Counting independent sets in hypergraphs and its applications

József Balogh
U. of Illinois at U.C.

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Transference theorems

Theorem (Conlon–Gowers [2009+], Schacht [2009+])

extremal result $\mathcal{R}$

$\mathcal{R} + \quad \Rightarrow \quad$ random analogue of $\mathcal{R}$.

supersaturation

Dr D. Conlon | Sir W.T. Gowers | Dr M. Schacht
Szemerédi’s theorem

Theorem (Szemerédi [1975])

For every $k \geq 3$, the largest subset of \( \{1, \ldots, n\} \) with no $k$-term AP has $o(n)$ elements.

Endre Szemerédi
Theorem (Kohayakawa–Łuczak–Rödl [1996])

For every $\delta > 0$, there exists a $C$ such that if $p(n) \geq Cn^{-1/2}$, then a.a.s.: the $p$-random subset $[n]_p$ satisfies:

Every $A \subseteq [n]_p$ with $|A| \geq \delta |[n]_p|$ contains a 3-term AP.
Theorem (Conlon–Gowers [2009+], Schacht [2009+])

extremal result $\mathcal{R}$

$\implies$ random analogue of $\mathcal{R}$

supersaturation
Transference theorems — corollary

Theorem (Conlon–Gowers [2009+], Schacht [2009+])

extremal result $\mathcal{R}$

+ $\implies$ random analogue of $\mathcal{R}$

supersaturation

Corollary (Random analogue of Szemerédi’s theorem)

For every $k \geq 3$ and $\delta > 0$, if $p(n) \geq C(k, \delta) \cdot n^{-\frac{1}{k-1}}$, then
a.a.s. $[n]_p$ satisfies that every $A \subseteq [n]_p$ with $|A| \geq \delta|\, [n]_p|$ contains a $k$-term AP.
Transference theorems — corollary

**Theorem (Turán [1941])**

For every $k \geq 3$,

$$ex(n, K_k) = e(T_{k-1}(n)) = \left(1 - \frac{1}{k-1} + o(1)\right) \binom{n}{2}.$$
Theorem (Turán [1941])

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Motivated by: Haxell, Kohayakawa, Łuczak, Rodl.
by many others later (or earlier): Babai, Gerke, Simonovits, Spencer, Steger, Szabó, Sudakov, Vu,\ldots
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Theorem (Conlon–Gowers [2009+], Schacht [2009+])
For $p = p(n) \gg n^{-\frac{2}{k+1}}$ a.a.s.: 
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**Theorem (Conlon–Gowers [2009+], Schacht [2009+])**

For $p = p(n) \gg n^{-\frac{2}{k+1}}$ a.a.s.:

$$ex(G(n, p), K_k) = \left(1 - \frac{1}{k-1} + o(1)\right) \cdot e(G(n, p)).$$

This is usually referred to as the random analogue of Turán’s theorem.
Authors I. [at the time of the submission of the paper]

W. Samotij

R. Morris
Authors II. [at the time of the submission of the paper]
Authors I. [at the time of the acceptance of the paper]

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Balogh–Morris–Samotij, Saxton–Thomason [2012+]

Certain hypergraphs have only few independent sets.
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Corollary (Random analogue of Szemerédi’s theorem)

For every $k \geq 3$ and $\delta > 0$, if $p(n) \geq C(k, \delta) \cdot n^{-\frac{1}{k-1}}$, then a.a.s. $[n]_p$ satisfies that every $A \subseteq [n]_p$ with $|A| \geq \delta |[n]_p|$ contains a $k$-term AP.
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Corollary (Counting analogue of Szemerédi’s theorem)

For every $k \geq 3$ and $\delta > 0$, if $m \geq C(k, \delta)n^{1-\frac{1}{k-1}}$, then

$$\# \text{m-subsets of } [n] \text{ with no } k\text{-term AP } \leq \binom{\delta n}{m}.$$
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**Theorem (Erdős–Kleitman–Rothschild [1976])**

There are at most $2^{(1+o(1)) \cdot \text{ex}(n,K_k)} K_k$-free graphs on $n$ vertices.
The Cameron–Erdős problem

**Question**
How many integers from \( \{1, \ldots, n\} \) can we select without creating a solution of

\[ x + y = z \]
The Cameron–Erdős problem

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- Set of odds is sum-free.
## The Cameron–Erdős problem

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- \( \{n/2 + 1, n/2 + 2, \ldots, n\} \) is sum-free.
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The Cameron–Erdős problem

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**Cameron – Erdős Conjecture (1990)**
The number of sum-free subsets of \([n]\) is \(O(2^{n/2})\).

**Remark**
The number of sum-free subsets of \([n]\) is more than \(2 \times 2^{n/2}\).
Any subset of \( \{n/2, n/2 + 1, \ldots, n - 1\} \) is sum-free, etc...
The Cameron–Erdős problem

Cameron – Erdős Conjecture (1990)

The number of sum-free subsets of \([n]\) is \(O(2^{n/2})\).


There are constants \(c_e\) and \(c_o\) s.t. the number of sum-free subsets of \([n]\) is

\[
(1 + o(1))c_e2^{n/2}, \quad (1 + o(1))c_o2^{n/2}
\]

depending on the parity of \(n\).
The Cameron–Erdős problem

Cameron – Erdős Conjecture (1999)

There is \( c > 0 \) that the number of maximal sum-free subsets of \([n]\) is

\[ O(2^{n/2-cn}). \]

There are at least \( 2^{n/4} \) maximal sum-free subsets of \([n]\).
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Most bipartite graphs are not maximal; there are only “few” complete bipartite graphs.
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- Let \(X := \{u_1v_1, \ldots, u_{n/4}v_{n/4}\}\) be a matching;
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- Most of these graphs will be maximal triangle-free.

Balogh–Petríčková [2014+]

There are at most $2^{n^2/8 + o(n^2)}$ maximal triangle-free graphs on $n$ vertices.
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**Balogh–Petříčková [2014+]**

There are at most $2^{n^2/8+o(n^2)}$ maximal triangle-free graphs on $n$ vertices.
New applications of the “Counting Method”:


Almost every maximal triangle-free graph has the above structure.
The number of triangle-free graphs: Regularity Lemma approach

**Theorem (Erdős–Kleitman–Rothschild [1976])**

The number of triangle-free graphs is $2^{n^2/4+o(n^2)}$.

- Apply Szemerédi Regularity Lemma for a $G_n$ triangle-free graph.
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- $C_n$ contains all but $o(n^2)$ edges of $G_n$. [Approximate Container]
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- \(C_n\) is triangle-free, hence \(e(C_n) \leq n^2/4\).
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- Number of choices for $C_n$ is $O(1) \cdot n^n$.
- Number of choices for $G_n$ is

$$O(1) \cdot n^n \cdot 2^{n^2/4} \cdot \left( \binom{n^2}{o(n^2)} \right) = 2^{n^2/4+o(n^2)}.$$
The number of triangle-free graphs: ‘New approach’

Theorem (Erdős–Kleitman–Rothschild [1976])

The number of triangle-free graphs is $2^{n^2/4+o(n^2)}$. 

Szemerédi container lemma

There is a $t = 2^{o(n^2)}$ and a set \{\(G_1, \ldots, G_t\}\) of graphs, each containing at most $o(n^3)$ triangles, such that for every triangle-free graph \(H\) there is an $i \in \{1, \ldots, t\}$ such that \(H \subseteq G_i\). 

For each \(F\) triangle-free graph there is an $i$ that $F \subset G_i$. 

\(e(G_i) \leq n^2/4+o(n^2)\).
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The number of triangle-free graphs is $2^{n^2/4+o(n^2)}$.

**Balogh–Morris–Samotij, Saxton–Thomason [2012+]**

There is a $t < 2^{O(\log n \cdot n^{3/2})}$ and a set $\{G_1, \ldots, G_t\}$ of graphs, each containing at most $o(n^3)$ triangles, such that for every triangle-free graph $H$ there is an $i \in [t]$ such that $H \subseteq G_i$. 
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The number of triangle-free graphs is $2^{n^2/4+o(n^2)}$.

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There is a $t < 2^{O(\log n \cdot n^{3/2})}$ and a set $\{G_1, \ldots, G_t\}$ of graphs, each containing at most $o(n^3)$ triangles, such that for every triangle-free graph $H$ there is an $i \in [t]$ such that $H \subseteq G_i$.

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Any graph $G_n$ with at most $o(n^3)$ triangles can be made triangle-free by removing at most $o(n^2)$ edges.

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- $|V| \leq n^2/4$; Hujter–Tuza gives $\leq 2^{n^2/8}$ choices.
New applications of the “Counting Method”:

Definition

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\( G[S] \) spans many edges \( \rightarrow \) \( G \) does not have `many` independent sets.
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Ellis (2011): Largest non-trivial intersecting permutation family has size at most $(1 - \frac{1}{e} + o(1))(n - 1)!$. 

Bollobás set-pair inequality: $|\Gamma| \leq \left(\frac{2}{n}\right)^n$. 

$(\frac{n!}{(2n)^n}) \cdot 2^{(1 - \frac{1}{e} + o(1))(n - 1)!} \ll 2^{\frac{n}{2}} \cdot 2^{2n} \cdot 2^{(1 - \frac{1}{e} + o(1))(n - 1)!} \ll 2^{n \log n}$. 

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Almost every intersecting permutation family is trivially intersecting.

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Example (Erdős–Turán problem)

- $V = \{1, \ldots, n\}$,
- $\mathcal{H} = k$-term APs in $[n]$. 

Example (Turán problem)

- $V$ are edges of $K_n$,
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Example (sum-free sets)

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General framework — examples

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For every $k$, $c$, $\varepsilon$ there is a $C$ that the following holds. Let $\mathcal{H} \subseteq \binom{V}{k}$ such that for $\ell \in [k]$, $p \in [0, 1]$

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Explain: Example of triangle-free graphs.
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**Explain**: Example of triangle-free graphs.
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- independent sets
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- Computation gives: Almost all $K_{r+1}$-free graph is almost $r$-partite.
  \[ \left( \frac{n^2/2}{Cn^{2-1/r}} \right) 2^{(1-1/r-c)n^2/2} \ll 2^{(1-1/r)n^2/2}. \]