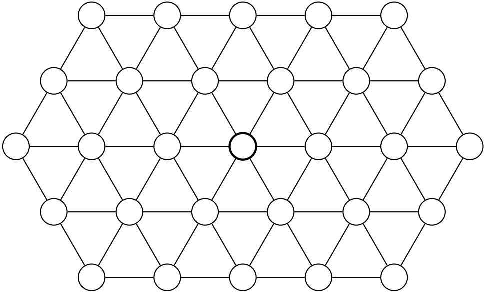
On Connected Identifying Codes for Infinite Lattices

F. Benevides[1] **V. Campos**[1] M. Dourado[2] R. Sampaio[1] A. Silva[1]

[1]ParGO, UFC, Brazil
[2]UFRJ, Brazil

Ball



Definition: $B_r(v)$

Set of vertices at distance at most r from v.

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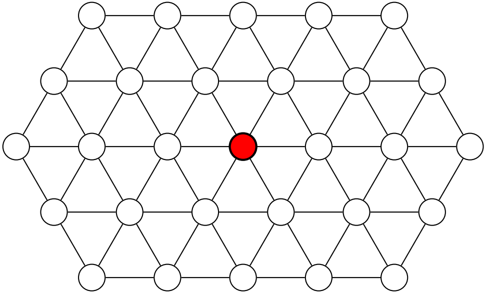

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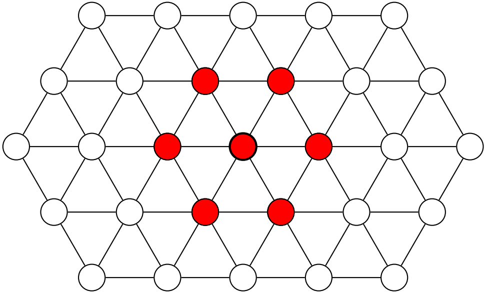

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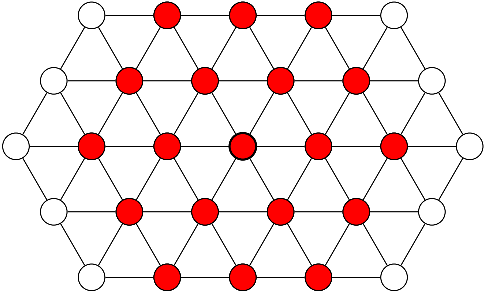

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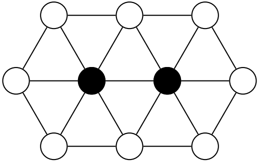

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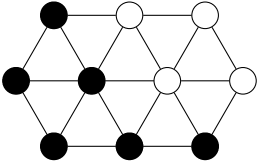

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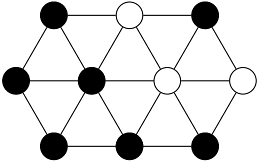

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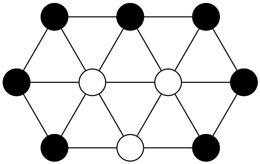

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- $id(G) = min\{D(C) | C \text{ is identifying code}\}$
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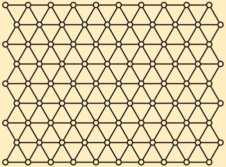
Results for Infinite Lattices

Triangular lattice L_T



- $id(L_T) = \frac{1}{4}$

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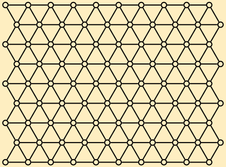

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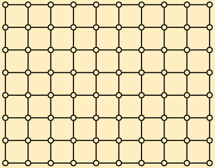
Square lattice L_S



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G. Cohen, S. Gravier, I. Honkala, A. Lobstein, M. Mollard, C. Payan and G. Zémor.
Improved identifying codes for the grid.
Electron. J. Combin. 1999.

Y. Ben-Haim and S. Litsyn.
Exact minimum density of codes identifying vertices in the square grid.
SIAM J. Discrete Math. 2005.

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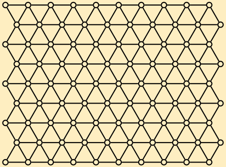

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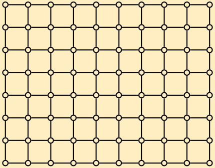

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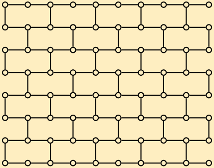
Hexagonal lattice L_H



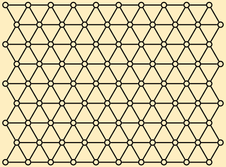
- $\frac{5}{12} \leq id(L_H) \leq \frac{3}{7}$

G. Cohen, I. Honkala, A. Lobstein and G. Zémor.
Bounds for codes identifying vertices in the hexagonal grid.
SIAM J. Discrete Math. 2000.

A. Cukierman and G. Yu.
New bounds on the minimum density of an identifying code for the infinite hexagonal grid.
Discrete Appl. Math. 2013.

Our Results

Triangular lattice L_T



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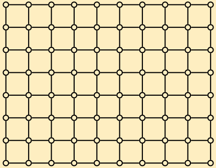
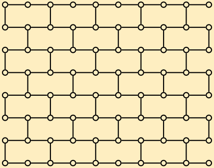


- $id(L_S) = \frac{7}{20}$
- $cid(L_H) = \frac{2}{5}$

Hexagonal lattice L_H



- $\frac{5}{12} \leq id(L_H) \leq \frac{3}{7}$
- $cid(L_H) = \frac{1}{2}$

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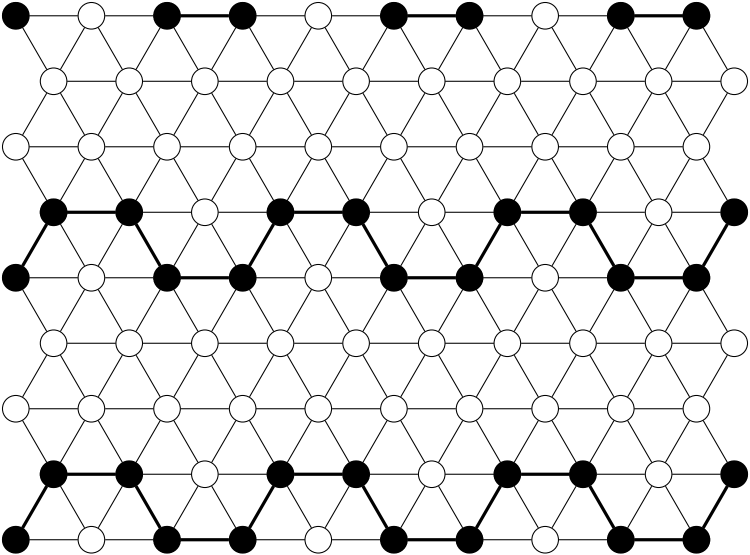
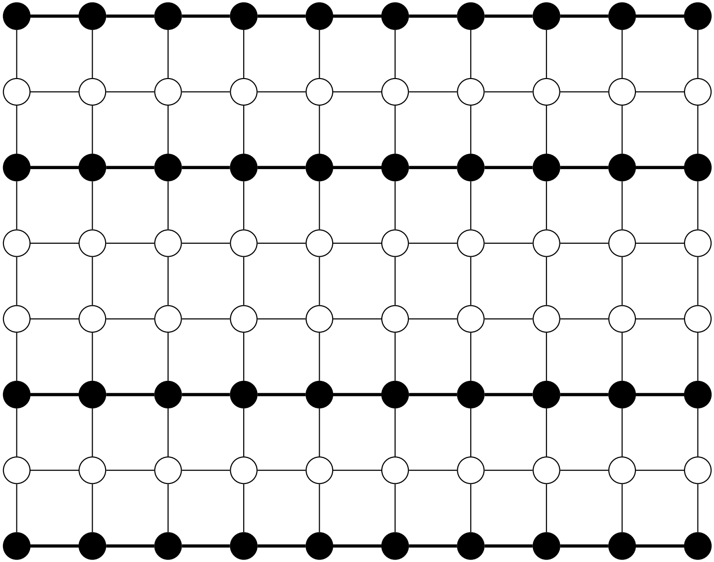
FoCM 2014.

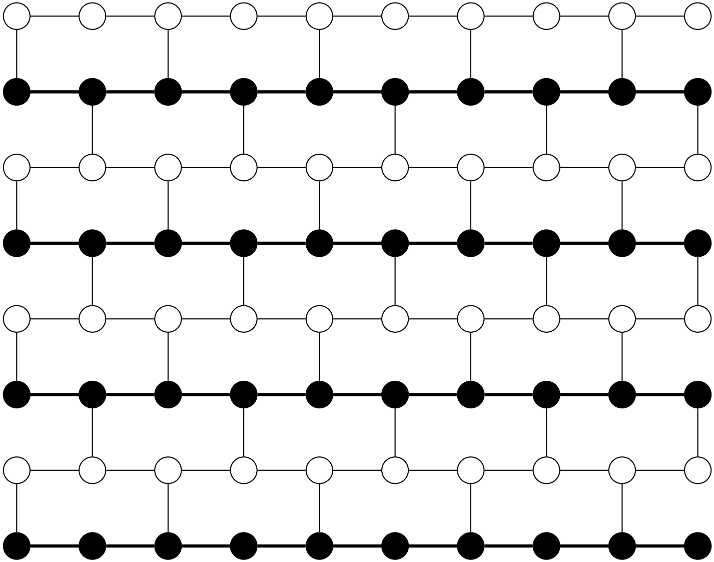
Upper bound: Triangular lattice



Upper bound: Square lattice



Upper bound: Hexagonal lattice



Lower Bounds

Theorem [KCL]

If G is finite with maximum degree Δ, then

$$id(G) \geq \frac{2}{\Delta + 2}.$$

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Theorem [BCDSS]

If G is finite with n vertices and maximum degree Δ, then

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Double counting proof

- N = number of pairs (c, v) with $c \in C$, $v \in V$ and $c \in I(v)$.

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- Reorganizing we get:

$$\frac{|C|}{n} \geq \frac{2}{\Delta + 1}(1 - \frac{1}{n})$$

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Careful application of Baranyai's Theorem.

Baranyai's Theorem (Particular case)

A complete graph on an even number of vertices can be decomposed into perfect matchings.

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Yes if Δ is even but no if Δ is odd.

Erdős-Gallai Theorem

A sequence of integers $d_1 \geq \cdots \geq d_n$ can be represented as the degree sequence of a simple graph iff $\sum d_i$ is even and

$$\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{n} \min\{d_i, k\}.$$

Tight lower bound theorem

Theorem [BCDSS]

If G is Δ-regular finite with n vertices and Δ is **even**, then

$$cid(G) \geq \frac{2}{\Delta+1}(1-\frac{1}{n}).$$

Theorem [BCDSS]

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Slight problem - Solved!

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Theorem [BCDSS]

If C is identifying code in L_S or L_H, respectively, then

$$\frac{|B_r(v) \cap C|}{|B_r(v)|} \geq \frac{2}{\Delta + 1} + o(1).$$

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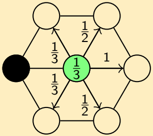
$$3|C| \geq |V|$$

Cont. Proof - Remaining charge



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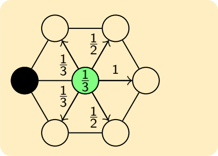
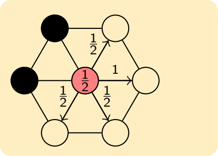
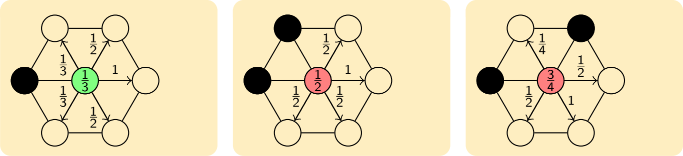




Cont. Proof - Remaining charge



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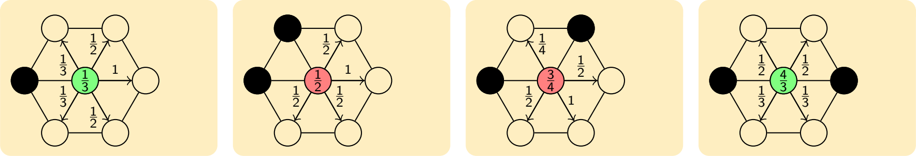

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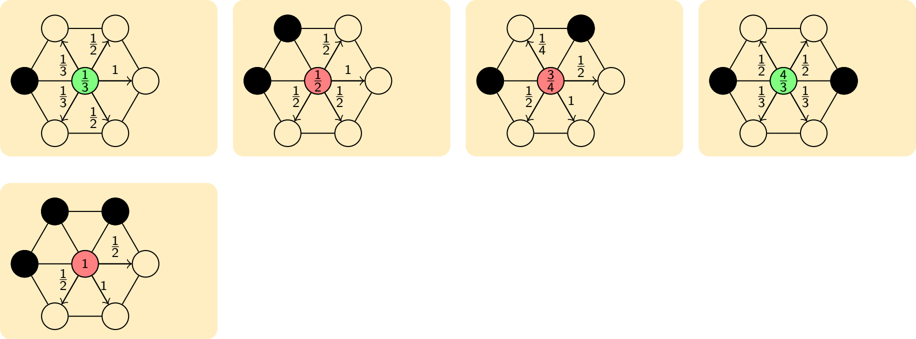

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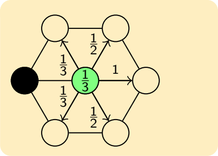

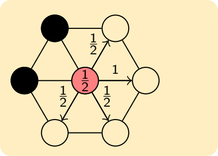

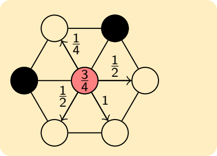

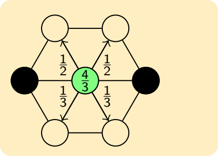

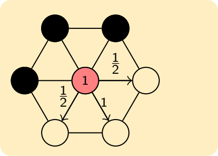

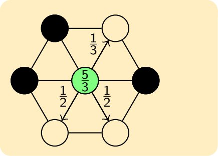

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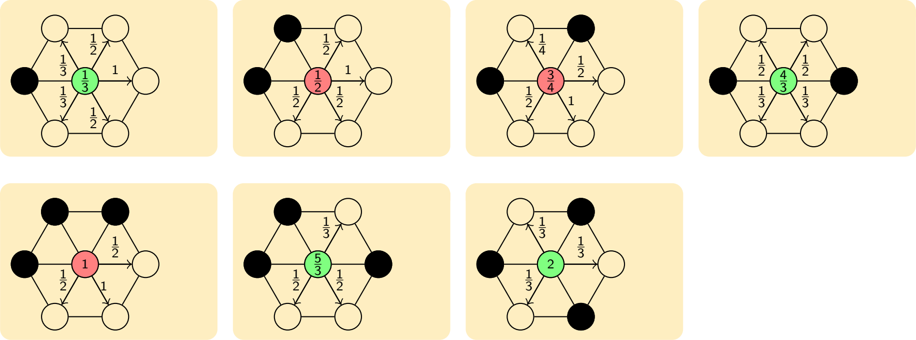

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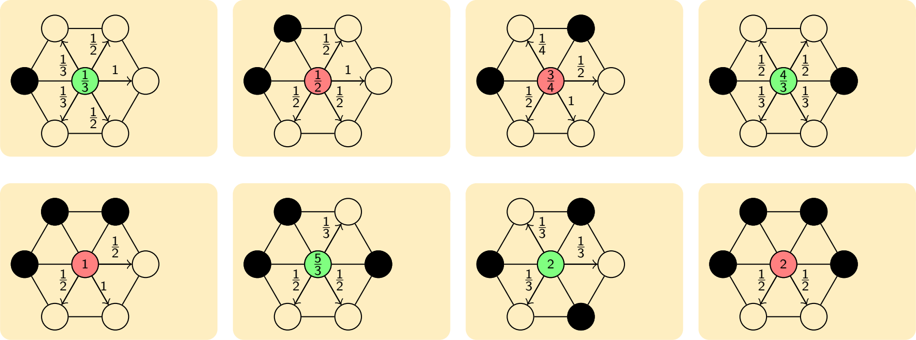

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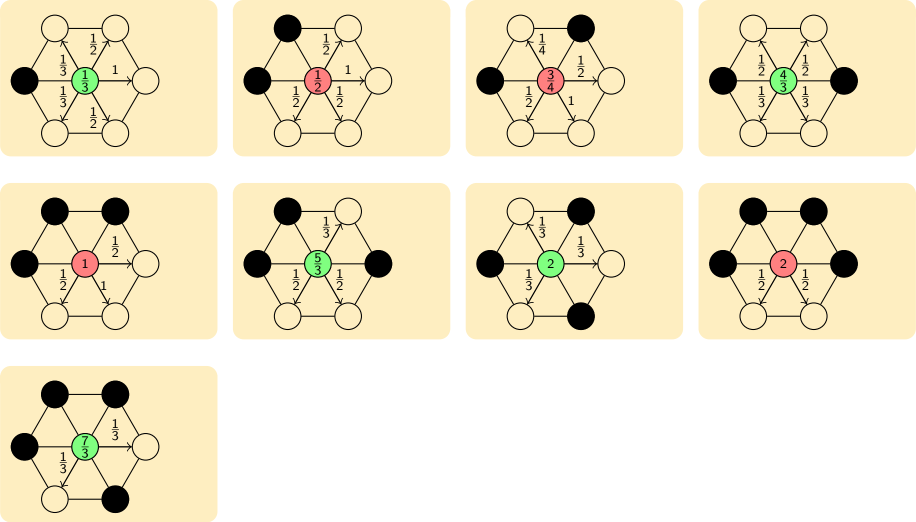

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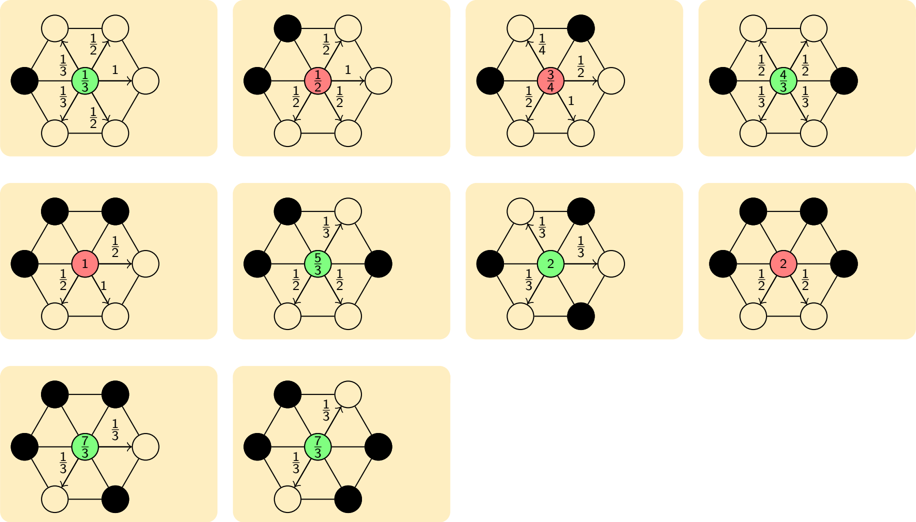

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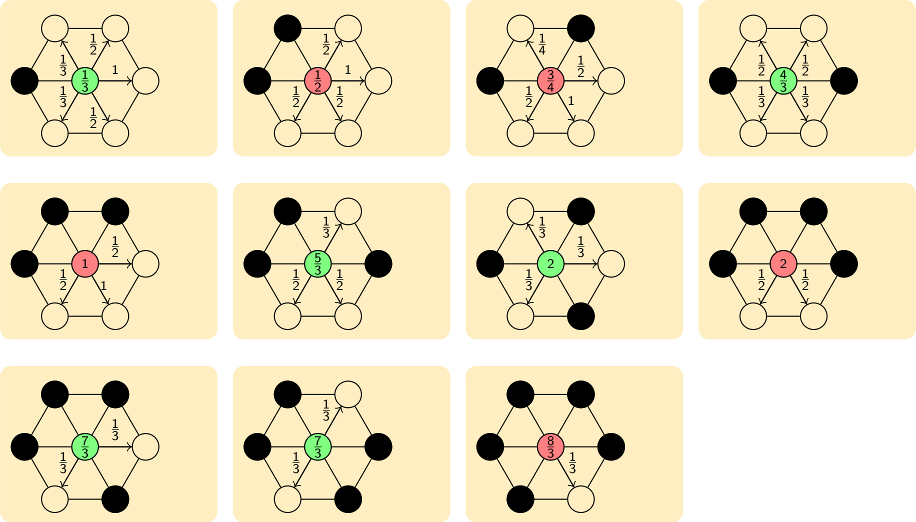

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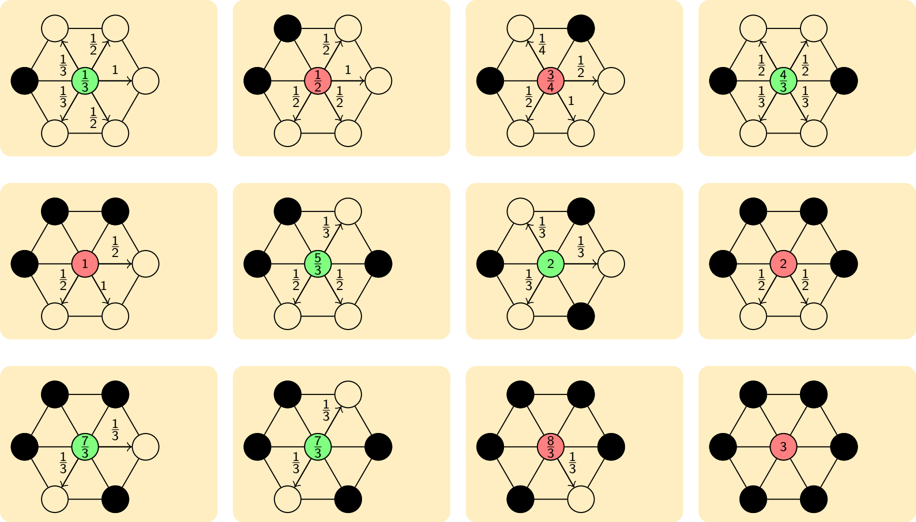

Open problem

King Grid?

The end

Thank you very much!