Toughness and Kronecker Product of Graphs

Authors: Dr. Daniel A. Jaume [1]

Departamento de Matemáticas
Universidad Nacional de San Luis, Argentina

—

FoCM 2014 Coference
Montevideo, Uruguay

December 11, 2014

[1] djaume@unsl.edu.ar

Vulnerability

Given a connected graph G we ask:

1. (minimun) size of a vertex cut set $S \subset V(G)$
2. number of remaining connected components $k(G - S)$
3. size of the largest connected component $m(G - S)$

Vulnerability

Given a connected graph G we ask:

1. (minimun) size of a vertex cut set $S \subset V(G)$
2. number of remaining connected components $k(G - S)$
3. size of the largest connected component $m(G - S)$

Vulnerability

Given a connected graph G we ask:

1. (minimun) size of a vertex cut set $S \subset V(G)$
2. number of remaining connected components $k(G - S)$
3. size of the largest connected component $m(G - S)$

(Vertex) Vulnerability parameters

192? **connectivity**, $\kappa(G)$, deal wiht 1

1973 Chvátal's **toughness**, $t(G)$, deal with 1 and 2.

1978 Jung's **scattering number**, $sc(G)$, deal with 1 and 2.

1987 Barefoot-Entringer-Swart's **integrity**, $I(G)$, deal with 1 and 3.

1992 Cozzens-Moazzami-Stueckle's **tenacity**, $T(G)$, deal with 1, 2 and 3.

2004 Li-Zhang-Li's **rupture degree**, $r(G)$, deal with 1, 2 and 3.

(Vertex) Vulnerability parameters

192? **connectivity**, $\kappa(G)$, deal wiht 1

1973 Chvátal's **toughness**, $t(G)$, deal with 1 and 2.

1978 Jung's **scattering number**, $sc(G)$, deal with 1 and 2.

1987 Barefoot-Entringer-Swart's **integrity**, $I(G)$, deal with 1 and 3.

1992 Cozzens-Moazzami-Stueckle's **tenacity**, $T(G)$, deal with 1, 2 and 3.

2004 Li-Zhang-Li's **rupture degree**, $r(G)$, deal with 1, 2 and 3.

(Vertex) Vulnerability parameters

192? **connectivity**, $\kappa(G)$, deal wiht 1

1973 Chvátal's **toughness**, $t(G)$, deal with 1 and 2.

1978 Jung's **scattering number**, $sc(G)$, deal with 1 and 2.

1987 Barefoot-Entringer-Swart's **integrity**, $I(G)$, deal with 1 and 3.

1992 Cozzens-Moazzami-Stueckle's **tenacity**, $T(G)$, deal with 1, 2 and 3.

2004 Li-Zhang-Li's **rupture degree**, $r(G)$, deal with 1, 2 and 3.

(Vertex) Vulnerability parameters

192? **connectivity**, $\kappa(G)$, deal wiht 1

1973 Chvátal's **toughness**, $t(G)$, deal with 1 and 2.

1978 Jung's **scattering number**, $sc(G)$, deal with 1 and 2.

1987 Barefoot-Entringer-Swart's **integrity**, $I(G)$, deal with 1 and 3.

1992 Cozzens-Moazzami-Stueckle's **tenacity**, $T(G)$, deal with 1, 2 and 3.

2004 Li-Zhang-Li's **rupture degree**, $r(G)$, deal with 1, 2 and 3.

(Vertex) Vulnerability parameters

192? **connectivity**, $\kappa(G)$, deal wiht 1

1973 Chvátal's **toughness**, $t(G)$, deal with 1 and 2.

1978 Jung's **scattering number**, $sc(G)$, deal with 1 and 2.

1987 Barefoot-Entringer-Swart's **integrity**, $I(G)$, deal with 1 and 3.

1992 Cozzens-Moazzami-Stueckle's **tenacity**, $T(G)$, deal with 1, 2 and 3.

2004 Li-Zhang-Li's **rupture degree**, $r(G)$, deal with 1, 2 and 3.

(Vertex) Vulnerability parameters

192? **connectivity**, $\kappa(G)$, deal wiht 1

1973 Chvátal's **toughness**, $t(G)$, deal with 1 and 2.

1978 Jung's **scattering number**, $sc(G)$, deal with 1 and 2.

1987 Barefoot-Entringer-Swart's **integrity**, $I(G)$, deal with 1 and 3.

1992 Cozzens-Moazzami-Stueckle's **tenacity**, $T(G)$, deal with 1, 2 and 3.

2004 Li-Zhang-Li's **rupture degree**, $r(G)$, deal with 1, 2 and 3.

Toughness

Definition

Given a connected non-complete graph G, the **toughness** of G is:

$$t(G) := \min \frac{|S|}{k(G-S)}$$

where the minimum is taken over all the vertex-cut sets $S \subset V(G)$.
By definition $t(K_n) := +\infty$

Example

- $t(C_n) = 1,\ n \geq 3$.
- $t(K_{k,n-k}) = \frac{k}{n-k}$, where $1 \leq k \leq \frac{n}{2}$.
- $t(\text{Petersen graph}) = \frac{4}{3}$.
- Every Hamiltonian graph have toughness at least 1.

Toughness

Definition

Given a connected non-complete graph G, the **toughness** of G is:

$$t(G) := \min \frac{|S|}{k(G-S)}$$

where the minimum is taken over all the vertex-cut sets $S \subset V(G)$.
By definition $t(K_n) := +\infty$

Example

- $t(C_n) = 1,\ n \geq 3$.
- $t(K_{k,n-k}) = \frac{k}{n-k}$, where $1 \leq k \leq \frac{n}{2}$.
- $t(\text{Petersen graph}) = \frac{4}{3}$.
- Every Hamiltonian graph have toughness at least 1.

Toughness

Definition

Given a connected non-complete graph G, the **toughness** of G is:

$$t(G) := \min \frac{|S|}{k(G-S)}$$

where the minimum is taken over all the vertex-cut sets $S \subset V(G)$.
By definition $t(K_n) := +\infty$

Example

- $t(C_n) = 1,\ n \geq 3$.
- $t(K_{k,n-k}) = \frac{k}{n-k}$, where $1 \leq k \leq \frac{n}{2}$.
- $t(\text{Petersen graph}) = \frac{4}{3}$.
- Every Hamiltonian graph have toughness at least 1.

Toughness

Definition

Given a connected non-complete graph G, the **toughness** of G is:

$$t(G) := \min \frac{|S|}{k(G-S)}$$

where the minimum is taken over all the vertex-cut sets $S \subset V(G)$.
By definition $t(K_n) := +\infty$

Example

- $t(C_n) = 1,\ n \geq 3$.
- $t(K_{k,n-k}) = \frac{k}{n-k}$, where $1 \leq k \leq \frac{n}{2}$.
- $t(\text{Petersen graph}) = \frac{4}{3}$.
- Every Hamiltonian graph have toughness at least 1.

Toughness

Definition

Given a connected non-complete graph G, the **toughness** of G is:

$$t(G) := \min \frac{|S|}{k(G-S)}$$

where the minimum is taken over all the vertex-cut sets $S \subset V(G)$.
By definition $t(K_n) := +\infty$

Example

- $t(C_n) = 1,\ n \geq 3$.
- $t(K_{k,n-k}) = \frac{k}{n-k}$, where $1 \leq k \leq \frac{n}{2}$.
- $t(\text{Petersen graph}) = \frac{4}{3}$.
- Every Hamiltonian graph have toughness at least 1.

Scattering number

Definition

Given a connected non-complete graph G, the **scattering number** of G is:

$$sc(G) := \max \{k(G - S) - |S|\}$$

where the maximum is taken over all the vertex-cut sets $S \subset V(G)$. By definition $sc(K_n) := -\infty$

Example

- $sc(C_n) = 0$, for $n \geq 4$.
- $sc(P_n) = 1$, for $n \geq 3$.
- $sc(K_{m,n}) = n - m$, if $m \leq n$ and $n \geq 2$.

Scattering number

Definition

Given a connected non-complete graph G, the **scattering number** of G is:

$$sc(G) := \max \{k(G - S) - |S|\}$$

where the maximum is taken over all the vertex-cut sets $S \subset V(G)$. By definition $sc(K_n) := -\infty$

Example

- $sc(C_n) = 0$, for $n \geq 4$.
- $sc(P_n) = 1$, for $n \geq 3$.
- $sc(K_{m,n}) = n - m$, if $m \leq n$ and $n \geq 2$.

Scattering number

Definition

Given a connected non-complete graph G, the **scattering number** of G is:

$$sc(G) := \max \{k(G - S) - |S|\}$$

where the maximum is taken over all the vertex-cut sets $S \subset V(G)$. By definition $sc(K_n) := -\infty$

Example

- $sc(C_n) = 0$, for $n \geq 4$.
- $sc(P_n) = 1$, for $n \geq 3$.
- $sc(K_{m,n}) = n - m$, if $m \leq n$ and $n \geq 2$.

Scattering number

Definition

Given a connected non-complete graph G, the **scattering number** of G is:

$$sc(G) := \max \{k(G - S) - |S|\}$$

where the maximum is taken over all the vertex-cut sets $S \subset V(G)$. By definition $sc(K_n) := -\infty$

Example

- $sc(C_n) = 0$, for $n \geq 4$.
- $sc(P_n) = 1$, for $n \geq 3$.
- $sc(K_{m,n}) = n - m$, if $m \leq n$ and $n \geq 2$.

Integrity

Definition

Given a connected non-complete graph G, the **integrity** of G is:

$$I(G) := \min\{|S| + m(G - S)\}$$

where the minimum is taken over all the vertex-cut sets $S \subset V(G)$. By definition $I(K_n) := n$.

Example

- $I(P_n) = \lceil 2\sqrt{n+1} \rceil - 2$.
- $I(K_{m,n}) = 1 + \min\{m, n\}$.

Integrity

Definition

Given a connected non-complete graph G, the **integrity** of G is:

$$I(G) := \min\{|S| + m(G - S)\}$$

where the minimum is taken over all the vertex-cut sets $S \subset V(G)$. By definition $I(K_n) := n$.

Example

- $I(P_n) = \lceil 2\sqrt{n+1} \rceil - 2$.
- $I(K_{m,n}) = 1 + \min\{m, n\}$.

Integrity

Definition

Given a connected non-complete graph G, the **integrity** of G is:

$$I(G) := \min\{|S| + m(G - S)\}$$

where the minimum is taken over all the vertex-cut sets $S \subset V(G)$. By definition $I(K_n) := n$.

Example

- $I(P_n) = \lceil 2\sqrt{n+1} \rceil - 2$.
- $I(K_{m,n}) = 1 + \min\{m, n\}$.

Tenacity

Definition

Given a connected non-complete graph G, the **tenacity** of G is

$$T(G) := \min \left\{ \frac{|S| + m(G - S)}{k(G - S)} \right\}$$

where the minimum is taken over all the vertex-cut sets $S \subset V(G)$. By definition $T(K_n) := n$.

Example

- $T(P_n) = 1$, if n is odd, and $T(P_n) = \frac{n+2}{n}$, if n is even.
- $T(C_n) = \frac{n+3}{n-1}$, if n is odd, and $T(P_n) = \frac{n+2}{n}$, if n is even.
- $T(K_{k,n-k}) = \frac{k+1}{n-k}$, for $1 \leq k \leq \frac{n}{2}$

Tenacity

Definition

Given a connected non-complete graph G, the **tenacity** of G is

$$T(G) := \min \left\{ \frac{|S| + m(G-S)}{k(G-S)} \right\}$$

where the minimum is taken over all the vertex-cut sets $S \subset V(G)$. By definition $T(K_n) := n$.

Example

- $T(P_n) = 1$, if n is odd, and $T(P_n) = \frac{n+2}{n}$, if n is even.
- $T(C_n) = \frac{n+3}{n-1}$, if n is odd, and $T(P_n) = \frac{n+2}{n}$, if n is even.
- $T(K_{k,n-k}) = \frac{k+1}{n-k}$, for $1 \leq k \leq \frac{n}{2}$

Tenacity

Definition

Given a connected non-complete graph G, the **tenacity** of G is

$$T(G) := \min \left\{ \frac{|S| + m(G-S)}{k(G-S)} \right\}$$

where the minimum is taken over all the vertex-cut sets $S \subset V(G)$. By definition $T(K_n) := n$.

Example

- $T(P_n) = 1$, if n is odd, and $T(P_n) = \frac{n+2}{n}$, if n is even.
- $T(C_n) = \frac{n+3}{n-1}$, if n is odd, and $T(P_n) = \frac{n+2}{n}$, if n is even.
- $T(K_{k,n-k}) = \frac{k+1}{n-k}$, for $1 \leq k \leq \frac{n}{2}$

Tenacity

Definition

Given a connected non-complete graph G, the **tenacity** of G is

$$T(G) := \min \left\{ \frac{|S| + m(G - S)}{k(G - S)} \right\}$$

where the minimum is taken over all the vertex-cut sets $S \subset V(G)$. By definition $T(K_n) := n$.

Example

- $T(P_n) = 1$, if n is odd, and $T(P_n) = \frac{n+2}{n}$, if n is even.
- $T(C_n) = \frac{n+3}{n-1}$, if n is odd, and $T(P_n) = \frac{n+2}{n}$, if n is even.
- $T(K_{k,n-k}) = \frac{k+1}{n-k}$, for $1 \leq k \leq \frac{n}{2}$

Rupture degree

Definition

Given a connected non-complete graph G, the **rupture degree** of G is

$$r(G) := \max \{k(G-S) - |S| - m(G-S)\}$$

where the maximun is taken over all the vertex-cut sets $S \subset V(G)$. By definition $r(K_n) := 1 - n$.

Example

- $r(P_n) = 0$, if n is odd, and $r(P_n) = -1$, if n is even.
- $r(C_n) = -2$, if n is odd, and $r(P_n) = -1$, if n is even.
- $r(K_{k,n-k}) = n - 2k - 1$, for $1 \leq k \leq \frac{n}{2}$

Rupture degree

Definition

Given a connected non-complete graph G, the **rupture degree** of G is

$$r(G) := \max \{k(G-S) - |S| - m(G-S)\}$$

where the maximun is taken over all the vertex-cut sets $S \subset V(G)$. By definition $r(K_n) := 1 - n$.

Example

- $r(P_n) = 0$, if n is odd, and $r(P_n) = -1$, if n is even.
- $r(C_n) = -2$, if n is odd, and $r(P_n) = -1$, if n is even.
- $r(K_{k,n-k}) = n - 2k - 1$, for $1 \leq k \leq \frac{n}{2}$

Rupture degree

Definition

Given a connected non-complete graph G, the **rupture degree** of G is

$$r(G) := \max \{k(G-S) - |S| - m(G-S)\}$$

where the maximun is taken over all the vertex-cut sets $S \subset V(G)$. By definition $r(K_n) := 1 - n$.

Example

- $r(P_n) = 0$, if n is odd, and $r(P_n) = -1$, if n is even.
- $r(C_n) = -2$, if n is odd, and $r(P_n) = -1$, if n is even.
- $r(K_{k,n-k}) = n - 2k - 1$, for $1 \leq k \leq \frac{n}{2}$

Rupture degree

Definition

Given a connected non-complete graph G, the **rupture degree** of G is

$$r(G) := \max \{k(G-S) - |S| - m(G-S)\}$$

where the maximun is taken over all the vertex-cut sets $S \subset V(G)$. By definition $r(K_n) := 1 - n$.

Example

- $r(P_n) = 0$, if n is odd, and $r(P_n) = -1$, if n is even.
- $r(C_n) = -2$, if n is odd, and $r(P_n) = -1$, if n is even.
- $r(K_{k,n-k}) = n - 2k - 1$, for $1 \leq k \leq \frac{n}{2}$

Kronecker Product

Definition

Let G, H be two graphs. The **Kronecker Product** of both is the graph (Nešetřil notation)

$$G \times H$$

with vertex set
$V(G \times H) = V(G) \times V(H)$
(Cartesian product of sets) and edge set
$E(G \times H) = \{\{(a, b), (a', b')\} | \{a, a'\} \in E(G), \{b, b'\} \in E(H)\}$

Results of Mamut-Vumar 2007, IPL

Theorem

Let n y m be integer with $n \geq m \geq 2$ and $n \geq 3$. Then:

1. $t(K_m \times K_n) = m - 1$
2. $sc(K_m \times K_n) = \begin{cases} 2 - (m-1)(n-1), & if\ m = n \\ 2n - mn, & otherwise \end{cases}$
3. $I(K_m \otimes K_n) = mn - n + 1$
4. $T(K_m \otimes K_n) = m + \frac{1}{n} - 1$
5. $r(K_m \otimes K_n) =???$

Results of Mamut-Vumar 2007, IPL

Theorem

Let n y m be integer with $n \geq m \geq 2$ and $n \geq 3$. Then:

1. $t(K_m \times K_n) = m - 1$
2. $sc(K_m \times K_n) = \begin{cases} 2 - (m-1)(n-1), & \text{if } m = n \\ 2n - mn, & \text{otherwise} \end{cases}$
3. $I(K_m \otimes K_n) = mn - n + 1$
4. $T(K_m \otimes K_n) = m + \frac{1}{n} - 1$
5. $r(K_m \otimes K_n) =???$

Results of Mamut-Vumar 2007, IPL

Theorem

Let n y m be integer with $n \geq m \geq 2$ and $n \geq 3$. Then:

1. $t(K_m \times K_n) = m - 1$
2. $sc(K_m \times K_n) = \begin{cases} 2 - (m-1)(n-1), & \text{if } m = n \\ 2n - mn, & \text{otherwise} \end{cases}$
3. $I(K_m \otimes K_n) = mn - n + 1$
4. $T(K_m \otimes K_n) = m + \frac{1}{n} - 1$
5. $r(K_m \otimes K_n) =???$

Results of Mamut-Vumar 2007, IPL

Theorem

Let n y m be integer with $n \geq m \geq 2$ and $n \geq 3$. Then:

1. $t(K_m \times K_n) = m - 1$
2. $sc(K_m \times K_n) = \begin{cases} 2 - (m-1)(n-1), & \text{if } m = n \\ 2n - mn, & \text{otherwise} \end{cases}$
3. $I(K_m \otimes K_n) = mn - n + 1$
4. $T(K_m \otimes K_n) = m + \frac{1}{n} - 1$
5. $r(K_m \otimes K_n) =???$

Results of Mamut-Vumar 2007, IPL

Theorem

Let n y m be integer with $n \geq m \geq 2$ and $n \geq 3$. Then:

1. $t(K_m \times K_n) = m - 1$
2. $sc(K_m \times K_n) = \begin{cases} 2 - (m-1)(n-1), & \text{if } m = n \\ 2n - mn, & \text{otherwise} \end{cases}$
3. $I(K_m \otimes K_n) = mn - n + 1$
4. $T(K_m \otimes K_n) = m + \frac{1}{n} - 1$
5. $r(K_m \otimes K_n) =$???

Mamut-Vumar-Problem

Problem (Mamut y Vumar (2007))

Determine good bounds for vunerability parameters of $G \times H$, with G and H arbitrary connected graphs.

Our little result

Theorem

Let G be a connected non-complete graph such that $t(G) \geq \frac{n}{2}$, with $n \geq 3$. Then

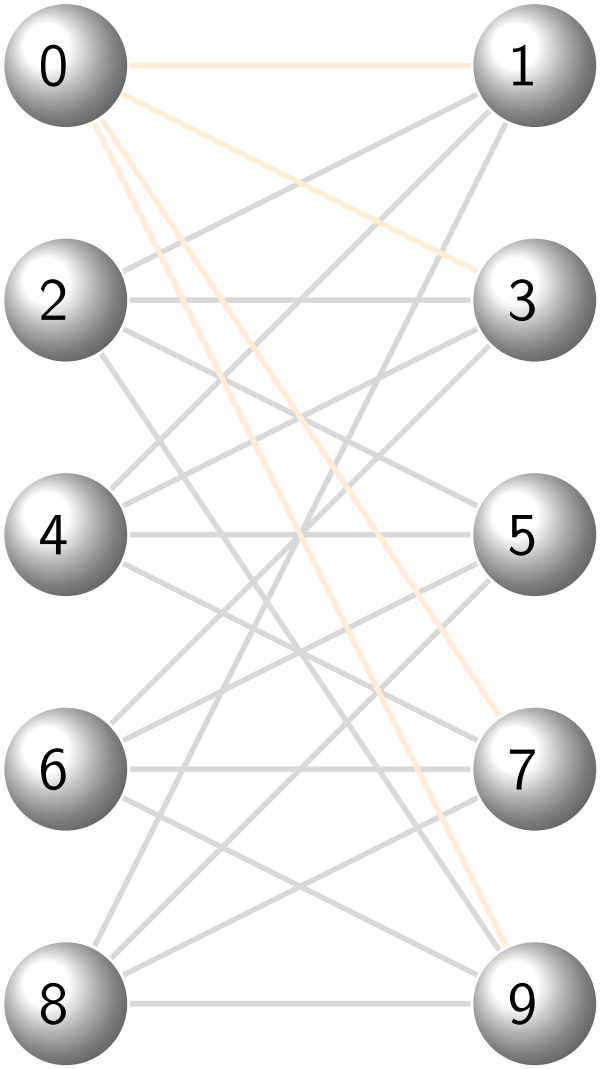
$$t(G \times K_n) = n - 1$$

Unitary Cayley Graph

Definition

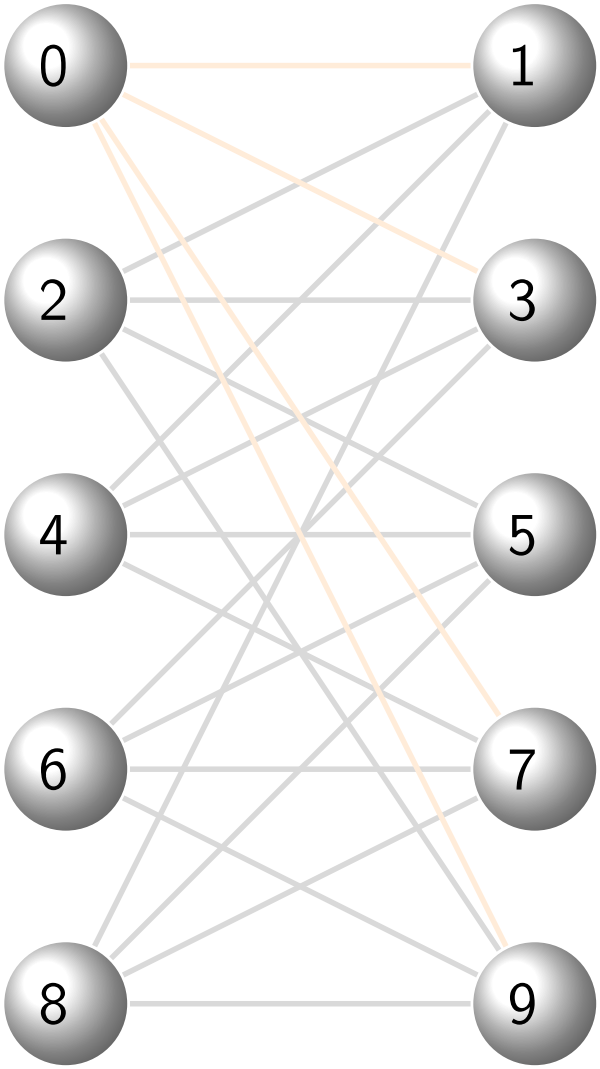
Let n be a positive integer greater than 1. The Unitary Cayley Graph X_n is defined as the graph with vertex set
$V(X_n) = \{0, 1, ..., n-1\}$
and edge set
$E(X_n) = \{(a, b) | \, a, b \in \mathbb{Z}_n, \quad \gcd(a-b, n) = 1\}$

Example: X_{10}



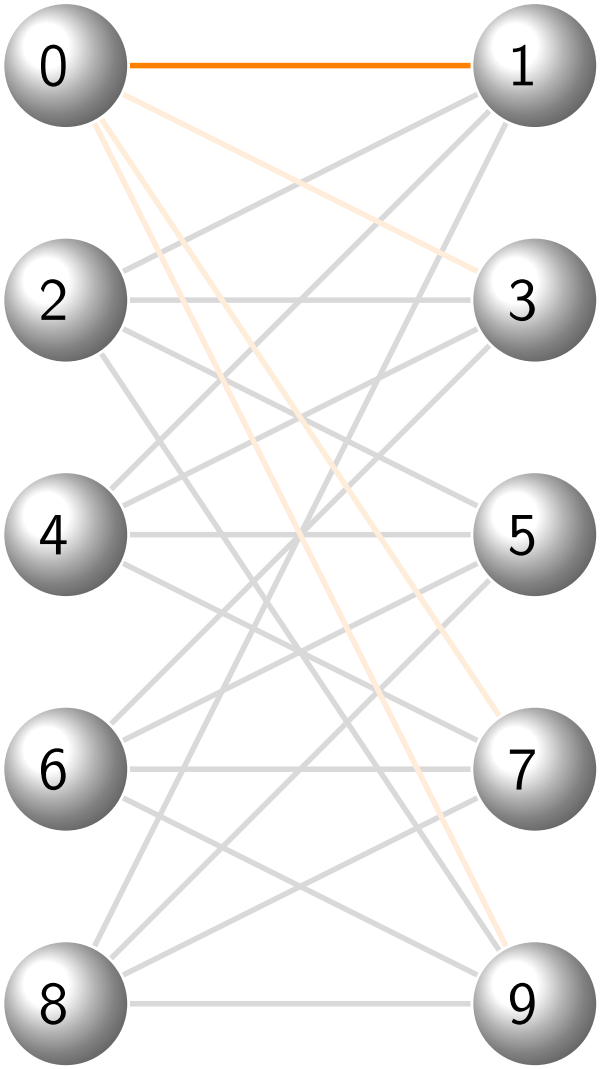
$$U_{10} = \{1, 3, 7, 9\}$$

Example: X_{10}



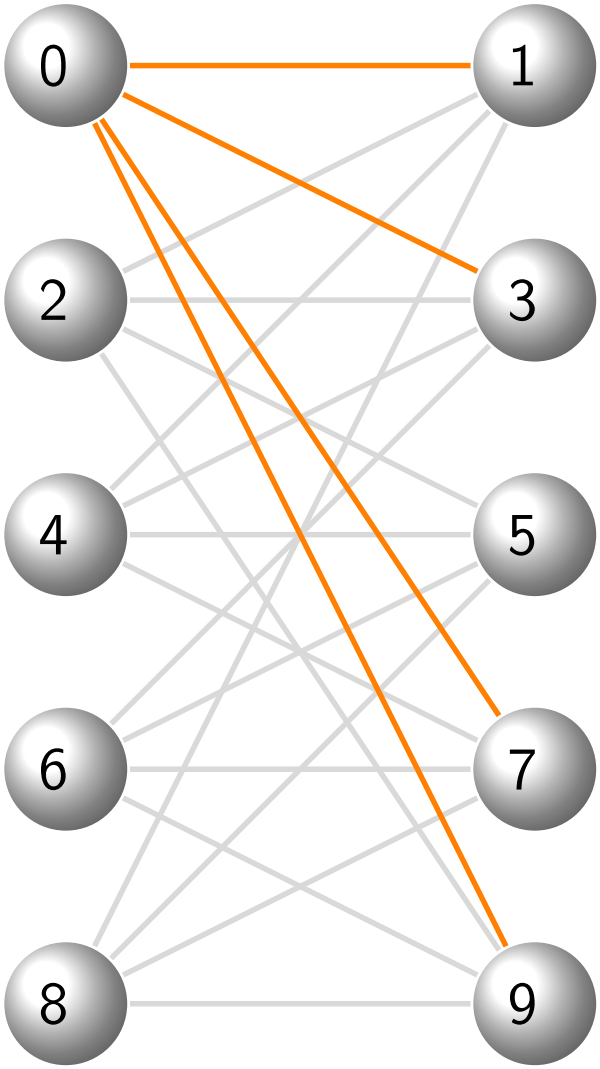
$$U_{10} = \{1, 3, 7, 9\}$$

Example: X_{10}



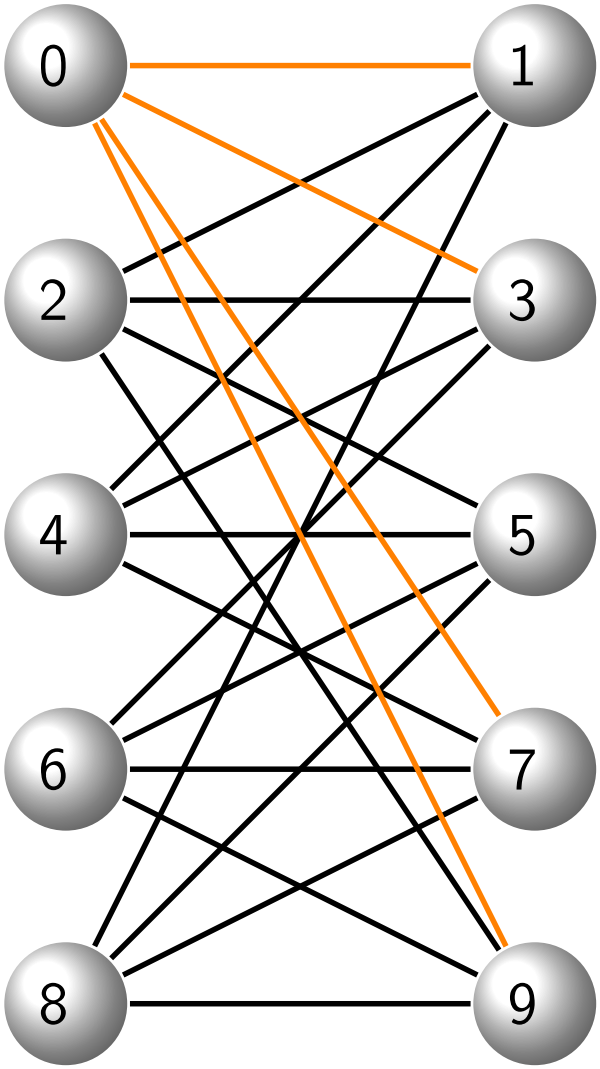
$$U_{10} = \{1, 3, 7, 9\}$$

Example: X_{10}



$$U_{10} = \{1, 3, 7, 9\}$$

Example: X_{10}



$$U_{10} = \{1, 3, 7, 9\}$$

Results by Klotz and Sander 2007, EJC

Some properties of unitary Cayley Graphs were determined by Klotz and Sander [4].
Here, $p_1(n)$ will denote the smallest prime number p such that $p|n$.

Theorem

Let X_n be a Unitary Cayley Graph of order n then

1. *the independence number* $\mathrm{ind}(X_n) = \dfrac{n}{p_1(n)}$.
2. *the vertex connectivity* $\kappa(X_n) = \varphi(n)$.

Results by Klotz and Sander 2007, EJC

Some properties of unitary Cayley Graphs were determined by Klotz and Sander [4].
Here, $p_1(n)$ will denote the smallest prime number p such that $p|n$.

Theorem

Let X_n be a Unitary Cayley Graph of order n then

1. *the independence number* $\mathrm{ind}(X_n) = \dfrac{n}{p_1(n)}$.
2. *the vertex connectivity* $\kappa(X_n) = \varphi(n)$.

Results by Klotz and Sander 2007, EJC

Some properties of unitary Cayley Graphs were determined by Klotz and Sander [4].
Here, $p_1(n)$ will denote the smallest prime number p such that $p|n$.

Theorem

Let X_n be a Unitary Cayley Graph of order n then

1. *the independence number* $\mathrm{ind}(X_n) = \dfrac{n}{p_1(n)}$.
2. *the vertex connectivity* $\kappa(X_n) = \varphi(n)$.

Results by Klotz and Sander 2007, EJC

Some properties of unitary Cayley Graphs were determined by Klotz and Sander [4].
Here, $p_1(n)$ will denote the smallest prime number p such that $p|n$.

Theorem

Let X_n be a Unitary Cayley Graph of order n then

1. *the independence number* $\mathrm{ind}(X_n) = \dfrac{n}{p_1(n)}$.
2. *the vertex connectivity* $\kappa(X_n) = \varphi(n)$.

Results by Klotz and Sander 2007, EJC

Some properties of unitary Cayley Graphs were determined by Klotz and Sander [4].
Here, $p_1(n)$ will denote the smallest prime number p such that $p|n$.

Theorem

Let X_n be a Unitary Cayley Graph of order n then

1. *the independence number* $\mathrm{ind}(X_n) = \dfrac{n}{p_1(n)}$.
2. *the vertex connectivity* $\kappa(X_n) = \varphi(n)$.

One result of Ramaswamy and Veena 2009, EJC

Theorem

Given $n \in \mathbb{Z}^+$, with prime factorization $= p_1^{\alpha_1} \dots p_k^{\alpha_k}$, where $p_1 < \dots < p_k$. Then

$$X_n \cong X_{p_1^{\alpha_1}} \times \dots \times X_{p_k^{\alpha_k}}$$

radical bound

Given $n \in \mathbb{Z}^+$, the **radical** of n, denoted $\mathrm{rad}(n)$, is the greastest square-free divisor of n.

Theorem

Let X_n be an Unitary Cayley Graph of order n. Then $t(X_n) \geq t(X_{\mathrm{rad}(n)})$

Proof.

- Take a t-vertex-cut-set $S \subset V(X_n)$.
- $Z := \{i \in \mathbb{Z}_{\mathrm{rad}(n)} |$ if $g \equiv i \mod \mathrm{rad}(n)$, then $g \in S\}$
  - $|S| \geq \frac{n}{\mathrm{rad}(n)}|Z|$.
  - $k(X_n - S) \leq k(X_{\mathrm{rad}(n)} - Z)$
- $$t(X_n) = \frac{|S|}{k(X_n - S)} \geq \frac{\frac{n}{\mathrm{rad}(n)}|Z|}{\frac{n}{\mathrm{rad}(n)}k(X_{rad(n)} - Z)} \geq t(X_{rad(n)})$$

radical bound

Given $n \in \mathbb{Z}^+$, the **radical** of n, denoted $\mathrm{rad}(n)$, is the greastest square-free divisor of n.

Theorem

Let X_n be an Unitary Cayley Graph of order n. Then $t(X_n) \geq t(X_{\mathrm{rad}(n)})$

Proof.

- Take a t-vertex-cut-set $S \subset V(X_n)$.
- $Z := \{i \in \mathbb{Z}_{\mathrm{rad}(n)} |$ if $g \equiv i \mod \mathrm{rad}(n)$, then $g \in S\}$
  - $|S| \geq \frac{n}{\mathrm{rad}(n)}|Z|$.
  - $k(X_n - S) \leq k(X_{\mathrm{rad}(n)} - Z)$
- $$t(X_n) = \frac{|S|}{k(X_n - S)} \geq \frac{\frac{n}{\mathrm{rad}(n)}|Z|}{\frac{n}{\mathrm{rad}(n)}k(X_{rad(n)} - Z)} \geq t(X_{rad(n)})$$

radical bound

Given $n \in \mathbb{Z}^+$, the **radical** of n, denoted $\mathrm{rad}(n)$, is the greastest square-free divisor of n.

Theorem

Let X_n be an Unitary Cayley Graph of order n. Then $t(X_n) \geq t(X_{\mathrm{rad}(n)})$

Proof.

- Take a t-vertex-cut-set $S \subset V(X_n)$.
- $Z := \{i \in \mathbb{Z}_{\mathrm{rad}(n)} |$ if $g \equiv i \mod \mathrm{rad}(n)$, then $g \in S\}$
  - $|S| \geq \frac{n}{\mathrm{rad}(n)}|Z|$.
  - $k(X_n - S) \leq k(X_{\mathrm{rad}(n)} - Z)$
- $$t(X_n) = \frac{|S|}{k(X_n - S)} \geq \frac{\frac{n}{\mathrm{rad}(n)}|Z|}{\frac{n}{\mathrm{rad}(n)}k(X_{rad(n)} - Z)} \geq t(X_{rad(n)})$$

radical bound

Given $n \in \mathbb{Z}^+$, the **radical** of n, denoted $\mathrm{rad}(n)$, is the greastest square-free divisor of n.

Theorem

Let X_n be an Unitary Cayley Graph of order n. Then $t(X_n) \geq t(X_{\mathrm{rad}(n)})$

Proof.

- Take a t-vertex-cut-set $S \subset V(X_n)$.
- $Z := \{i \in \mathbb{Z}_{\mathrm{rad}(n)} |$ if $g \equiv i \mod \mathrm{rad}(n)$, then $g \in S\}$
  - $|S| \geq \frac{n}{\mathrm{rad}(n)}|Z|$.
  - $k(X_n - S) \leq k(X_{\mathrm{rad}(n)} - Z)$
- $$t(X_n) = \frac{|S|}{k(X_n - S)} \geq \frac{\frac{n}{\mathrm{rad}(n)}|Z|}{\frac{n}{\mathrm{rad}(n)}k(X_{rad(n)} - Z)} \geq t(X_{rad(n)})$$

radical bound

Given $n \in \mathbb{Z}^+$, the **radical** of n, denoted $\mathrm{rad}(n)$, is the greastest square-free divisor of n.

Theorem

Let X_n be an Unitary Cayley Graph of order n. Then $t(X_n) \geq t(X_{\mathrm{rad}(n)})$

Proof.

- Take a t-vertex-cut-set $S \subset V(X_n)$.
- $Z := \{i \in \mathbb{Z}_{\mathrm{rad}(n)} |$ if $g \equiv i \mod \mathrm{rad}(n)$, then $g \in S\}$
  - $|S| \geq \frac{n}{\mathrm{rad}(n)}|Z|$.
  - $k(X_n - S) \leq k(X_{\mathrm{rad}(n)} - Z)$
- $$t(X_n) = \frac{|S|}{k(X_n - S)} \geq \frac{\frac{n}{\mathrm{rad}(n)}|Z|}{\frac{n}{\mathrm{rad}(n)}k(X_{rad(n)} - Z)} \geq t(X_{rad(n)})$$

radical bound

Given $n \in \mathbb{Z}^+$, the **radical** of n, denoted $\text{rad}(n)$, is the greastest square-free divisor of n.

Theorem

Let X_n be an Unitary Cayley Graph of order n. Then $t(X_n) \geq t(X_{\text{rad}(n)})$

Proof.

- Take a t-vertex-cut-set $S \subset V(X_n)$.
- $Z := \{i \in \mathbb{Z}_{\text{rad}(n)} |$ if $g \equiv i \mod \text{rad}(n)$, then $g \in S\}$
  - $|S| \geq \frac{n}{\text{rad}(n)}|Z|$.
  - $k(X_n - S) \leq k(X_{\text{rad}(n)} - Z)$
- $$t(X_n) = \frac{|S|}{k(X_n - S)} \geq \frac{\frac{n}{\text{rad}(n)}|Z|}{\frac{n}{\text{rad}(n)}k(X_{rad(n)} - Z)} \geq t(X_{rad(n)})$$

Tools

First a very well known result about toughness

Theorem

For every noncomplete graph G,

$$\frac{\kappa(G)}{ind(G)} \leq t(G) \leq \frac{\kappa(G)}{2}$$

Now, a humble result of us:

Theorem

Let $p < q$ be prime numbers and S vertex-cut set with $|S| < pq - q$ then $k(X_{pq} - S) = 2$.

Tools

First a very well known result about toughness

Theorem

For every noncomplete graph G,

$$\frac{\kappa(G)}{ind(G)} \leq t(G) \leq \frac{\kappa(G)}{2}$$

Now, a humble result of us:

Theorem

Let $p < q$ be prime numbers and S vertex-cut set with $|S| < pq - q$ then $k(X_{pq} - S) = 2$.

Tools

First a very well known result about toughness

Theorem

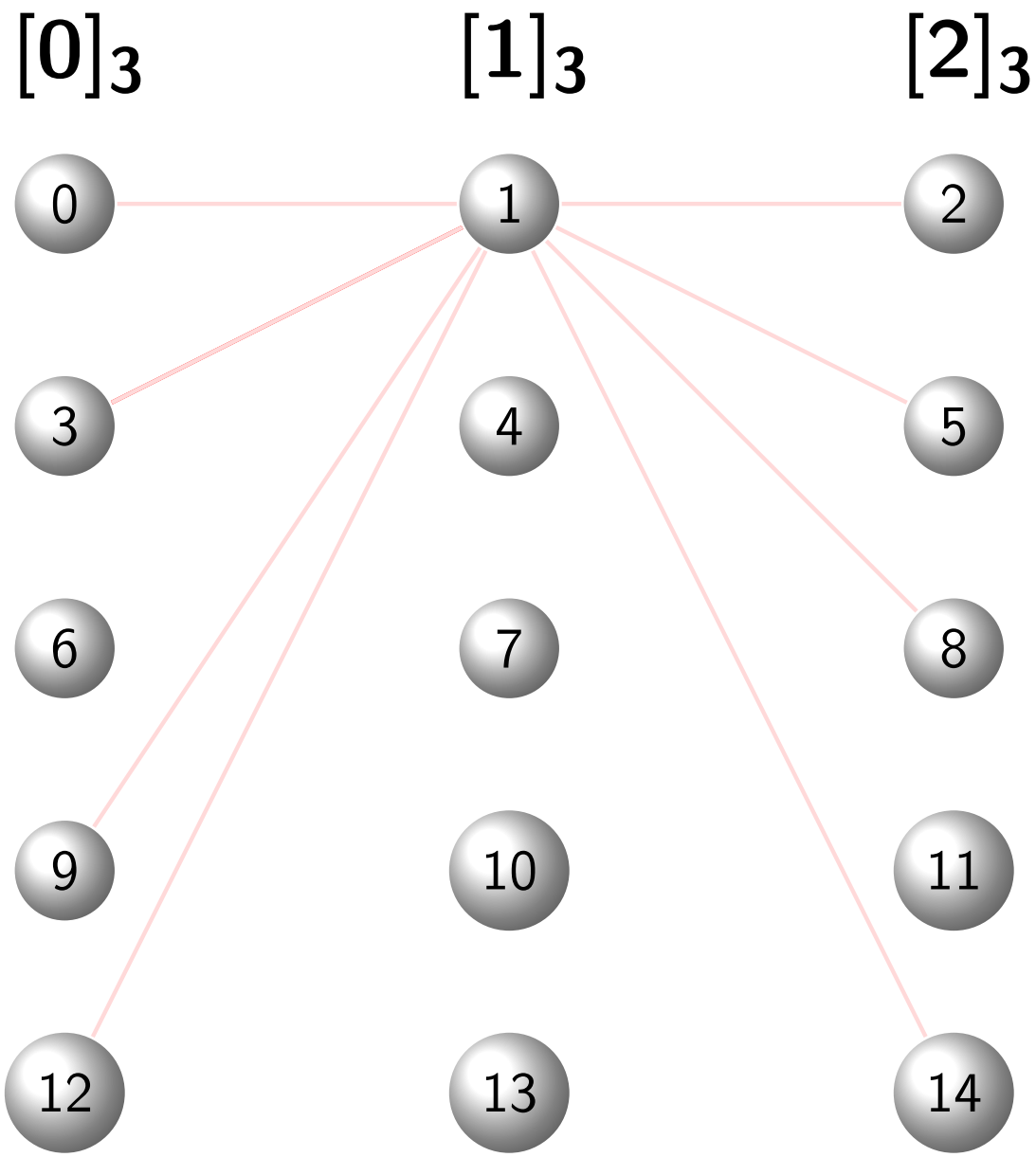
For every noncomplete graph G,

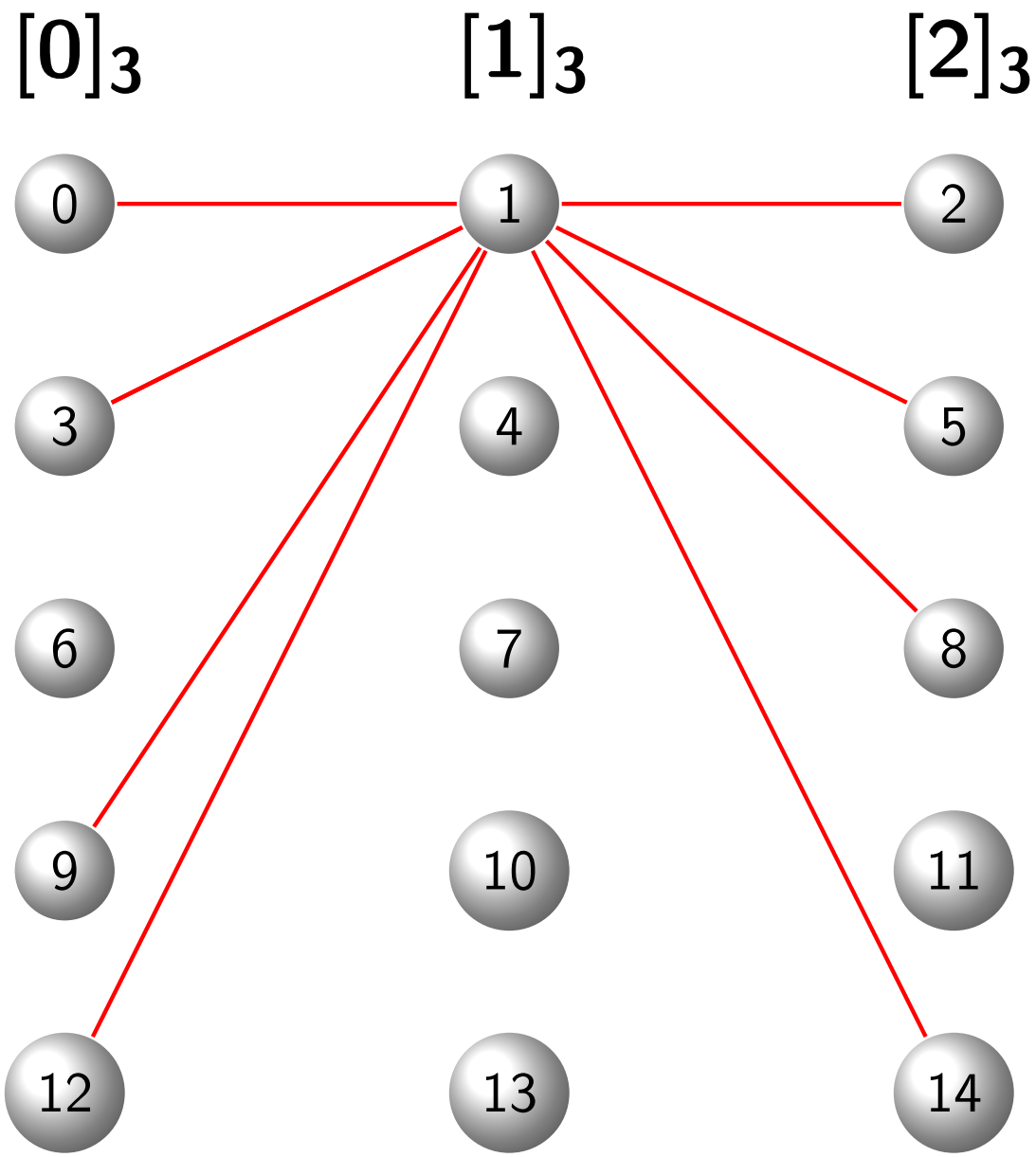
$$\frac{\kappa(G)}{ind(G)} \leq t(G) \leq \frac{\kappa(G)}{2}$$

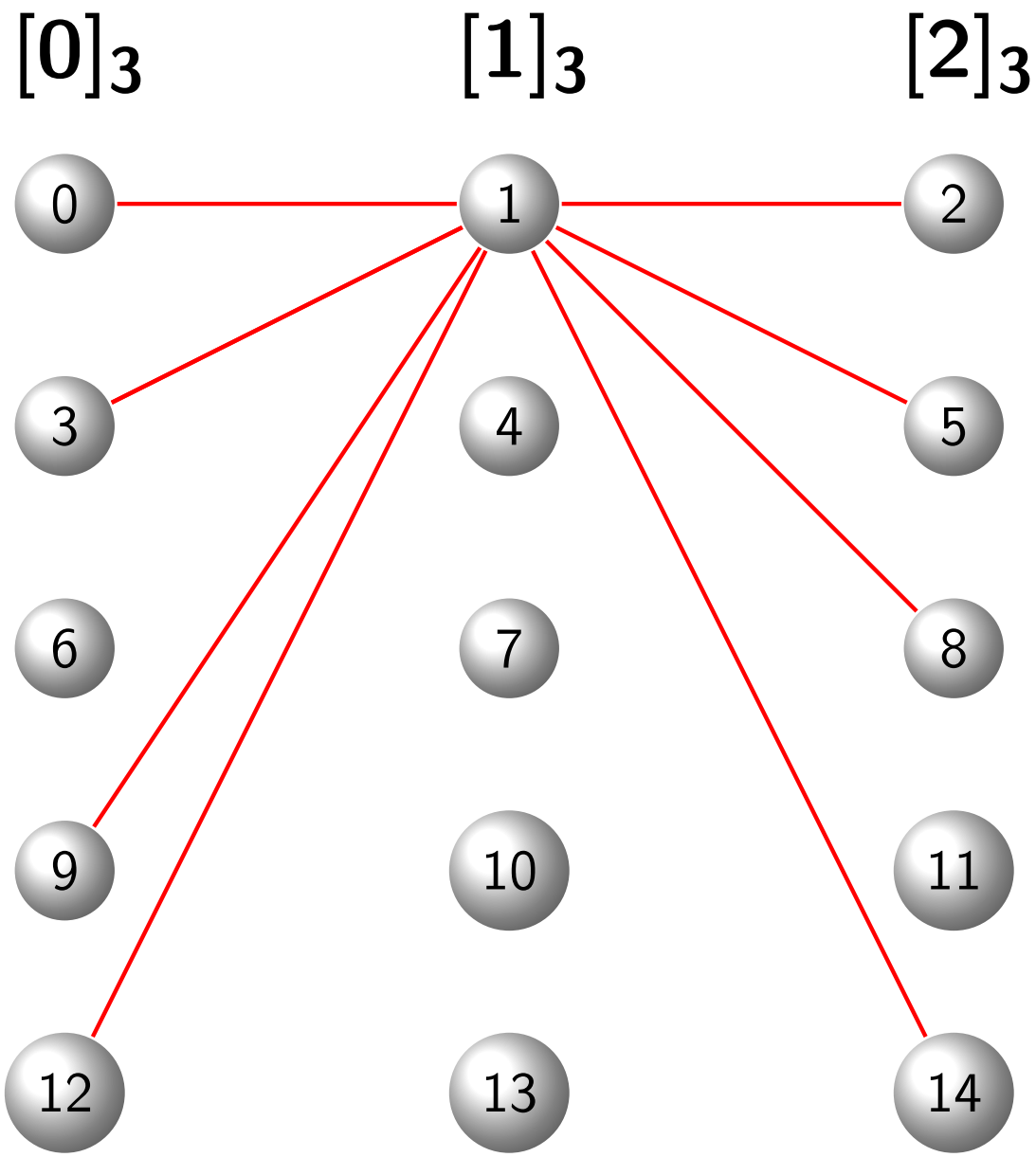
Now, a humble result of us:

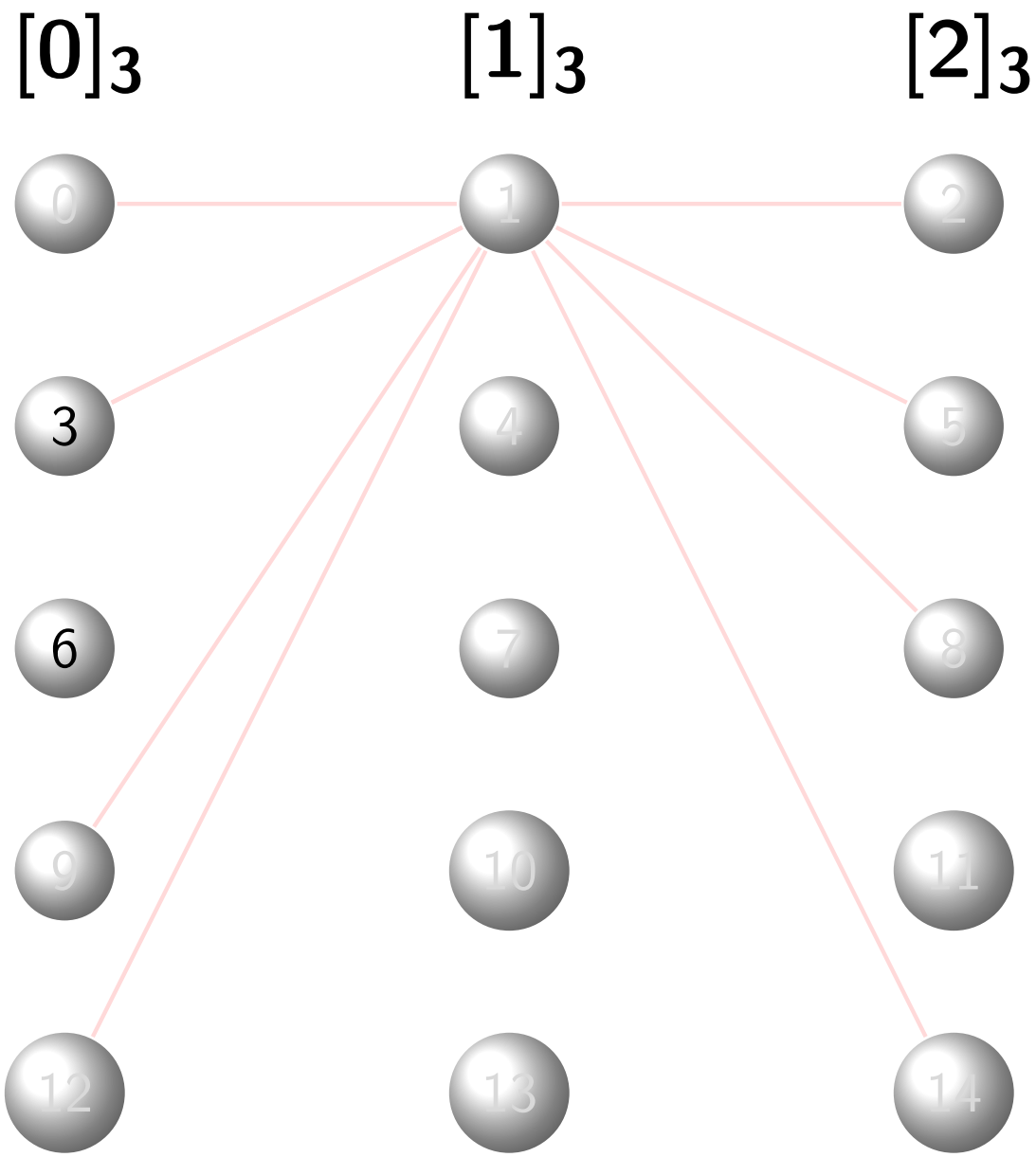
Theorem

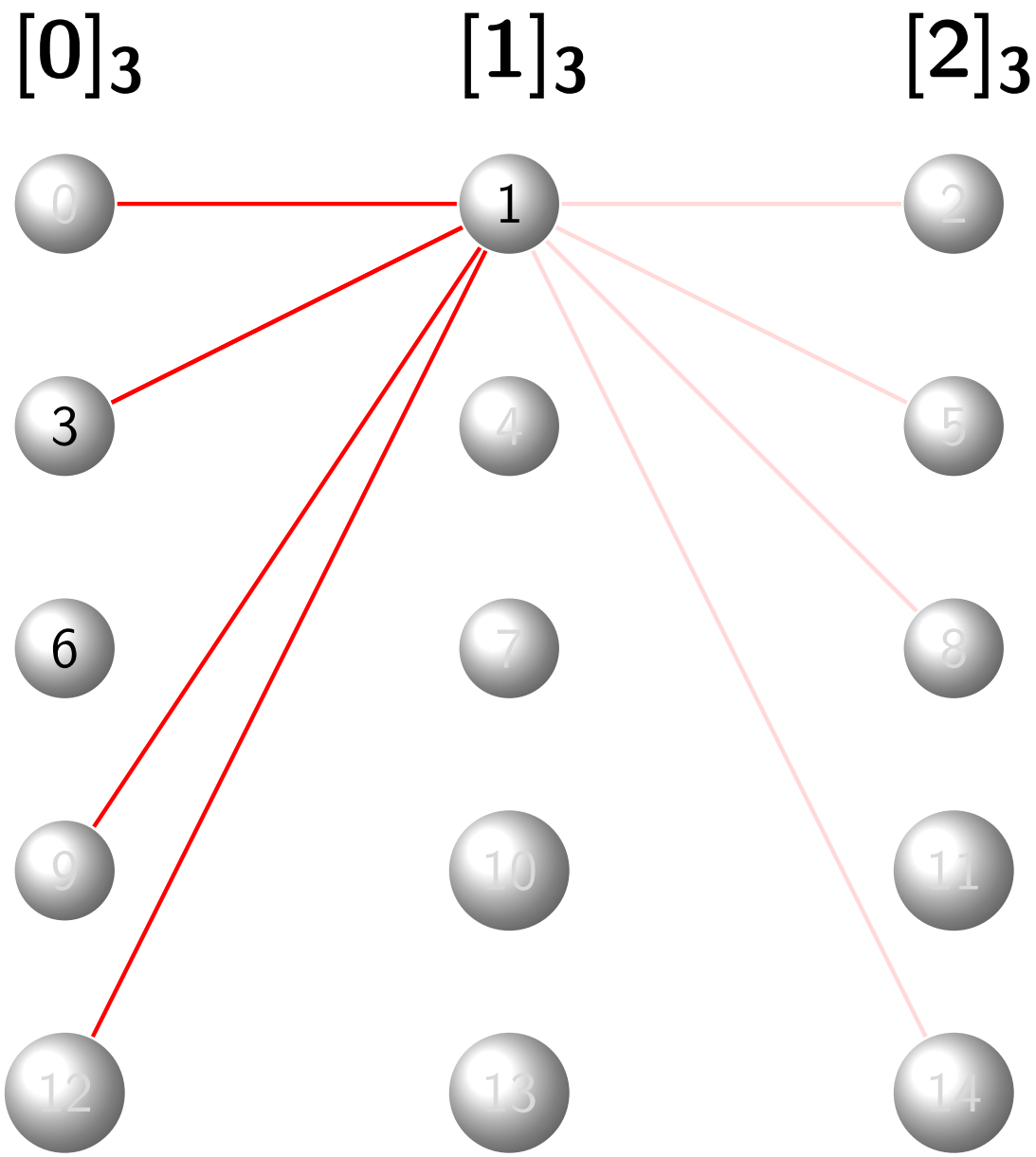
Let $p < q$ be prime numbers and S vertex-cut set with $|S| < pq - q$ then $k(X_{pq} - S) = 2$.


$[0]_3$
$[1]_3$
$[2]_3$
0
1
2
3
4
5
6
7
8
9
10
11
12
13
14


$[0]_3$
$[1]_3$
$[2]_3$
0
1
2
3
4
5
6
7
8
9
10
11
12
13
14


$[0]_3$
$[1]_3$
$[2]_3$
0
1
2
3
4
5
6
7
8
9
10
11
12
13
14


$[0]_3$
$[1]_3$
$[2]_3$
0
1
2
3
4
5
6
7
8
9
10
11
12
13
14


$[0]_3$
$[1]_3$
$[2]_3$
0
1
2
3
4
5
6
7
8
9
10
11
12
13
14

Toughness of UCG

Theorem

Let X_n be an Unitary Cayley Graph of order n. Then $t(X_n) = p_1(n) - 1$

Proof.

Assume that n is square-free.
Induction over the number of prime divisor of n

- $n = p_1 p_2$, and $S \subset V(X_n)$ a vertes-cut of X_n
- If $|S| < p_1 p_2 - p_2$, then $k(X_{p_1 p_2} - S) = 2$.
- $$\frac{|S|}{k(X_{p_1 p_2} - S)} \geq \frac{\kappa(X_{p_1 p_2})}{2} = \frac{(p_1 - 1)(p_2 - 1)}{2} \geq p_1 - 1$$

$\square$

Toughness of UCG

Theorem

Let X_n be an Unitary Cayley Graph of order n. Then $t(X_n) = p_1(n) - 1$

Proof.

Assume that n is square-free.
Induction over the number of prime divisor of n

- $n = p_1 p_2$, and $S \subset V(X_n)$ a vertes-cut of X_n
- If $|S| < p_1 p_2 - p_2$, then $k(X_{p_1 p_2} - S) = 2$.
- $$\frac{|S|}{k(X_{p_1 p_2} - S)} \geq \frac{\kappa(X_{p_1 p_2})}{2} = \frac{(p_1 - 1)(p_2 - 1)}{2} \geq p_1 - 1$$

$\square$

Toughness of UCG

Theorem

Let X_n be an Unitary Cayley Graph of order n. Then $t(X_n) = p_1(n) - 1$

Proof.

Assume that n is square-free.
Induction over the number of prime divisor of n

- $n = p_1 p_2$, and $S \subset V(X_n)$ a vertes-cut of X_n
- If $|S| < p_1 p_2 - p_2$, then $k(X_{p_1 p_2} - S) = 2$.
- $$\frac{|S|}{k(X_{p_1 p_2} - S)} \geq \frac{\kappa(X_{p_1 p_2})}{2} = \frac{(p_1 - 1)(p_2 - 1)}{2} \geq p_1 - 1$$

$\square$

Toughness of UCG

Theorem

Let X_n be an Unitary Cayley Graph of order n. Then $t(X_n) = p_1(n) - 1$

Proof.

Assume that n is square-free.
Induction over the number of prime divisor of n

- $n = p_1 p_2$, and $S \subset V(X_n)$ a vertes-cut of X_n
- If $|S| < p_1 p_2 - p_2$, then $k(X_{p_1 p_2} - S) = 2$.
- $$\frac{|S|}{k(X_{p_1 p_2} - S)} \geq \frac{\kappa(X_{p_1 p_2})}{2} = \frac{(p_1 - 1)(p_2 - 1)}{2} \geq p_1 - 1$$

$\square$

proof

Proof.

- If $|S| \geq p_1 p_2 - p_2$, then

$$\frac{|S|}{k(X_{p_1 p_2} - S)} \geq \frac{p_1 p_2 - p_2}{ind(X_{p_1 p_2})} = p_1 - 1$$

- $n = p_1 p_2 \ldots p_{k+1}$ con $p_1 < \ldots < p_{k+1}$
- $X_n \cong X_{p_1} \times G$, where $G = X_{p_2} \times \ldots \times X_{p_{k+1}}$
- But $t(G) \geq p_2 - 1$ by induction hypothesis.
- Then, $t(X_{p_1} \times G) = p_1 - 1$.
- If n is non-square-free, as $t(X_{\mathrm{rad}(n)}) \leq t(X_n) \leq p_1(n) - 1$, we are done!.

□

proof

Proof.

- If $|S| \geq p_1 p_2 - p_2$, then

$$\frac{|S|}{k(X_{p_1 p_2} - S)} \geq \frac{p_1 p_2 - p_2}{ind(X_{p_1 p_2})} = p_1 - 1$$

- $n = p_1 p_2 \ldots p_{k+1}$ con $p_1 < \ldots < p_{k+1}$
- $X_n \cong X_{p_1} \times G$, where $G = X_{p_2} \times \ldots \times X_{p_{k+1}}$
- But $t(G) \geq p_2 - 1$ by induction hypothesis.
- Then, $t(X_{p_1} \times G) = p_1 - 1$.
- If n is non-square-free, as $t(X_{\mathrm{rad}(n)}) \leq t(X_n) \leq p_1(n) - 1$, we are done!.

□

proof

Proof.

- If $|S| \geq p_1 p_2 - p_2$, then

$$\frac{|S|}{k(X_{p_1 p_2} - S)} \geq \frac{p_1 p_2 - p_2}{ind(X_{p_1 p_2})} = p_1 - 1$$

- $n = p_1 p_2 \ldots p_{k+1}$ con $p_1 < \ldots < p_{k+1}$
- $X_n \cong X_{p_1} \times G$, where $G = X_{p_2} \times \ldots \times X_{p_{k+1}}$
- But $t(G) \geq p_2 - 1$ by induction hypothesis.
- Then, $t(X_{p_1} \times G) = p_1 - 1$.
- If n is non-square-free, as $t(X_{\mathrm{rad}(n)}) \leq t(X_n) \leq p_1(n) - 1$, we are done!.

□

proof

Proof.

- If $|S| \geq p_1 p_2 - p_2$, then

$$\frac{|S|}{k(X_{p_1 p_2} - S)} \geq \frac{p_1 p_2 - p_2}{ind(X_{p_1 p_2})} = p_1 - 1$$

- $n = p_1 p_2 \ldots p_{k+1}$ con $p_1 < \ldots < p_{k+1}$
- $X_n \cong X_{p_1} \times G$, where $G = X_{p_2} \times \ldots \times X_{p_{k+1}}$
- But $t(G) \geq p_2 - 1$ by induction hypothesis.
- Then, $t(X_{p_1} \times G) = p_1 - 1$.
- If n is non-square-free, as $t(X_{\mathrm{rad}(n)}) \leq t(X_n) \leq p_1(n) - 1$, we are done!.

□

proof

Proof.

- If $|S| \geq p_1 p_2 - p_2$, then

$$\frac{|S|}{k(X_{p_1 p_2} - S)} \geq \frac{p_1 p_2 - p_2}{ind(X_{p_1 p_2})} = p_1 - 1$$

- $n = p_1 p_2 \ldots p_{k+1}$ con $p_1 < \ldots < p_{k+1}$
- $X_n \cong X_{p_1} \times G$, where $G = X_{p_2} \times \ldots \times X_{p_{k+1}}$
- But $t(G) \geq p_2 - 1$ by induction hypothesis.
- Then, $t(X_{p_1} \times G) = p_1 - 1$.
- If n is non-square-free, as $t(X_{\mathrm{rad}(n)}) \leq t(X_n) \leq p_1(n) - 1$, we are done!.

□

proof

Proof.

- If $|S| \geq p_1 p_2 - p_2$, then

$$\frac{|S|}{k(X_{p_1 p_2} - S)} \geq \frac{p_1 p_2 - p_2}{ind(X_{p_1 p_2})} = p_1 - 1$$

- $n = p_1 p_2 \ldots p_{k+1}$ con $p_1 < \ldots < p_{k+1}$
- $X_n \cong X_{p_1} \times G$, where $G = X_{p_2} \times \ldots \times X_{p_{k+1}}$
- But $t(G) \geq p_2 - 1$ by induction hypothesis.
- Then, $t(X_{p_1} \times G) = p_1 - 1$.
- If n is non-square-free, as $t(X_{\mathrm{rad}(n)}) \leq t(X_n) \leq p_1(n) - 1$, we are done!.

$\square$

Tenacity

It is well known that:

Theorem

Let G be a non-complete connected graph. Then

$$t(G) + \frac{1}{ind(G)} \leq T(G) \leq \frac{|V(G)| - ind(G) + 1}{ind(G)}$$

With this we can prove that

Corollary

Let X_n a Unitary Cayley Graph of order n. Then

$$T(X_n) = p_1(n)\left(1 + \frac{1}{n}\right) - 1$$

Tenacity

It is well known that:

Theorem

Let G be a non-complete connected graph. Then

$$t(G) + \frac{1}{ind(G)} \leq T(G) \leq \frac{|V(G)| - ind(G) + 1}{ind(G)}$$

With this we can prove that

Corollary

Let X_n a Unitary Cayley Graph of order n. Then

$$T(X_n) = p_1(n)\left(1 + \frac{1}{n}\right) - 1$$

Integrity

Remember

Definition

Given a non-complete connected graph G, the **integrity** of G, $I(G)$ is defined as:

$$\min\{|S| + m(G - S)\}$$

where the minimum is taken over all the $S \subset V(G)$.

A Known Result on Integrity

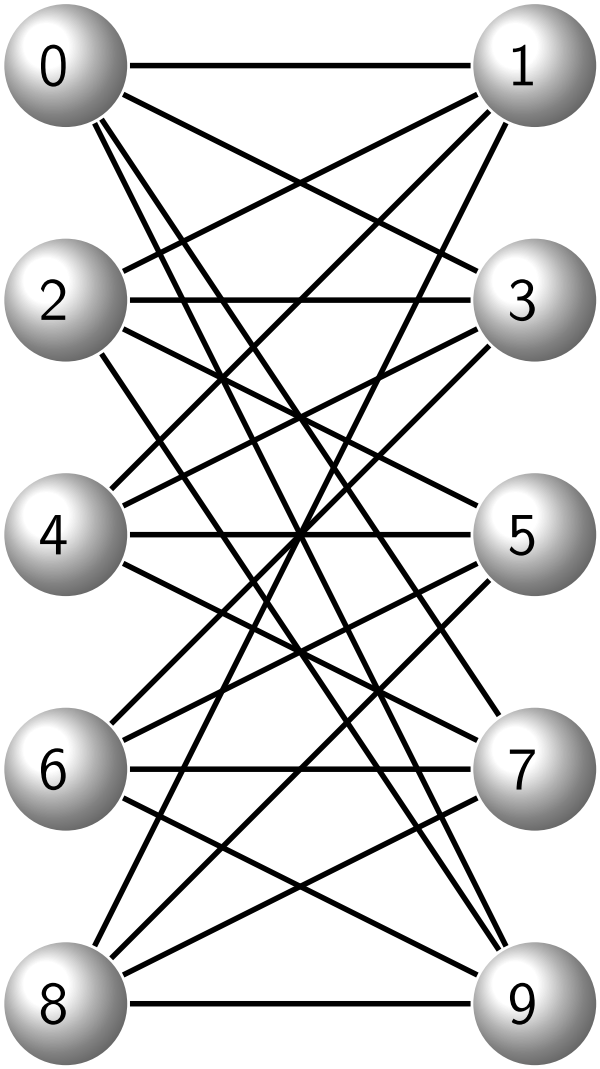
We will make use of the following known result about integrity:

Theorem

Let G be a k-regular Graph of order n. Then

$$k + 1 \leq I(G) \leq n - ind(G) + 1$$

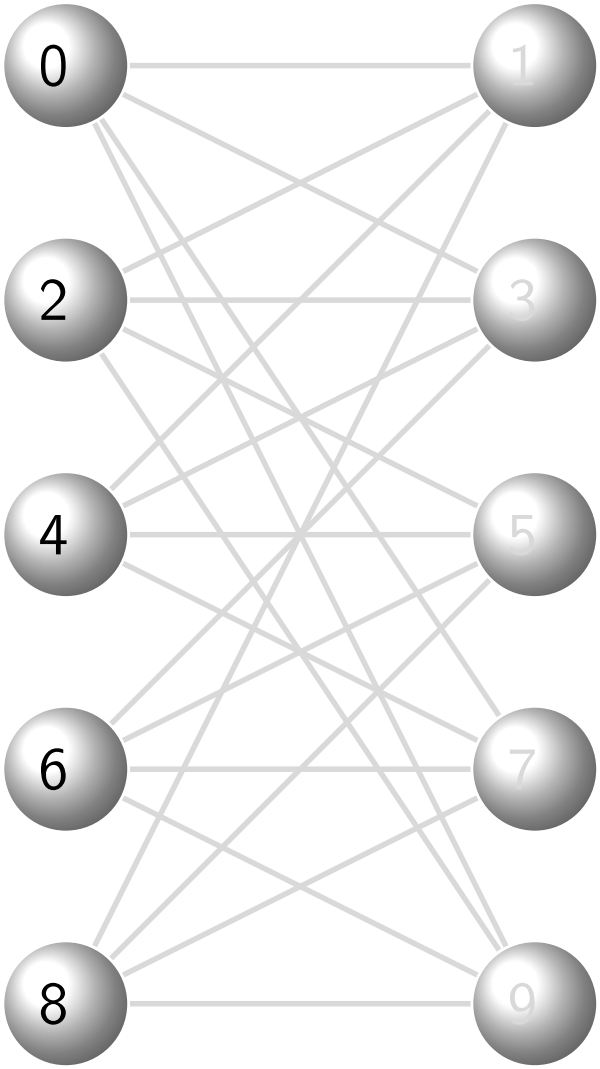
Example: X_{10}



$t(X_{10}) = 1$

$I(X_{10}) = 6$

Example: X_{10}



$t(X_{10}) = 1$

$I(X_{10}) = 6$

Before going on with the results on integrity we will write the bounds for integrity in the language of Unitary Cayley Graphs:

$$\varphi(n) + 1 \leq I(X_n) \leq n - \frac{n}{p_1(n)} + 1$$

Theorem

Let p be a prime number and $k \in \mathbb{Z}^+$. Then
$I(X_{p^k}) = p^k - p^{k-1} + 1$

Proof.

We know that

$$I(X_{p^k}) \geq \varphi(p^k) + 1 = p^k - p^{k-1} + 1$$

But we also have that:

$$I(X_{p^k}) \leq p^k - \frac{p^k}{p} + 1$$

$$= p^k - p^{k-1} + 1$$

Thus $I(X_{p^k}) = p^k - p^{k-1} + 1$ as we wanted to prove. □

Theorem

Let p be a prime number and $k \in \mathbb{Z}^+$. Then
$I(X_{p^k}) = p^k - p^{k-1} + 1$

Proof.

We know that

$$I(X_{p^k}) \geq \varphi(p^k) + 1 = p^k - p^{k-1} + 1$$

But we also have that:

$$I(X_{p^k}) \leq p^k - \frac{p^k}{p} + 1$$

$$= p^k - p^{k-1} + 1$$

Thus $I(X_{p^k}) = p^k - p^{k-1} + 1$ as we wanted to prove. □

Theorem

Let $p < q$ be prime numbers, then $I(X_{pq}) = pq - q + 1$

Proof.

Let S be a vertex-cut set such that $|S| < pq - q$. We have that $k(X_{pq} - S) = 2$. Using the pigeonhole principle we have that:

$$m(X_{pq} - S) \geq \frac{pq - |S|}{2}$$

And so we have, using that $p < q$ and that, as S is a vertex-cut, $|S| \geq \kappa(X_{pq}) = (p-1)(q-1)$:

$$|S| + m(X_{pq} - S) \geq |S| + \frac{pq - |S|}{2}$$

$$\geq \frac{|S| + pq}{2} \geq \frac{(p-1)(q-1) + pq}{2}$$

Proof.

$$|S| + m(X_{pq} - S) \geq \frac{(p-1)(q-1)+pq}{2}$$

$$\geq pq - \frac{p+q}{2} + \frac{1}{2} > pq - q$$

Assume now that $|S| \geq pq - q$, then:

$$|S| + m(X_{pq} - S) \geq pq - q + 1$$

And so $I(X_{pq}) = pq - q + 1$. □

Notice that in both cases the integrity is equal to the upper bound $n - \frac{n}{p_1(n)} + 1$, we conjecture that this is the case for every Unitary Cayley Graph.

GRACIAS!!!!!!!!

Bibliography

- N. Biggs, *Algebraic graph theory, Second Edition*, Chematical library. Cambridge University Press, 1993.
- G. Chartrand and L. Lesniak, *Graphs and Digraphs, Third Edition*,Chapman and Hall, 1996.
- C. Godsil and R. Royle, *Algebraic graph theory*, Graduate Text in Mathematics. Springer, 2001.
- W. Klotz and T. Sander, *Some properties of unitary Cayley graphs*, The Electronic Journal of Combiantorics 14 (2007), R45, pp. 1-12.