An Erdős-Lovász-Spencer Theorem for permutations and its consequences for parameter testing

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This is joint work with
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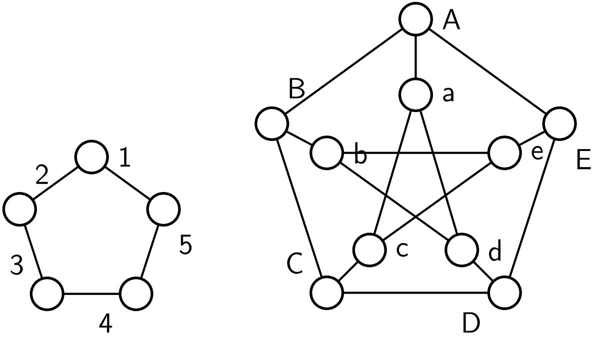
Preliminaries

We look at a copies of a subgraph $H = (W, F)$ in a graph $G = (V, E)$ as injective functions $f : W \to V$ such that

$$\{u, v\} \in F \iff \{f(u), f(v)\} \in E.$$



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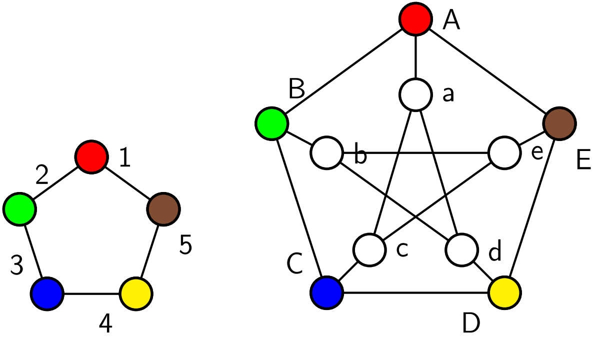

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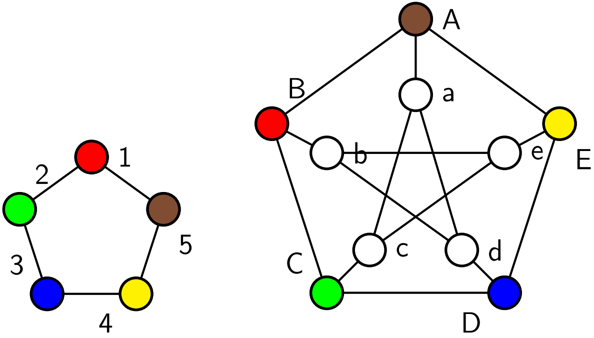

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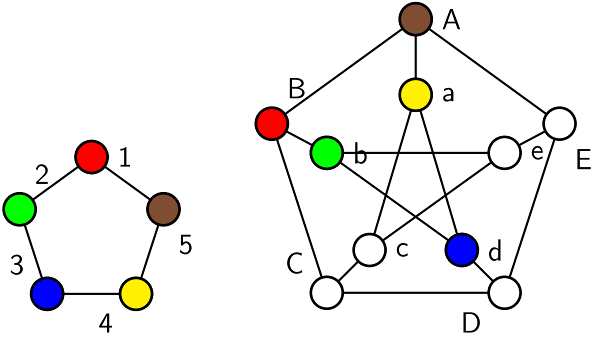


Preliminaries

Given graphs H and G, let $\Lambda(H, G)$ denote the number of copies of H in G. The density of H as an induced subgraph of G is given by

$$t(H, G) = \frac{\Lambda(H, G)}{\binom{n}{k}},$$

where $n = |V(G)|$ and $k = |V(H)|$.

Example: $t(K_2, I_2) = 1/3$, $t(K_3, I_2) = 0$

Preliminaries

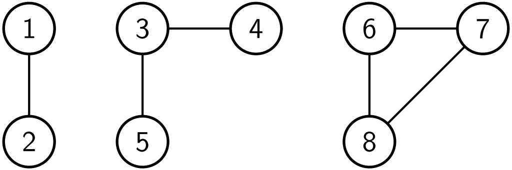
Let $\mathcal{G}$ denote the set of all finite graphs. For a fixed family of graphs $H_1, \ldots, H_r$, consider the function $\mathcal{F}\colon \mathcal{G} \to [0,1]^r$ given by

$$\mathcal{F}(G) = (t(H_1, G), \cdots, t(H_r, G)).$$

Here, we will consider $H_1, \ldots, H_r$ to be the set of all (unlabelled) connected graphs with $2 \leq |V(H_i)| \leq k$ for some $k \in \mathbb{N}$.

Preliminaries

For $k = 3$, the graphs are



$$K_4 \to (1, 0, 1), \quad P_3 \to (2/3, 1, 0)$$

The image of $\mathcal{F}$

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Consider the set

$$S_r = \{\mathbf{v} \in [0,1]^r : \exists \textit{ sequence } G_n, |V(G_n)| \to \infty, \mathcal{F}(G_n) \to \mathbf{v}\}.$$

This is the set of accumulation points of the image of $\mathcal{F}$.

The image of Φ

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$$S_r = \{\mathbf{v} \in [0,1]^r : \exists \textit{ sequence } G_n, |V(G_n)| \to \infty, \mathcal{F}(G_n) \to \mathbf{v}\}.$$

- For $k = 3$, $(2/3, 1, 0) \notin S_r$, even though there is a graph G such that $\mathcal{F}(G) = (2/3, 1, 0)$.
- S_r may contain points with irrational coordinates.

A theorem of Erdős, Lovász and Spencer

Theorem (Erdős, Lovász, Spencer'79)

For all $k \geq 2$, the set S_r has an interior point. In particular, the set S_r is r-dimensional.

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For all $k \geq 2$, the set S_r has an interior point. In particular, the set S_r is r-dimensional.

However, the densities of disconnected subgraphs are asymptotically determined by the densities of their connected components.

Permutations

A permutation σ on $[n] = \{1, 2, \ldots, n\}$ is a bijective function of the set $[n]$ into itself.

$(4, 5, 2, 3, 6, 1)$ is a permutation on $[6]$.

Parameter testing

We consider general (quantitative) properties of a permutation:

- How many fixed points does it have?
- What is the size of the longest increasing subpermutation?

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- How many fixed points does it have?
- What is the size of the longest increasing subpermutation?

Question: Can one estimate the answer of such a question accurately by looking only at a randomly chosen substructure of sufficiently large, but constant size?

Subpermutations

A subpermutation of a permutation σ on $[n]$ is a permutation τ on $[k]$ such that there is an k-tuple $x_1 < \cdots < x_k \in [n]^k$ such that $\tau(i) < \tau(j)$ if and only if $\sigma(x_i) < \sigma(x_j)$ for every $(i, j) \in [k]^2$.

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Example: $\tau = (3, 1, 4, 2)$, $\sigma = (5, 6, 2, 4, 7, 1, 3)$.

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Example: For $\tau = (2, 1)$ and $\sigma = (3, 1, 2)$,

$$t(\tau, \sigma) = \binom{3}{2}^{-1} \cdot 2 = \frac{2}{3}$$

Random subpermutations

Let $k \leq n$ be positive integers and let σ be a permutation on $[n]$. A random subpermutation $\mathrm{sub}(k, \sigma)$ of σ is obtained as follows:

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$$k = 3,\ n = 10,\ \sigma = (5, 7, 2, 10, 1, 4, 8, 6, 3, 9)$$

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$$X = \{2, 5, 9\} \rightarrow \sigma = (5, 7, 2, 10, 1, 4, 8, 6, 3, 9)$$
$$\rightarrow \mathrm{sub}(k, \sigma) = (3, 1, 2)$$

Permutation parameters

A permutation parameter is a function $f : \bigcup_n \mathfrak{S}_n \to \mathbb{R}$, where $\mathfrak{S}_n = \{\text{permutations on } [n]\}$.

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A parameter f is testable (through subpermutations) if, for every $\epsilon > 0$, there exists a positive integer $k = k(\epsilon)$ with the following property. If σ is a permutation of length $n > k$, then

$$\mathbb{P}\Big(|f(\sigma) - f(\mathrm{sub}(k,\sigma))| > \epsilon\Big) \ \leq \ \epsilon.$$

Limits of permutation sequences

Borgs, Chayes, Lovász, Sós, Szegedy and Vesztergombi (2006) proved that, for graphs, testable parameters are characterized by graph limits. A theory of convergence has been devised for permutation sequences.

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A sequence of permutations (σ_n) is said to converge if, for every fixed permutation τ, the real sequence $(t(\tau, \sigma_n))_{n\in\mathbb{N}}$ converges.

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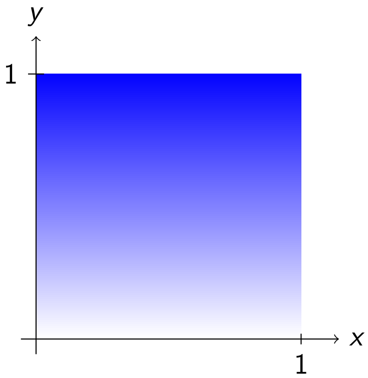
A permuton (or permutation limit) is a probability measure Φ on the unit square $[0,1]^2$ such that Φ has uniform marginals:

$$\Phi([\alpha, \beta] \times [0,1]) = \Phi([0,1] \times [\alpha, \beta]) = \beta - \alpha$$

for every $0 \leq \alpha \leq \beta \leq 1$.

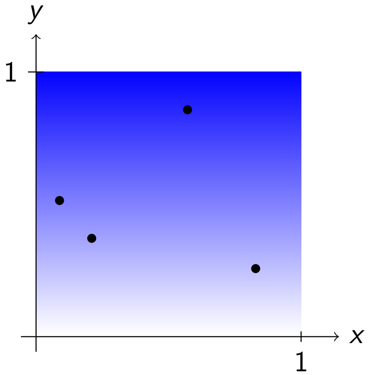
Random permutations

For a permuton Φ, a Φ-random permutation of order n is a permutation $\sigma_{\Phi,n}$ generated as follows.



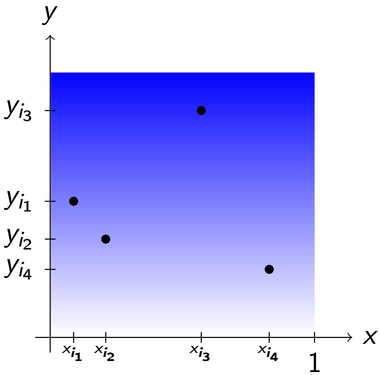
Random permutations

Sample n points $(x_1, y_1), \dots, (x_n, y_n)$ in $[0, 1]^2$ independently with the distribution given by Φ.



Random permutations

Let $i_1, \ldots, i_n$ such that $x_{i_1} < x_{i_2} < \cdots < x_{i_n}$. The permutation is given by the relative order of the y_{i_j}.



(3,2,4,1)

Existence of a limit

If Φ is a permuton and τ is a permutation on $[k]$, we define

$$t(\tau, \Phi) = \mathbb{P}(\sigma_{\Phi,k} = \tau).$$

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Theorem (H., Kohayakawa, Moreira, Ráth and Sampaio'10)

Given a convergent permutation sequence (σ_n)*, there exists a permuton* $\Phi : [0,1]^2 \to [0,1]$ *such that*

$$\lim_{n\to\infty} t(\tau, \sigma_n) = t(\tau, \Phi)$$

for every permutation τ*.*

Characterization of testable parameters

A permutation parameter f is bounded if there is a constant M such that $|f(\sigma)| < M$ for every permutation σ.

Theorem (H., Kohayakawa, Moreira and Sampaio'10)

A bounded permutation parameter is testable if and only if the sequence $(f(\sigma_n))$ converges for every convergent permutation sequence (σ_n).

Finite forcibility

A permutation parameter is finitely forcible if there exists a finite family of permutations $\mathcal{A}$ which determines the value of the parameter.

Formally, for every $\epsilon > 0$, there exist an integer n_0 and a constant $\delta > 0$ such that if σ and π are permutations on $[n]$, where $n \geq n_0$, satisfying $|t(\tau, \sigma) - t(\tau, \pi)| < \delta$ for every $\tau \in \mathcal{A}$, then $|f(\sigma) - f(\pi)| < \epsilon$.

Finite approximability

A permutation parameter is finitely approximable if, for every $\epsilon > 0$, there exist a finite family of permutations $\mathcal{A}_\epsilon$, an integer n_0 and a constant $\delta > 0$ such that if σ and π are permutations on $[n]$, where $n \geq n_0$, satisfying $|t(\tau, \sigma) - t(\tau, \pi)| < \delta$ for every $\tau \in \mathcal{A}$, then $|f(\sigma) - f(\pi)| < \epsilon$.

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[H., Kohayakawa, Moreira and Sampaio] f is testable if and only if f is finitely approximable. Is there a testable parameter that is not finitely forcible?

Connected permutations

A permutation on $[n]$ is connected if there is no $m < n$ such that $\sigma([m]) = [m]$.

Example: $(2,1)$ is the single connected permutation on $[2]$ and $(3,1,2)$, $(2,3,1)$ and $(3,2,1)$ are the connected permutations on $[3]$.

Erdős-Lovász-Spencer Theorem for permutations

- Given $k \geq 2$, let $\tau_1, \ldots, \tau_r$ be the set of connected permutations on $[j]$, where $2 \leq j \leq k$.
- Consider the function $\sigma \to (t(\tau_1, \sigma), \ldots, t(\tau_r, \sigma))$ and the set S_r of its accumulation points. In fact,

$$S_r = \{(t(\tau_1, \Phi), \ldots, t(\tau_r, \Phi)) \colon \Phi \textit{ permuton}\}$$

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Theorem (Glebov, H., Klimošová, Kohayakawa, Král, Liu'14?)

For all $k \geq 2$, the set S_r has an interior point.

A testable permutation parameter that is not finitely forcible

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For all $k \geq 2$, the set S_r has an interior point.

This result implies that, for any finite family of permutations $\mathcal{A}$, there exist a permutation τ and permutons Φ and Φ' such that

- $t(\pi, \Phi) = t(\pi, \Phi')$ for every $\pi \in \mathcal{A}$
- $t(\tau, \Phi) \neq t(\tau, \Phi')$

A testable permutation parameter that is not finitely forcible

- Using this theorem inductively, fix a sequence (τ_i) of permutations of strictly increasing orders such that, for every $k > 1$, there exist permutons Φ_k and Φ'_k satisfying the following:
  - $t(\sigma, \Phi_k) = t(\sigma, \Phi'_k)$ for every σ such that $|\sigma| \leq |\tau_{k-1}|$
  - $t(\tau_k, \Phi_k) > t(\tau_k, \Phi'_k)$

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  - $t(\tau_k, \Phi_k) > t(\tau_k, \Phi'_k)$
- With this, we may easily find a sequence α_i of positive reals such that

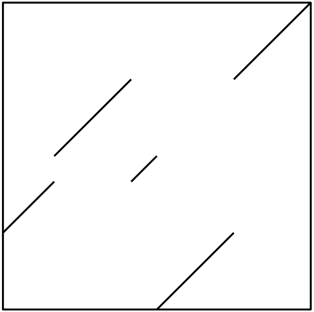
$$f_\bullet(\sigma) = \sum_{i=1}^{\infty} \alpha_i t(\tau_i, \sigma)$$

is testable, but not finitely forcible.

Sketch of the proof our main theorem

We define two constructions on permutons:

- Step-up permuton: Given a permutation σ on $[n]$ and a vector $\mathbf{y} = (y_1, \ldots, y_n) \in [0,1]^n$ such that $\sum y_i \leq 1$, we define the step-up permuton $\Phi_\sigma^{\mathbf{y}}$.

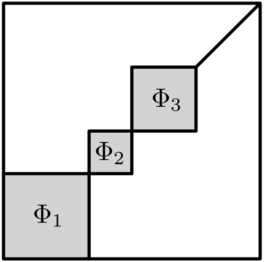


$\sigma = (2,4,3,1)$, $\mathbf{y} = (1/6, 1/4, 1/12, 1/4)$

Sketch of the proof our main theorem

We define two constructions on permutons:

- Composed permuton: given permutons $\Phi_1, \ldots, \Phi_\ell$, and a vector $\mathbf{y} = (y_1, \ldots, y_\ell) \in [0,1]^\ell$ such that $\sum y_i \leq 1$, we define the composed permuton $\bigoplus(y_i, \Phi_i)$.



The permuton $(1/3, \Phi_1) \oplus (1/6, \Phi_2) \oplus (1/4, \Phi_3)$.

Sketch of the proof our main theorem

- Given $\mathbf{x} = (x_1, \ldots, x_n)$, we find formulas for $t(\tau, \Phi_\sigma^{\mathbf{x}})$ and $t(\tau, \bigoplus(x_i, \Phi_i))$, which are homogeneous polynomials of degree $|\tau|$ on the indeterminates $\mathbf{x}$.
- If $\tau_1, \ldots, \tau_r$ are the nontrivial connected permutations of order up to $k \geq 2$, and $c_1, \ldots, c_r$ are real number, not all of which are zero, we show that there is a permuton Φ such that $\sum_{i=1}^{r} c_i t(\tau_i, \Phi) \neq 0$. In particular, S_r contains elements that span $\mathbb{R}^r$.

Sketch of the proof our main theorem

- Let $\Phi_1, \ldots, \Phi_r$ be permutons such that $\{(t(\tau_1, \Phi_i), \ldots, t(\tau_r, \Phi_i))\}$ span $\mathbb{R}^r$. Let $\Psi : \mathbb{R}^r \to \mathbb{R}^r$ be the map:

$$\Psi_j : (x_1, \ldots, x_r) \to \sum_{i=1}^{r} x_i^{|\tau_j|} t(\tau_j, \Phi_i).$$

- Our formulas show that $\Psi((0, 1/r)^r) \subset S_r$. Moreover, there exists $\mathbf{x} \in (0, 1/r)^r$ such that $\mathrm{Jac}(\Psi)(\mathbf{x}) \neq 0$. Then S_r contains an open ball around $\Psi(\mathbf{x})$.

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