Forbidden induced subgraph characterizations of graph classes related to perfect graphs

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Main objective

- The main objective of this talk is to show different forbidden induced subgraph partial characterizations of graph classes related to perfect graphs obtained for our group in the last 10 years.
- We will focus in some variations and subclasses of perfect graphs.

Classes of graphs to be reviewed

1. Perfect graphs
2. Clique-perfect graphs
3. Balanced graphs
4. Coordinated graphs
5. Neighborhood-Perfect Graphs

Intersection graphs

- Consider a finite family $\mathcal{F}$ of non-empty sets. The intersection graph of $\mathcal{F}$ is obtained by representing each set by a vertex, two vertices being connected by an edge if and only if the corresponding sets intersect.
- A circular-arc graph is the intersection graph of a finite family of arcs in a circle (such a family is called a circular-arc model of the graph).

Clique graph

- A clique in a graph is a maximal set of pairwise adjacent vertices (a maximal complete).
- The clique graph $K(G)$ of a graph G is the intersection graph of its cliques.

  Example:

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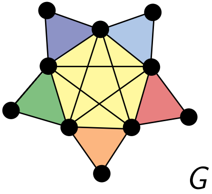

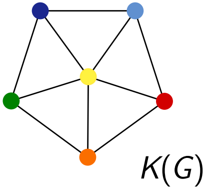

Line graph

- The line graph $L(G)$ of a graph G is the intersection graph of its edges.

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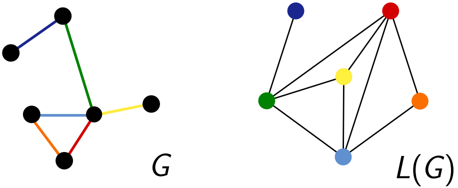

- When the graph G has no triangles and no isolated vertices, then the cliques of G are its edges, and $L(G) = K(G)$.

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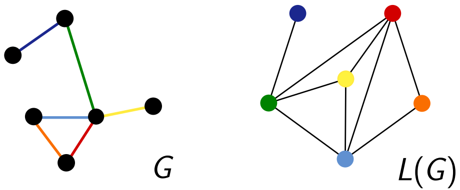

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The Helly property

- A family of sets $\mathcal{F}$ is said to satisfy the Helly property if every subfamily of $\mathcal{F}$, consisting of pairwise intersecting sets, has a common element.
- A graph is clique-Helly (CH) if its cliques satisfy the Helly property, and it is hereditary clique-Helly (HCH) if all its induced subgraphs are clique-Helly.

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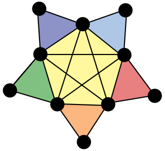
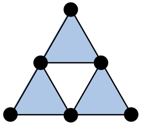


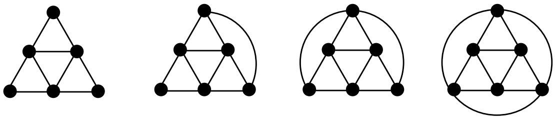
clique-Helly



not clique-Helly

The Helly property

- Theorem (Prisner, 1993): A graph is hereditary clique-Helly iff it contains none of the following graphs as an induced subgraph.



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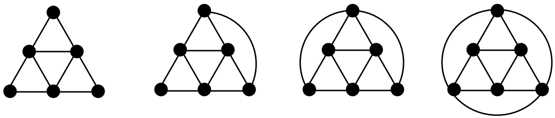

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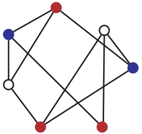
Chromatic number and maximum clique

- Coloring a graph consists of assigning colors to its vertices in such a way that no two adjacent vertices are given the same color. The minimum number of different colors needed to color a graph G is called the chromatic number of G and is denoted by $\chi(G)$.



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- In a coloring of G, the vertices of G having the same color must be pairwise non-adjacent. A set of pairwise non-adjacent vertices is called a stable set.

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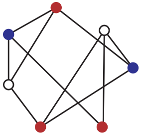

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Chromatic number and maximum clique

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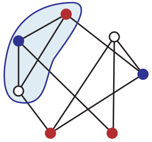

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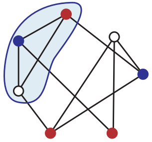

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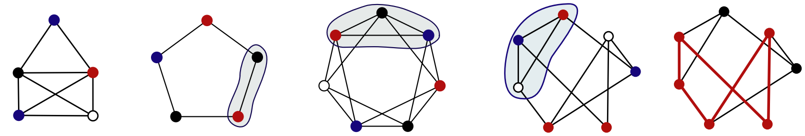
Perfect graphs

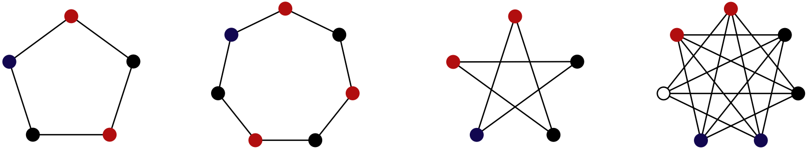
Berge defined perfect graphs in 1961. A graph G is perfect when $\chi(H) = \omega(H)$ for every induced subgraph H of G.

Examples:



Holes and antiholes

- A hole C_n is a chordless cycle of length $n \geq 4$.
- An antihole is the complement of a hole.
- A hole or antihole is odd if it has an odd number of vertices (if n is odd).
- Odd holes and odd antiholes are not perfect:



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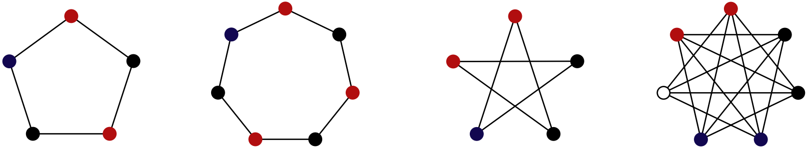
  - $\chi(\overline{C_{2k+1}}) = k+1$ and $\omega(\overline{C_{2k+1}}) = k$, $k \geq 2$



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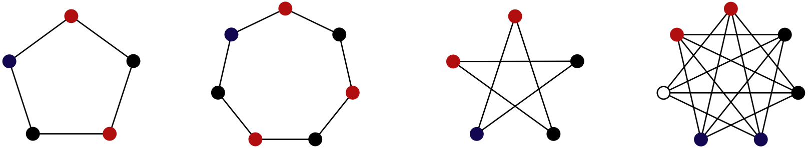

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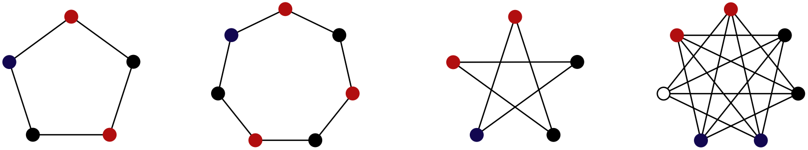

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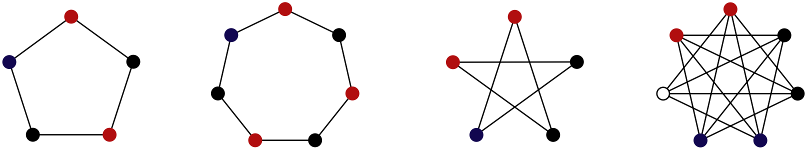

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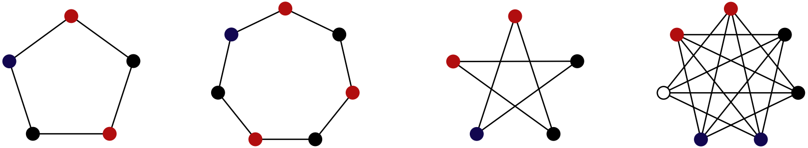


Berge conjectures (1961), now theorems

First conjecture: a graph is perfect iff its complement is perfect.

Perfect Graph Theorem (Lóvasz, 1972)

A graph is perfect iff its complement is perfect.

Lóvasz

Berge conjectures (1961), now theorems

Remark:

An alternative proof of the PGT using polyhedral theory was given by Fulkerson in 1973.

Fulkerson

Berge conjectures, now theorems

The second conjecture was that the only minimal imperfect graphs are odd holes and their complements.

Strong Perfect Graph Theorem (SPGT)
(Chudnovsky, Robertson, Seymour and Thomas, 2002)

A graph is perfect iff it neither contains an odd hole nor an odd antihole as an induced subgraph.


Chudnovsky


Robertson


Seymour


Thomas

Polynomial recognition

Perfect graphs can be recognized in polynomial time (Chudnovsky, Cornuéjols, Liu, Seymour and Vušković, 2003).

Chudnovsky

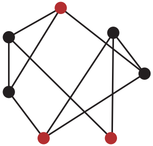
Cornuéjols

Seymour

Vušković

Stable set and clique cover

- The stability number $\alpha(G)$ is the cardinality of a maximum stable set of G. It holds $\alpha(G) = \omega(\overline{G})$.
- A clique cover of a graph G is a subset of completes covering all the vertices of G. The clique-covering number $k(G)$ is the cardinality of a minimum clique cover of G. It holds $k(G) = \chi(\overline{G})$. So, $k(G) \geq \alpha(G)$.
- By the PGT, a graph G is perfect if and only if $\alpha(H) = k(H)$ for every induced subgraph H of G.



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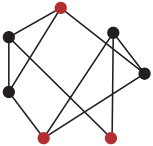

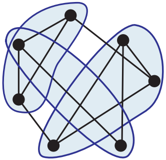

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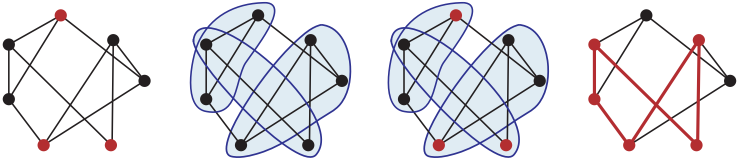

Clique-independent sets and clique-transversals

- A clique-independent set is a collection of pairwise vertex-disjoint cliques. The clique-independence number $\alpha_c(G)$ is the size of a maximum clique-independent set of G.
- A clique-transversal of a graph G is a subset of vertices that meets all the cliques of G. The clique-transversal number $\tau_c(G)$ is the size of a minimum clique-transversal of G.
- Disjoint cliques must be covered with different vertices, so

$$\tau_c(G) \geq \alpha_c(G)$$

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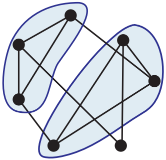

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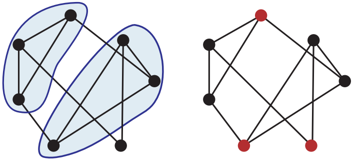

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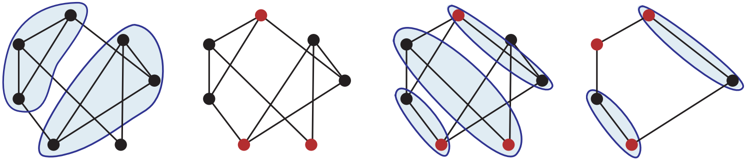

Clique-perfect graphs

- A graph G is clique-perfect when $\alpha_c(H) = \tau_c(H)$ for every induced subgraph H of G.
- The terminology "clique-perfect" has been introduced by Guruswami and Pandu Rangan in 2000, but the equality of the parameters α_c and τ_c was previously studied by Berge and Las Vergnas in the seventies in the context of balanced hypergraphs.

Guruswami

Pandu Rangan

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Guruswami

Pandu Rangan

Open problems on clique-perfect graphs

- The complete list of minimal clique-imperfect graphs is still not known.
- Another open question concerning clique-perfect graphs is the complexity of the recognition problem.

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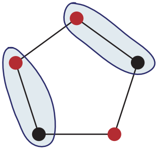
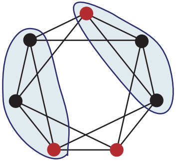
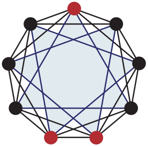
Question: is there some relation between clique-perfect graphs and perfect graphs?

- Odd holes $C_{2k+1}, k \geq 2$, are not clique-perfect: $\alpha_c(C_{2k+1}) = k$ and $\tau_c(C_{2k+1}) = k+1$.
- Antiholes $\overline{C_n}, n \geq 5$, are clique-perfect if and only if $n \equiv 0(3)$ (Reed, 2000): $\tau_c(\overline{C_n}) = 3$ and $\alpha_c(\overline{C_n}) = 2$ or 3, being 3 only if n is divisible by three.

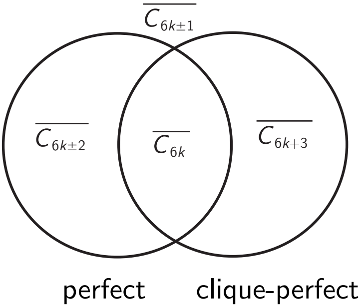

C_5


$\overline{C_7}$


$\overline{C_9}$

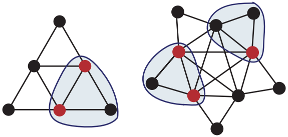
Relation with perfect graphs

So the classes overlap and we have the following scheme of relation between perfect graphs and clique-perfect graphs:



Odd suns

An *r*-sun is a chordal graph with a cycle of length r and r vertices, each one of them is adjacent to the endpoints of an edge of the cycle.



3-sun 5-sun

Odd suns are not clique-perfect: they verify $\alpha_c((2k+1)\text{-sun}) = k$ and $\tau_c((2k+1)\text{-sun}) = k+1$ (analogous to odd holes).

Partial advances on a forbidden induced subgraph characterization of clique-perfect graphs

Partial characterizations of clique-perfect graphs were obtained within the following classes:

- Chordal graphs (Lehel and Tuza, 1986)
- Diamond-free graphs
- Line graphs
- Complements of line graphs
- HCH claw-free graphs
- Helly circular-arc graphs
- Paw-free graphs
- $\{$gem, W_4, bull$\}$-free graphs
- P_4-tidy graphs

Characterization of clique-perfect graphs for line graphs

Theorem

Let G be a line graph. Then the following are equivalent:

1. G is clique-perfect.
2. No induced subgraph of G is and odd hole, or a 3-sun.
3. G is perfect and it does not contain a 3-sun.

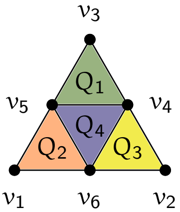
The recognition of clique-perfect line graphs can be reduced to the recognition of perfect graphs with no 3-sun, which is solvable in polynomial time (but we can use the proof of the Theorem to deduce a linear time algorithm).

Sketch of proof

$2 \Leftrightarrow 3$ is a corollary of SPGT, because line graphs cannot contain antiholes $\overline{C_n}$ with $n \geq 7$ as induced subgraphs.
$1 \Rightarrow 2$ is easy.
To prove $2 \Rightarrow 1$, we prove that line graphs with neither 3-sun nor odd holes are K-perfect. Then the result is proved by induction, taking as a basic case when the graph is hereditary clique-Helly.

Clique matrix

- The clique matrix A_G of a graph G has a row for each clique of G and a column for each vertex of G. $A_G(i,j) = 1$ if vertex j belongs to clique i and 0 otherwise.



$$
\begin{array}{c} \\ Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{array}
\begin{array}{c}
\begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} \\
\begin{pmatrix}
0 & 0 & 1 & 1 & 1 & 0 \\
1 & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 1
\end{pmatrix}
\end{array}
$$

Perfect and balanced matrices

- A 0-1 matrix A is perfect if all the extreme points of
$$\{\mathbf{0} \leq x \leq \mathbf{1} : \ Ax \leq \mathbf{1}\} \text{ are integer.}$$

Theorem (Chvátal, 1975)

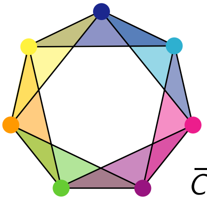
A graph G is perfect iff the clique matrix A_G is perfect.

- A 0-1 matrix is balanced if it does not contain an odd square submatrix with exactly two 1's per row and per column.

Balanced graphs

- A graph is balanced iff its clique matrix is balanced (Dahlhaus, Manuel, Miller, 1998).
- The polynomial algorithm for recognizing balanced matrices given by Conforti, Cornuéjols and Rao (1999) can be used to design a polynomial algorithm for balanced graphs.

Example:



$\overline{C}_7$

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

Balanced graphs

- Balanced matrices are perfect matrices (Berge, 1972), so balanced graphs are perfect graphs.
- Balanced graphs are *HCH* (Prisner 1993).
- Moreover, balanced graphs are K-perfect. Since balanced graphs are hereditary on induced subgraphs then balanced graphs are hereditary K-perfect.
- In conclusion, balanced graphs belong to the intersection between perfect, clique-perfect and hereditary K-perfect graphs and, besides, they are HCH.

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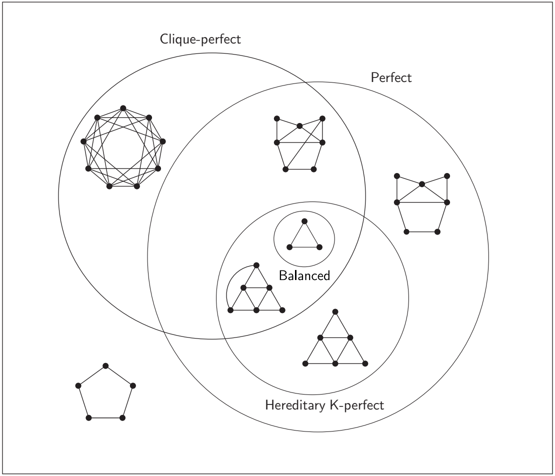
Relation between the classes



Characterization of balanced graphs by forbidden induced subgraphs

- A sun (or trampoline) is a chordal graph G on $2r$ vertices whose vertex set can be partitioned into two sets, $W = \{w_1, \ldots, w_r\}$ and $U = \{u_1, \ldots, u_r\}$, such that W is a stable set and for each i and j, w_j is adjacent to u_i if and only if $i = j$ or $i \equiv j + 1 \pmod{r}$. A sun is odd if r is odd.
- An extended odd sun is an odd cycle C and a subset of pairwise adjacent vertices $W_e \subseteq N_G(e) \setminus C$ for each edge e of C, such that $N_G(W_e) \cap N_G(e) \cap C = \emptyset$ and $|W_e| \leq |N_G(e) \cap C|$.
- Clearly, odd suns and odd holes are extended odd suns. The smallest extended odd suns are the pyramid and C_5.

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Characterization of balanced graphs by forbidden induced subgraphs

- Balanced graphs were characterized by forbidden induced subgraphs.

Theorem

A graph G is balanced iff G contains no induced extended odd sun.

- Unfortunately, this characterization is not by minimal forbidden induced subgraphs; i.e., there are extended odd suns which contain properly induced extended odd suns.

non-minimal extended odd sun

- Main open problem on this class: the list of all minimal extended odd suns is not known.

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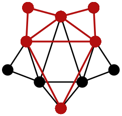

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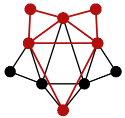

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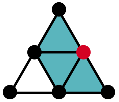
Partial advances on a minimal forbidden induced subgraph characterization of balanced graphs

Partial characterizations of balanced graphs by minimal forbidden induced subgraphs were obtained within the following classes:

- Paw-free graphs
- P_4-tidy graphs
- Line graphs
- Complements of line graphs
- Helly circular-arc graphs
- Gem-free circular-arc graphs
- Claw-free circular-arc graphs

Coordinated graphs

- A Helly K-complete of a graph G is a collection of cliques of G with common intersection. The Helly K-clique number $M(G)$ of G is the size of a maximum Helly K-complete.
- The K-chromatic number $F(G)$ of a graph G is the smallest number of colors that can be assigned to the cliques of G in such a way that no two cliques with non-empty intersection receive the same color; i.e., $F(G) = \chi(K(G))$.
- Clearly $F(G) \geq M(G)$, for every graph G. A graph G is coordinated when $F(H) = M(H)$ for every induced subgraph H of G.



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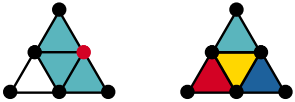

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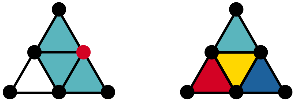

Coordinated graphs

- We proved that coordinated graphs are a subclass of perfect graphs and, as a corollary of results by Lóvasz (1972) about normal hypergraphs, it can be deduced that they are a superclass of balanced graphs.
- The complete list of minimally non-coordinated graphs is still not known.
- The coordinated graph recognition problem is NP-hard, and it is NP-complete even restricted to some subclasses of graphs with $M = 3$ (Soulignac, Sueiro, 2006).
- But in this case we do not know the computational complexity of verifying the equality of the parameters.

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Partial advances on a forbidden induced subgraph characterization of coordinated graphs

Partial characterizations of coordinated graphs were obtained within the following classes:

- Paw-free graphs
- {gem, W_4,bull}-free graphs
- Line graphs
- Complements of forests

Neighborhood-covering sets

- A neighborhood set of a graph G is a set $C \subseteq V(G)$ such that every edge and vertex of G belongs to a $G[N[v]]$, for at least one vertex $v \in C$. The neighborhood-covering number $\rho_{\mathrm{n}}(G)$ is the size of a minimum neighborhood-covering set of G.

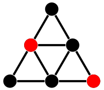

Figure: Example of a minimum neighborhood-covering set in red

Neighborhood-independent sets

- Two elements $s, t \in V(G) \cup E(G)$ are said to be neighborhood-independent if there is no vertex v such that $s, t \in G[N[v]]$. A neighborhood-independent set of a graph G is a set $I \subseteq E(G) \cup V(G)$ formed by pairwise neighborhood-independent elements. The neighborhood-independence number $\alpha_{\mathrm{n}}(G)$ is the size of a maximum neighborhood-independent set of G.

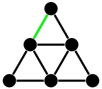

Figure: Example of a maximum neighborhood-independent set in green

Neighborhood-perfect graphs

- Neighborhood-independent elements must be covered with different vertices, so

$$\rho_{\mathrm{n}}(G) \geq \alpha_{\mathrm{n}}(G)$$

- A graph G is neighborhood-perfect when $\alpha_{\mathrm{n}}(H) = \rho_{\mathrm{n}}(H)$ for every induced subgraph H of G.
- The SPGT implies that neighborhood-perfect graphs are perfect.
- The term "neighborhood-perfect" was defined by Lehel and Tuza in 1986.

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Open problems on neighborhood-perfect graphs

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Partial advances on a forbidden induced subgraph characterization of neighborhood-perfect graphs

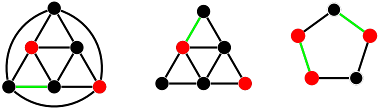
Partial characterizations of neighborhood-perfect graphs were obtained within the following classes:

- Chordal graphs (Lehel and Tuza, 1986)
- Line graphs (Lehel et al. 1994)
- Cographs (Gyárfás et al. 1996)
- P_4-tidy graphs
- tree-cographs
- Helly circular-arc graphs
- gem-free circular-arc graphs
- diamond-free graphs
- HCH claw-free graphs

Characterization of neighborhood-perfect graphs restricted to P_4-tidy graphs

Theorem

Let G be a P_4-tidy graph, then it is neighborhood-perfect if and only if it is $\{\overline{3K_2}, \text{3-sun}, C_5\}$-free.



$\overline{3K_2}$ 3-sun C_5

Figure: Forbidden graphs, with colored neighborhood-independent sets and neighborhood-covering sets.

Works of our group about these topics

- G. Durán, M.D. Safe and X. Warnes, Neighborhood covering and independence on two superclasses of cographs, working paper (2014).
- F. Bonomo, G. Durán, M.D. Safe and A.K. Wagler, Clique-perfectness and balancedness of some graph classes, International Journal of Computer Mathematics, (2014) in press.
- F. Bonomo, G. Durán, M.D. Safe and A.K. Wagler, Balancedness of some subclasses of circular-arc graphs, Discrete Mathematics and Theoretical Computer Science 16 (3) (2014), 1-22.
- F. Bonomo, G. Durán, M.D. Safe and A.K. Wagler, On minimal forbidden subgraph characterizations of balanced graphs, Discrete Applied Mathematics 161 (13-14) (2013), 1925-1942.
- F. Bonomo, G. Durán, F. Soulignac and G. Sueiro, Partial characterizations of clique-perfect and coordinated graphs: superclasses of triangle-free graphs, Discrete Applied Mathematics 157 (17) (2009), 3511-3518.
- F. Bonomo, G. Durán, F. Soulignac and G. Sueiro, Partial characterizations of coordinated graphs: line graphs and complements of forests, Mathematical Methods of Operations Research, 69(2) (2009), 251-270.

Works of our group about these topics

- F Bonomo, M. Chudnovsky and G. Durï¿½n, Partial characterizations of clique-perfect graphs II: diamond-free and Helly circular-arc graphs, Discrete Mathematics, 309(11) (2009), 3485-3499.
- F. Bonomo, M. Chudnovsky and G. Durï¿½n, Partial characterizations of clique-perfect graphs I: subclasses of claw-free graphs, Discrete Applied Mathematics 156(7) (2008), 1058-1082.
- F. Bonomo, G. Durán and M. Groshaus, Coordinated graphs and clique graphs of clique-Helly perfect graphs, Utilitas Mathematica 72 (2007), 175-191.
- F. Bonomo, G. Durán, M. Lin and J. Szwarcfiter, On Balanced Graphs, Mathematical Programming 105 (2006), 233-250.
- F. Bonomo, G. Durán, M. Groshaus and J. Szwarcfiter, On clique-perfect and K-perfect graphs, Ars Combinatoria 80 (2006), 97-112.
- G. Durán, M. Lin and J. Szwarcfiter, "On clique-transversals and clique-independent sets", Annals of Operations Research 116 (2002), 71-77.