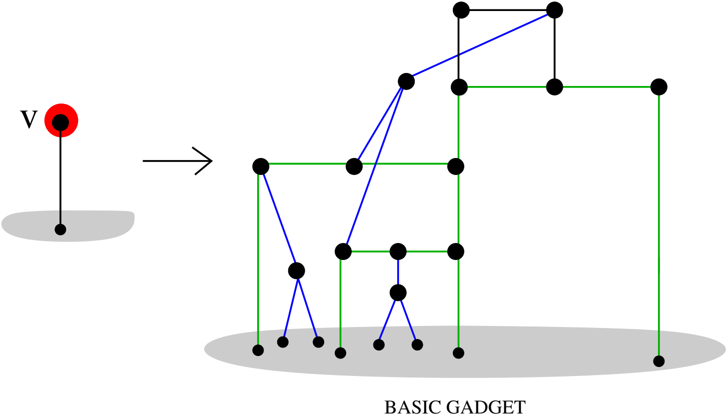

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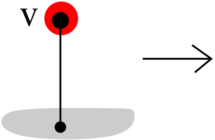

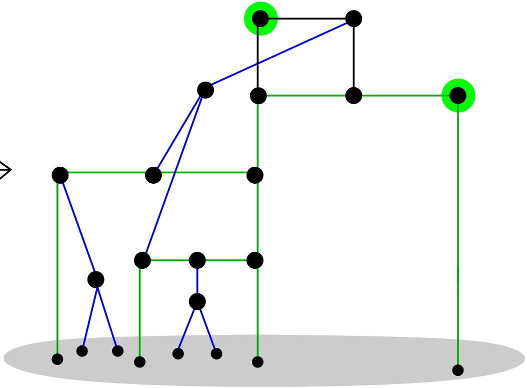

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