Characterizing and Recognizing Normal Helly Circular-Arc Graphs

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Joint work with: Yixin Cao and Martín Safe

Outline

Interval graphs

- An interval graph is a graph whose vertices can be assigned to a finite family of intervals on a line such that two vertices are adjacent if and only if their corresponding intervals intersect.

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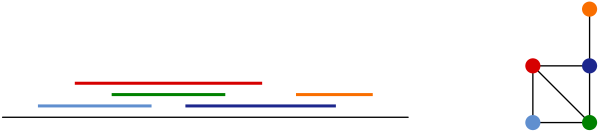

- Clearly, the class of interval graphs is a hereditary class.

Forbidden induced subgraph characterization

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Theorem (Lekkerkerker and Boland, 1962)

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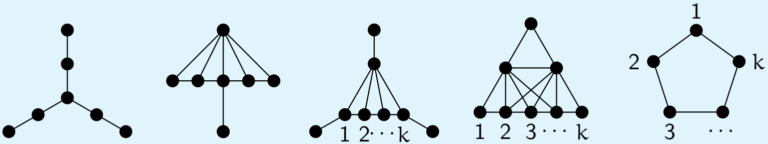

bipartite claw | umbrella | k-net, $k \geqslant 2$ | k-tent, $k \geqslant 3$ | C_k, $k \geqslant 4$

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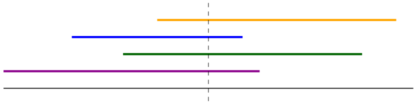

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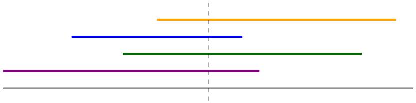

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Theorem (Roberts, 1969)

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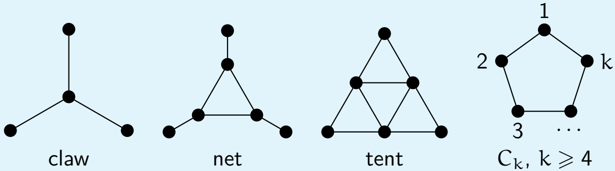

Circular-arc Graphs

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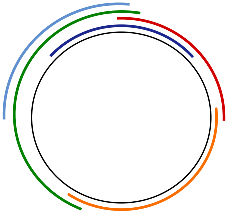

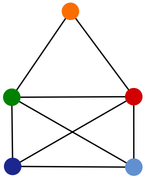

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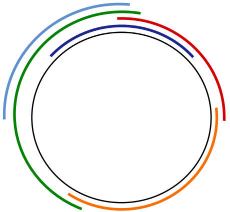

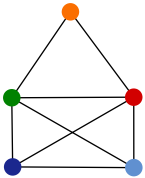

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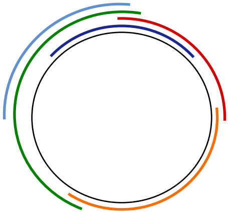

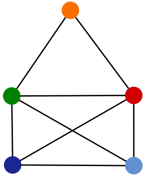

- Clearly, the class of circular-arc graphs generalizes that of interval graphs and is also hereditary.
- A family of arcs on a circle not necessary has the Helly property.

Characterizations of some subclasses of circular-arc graphs

- Despite their similarity in definition to interval graphs, characterizing circular-arc graphs by forbidden induced subgraphs is a long-standing open problem (Hadwiger and Debrunner, 1964; Klee, 1969). However Mathew, Hell and Stacho, characterized them in terms of some obstructions called quadruples, giving a characterization similar to those well-known characterization of interval graphs in terms of asteroidal triples (2014).

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- Tucker pioneered the study of circular-arc graphs and some important subclasses. In 1974, he found the minimal forbidden induced subgraph characterizations of both unit circular-arc graphs (defined analogously to unit interval graphs) and proper circular-arc graphs, which are those intersection graphs of finite sets of arcs on a circle such that none of the arcs is contained in another of the arcs.

Characterizations of some subclasses of circular-arc graphs

Since then, the problem of characterizing circular-arc graphs and some of its subclasses by forbidden induced subgraphs or some other kinds of obstructions has attracted considerable attention:

- co-bipartite circular-arc graphs (Trotter and Moore, 1976; Feder, Hell, and Huang, 1999)
- chordal proper circular-arc graphs (Bang-Jensen and Hell, 1994)
- Helly circular-arc graphs (Lin and Szwarcfiter, 2006; Joeris, Lin, McConnell, Spinrad, and Szwarcfiter, 2011)
- proper Helly circular-arc graphs (Lin, Soulignac, and Szwarcfiter, 2007)
- partial characterizations of circular-arc graphs (Bonomo, Durán, G., and Safe, 2009)
- normal Helly circular-arc graphs (Lin, Soulignac, and Szwarcfiter, 2011)

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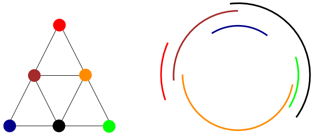

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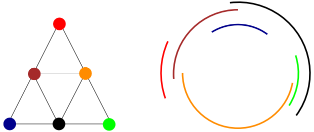

- No characterization of normal circular-arc graphs by forbidden induced subgraphs is known, but there are some partial characterizations (Bonomo et al., 2009).

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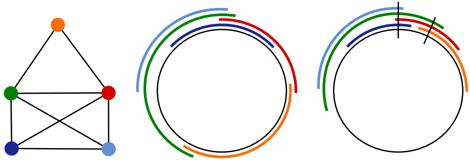

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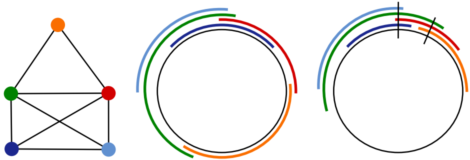

- There is a characterization by forbidden induced subgraphs by Lin and Szwarcfiter (2006) by restricting themselves to circular-arc graphs.

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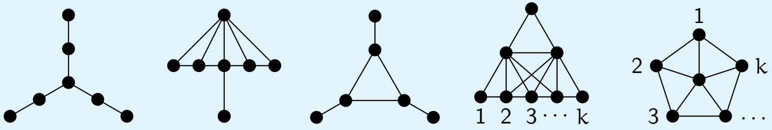
- Bhowmick and Chandran (2011) showed that NHCA graphs have boxicity at most 3 (3 is the minimum k such that G is the intersection graph of a family of k-dimensional boxes).

Characterization of normal Helly circular-arc graphs

Lin, Soulignac, and Szwarcfiter (2011) drew several parallels between normal Helly circular-arc graphs. In particular, they proved the following characterization of normal Helly circular-arc graphs by minimal forbidden induced subgraphs, by restricting themselves to circular-arc graphs.

Theorem (Lin, Soulignac, and Szwarcfiter, 2011)

Let G be a circular-arc graph. Then, G is a normal Helly circular-arc graph if and only if G contains no induced: bipartite claw, umbrella, net, k-tent for some $k \geqslant 3$, or k-wheel some $k \geqslant 4$.



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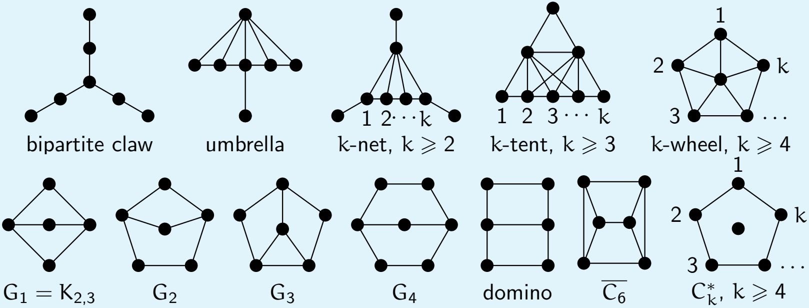
Complete characterization

Theorem

A graph is a normal Helly circular-arc graph if and only if it contains none of the following graphs as an induced subgraph:



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Let G be a graph containing none of the graphs displayed on the statement of the theorem as an induced subgraph.
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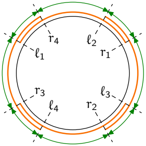

We start with a circular-arc model of C consisting of closed arcs.Its $2n$ endpoints divide the circle into $2n$ portions.

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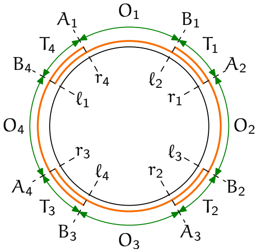

By looking at the neighborhoods in the cycle C of those vertices outside C, we determine for each $i = 1, \dots, n$:

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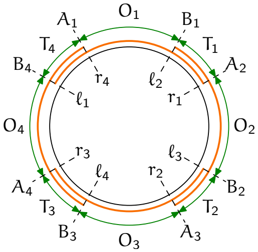

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- A set A_i of vertices whose circular-arcs must cross r_{i-1}.

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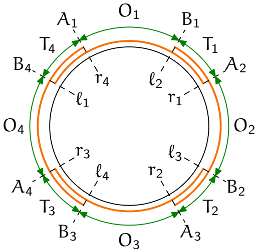

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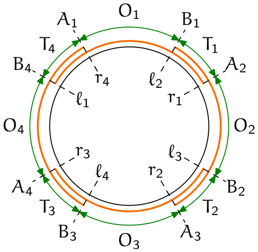

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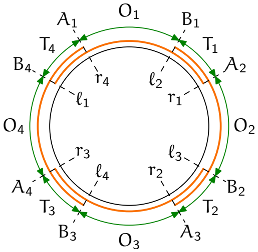

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- A set T_i of vertices whose circular-arcs can be assumed to lie within $[\ell_{i+1}, r_i]$.

Left-right sets

A pair A, B of vertex sets of an interval graph is left-right if there is an interval model such that all vertices of A have the same left endpoint and no other endpoints are further to the left and all vertices of B have the same right endpoint and no other endpoints are further to the right.

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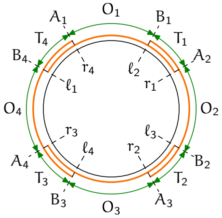

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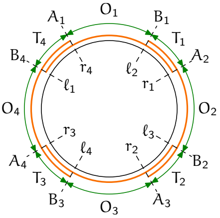

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- $G[A_i \cup O_i \cup B_i]$ is an interval graph with left-right pair A_i, B_i.

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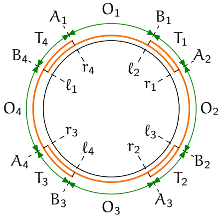

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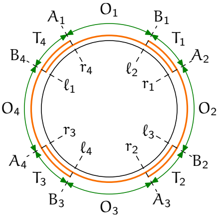

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We show that all these interval models can be glued together to build a circular-arc model of G. □

The auxiliary graph $\mho(G)$

Theorem

If G is NHCA, then $\mho(G)$ is an interval graphs. In addition, if $\mho(G)$ is an interval graph, then we can in build in linear-time a circular-arc model of G, otherwise a minimal non-NHCA graph can be obtained from a minimal noninterval graph of $\mho(G)$.

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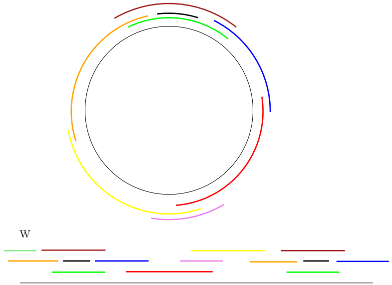

The linear-time recognition algorithm

1. Test if G is chordal and find a chordless cycle if it is not (Tarjan and Yannakakis, 1985);
   if G is chordal **then** verify whether G is an interval graph or not (Lindzey and McConnell, 2014);
2. Construct the auxiliary graph $\mho(G)$;
3. **if** $\mho(G)$ is not an interval graph **then** find a minimal non-NHCA graph of G.
4. Construct a circular-arc model $\mathcal{A}$ of G;
5. verify whether $\mathcal{A}$ is NHCA (Lin et al., 2013). In case, $\mathcal{A}$ is not NHCA, find a minimal non-NHCA (Cao, 2014).

Thank you for your attention!