Counting independent sets in hypergraphs and its applications

József Balogh
U. of Illinois at U.C.

2014

Transference theorems

Theorem (Conlon–Gowers [2009+], Schacht [2009+])

extremal result $\mathcal{R}$
$+$
supersaturation $\Longrightarrow$ random analogue of $\mathcal{R}$.

Dr D. Conlon

Sir W.T. Gowers

Dr M. Schacht

Szemerédi's theorem

Theorem (Szemerédi [1975])

For every $k \geqslant 3$, the largest subset of $\{1, \ldots, n\}$ with no k-term AP has $o(n)$ elements.

Endre Szemerédi

Random analogue of Szemerédi's theorem

Theorem (Kohayakawa–Łuczak–Rödl [1996])

For every $\delta > 0$, there exists a C such that if $p(n) \geqslant Cn^{-1/2}$, then a.a.s.: the p-random subset $[n_p]$ satisfies:

Every $A \subseteq [n]_p$ with $|A| \geqslant \delta |[n]_p|$ contains a 3-term AP.

Y. Kohayakawa

T. Łuczak

V. Rödl

Transference theorems — corollary

Theorem (Conlon–Gowers [2009+], Schacht [2009+])

$$\begin{array}{c}\text{extremal result } \mathcal{R} \\ + \\ \text{supersaturation}\end{array} \implies \text{random analogue of } \mathcal{R}$$

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Corollary (Random analogue of Szemerédi's theorem)

For every $k \geqslant 3$ and $\delta > 0$, if $p(n) \geqslant C(k, \delta) \cdot n^{-\frac{1}{k-1}}$, then a.a.s. $[n]_p$ satisfies that every $A \subseteq [n]_p$ with $|A| \geqslant \delta |[n]_p|$ contains a k-term AP.

Transference theorems — corollary

Theorem (Turán [1941])

For every $k \geq 3$,

$$\mathrm{ex}(n, K_k) = e(T_{k-1}(n)) \;=\; \left(1 - \frac{1}{k-1} + o(1)\right)\binom{n}{2}.$$

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by many others later (or earlier): Babai, Gerke, Simonovits, Spencer, Steger, Szabó, Sudakov, Vu,. . .

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For $p = p(n) \gg n^{-\frac{2}{k+1}}$ a.a.s.:

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Theorem (Conlon–Gowers [2009+], Schacht [2009+])

For $p = p(n) \gg n^{-\frac{2}{k+1}}$ a.a.s.:

$$\mathrm{ex}\big(G(n,p), K_k\big) \;=\; \left(1 - \frac{1}{k-1} + o(1)\right) \cdot e\big(G(n,p)\big).$$

This is usually referred to as the random analogue of Turán's theorem.

Authors I. [at the time of the submission of the paper]

W. Samotij

R. Morris

Authors II. [at the time of the submission of the paper]

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Counting Independent sets in Hypergraphs

Balogh–Morris–Samotij, Saxton–Thomason [2012+]

Certain hypergraphs have only few independent sets.

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For every $k \geq 3$ and $\delta > 0$, if $m \geq C(k, \delta)n^{1-\frac{1}{k-1}}$, then

$$\#m\text{-subsets of } [n] \text{ with no } k\text{-term AP} \leq \binom{\delta n}{m}.$$

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Theorem (Erdős–Kleitman–Rothschild [1976])

There are at most $2^{(1+o(1)) \cdot \mathrm{ex}(n, K_k)}$ K_k-free graphs on n vertices.

The Cameron–Erdős problem

Question

How many integers from $\{1, \ldots, n\}$ can we select without creating a solution of

$$x + y = z?$$

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The number of sum-free subsets of $[n]$ is $O(2^{n/2})$.

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Remark

The number of sum-free subsets of $[n]$ is more than $2 \times 2^{n/2}$.
Any subset of $\{n/2, n/2 + 1, \ldots, n - 1\}$ is sum-free, etc...

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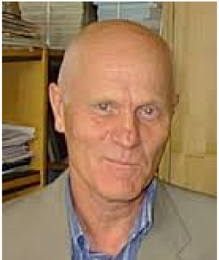
Green (2004), Sapozhenko (2003)

There are constans c_e and c_o s.t. the number of sum-free subsets of $[n]$ is

$$(1 + o(1))c_e 2^{n/2}, \quad (1 + o(1))c_o 2^{n/2}$$

depending on the parity of n.



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There is $c > 0$ that the number of **maximal** sum-free subsets of $[n]$ is

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There are at least $2^{n/4}$ **maximal** sum-free subsets of $[n]$.

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Almost all triangle-free graphs are bipartite.

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Balogh–Petříčková [2014+]

There are at most $2^{n^2/8+o(n^2)}$ maximal triangle-free graphs on n vertices.

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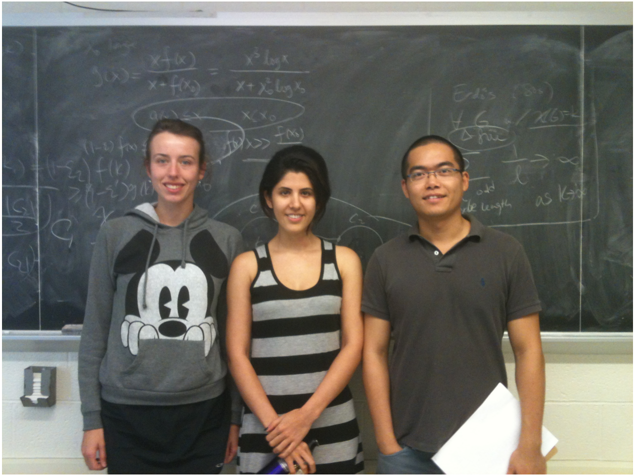
Balogh–H. Liu–Petrickova–Sharifzadeh [2014+++]

Almost every maximal triangle-free graph has the above structure.



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$$O(1) \cdot n^n \cdot 2^{n^2/4} \cdot \binom{n^2}{o(n^2)} = 2^{n^2/4+o(n^2)}.$$

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Any triangle-free graph T_N has at most $2^{N/2}$ maximal independent sets. Sharpness is by a perfect matching.

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- $|V| \ \leqslant \ n^2/4$; Hujter–Tuza gives $\leqslant \ 2^{n^2/8}$ choices.

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General framework — examples

Example (Erdős–Turán problem)

- $V = \{1, \ldots, n\}$,
- $\mathcal{H} =$ k-term APs in $[n]$.

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General framework — examples

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Example (sum-free sets)

- $V =$ an Abelian group,
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For every k, c, ε there is a C that the following holds. Let $\mathcal{H} \subseteq \binom{V}{k}$ such that for $\ell \in [k]$, $p \in [0,1]$

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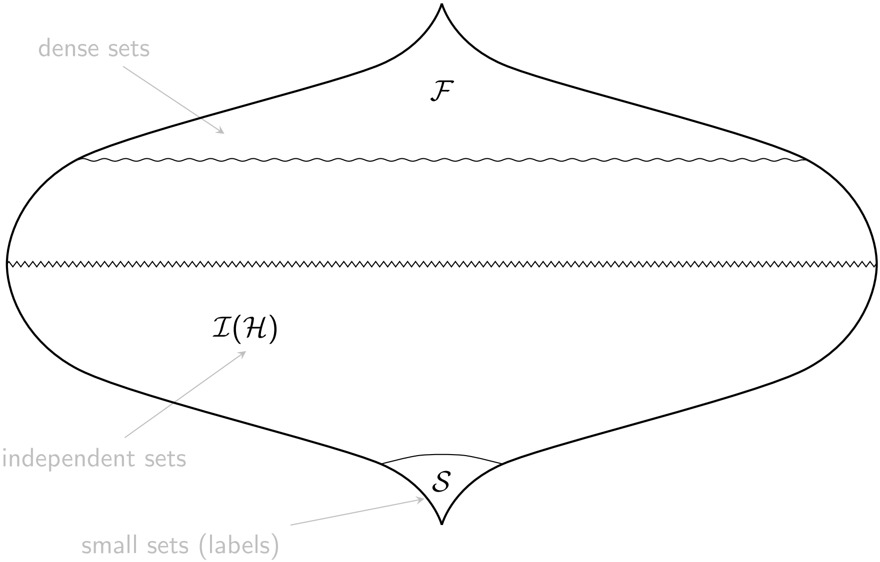
Similar result was obtained independently by Saxton and Thomason.

Explain: Example of triangle-free graphs.

Transference Theorem: — illustration



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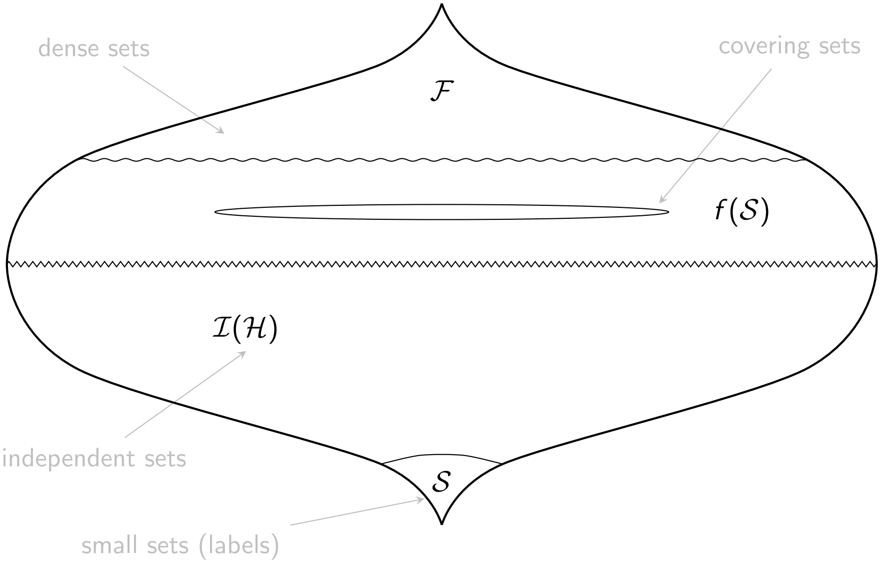

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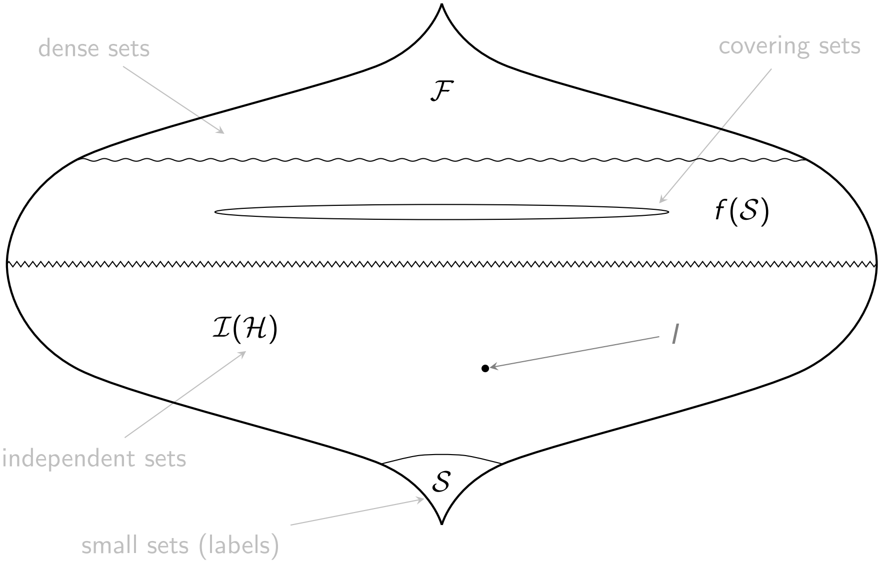

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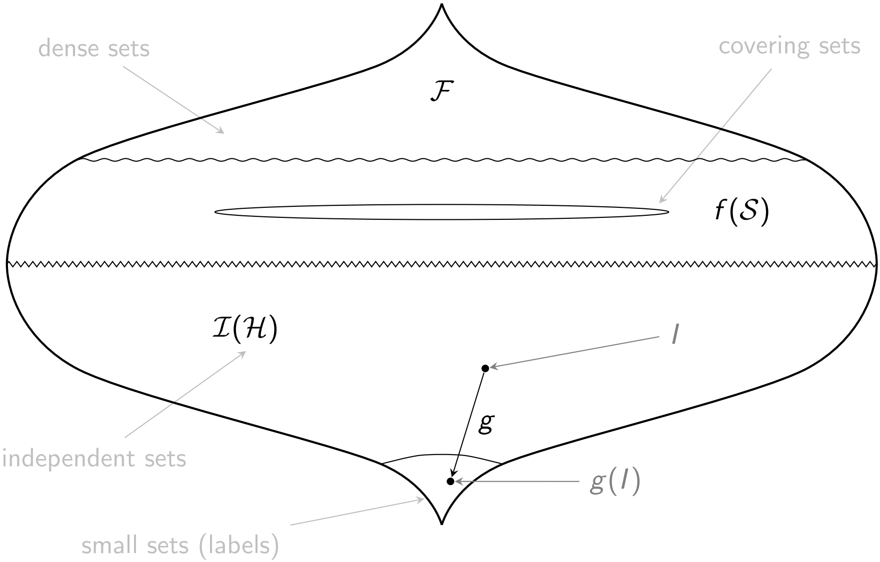

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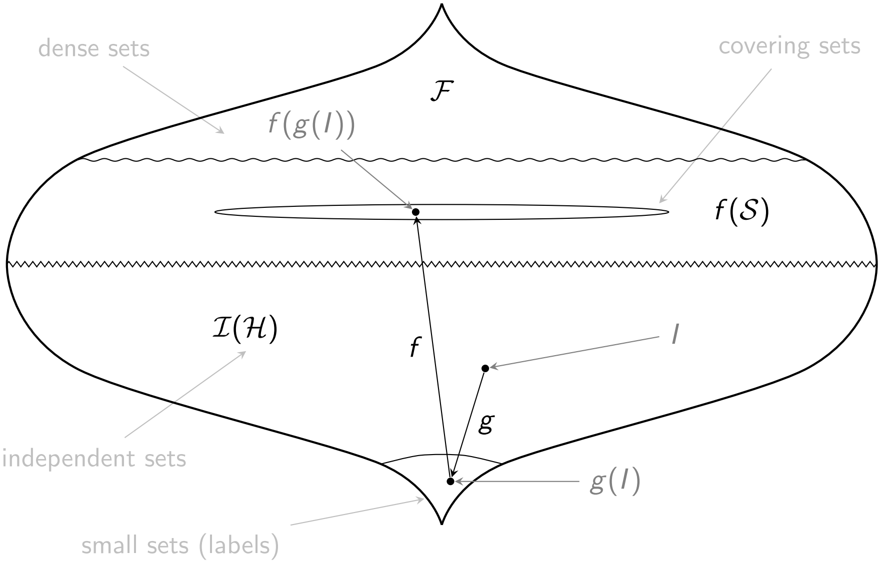

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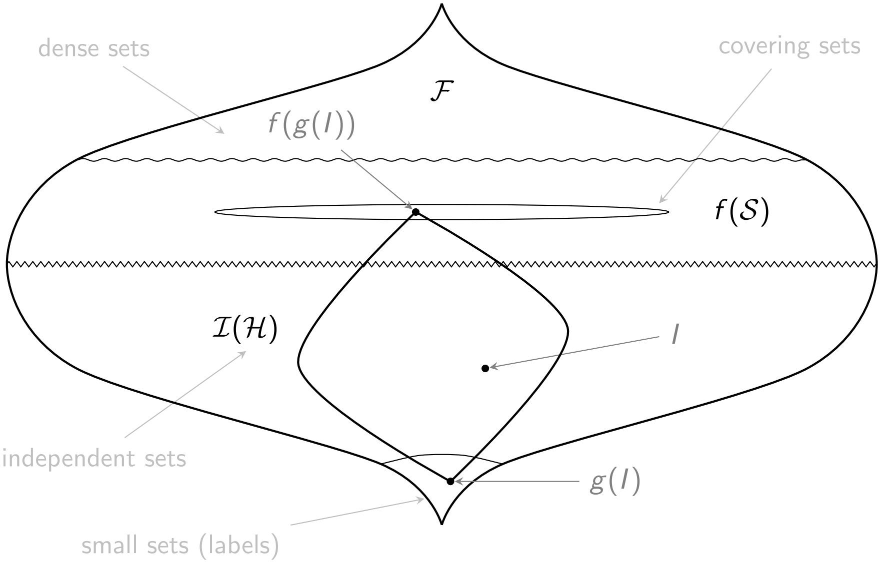

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- Computation gives: Almost all K_{r+1}-free graph is almost r-partite.

$$\binom{n^2/2}{Cn^{2-1/r}} 2^{(1-1/r-c)n^2/2} \ll 2^{(1-1/r)n^2/2}.$$