Toughness and Kronecker Product of Graphs

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Given a connected graph $G$ we ask:

1. (Minimum) size of a vertex cut set $S \subset V(G)$
2. Number of remaining connected components $k(G - S)$
3. Size of the largest connected component $m(G - S)$
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(Vertex) Vulnerability parameters

192? **connectivity**, \( \kappa(G) \), deal with 1

1973 Chvátal’s **toughness**, \( t(G) \), deal with 1 and 2.

1978 Jung’s **scattering number**, \( sc(G) \), deal with 1 and 2.

1987 Barefoot-Entringer-Swart’s **integrity**, \( I(G) \), deal with 1 and 3.

1992 Cozzens-Moazzami-Stueckle’s **tenacity**, \( T(G) \), deal with 1, 2 and 3.

2004 Li-Zhang-Li’s **rupture degree**, \( r(G) \), deal with 1, 2 and 3.
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Toughness

Definition

Given a connected non-complete graph $G$, the toughness of $G$ is:

$$t(G) := \min \frac{|S|}{k(G - S)}$$

where the minimum is taken over all the vertex-cut sets $S \subset V(G)$. By definition $t(K_n) := +\infty$

Example

- $t(C_n) = 1$, $n \geq 3$.
- $t(K_{k, n-k}) = \frac{k}{n-k}$, where $1 \leq k \leq \frac{n}{2}$.
- $t($Petersen graph$) = \frac{4}{3}$.
- Every Hamiltonian graph have toughness at least 1.
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### Scattering number

#### Definition

Given a connected non-complete graph $G$, the **scattering number** of $G$ is:

$$sc(G) := \max \{ k(G - S) - |S| \}$$

where the maximum is taken over all the vertex-cut sets $S \subset V(G)$. By definition $sc(K_n) := -\infty$.

#### Example

- $sc(C_n) = 0$, for $n \geq 4$.
- $sc(P_n) = 1$, for $n \geq 3$.
- $sc(K_{m,n}) = n - m$, if $m \leq n$ and $n \geq 2$.
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Integrity

Definition

Given a connected non-complete graph $G$, the integrity of $G$ is:

$$I(G) := \min\{|S| + m(G - S)|$$

where the minimum is taken over all the vertex-cut sets $S \subset V(G)$. By definition $I(K_n) := n$.

Example

- $I(P_n) = \lceil 2\sqrt{n+1} \rceil - 2$.
- $I(K_{m,n}) = 1 + \min\{m, n\}$. 
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Tenacity

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Given a connected non-complete graph $G$, the *tenacity* of $G$ is

$$T(G) := \min \left\{ \frac{|S| + m(G - S)}{k(G - S)} \right\}$$

where the minimum is taken over all the vertex-cut sets $S \subseteq V(G)$. By definition $T(K_n) := n$.

**Example**

- $T(P_n) = 1$, if $n$ is odd, and $T(P_n) = \frac{n+2}{n}$, if $n$ is even.
- $T(C_n) = \frac{n+3}{n-1}$, if $n$ is odd, and $T(P_n) = \frac{n+2}{n}$, if $n$ is even.
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Rupture degree

Definition

Given a connected non-complete graph $G$, the **rupture degree** of $G$ is

$$r(G) := \max \{ k(G - S) - |S| - m(G - S) \}$$

where the maximum is taken over all the vertex-cut sets $S \subset V(G)$. By definition $r(K_n) := 1 - n$.

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Kronecker Product

Definition

Let $G, H$ be two graphs. The **Kronecker Product** of both is the graph (Nešetřil notation)

$$G \times H$$

with vertex set

$$V(G \times H) = V(G) \times V(H)$$

(Cartesian product of sets) and edge set

$$E(G \times H) = \{ \{(a, b), (a', b')\} \mid \{a, a'\} \in E(G), \{b, b'\} \in E(H) \}$$
Let \( n \) and \( m \) be integer with \( n \geq m \geq 2 \) and \( n \geq 3 \). Then:

1. \( t(K_m \times K_n) = m - 1 \)
2. \( sc(K_m \times K_n) = \begin{cases} 2 - (m - 1)(n - 1), & \text{if } m = n \\ 2n - mn, & \text{otherwise} \end{cases} \)
3. \( l(K_m \otimes K_n) = mn - n + 1 \)
4. \( T(K_m \otimes K_n) = m + \frac{1}{n} - 1 \)
5. \( r(K_m \otimes K_n) = ??? \)
Theorem

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Mamut-Vumar-Problem

Problem (Mamut y Vumar (2007))

Determine good bounds for vulnerability parameters of $G \times H$, with $G$ and $H$ arbitrary connected graphs.
Our little result

Theorem

Let $G$ be a connected non-complete graph such that $t(G) \geq \frac{n}{2}$, with $n \geq 3$. Then

$$t(G \times K_n) = n - 1$$
Definition

Let $n$ be a positive integer greater than 1. The Unitary Cayley Graph $X_n$ is defined as the graph with vertex set $V(X_n) = \{0, 1, \ldots, n - 1\}$ and edge set $E(X_n) = \{(a, b) \mid a, b \in \mathbb{Z}_n, \quad \gcd(a - b, n) = 1\}$.
Example: $X_{10}$

$U_{10} = \{1, 3, 7, 9\}$
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Some properties of unitary Cayley Graphs were determined by Klotz and Sander [4]. Here, \( p_1(n) \) will denote the smallest prime number \( p \) such that \( p | n \).

**Theorem**

Let \( X_n \) be a Unitary Cayley Graph of order \( n \) then

1. the independence number \( \text{ind}(X_n) = \frac{n}{p_1(n)} \).
2. the vertex connectivity \( \kappa(X_n) = \varphi(n) \).
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Theorem

Given \( n \in \mathbb{Z}^+ \), with prime factorization \( = p_1^{\alpha_1} \ldots p_k^{\alpha_k} \), where \( p_1 < \ldots < p_k \). Then

\[
X_n \cong X_{p_1^{\alpha_1}} \times \ldots \times X_{p_k^{\alpha_k}}
\]
Given \( n \in \mathbb{Z}^+ \), the \textbf{radical} of \( n \), denoted \( \text{rad}(n) \), is the greatest square-free divisor of \( n \).

**Theorem**

Let \( X_n \) be an Unitary Cayley Graph of order \( n \). Then

\[
t(X_n) \geq t(X_{\text{rad}(n)})
\]

**Proof.**

- Take a \( t \)-vertex-cut-set \( S \subset V(X_n) \).
- \( Z := \{ i \in \mathbb{Z}_{\text{rad}(n)} \mid \text{if } g \equiv i \mod \text{rad}(n), \text{then } g \in S \} \)
  - \( |S| \geq \frac{n}{\text{rad}(n)} |Z| \).
  - \( k(X_n - S) \leq k(X_{\text{rad}(n)} - Z) \).

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t(X_n) = \frac{|S|}{k(X_n - S)} \geq \frac{n}{\text{rad}(n)} \frac{|Z|}{k(X_{\text{rad}(n)} - Z)} \geq t(X_{\text{rad}(n)})
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radical bound

Given $n \in \mathbb{Z}^+$, the radical of $n$, denoted $\text{rad}(n)$, is the greatest square-free divisor of $n$.

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Let \( X_n \) be an Unitary Cayley Graph of order \( n \). Then

\[
t(X_n) \geq t(X_{\text{rad}(n)})
\]

**Proof.**

- Take a \( t \)-vertex-cut-set \( S \subset V(X_n) \).
- \( Z := \{ i \in \mathbb{Z}_{\text{rad}(n)} \mid \text{if } g \equiv i \mod \text{rad}(n), \text{then } g \in S \} \)
  - \( |S| \geq \frac{n}{\text{rad}(n)} |Z| \).
  - \( k(X_n - S) \leq k(X_{\text{rad}(n)} - Z) \)

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First a very well known result about toughness

**Theorem**

For every non-complete graph $G$, \[ \frac{\kappa(G)}{\text{ind}(G)} \leq t(G) \leq \frac{\kappa(G)}{2} \]

Now, a humble result of us:

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Let $p < q$ be prime numbers and $S$ vertex-cut set with $|S| < pq - q$ then $k(X_{pq} - S) = 2$. 
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Daniel A. Jaume, Adrián Pastine, Denis E. Videla

Vulnerability of Unitary Cayley Graphs
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Vulnerability of Unitary Cayley Graphs
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Vulnerability of Unitary Cayley Graphs
Toughness of UCG

Theorem

Let $X_n$ be an Unitary Cayley Graph of order $n$. Then $t(X_n) = p_1(n) - 1$

Proof.

Assume that $n$ is square-free.

Induction over the number of prime divisors of $n$

- $n = p_1 p_2$, and $S \subset V(X_n)$ a vertex-cut of $X_n$
- If $|S| < p_1 p_2 - p_2$, then $k(X_{p_1 p_2} - S) = 2$.

\[
\frac{|S|}{k(X_{p_1 p_2} - S)} \geq \frac{\kappa(X_{p_1 p_2})}{2} = \frac{(p_1 - 1)(p_2 - 1)}{2} \geq p_1 - 1
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- If $|S| \geq p_1p_2 - p_2$, then
  
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- $n = p_1p_2 \ldots p_{k+1}$ con $p_1 < \ldots < p_{k+1}$
- $X_n \cong X_{p_1} \times G$, where $G = X_{p_2} \times \ldots \times X_{p_{k+1}}$
- But $t(G) \geq p_2 - 1$ by induction hypothesis.
- Then, $t(X_{p_1} \times G) = p_1 - 1$.
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Tenacity

It is well known that:

**Theorem**

Let $G$ be a non-complete connected graph. Then

$$t(G) + \frac{1}{\text{ind}(G)} \leq T(G) \leq \frac{|V(G)| - \text{ind}(G) + 1}{\text{ind}(G)}$$

With this we can prove that

**Corollary**

Let $X_n$ a Unitary Cayley Graph of order $n$. Then

$$T(X_n) = p_1(n) \left(1 + \frac{1}{n}\right) - 1$$
Tenacity

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Remember

**Definition**

Given a non-complete connected graph $G$, the **integrity** of $G$, $I(G)$ is defined as:

$$\min\{|S| + m(G - S)\}$$

where the minimum is taken over all the $S \subset V(G)$.
We will make use of the following known result about integrity:

**Theorem**

Let $G$ be a $k$-regular Graph of order $n$. Then

$$k + 1 \leq I(G) \leq n - \text{ind}(G) + 1$$
Example: $X_{10}$

\[ t(X_{10}) = 1 \]

\[ I(X_{10}) = 6 \]
Example: $X_{10}$

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Before going on with the results on integrity we will write the bounds for integrity in the language of Unitary Cayley Graphs:

$$\varphi(n) + 1 \leq I(X_n) \leq n - \frac{n}{p_1(n)} + 1$$
Theorem

Let \( p \) be a prime number and \( k \in \mathbb{Z}^+ \). Then

\[
I(X_{p^k}) = p^k - p^{k-1} + 1
\]

Proof.

We know that

\[
I(X_{p^k}) \geq \varphi(p^k) + 1 = p^k - p^{k-1} + 1
\]

But we also have that:

\[
I(X_{p^k}) \leq p^k - \frac{p^k}{p} + 1
\]

\[
= p^k - p^{k-1} + 1
\]

Thus \( I(X_{p^k}) = p^k - p^{k-1} + 1 \) as we wanted to prove.
Theorem

Let $p$ be a prime number and $k \in \mathbb{Z}^+$. Then

$$l(X_{p^k}) = p^k - p^{k-1} + 1$$

Proof.

We know that

$$l(X_{p^k}) \geq \varphi(p^k) + 1 = p^k - p^{k-1} + 1$$

But we also have that:

$$l(X_{p^k}) \leq p^k - \frac{p^k}{p} + 1$$

$$= p^k - p^{k-1} + 1$$

Thus $l(X_{p^k}) = p^k - p^{k-1} + 1$ as we wanted to prove.
Theorem

Let $p < q$ be prime numbers, then $I(X_{pq}) = pq - q + 1$

Proof.

Let $S$ be a vertex-cut set such that $|S| < pq - q$. We have that $k(X_{pq} - S) = 2$. Using the pigeonhole principle we have that:

$$m(X_{pq} - S) \geq \frac{pq - |S|}{2}$$

And so we have, using that $p < q$ and that, as $S$ is a vertex-cut, $|S| \geq \kappa(X_{pq}) = (p - 1)(q - 1)$:

$$|S| + m(X_{pq} - S) \geq |S| + \frac{pq - |S|}{2}$$

$$\geq \frac{|S| + pq}{2} \geq \frac{(p - 1)(q - 1) + pq}{2}$$
Proof.

\[ |S| + m(X_{pq} - S) \geq \frac{(p - 1)(q - 1) + pq}{2} \]

\[ \geq pq - \frac{p + q}{2} + \frac{1}{2} > pq - q \]

Assume now that \( |S| \geq pq - q \), then:

\[ |S| + m(X_{pq} - S) \geq pq - q + 1 \]

And so \( I(X_{pq}) = pq - q + 1 \).
Notice that in both cases the integrity is equal to the upper bound $n - \frac{n}{p_1(n)} + 1$, we conjecture that this is the case for every Unitary Cayley Graph.
GRACIAS!!!!!!!!
Bibliography


