Dancing Links

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My purpose is to discuss an extremely simple technique that deserves to be better known. Suppose x points to an element of a doubly linked list; let $L[x]$ and $R[x]$ point to the predecessor and successor of that element. Then the operations

$$L\big[R[x]\big] \leftarrow L[x], \qquad R\big[L[x]\big] \leftarrow R[x] \tag{1}$$

remove x from the list; every programmer knows this. But comparatively few programmers have realized that the subsequent operations

$$L\big[R[x]\big] \leftarrow x, \qquad R\big[L[x]\big] \leftarrow x \tag{2}$$

will put x back into the list again.

This fact is, of course, obvious, once it has been pointed out. Yet I remember feeling a definite sense of "Aha!" when I first realized that (2) would work, because the values of $L[x]$ and $R[x]$ no longer have their former semantic significance after x has been removed from its list. Indeed, a tidy programmer might want to clean up the data structure by setting $L[x]$ and $R[x]$ both equal to x, or to some null value, after x has been deleted. Danger sometimes lurks when objects are allowed to point into a list from the outside; such pointers can, for example, interfere with garbage collection.

Why, therefore, am I sufficiently fond of operation (2) that I am motivated to write an entire paper about it? The element denoted by x has been deleted from its list; why would anybody want to put it back again? Well, I admit that updates to a data structure are usually intended to be permanent. But there are also many occasions when they are not. For example, an interactive program may need to revert to a former state when the user wants to undo an operation or a sequence of operations. Another typical application arises in *backtrack programs* [16], which enumerate all solutions to a given set of constraints. Backtracking, also called *depth-first search*, will be the focus of the present paper.

The idea of (2) was introduced in 1979 by Hitotumatu and Noshita [22], who showed that it makes Dijkstra's well-known program for the N queens problem [6, pages 72–82] run nearly twice as fast without making the program significantly more complicated.

Floyd's elegant discussion of the connection between backtracking and nondeterministic algorithms [11] includes a precise method for updating data structures before choosing between alternative lines of computation, and for downdating the data when it is time to explore another line. In general, the key problem of backtrack programming can be regarded as the task of deciding how to narrow the search and at the same time to organize the data that controls the decisions. Each step in the solution to a multistep problem changes the remaining problem to be solved.

In simple situations we can simply maintain a stack that contains snapshots of the relevant state information at all ancestors of the current node in the search tree. But the task of copying the entire state at each level might take too much time. Therefore we often need to work with global data structures, which are modified whenever the search enters a new level and restored when the search returns to a previous level.

For example, Dijkstra's recursive procedure for the queens problem kept the current state in three global Boolean arrays, representing the columns, the diagonals, and the reverse diagonals of a chessboard; Hitotumatu and Noshita's program kept it in a doubly linked list of available columns together with Boolean arrays for both kinds of diagonals. When Dijkstra tentatively placed a queen, he changed one entry of each Boolean array from true to false; then he made the entry true again when backtracking. Hitotumatu and Noshita used (1) to remove a column and (2) to restore it again; this meant that they could find an empty column without having to search for it. Each program strove to record the state information in such a way that the placing and subsequent unplacing of a queen would be efficient.

The beauty of (2) is that operation (1) can be undone by knowing only the value of x. General schemes for undoing assignments require us to record the identity of the left-hand side together with its previous value (see [11]; see also [25], pages 268–284). But in this case only the single quantity x is needed, and backtrack programs often know the value of x implicitly as a byproduct of their normal operation.

We can apply (1) and (2) repeatedly in complex data structures that involve large numbers of interacting doubly linked lists. The program logic that traverses those lists and decides what elements should be deleted can often be run in reverse, thereby deciding what elements should be undeleted. And undeletion restores links that allow us to continue running the program logic backwards until we're ready to go forward again.

This process causes the pointer variables inside the global data structure to execute an exquisitely choreographed dance; hence I like to call (1) and (2) the technique of *dancing links*.

The exact cover problem. One way to illustrate the power of dancing links is to consider a general problem that can be described abstractly as follows: Given a matrix of 0s and 1s, does it have a set of rows containing exactly one 1 in each column?

For example, the matrix

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{pmatrix} \tag{3}$$

has such a set (rows 1, 4, and 5). We can think of the columns as elements of a universe, and the rows as subsets of the universe; then the problem is to cover the universe with disjoint subsets. Or we can think of the rows as elements of a universe, and the columns as subsets of that universe; then the problem is to find a collection of elements that intersect each subset in exactly one point. Either way, it's a potentially tough problem, well known to be NP-complete even when each row contains exactly three 1s [13, page 221]. And it is a natural candidate for backtracking.

Dana Scott conducted one of the first experiments on backtrack programming in 1958, when he was a graduate student at Princeton University [34]. His program, written for the IAS "MANIAC" computer with the help of Hale F. Trotter, produced the first listing of all ways to place the 12 pentominoes into a chessboard leaving the center four squares vacant. For example, one of the 65 solutions is shown in Figure 1. (Pentominoes are the case $n = 5$ of n-ominoes, which are connected n-square subsets of an infinite board; see [15]. Scott was probably inspired by Golomb's paper [14] and some extensions reported by Martin Gardner [12].)

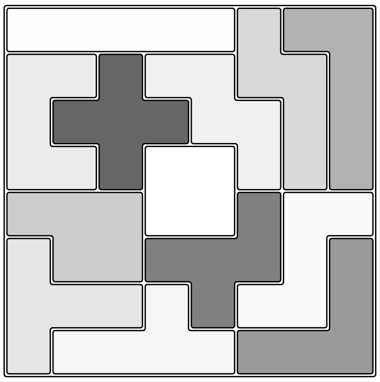


FIGURE 1. Scott's pentomino problem.

This problem is a special case of the exact cover problem. Imagine a matrix that has 72 columns, one for each of the 12 pentominoes and one for each of the 60 cells of the chessboard-minus-its-center. Construct all possible rows representing a way to place a pentomino on the board; each row contains a 1 in the column identifying the piece, and five 1s in the columns identifying its positions. (There are exactly 1568 such rows.) We can name the first twelve columns F I L P N T U V W X Y Z, following Golomb's recommended names for the pentominoes [15, page 7], and we can use two digits ij to name the column corresponding to rank i and file j of the board; each row is conveniently represented by giving the names of the columns where 1s appear. For example, Figure 1 is the exact cover corresponding to the twelve rows

I	11	12	13	14	15
N	16	26	27	37	47
L	17	18	28	38	48
U	21	22	31	41	42
X	23	32	33	34	43
W	24	25	35	36	46
P	51	52	53	62	63
F	56	64	65	66	75
Z	57	58	67	76	77
T	61	71	72	73	81
V	68	78	86	87	88
Y	74	82	83	84	85 .

Solving an exact cover problem. The following nondeterministic algorithm, which I will call algorithm X for lack of a better name, finds all solutions to the exact cover problem defined by any given matrix A of 0s and 1s. Algorithm X is simply a statement of the obvious trial-and-error approach. (Indeed, I can't think of any other reasonable way to do the job, in general.)

If A is empty, the problem is solved; terminate successfully.
Otherwise choose a column, c (deterministically).
Choose a row, r, such that $A[r, c] = 1$ (nondeterministically).
Include r in the partial solution.
For each j such that $A[r, j] = 1$,
    delete column j from matrix A;
    for each i such that $A[i, j] = 1$,
        delete row i from matrix A.
Repeat this algorithm recursively on the reduced matrix A.

The nondeterministic choice of r means that the algorithm essentially clones itself into independent subalgorithms; each subalgorithm inherits the current matrix A, but reduces it with respect to a different row r. If column c is entirely zero, there are no subalgorithms and the process terminates unsuccessfully.

The subalgorithms form a *search tree* in a natural way, with the original problem at the root and with level k containing each subalgorithm that corresponds to k chosen rows. Backtracking is the process of traversing the tree in preorder, "depth first."

Any systematic rule for choosing column c in this procedure will find all solutions, but some rules work much better than others. For example, Scott [34] said that his initial inclination was to place the first pentomino first, then the second pentomino, and so on; this would correspond to choosing column F first, then column I, etc., in the corresponding exact cover problem. But he soon realized that such an approach would be hopelessly slow: There are 192 ways to place the F, and for each of these there are approximately 34 ways to place the I. The Monte Carlo estimation procedure described in [24] suggests that the search tree for this scheme has roughly 2×10^{12} nodes! By contrast, the alternative of choosing column 11 first (the column corresponding to rank 1 and file 1 of the board), and in general choosing the lexicographically first uncovered column, leads to a search tree with 9,015,751 nodes.

Even better is the strategy that Scott finally adopted [34]: He realized that piece X has only 3 essentially different positions, namely centered at 23, 24, and 33. Furthermore, if the X is at 33, we can assume that the P pentomino is not "turned over," so that it takes only four of its eight orientations. Then we get each of the 65 essentially different solutions exactly once, and the full set of $8 \times 65 = 520$ solutions is easily obtained by rotation and reflection. These constraints on X and P lead to three independent problems, with

103,005 nodes and 19 solutions (X at 23);
106,232 nodes and 20 solutions (X at 24);
126,636 nodes and 26 solutions (X at 33, P not flipped),

when columns are chosen lexicographically.

Golomb and Baumert [16] suggested choosing, at each stage of a backtrack procedure, a subproblem that leads to the fewest branches, whenever this can be done efficiently. In the case of an exact cover problem, this means that we want to choose at each stage a column with fewest 1s in the current matrix A. Fortunately we will see that the technique of dancing links allows us to do this quite nicely; the search trees for Scott's pentomino problem then have only

$$\begin{aligned} &10{,}421 \text{ nodes} && (\text{X at } 23);\\ &12{,}900 \text{ nodes} && (\text{X at } 24);\\ &14{,}045 \text{ nodes} && (\text{X at } 33,\ \text{P not flipped}), \end{aligned}$$

respectively.

The dance steps. One good way to implement algorithm X is to represent each 1 in the matrix A as a *data object* x with five fields $L[x], R[x], U[x], D[x], C[x]$. Rows of the matrix are doubly linked as circular lists via the L and R fields ("left" and "right"); columns are doubly linked as circular lists via the U and D fields ("up" and "down"). Each column list also includes a special data object called its *list header*.

The list headers are part of a larger object called a *column object*. Each column object y contains the fields $L[y], R[y], U[y], D[y]$, and $C[y]$ of a data object and two additional fields, $S[y]$ ("size") and $N[y]$ ("name"); the size is the number of 1s in the column, and the name is a symbolic identifier for printing the answers. The C field of each object points to the column object at the head of the relevant column.

The L and R fields of the list headers link together all columns that still need to be covered. This circular list also includes a special column object called the *root*, h, which serves as a master header for all the active headers. The fields $U[h], D[h], C[h], S[h]$, and $N[h]$ are not used.

For example, the 0-1 matrix of (3) would be represented by the objects shown in Figure 2, if we name the columns A, B, C, D, E, F, and G. (This diagram "wraps around" toroidally at the top, bottom, left, and right. The C links are not shown because they would clutter up the picture; each C field points to the topmost element in its column.)

Our nondeterministic algorithm to find all exact covers can now be cast in the following explicit, deterministic form as a recursive procedure *search*(k), which is invoked initially with $k = 0$:

> If $R[h] = h$, print the current solution (see below) and return.
> Otherwise choose a column object c (see below).
> Cover column c (see below).
> For each $r \leftarrow D[c]$, $D\bigl[D[c]\bigr]$, $\ldots$, while $r \neq c$,
>   set $O_k \leftarrow r$;
>   for each $j \leftarrow R[r]$, $R\bigl[R[r]\bigr]$, $\ldots$, while $j \neq r$,
>   cover column $C[j]$ (see below);
>   *search*$(k+1)$;
>   set $r \leftarrow O_k$ and $c \leftarrow C[r]$;
>   for each $j \leftarrow L[r]$, $L\bigl[L[r]\bigr]$, $\ldots$, while $j \neq r$,
>   uncover column $C[j]$ (see below).
> Uncover column c (see below) and return.

The operation of printing the current solution is easy: We successively print the rows containing O_0, O_1, $\ldots$, O_{k-1}, where the row containing data object O is printed by printing $N\bigl[C[O]\bigr]$, $N\bigl[C[R[O]]\bigr]$, $N\bigl[C[R[R[O]]]\bigr]$, etc.

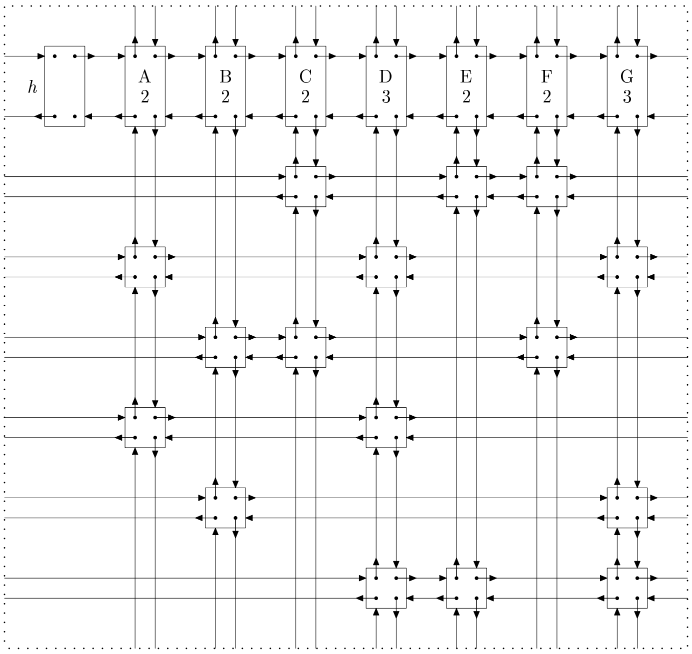


FIGURE 2. Four-way-linked representation of the exact cover problem (3).

To choose a column object c, we could simply set $c \leftarrow R[h]$; this is the leftmost uncovered column. Or if we want to minimize the branching factor, we could set $s \leftarrow \infty$ and then

$$\begin{gathered}\text{for each } j \leftarrow R[h],\ R\bigl[R[h]\bigr],\ \ldots,\ \text{while } j \neq h,\\ \text{if } S[j] < s \quad \text{set } c \leftarrow j \text{ and } s \leftarrow S[j].\end{gathered}$$

Then c is a column with the smallest number of 1s. (The S fields are not needed unless we want to minimize branching in this way.)

The operation of covering column c is more interesting: It removes c from the header list and removes all rows in c's own list from the other column lists they are in.

Set $L\bigl[R[c]\bigr] \leftarrow L[c]$ and $R\bigl[L[c]\bigr] \leftarrow R[c]$.
For each $i \leftarrow D[c],\ D\bigl[D[c]\bigr],\ \ldots,$ while $i \neq c$,
  for each $j \leftarrow R[i],\ R\bigl[R[i]\bigr],\ \ldots,$ while $j \neq i$,
    set $U\bigl[D[j]\bigr] \leftarrow U[j],\ \ D\bigl[U[j]\bigr] \leftarrow D[j]$,
    and set $S\bigl[C[j]\bigr] \leftarrow S\bigl[C[j]\bigr] - 1$.

Operation (1), which I mentioned at the outset of this paper, is used here to remove objects in both the horizontal and vertical directions.

Finally, we get to the point of this whole algorithm, the operation of *uncovering* a given column c. Here is where the links do their dance:

For each $i = U[c],\ U\bigl[U[c]\bigr],\ \ldots,$ while $i \neq c$,
  for each $j \leftarrow L[i],\ L\bigl[L[i]\bigr],\ \ldots,$ while $j \neq i$,
    set $S\bigl[C[j]\bigr] \leftarrow S\bigl[C[j]\bigr] + 1$,
    and set $U\bigl[D[j]\bigr] \leftarrow j,\ \ D\bigl[U[j]\bigr] \leftarrow j$.
Set $L\bigl[R[c]\bigr] \leftarrow c$ and $R\bigl[L[c]\bigr] \leftarrow c$.

Notice that uncovering takes place in precisely the reverse order of the covering operation, using the fact that (2) undoes (1). (Actually we need not adhere so strictly to the principle of "last done, first undone" in this case, since j could run through row i in any order. But we must be careful to unremove the rows from bottom to top, because we removed them from top to bottom. Similarly, it is important to uncover the columns of row r from right to left, because we covered them from left to right.)

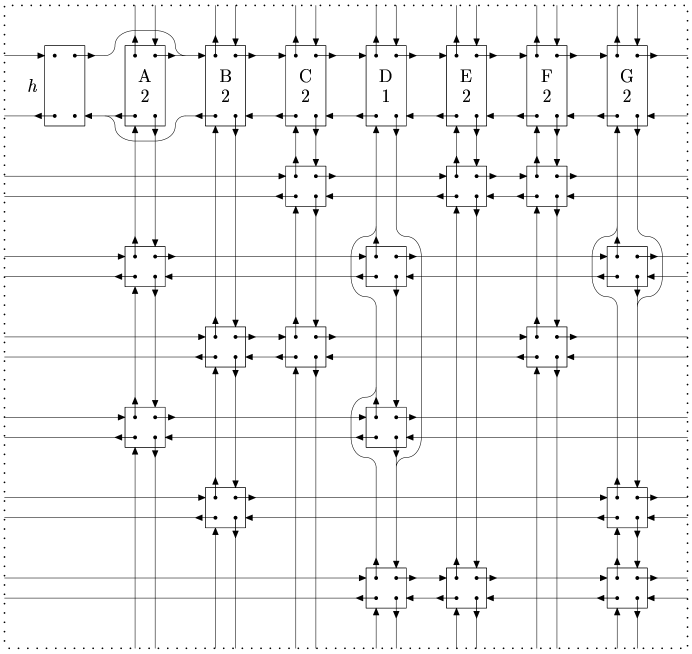


FIGURE 3. The links after column A in Figure 2 has been covered.

Consider, for example, what happens when *search*(0) is applied to the data of (3) as represented by Figure 2. Column A is covered by removing both of its rows from their other columns; the structure now takes the form of Figure 3. Notice the asymmetry of the links that now appear in column D: The upper element was deleted first, so it still points to its original neighbors, but the other deleted element points upward to the column header.

Continuing *search*(0), when r points to the A element of row $(\mathrm{A}, \mathrm{D}, \mathrm{G})$, we also cover columns D and G. Figure 4 shows the status as we enter *search*(1); this data structure represents the reduced matrix

$$\begin{array}{cccc} \mathrm{B} & \mathrm{C} & \mathrm{E} & \mathrm{F} \end{array}$$
$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{pmatrix}. \tag{4}$$

Now *search*(1) will cover column B, and there will be no 1s left in column E. So *search*(2) will find nothing. Then *search*(1) will return, having found no solutions, and the state of Figure 4 will be restored. The outer level routine, *search*(0), will proceed to convert Figure 4 back to Figure 3, and it will advance r to the A element of row (A, D).

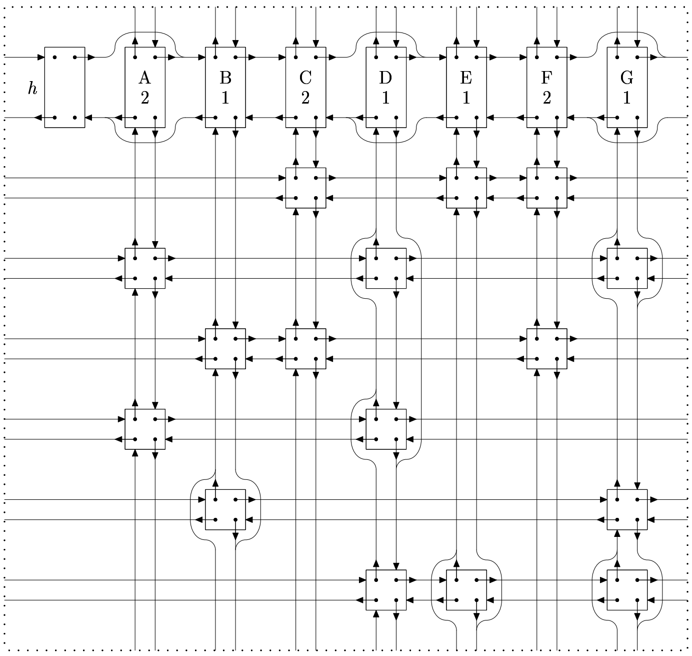


FIGURE 4. The links after columns D and G in Figure 3 have been covered.

Soon the solution will be found. It will be printed as

A D
B G
C E F

if the S fields are ignored in the choice of c, or as

A D
E F C
B G

if the shortest column is chosen at each step. (The first item printed in each row list is the name of the column on which branching was done.) Readers who play through the action of this algorithm on some examples will understand why I chose the title of this paper.

Efficiency considerations. When algorithm X is implemented in terms of dancing links, let's call it algorithm DLX. The running time of algorithm DLX is essentially proportional to the number of times it applies operation (1) to remove an object from a list; this is also the number of times it applies operation (2) to unremove an object. Let's say that this quantity is the number of *updates*. A total of 28 updates are performed during the solution of (3) if we repeatedly choose the shortest column: 10 updates are made on level 0, 14 on level 1, and 4 on level 2. Alternatively, if we ignore the S heuristic, the algorithm makes 16 updates on level 1 and 7 updates on level 2, for a total of 33. But in the latter case each update will go noticeably faster, since the statements $S\bigl[C[j]\bigr] \leftarrow S\bigl[C[j]\bigr] \pm 1$ can be omitted; hence the overall running time will probably be less. Of course we need to

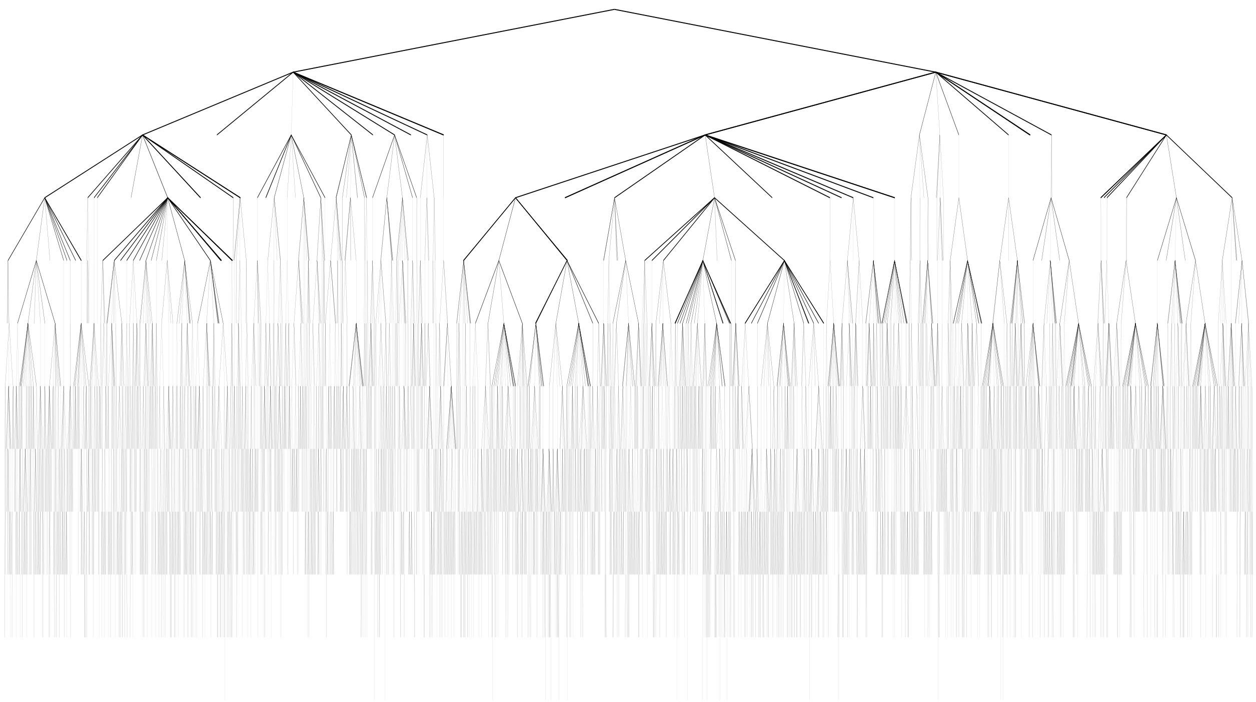


FIGURE 5. The search tree for one case of Scott's pentomino problem.

study larger examples before drawing any general conclusions about the desirability of the S heuristic.

A backtrack program usually spends most of its time on only a few levels of the search tree (see [24]). For example, Figure 5 shows the search tree for the case X = 23 of Dana Scott's pentomino problem using the S heuristic; it has the following profile:

Level	Nodes	Updates	Updates per node
0	1 (0%)	2,031 (0%)	2031.0
1	2 (0%)	1,676 (0%)	838.0
2	22 (0%)	28,492 (1%)	1295.1
3	77 (1%)	77,687 (2%)	1008.9
4	219 (2%)	152,957 (4%)	698.4
5	518 (5%)	367,939 (10%)	710.3
6	1,395 (13%)	853,788 (24%)	612.0
7	2,483 (24%)	941,265 (26%)	379.1
8	2,574 (25%)	740,523 (20%)	287.7
9	2,475 (24%)	418,334 (12%)	169.0
10	636 (6%)	32,205 (1%)	50.6
11	19 (0%)	826 (0%)	43.5
Total	10,421 (100%)	3,617,723 (100%)	347.2

(The number of updates shown for level k is the number of times an element was removed from a doubly linked list during the calculations between levels $k-1$ and k. The 2,031 updates on level 0 correspond to removing column X from the header list and then removing $2030/5 = 406$ rows from their other columns; these are the rows that overlap with the placement of X at 23. A slight optimization was made when tabulating this data: Column c was not covered and uncovered in trivial cases when it contained no rows.) Notice that more than half of the nodes lie on levels ≥ 8, but more than half of the updates occur on the way to level 7. Extra work on the lower levels has reduced the need for hard work at the higher levels.

The corresponding statistics look like this when the same problem is run *without* the ordering heuristic based on S fields:

Level	Nodes		Updates		Updates per node
0	1	(0%)	2,031	(0%)	2031.0
1	6	(0%)	5,606	(0%)	934.3
2	24	(0%)	30,111	(0%)	1254.6
3	256	(0%)	249,904	(1%)	976.2
4	581	(1%)	432,471	(2%)	744.4
5	1,533	(1%)	1,256,556	(7%)	819.7
6	3,422	(3%)	2,290,338	(13%)	669.3
7	10,381	(10%)	4,442,572	(25%)	428.0
8	26,238	(25%)	5,804,161	(33%)	221.2
9	46,609	(45%)	3,006,418	(17%)	64.5
10	13,935	(14%)	284,459	(2%)	20.4
11	19	(0%)	14,125	(0%)	743.4
Total	103,005	(100%)	17,818,752	(100%)	173.0

Each update involves about 14 memory accesses when the S heuristic is used, and about 8 accesses when S is ignored. Thus the S heuristic multiplies the total number of memory accesses by a factor of approximately $(14 \times 3{,}617{,}723)/(8 \times 17{,}818{,}752) \approx 36\%$ in this example. The heuristic is even more effective in larger problems, because it tends to reduce the total number of nodes by a factor that is exponential in the number of levels while the cost of applying it grows only linearly.

Assuming that the S heuristic is good in large trees but not so good in small ones, I tried a hybrid scheme that uses S at low levels but not at high levels. This experiment was, however, unsuccessful. If, for example, S was ignored after level 7, the statistics for levels 8–11 were as follows:

Level	Nodes	Updates
8	18,300	5,672,258
9	28,624	2,654,310
10	9,989	213,944
11	19	10,179

And if the change was applied after level 8, the statistics were

Level	Nodes	Updates
9	11,562	1,495,054
10	6,113	148,162
11	19	6,303

Therefore I decided to retain the S heuristic at all levels of algorithm DLX.

My trusty old SPARCstation 2 computer, vintage 1992, is able to perform approximately 0.39 mega-updates per second when working on large problems and maintaining the S fields. The 120 MHz Pentium I computer that Stanford computer science faculty were given in 1996 did 1.21 mega-updates per second, and my new 500 MHz Pentium III does 5.94. Thus the running time decreases as technology advances; but it remains essentially proportional to the number of updates, which is the number of times the links do their dance. Therefore I prefer to measure the performance of algorithm DLX by counting the number of updates, not by counting the number of elapsed seconds.

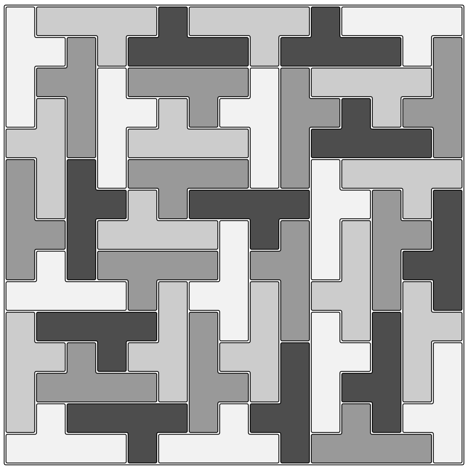
Scott [34] was pleased to discover that his program for the MANIAC solved the pentomino problem in about 3.5 hours. The MANIAC executed approximately 4000 instructions per second, so this represented roughly 50 million instructions. He and H. F. Trotter found a nice way to use the "bitwise-and" instructions of the MANIAC, which had 40-bit registers. Their code, which executed about $50{,}000{,}000/(103{,}005+106{,}232+154{,}921) \approx 140$ instructions per node of the search tree, was quite efficient in spite of the fact that they had to deal with about ten times as many nodes as would be produced by the ordering heuristic. Indeed, the linked-list approach of algorithm DLX performs a total of $3{,}617{,}723 + 4{,}547{,}186 + 5{,}526{,}988 = 13{,}691{,}897$ updates, or about 192 million memory accesses; and it would never fit in the 5120-byte memory of the MANIAC! From this standpoint the technique of dancing links is actually a step backward from Scott's 40-year-old method, although of course that method works only for very special types of exact cover problems in which simple geometric structure can be exploited.

The task of finding all ways to pack the set of pentominoes into a 6×10 rectangle is more difficult than Scott's $8 \times 8 - 2 \times 2$ problem, because the backtrack tree for the 6×10 problem is larger and there are 2339 essentially different solutions [21]. In this case we limit the X pentomino to the upper left quarter of the board; our linked-memory algorithm generates 902,631 nodes and 309,134,131 updates (or 28,320,810 nodes and 4,107,105,935 updates without the S heuristic). This solves the problem in less than a minute on a Pentium III; however, again I should point out that the special characteristics of pentominoes allow a faster approach.

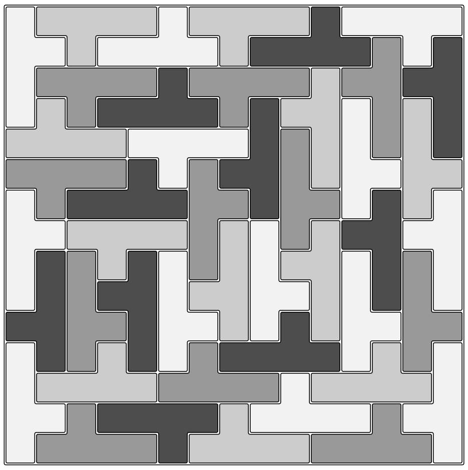
John G. Fletcher needed only ten minutes to solve the 6×10 problem on an IBM 7094 in 1965, using a highly optimized program that had 765 instructions in its inner loop [10]. The 7094 had a clock rate of 0.7 MHz, and it could access two 36-bit words in a single clock cycle. Fletcher's program required only about $600 \times 700{,}000/28{,}320{,}810 \approx 15$ clock cycles per node of the search tree; so it was superior to the bitwise method of Scott and Trotter, and it remains the fastest algorithm known for problems that involve placing the twelve pentominoes. (N. G. de Bruijn discovered an almost identical method independently; see [7].)

With a few extensions to the 0-1 matrix for Dana Scott's problem, we can solve the more general problem of covering a chessboard with twelve pentominoes and one square tetromino, without insisting that the tetromino occupy the center. This is essentially the classic problem of Dudeney, who invented pentominoes in 1907 [9]. The total number of such chessboard dissections has apparently never appeared in the literature; algorithm DLX needs 1,526,279,783 updates to determine that it is exactly 16,146.

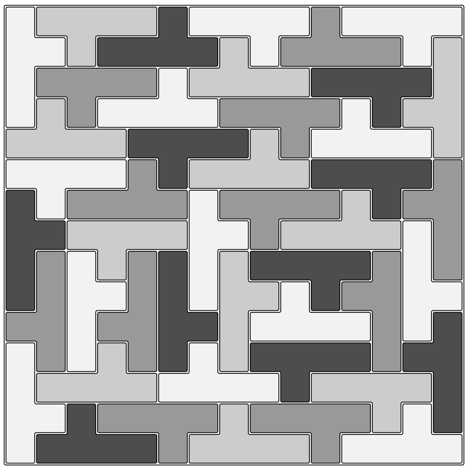
Many people have written about polyomino problems, including distinguished mathematicians such as Golomb [15], de Bruijn [7, 8], Berlekamp, Conway and Guy [4]. Their



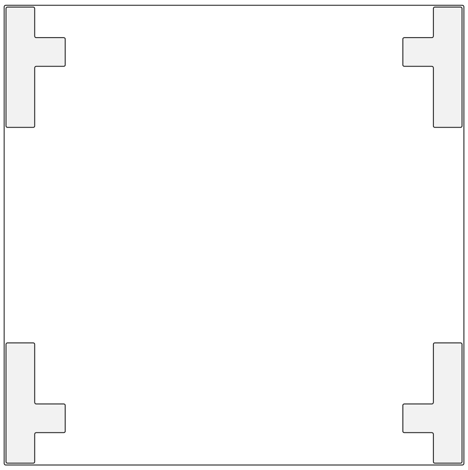
92 solutions, 14,352,556 nodes, 1,764,631,796 updates



100 solutions, 10,258,180 nodes, 1,318,478,396 updates



20 solutions, 6,375,335 nodes, 806,699,079 updates



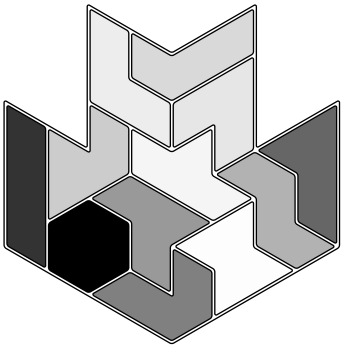
0 solutions, 1,234,485 nodes, 162,017,125 updates

FIGURE 6. Packing 45 Y pentominoes into a square.

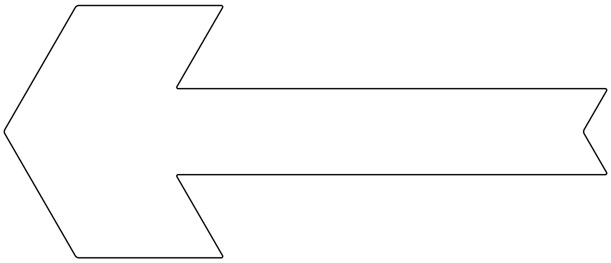
arguments for placing the pieces are sometimes based on enumerating the number of ways a certain cell on the board can be filled, sometimes on the number of ways a certain piece can be placed. But as far as I know, nobody has previously pointed out that such problems are actually exact cover problems, in which there is perfect symmetry between cells and pieces. Algorithm DLX will branch on the ways to fill a cell if some cell is difficult to fill, or on the ways to place a piece if some piece is difficult to place. It knows no difference, because pieces and cells are simply columns of the given input matrix.

Algorithm DLX begins to outperform other pentomino-placing procedures in problems where the search tree has many levels. For example, let's consider the problem of packing 45 Y pentominoes into a 15×15 square. Jenifer Haselgrove studied this with the help of a machine called the ICS Multum—which qualified as a "fast minicomputer" in 1973 [20]. The Multum produced an answer after more than an hour, but she remained uncertain whether other solutions were possible. Now, with the dancing links approach described above, we can obtain several solutions almost instantly, and the total number of solutions turns out to be 212. The solutions fall into four classes, depending on the behavior at the four corners; representatives of each achievable class are shown in Figure 6.

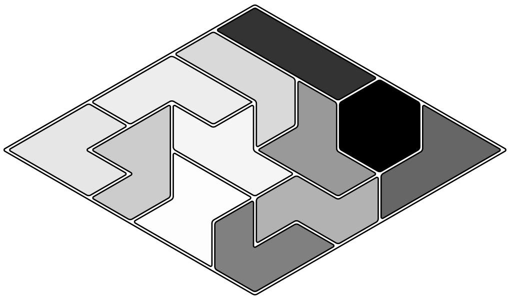
Applications to hexiamonds. In the late 1950s, T. H. O'Beirne introduced a pleasant variation on polyominoes by substituting triangles for squares. He named the resulting shapes *polyiamonds*: moniamonds, diamonds, triamonds, tetriamonds, pentiamonds, hexiamonds, etc. The twelve hexiamonds were independently discovered by J. E. Reeve and J. A. Tyrell [32], who found more than forty ways to arrange them into a 6×6 rhombus. Figure 7 shows one such arrangement, together with some arrow dissections that I couldn't resist trying when I first learned about hexiamonds. The 6×6 rhombus can be tiled by the twelve hexiamonds in exactly 156 ways. (This fact was first proved by P. J. Torbijn [35], who worked without a computer; algorithm DLX confirms his result after making 37,313,405 updates, if we restrict the "sphinx" to only 3 of its 12 orientations.)



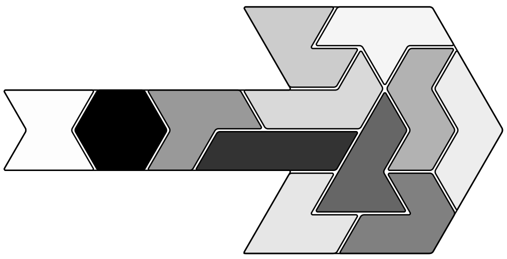
4 solutions, 6,677 nodes, 4,687,159 updates



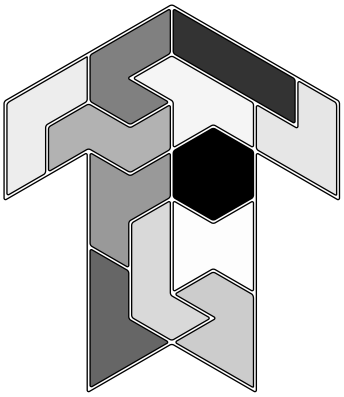
0 solutions, 7,603 nodes, 3,115,387 updates



156 solutions, 70,505 nodes, 37,313,405 updates



41 solutions, 35,332 nodes, 14,948,759 updates



3 solutions, 5546 nodes, 3,604,817 updates

FIGURE 7. The twelve hexiamonds, packed into a rhombus and into various arrowlike shapes.

O'Beirne was particularly fascinated by the fact that seven of the twelve hexiamonds have different shapes when they are flipped over, and that the resulting 19 *one-sided hexiamonds* have the correct number of triangles to form a hexagon: a hexagon of hexiamonds (see Figure 8). In November of 1959, after three months of trials, he found a solution; and two years later he challenged the readers of *New Scientist* to match this feat [28, 29, 30].

Meanwhile he had shown the puzzle to Richard Guy and his family. The Guys published several solutions in a journal published in Singapore, where Richard was a professor [17]. Guy, who has told the story of this fascinating recreation in [18], says that when O'Beirne first described the puzzle, "Everyone wanted to try it at once. No one went to bed for about 48 hours."

A 19-level backtrack tree with many possibilities at each level makes an excellent test case for the dancing links approach to covering, so I fed O'Beirne's problem to my program. I broke the general case into seven subcases, depending on the distance of the hexagon piece from the center; furthermore, when that distance was zero, I considered two subcases depending on the position of the "crown." Figure 8 shows a representative of each of the seven cases, together with statistics about the search. The total number of updates performed was 134,425,768,494.

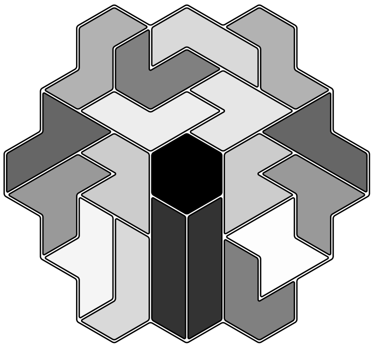
My goal was not only to count the solutions, but also to find arrangements that were as symmetrical as possible—in response to a problem that was stated in Berlekamp, Guy, and Conway's book *Winning Ways* [4, page 788]. Let us define the *horizontal symmetry* of a configuration to be the number of edges between pieces that also are edges between pieces in the left-right reflection of that configuration. The overall hexagon has 156 internal edges, and the 19 one-sided hexiamonds have 96 internal non-edges. Therefore if an arrangement were perfectly symmetrical—unchanged by left-right reflection—its horizontal symmetry would be 60. But no such perfectly symmetric solution is possible.

The vertical symmetry of a configuration is defined similarly, but with respect to top-bottom reflection. A solution to the hexiamond problem is *maximally symmetric* if it has the highest horizontal or vertical symmetry score, and if the smaller score is as large as possible consistent with the larger score. Each of the solutions shown in Figure 8 is, in fact, maximally symmetric in its class. (And so is the solution to Dana Scott's problem that is shown in Figure 1: It has vertical symmetry 36 and horizontal symmetry 30.)

The largest possible vertical symmetry score is 50; it is achieved in Figure 8(c), and in seven other solutions obtained by independently rearranging three of its symmetrical subparts. Four of the eight have a horizontal symmetry score of 32; the others have horizontal symmetry 24. John Conway found these solutions by hand in 1964 and conjectured that they were maximally symmetric overall. But that honor belongs uniquely to the solution in Figure 8(f), at least by my definition, because Figure 8(f) has horizontal symmetry 52 and vertical symmetry 27. The only other ways to achieve horizontal symmetry 52 have vertical symmetry scores of 20, 22, and 24. (Two of those other ways do, however, have the surprising property that 13 of their 19 pieces are unchanged by horizontal reflection; this is symmetry of entire pieces, not just of edges.)

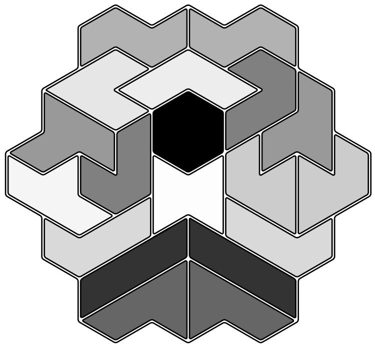
After I had done this enumeration, I read Guy's paper [18] for the first time and learned that Marc M. Paulhus had already enumerated all solutions in May 1996 [31]. Good, our independent computations would confirm the results. But no—my program found 124,519 solutions, while his had found 124,518! He reran his program in 1999 and now we agree.

(a)



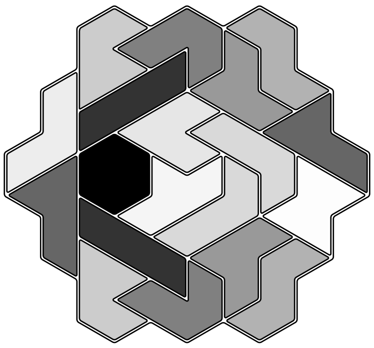
(hsym = 51, vsym = 24)
1,914 solutions, 4,239,132 nodes
2,142,276,414 updates

(b)



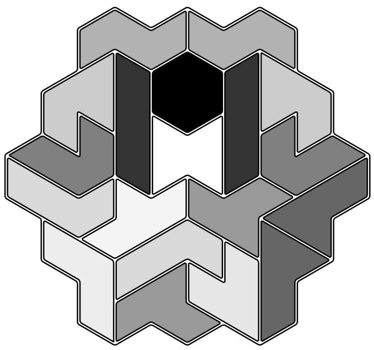
(hsym = 52, vsym = 24)
5,727 solutions, 21,583,173 nodes
11,020,236,507 updates

(c)



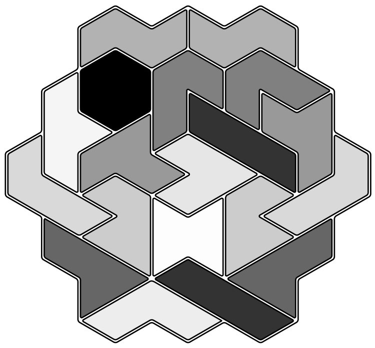
(hsym = 32, vsym = 50)
11,447 solutions, 20,737,702 nodes
10,315,775,812 updates

(d)



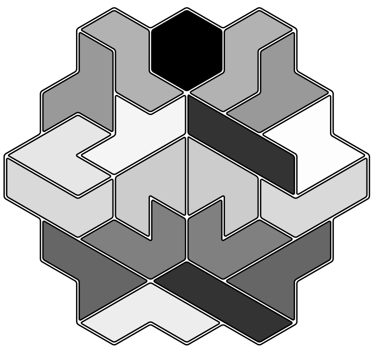
(hsym = 51, vsym = 22)
7,549 solutions, 24,597,239 nodes
12,639,698,345 updates

(e)



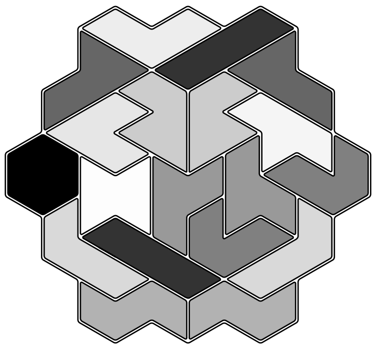
(hsym = 48, vsym = 30)
6,675 solutions, 17,277,362 nodes
8,976,245,858 updates

(f)



(hsym = 52, vsym = 27)
15,717 solutions, 43,265,607 nodes
21,607,912,011 updates

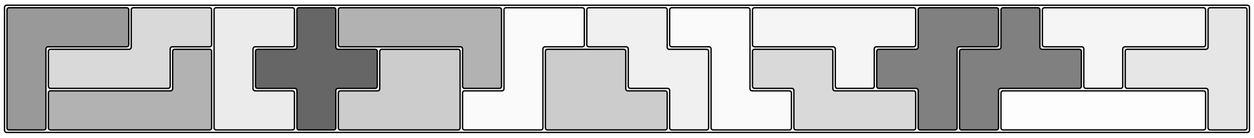
(g)



(hsym = 48, vsym = 29)
75,490 solutions, 137,594,347 nodes
67,723,623,547 updates

FIGURE 8. Solutions to O'Beirne's hexiamond hexagon problem, with the small hexagon at various distances from the center of the large one.

O'Beirne [29] also suggested an analogous problem for pentominoes, since there are 18 one-sided pentominoes. He asked if they can be put into a 9×10 rectangle, and Golomb provided an example in [15, Chapter 6]. Jenifer Leech wrote a program to prove that there are exactly 46 different ways to pack the one-sided pentominoes in a 3×30 rectangle; see [26]. Figure 9 shows a maximally symmetric example (which isn't really very symmetrical).

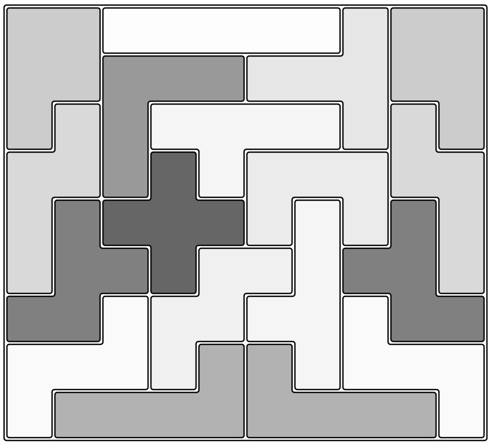


46 solutions, 605,440 nodes, 190,311,749 updates, hsym = 51, vsym = 48

FIGURE 9. The one-sided pentominoes, packed into a 3×30 rectangle.

I set out to count the solutions to the 9×10, figuring that an 18-stage exact cover problem with six 1s per row would be simpler than a 19-stage problem with seven 1s per row. But I soon found that the task would be hopeless, unless I invented a much better algorithm. The Monte Carlo estimation procedure of [24] suggests that about 19 quadrillion updates will be needed, with 64 trillion nodes in the search trees. If that estimate is correct, I could have the result in a few months; but I'd rather try for a new Mersenne prime.

I do, however, have a conjecture about the solution that will have maximum horizontal symmetry; see Figure 10.



hsym = 74, vsym = 49

FIGURE 10. Is this the most symmetrical way to pack one-sided pentominoes into a rectangle?

A failed experiment. Special arguments based on "coloring" often give important insights into tiling problems. For example, it is well known [5, pages 142 and 394] that if we remove two cells from opposite corners of a chessboard, there is no way to cover the remaining 62 cells with dominoes. The reason is that the mutilated chessboard has, say, 32 white cells and 30 black cells, but each individual domino covers one cell of each color. If we present such a covering problem to algorithm DLX, it makes 4,780,846 updates (and finds 13,922 ways to place 30 of the 31 dominoes) before concluding that there is no solution.

The cells of the hexiamond-hexagon problem can be colored black and white in a similar fashion: All triangles that point left are black, say, and all that point right are white. Then fifteen of the one-sided hexiamonds cover three triangles of each color; but the remaining four, namely the "sphinx" and the "yacht" and their mirror images, each have a four-to-two color bias. Therefore every solution to the problem must put exactly two of those four pieces into positions that favor black.

I thought I'd speed things up by dividing the problem into six subproblems, one for each way to choose the two pieces that will favor black. Each of the subproblems was expected to have about 1/6 as many solutions as the overall problem, and each subproblem was simpler because it gave four of the pieces only half as many options as before. Thus I expected the subproblems to run up to 16 times as fast as the original problem, and I expected the extra information about impossible correlations of piece placement to help algorithm DLX make intelligent choices.

But this turned out to be a case where mathematics gave me bad advice. The overall problem had 6675 solutions and required 8,976,245,858 updates (Figure 8(c)). The six subproblems turned out to have respectively 955, 1208, 1164, 1106, 1272, and 970 solutions, roughly as expected; but they each required between 1.7 and 2.2 billion updates, and the total work to solve all six subproblems was 11,519,571,784. So much for *that* bright idea.

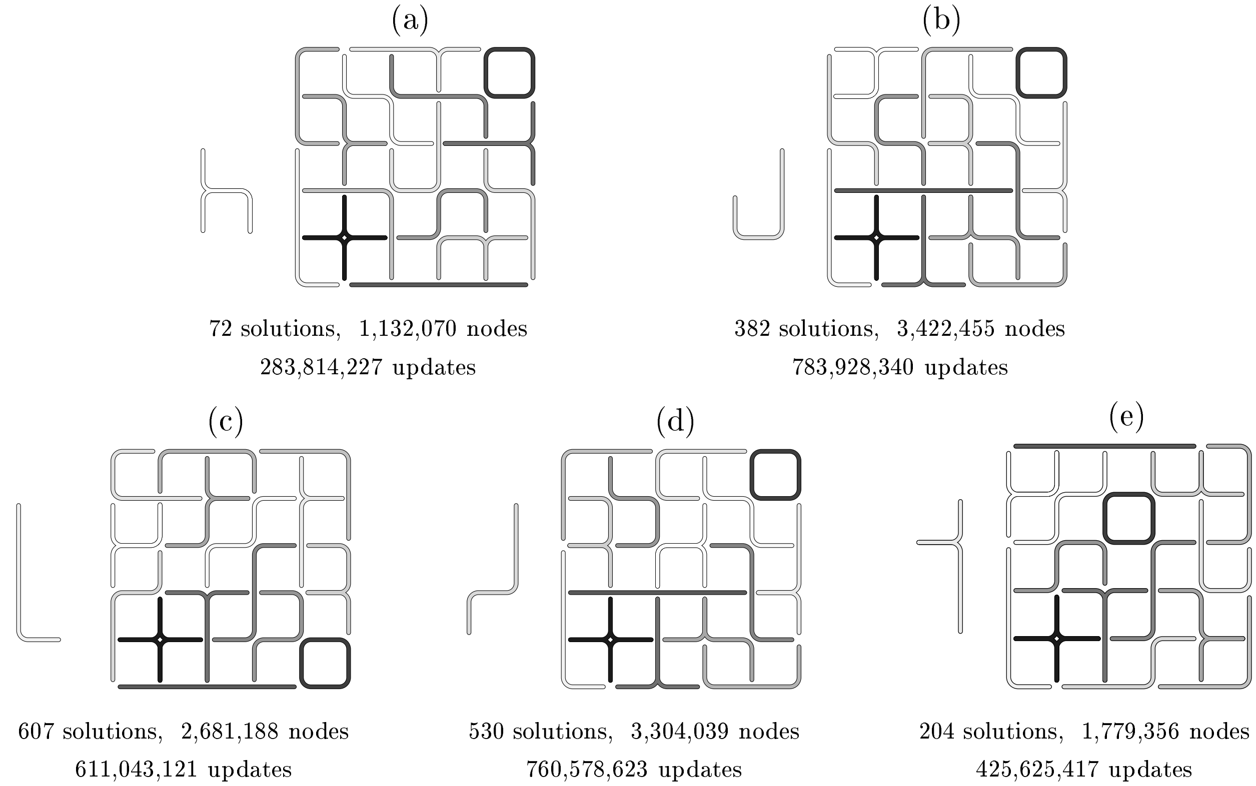
Applications to tetrasticks. Instead of making pieces by joining squares or triangles together, Brian Barwell [3] considered making them from line segments or sticks. He called the resulting objects *polysticks*, and noted that there are 2 disticks, 5 tristicks, and 16 tetrasticks. The tetrasticks are especially interesting from a recreational standpoint; I received an attractive puzzle in 1993 that was equivalent to placing ten of the tetrasticks in a 4×4 square [1], and I spent many hours trying to psych it out.

Barwell proved that the sixteen tetrasticks cannot be assembled into any symmetrical shape. But by leaving out any one of the five tetrasticks that have an excess of horizontal or vertical line segments, he found ways to fill a 5×5 square. (See Figure 11.) Such puzzles are quite difficult to do by hand, and he had found only five solutions at the time he wrote his paper; he conjectured that fewer than a hundred solutions would actually exist. (The set of all solutions was first found by Wiezorke and Haubrich [37], who invented the puzzle independently after seeing [1].)

Polysticks introduce a new feature that is not present in the polyomino and polyiamond problems: *The pieces must not cross each other.* For example, Figure 12 shows a non-solution to the problem considered in Figure 11(c). Every line segment in the grid of 5×5 squares is covered, but the 'V' tetrastick crosses the 'Z'.

We can handle this extra complication by generalizing the exact cover problem. Instead of requiring all columns of a given 0-1 matrix to be covered by disjoint rows, we will distinguish two kinds of columns: *primary* and *secondary*. The generalized problem asks for a set of rows that covers every primary column exactly once and every secondary column *at most* once.

The tetrastick problem of Figure 11(c) can be set up as a generalized cover problem in a natural way. First we introduce primary columns F, H, I, J, N, O, P, R, S, U, V,



(a) 72 solutions, 1,132,070 nodes
283,814,227 updates

(b) 382 solutions, 3,422,455 nodes
783,928,340 updates

(c) 607 solutions, 2,681,188 nodes
611,043,121 updates

(d) 530 solutions, 3,304,039 nodes
760,578,623 updates

(e) 204 solutions, 1,779,356 nodes
425,625,417 updates

FIGURE 11. Filling a 5×5 grid with 15 of the 16 tetrasticks; we must leave out either the H, the J, the L, the N, or the Y.

W, X, Y, Z representing the fifteen tetrasticks (excluding L), as well as columns Hxy representing the horizontal segments (x, y) — $(x+1, y)$ and Vxy representing the vertical segments (x, y) — $(x, y+1)$, for $0 \le x, y < 5$. We also need secondary columns Ixy to represent interior junction points (x, y), for $0 < x, y < 5$. Each row represents a possible placement of a piece, as in the polyomino and polyiamond problems; but if a piece has two consecutive horizontal or vertical segments and does not lie on the edge of the diagram, it should include the corresponding interior junction point as well.

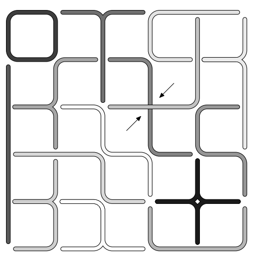


FIGURE 12. Polysticks are not supposed to cross each other as they do here.

For example, the two rows corresponding to the placement of V and Z in Figure 12 are

V	H23	I33	H33	V43	I44	V44
Z	H24	V33	I33	V32	H32	

The common interior point I33 means that these rows cross each other. On the other hand, I33 is not a primary column, because we do not necessarily need to cover it. The solution in Figure 11(c) covers only the interior points I14, I21, I32, and I41.

Fortunately, we can solve the generalized cover problem by using almost the same algorithm as before. The only difference is that we initialize the data structure by making a circular list of the column headers for the primary columns only. The header for each secondary column should have L and R fields that simply point to itself. The remainder of the algorithm proceeds exactly as before, so we will still call it algorithm DLX.

A generalized cover problem can be converted to an equivalent exact cover problem if we simply append one row for each secondary column, containing a single 1 in that column. But we are better off working with the generalized problem, because the generalized algorithm is simpler and faster.

I decided to experiment with the subset of *welded tetrasticks*, namely those that do not form a simple connected path because they contain junction points: F, H, R, T, X, Y. There are ten *one-sided* welded tetrasticks if we add the mirror images of the unsymmetrical pieces as we did for one-sided hexiamonds and pentominoes. And—aha—these ten tetrasticks can be arranged in a 4×4 grid. (See Figure 13.) Only three solutions are possible, including the two perfectly symmetric solutions shown. I've decided not to show the third solution, which has the X piece in the middle, because I want readers to have the pleasure of finding it for themselves.

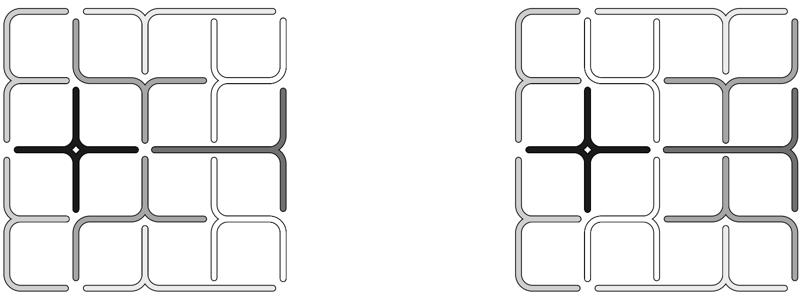


FIGURE 13. Two of the three ways to pack the one-sided welded tetrasticks into a square.

There are fifteen one-sided *unwelded tetrasticks*, and I thought they would surely fit into a 5×5 grid in a similar way; but this turned out to be impossible. The reason is that if, say, piece I is placed vertically, four of the six pieces J, J′, L, L′, N, N′ must be placed to favor the horizontal direction, and this severely limits the possibilities. In fact, I have been unable to pack those fifteen pieces into any simple symmetrical shape; my best effort so far is the "oboe" shown in Figure 14.

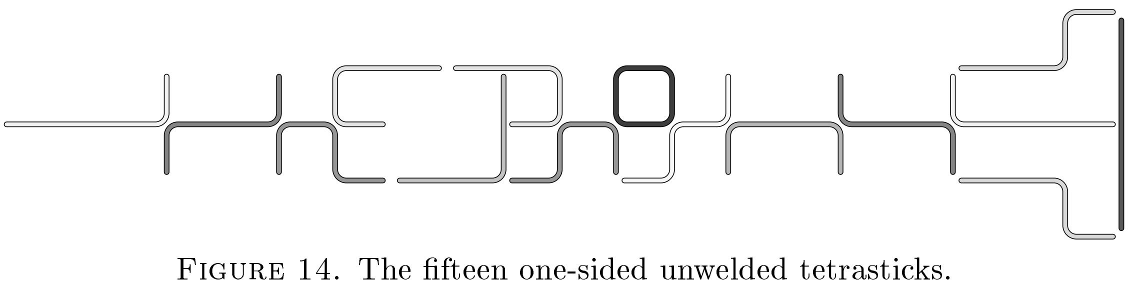


FIGURE 14. The fifteen one-sided unwelded tetrasticks.

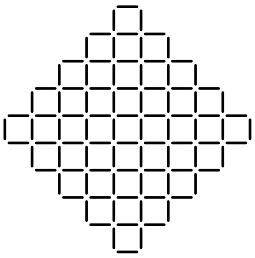


FIGURE 15. Do all 25 one-sided tetrasticks fit in this shape?

I also tried unsuccessfully to pack all 25 of the one-sided tetrasticks into the Aztec diamond pattern of Figure 15; but I see no way to prove that a solution is impossible. An exhaustive search seems out of the question at the present time.

Applications to queens. Now we can return to the problem that led Hitotumatu and Noshita to introduce dancing links in the first place, namely the N queens problem, because that problem is actually a special case of the generalized cover problem in the previous section. For example, the 4 queens problem is just the task of covering eight primary columns (R0, R1, R2, R3, F0, F1, F2, F3) corresponding to ranks and files, while using at most one element in each of the secondary columns (A0, A1, A2, A3, A4, A5, A6, B0, B1, B2, B3, B4, B5, B6) corresponding to diagonals, given the sixteen rows

R0	F0	A0	B3
R0	F1	A1	B4
R0	F2	A2	B5
R0	F3	A3	B6
R1	F0	A1	B2
R1	F1	A2	B3
R1	F2	A3	B4
R1	F3	A4	B5
R2	F0	A2	B1
R2	F1	A3	B2
R2	F2	A4	B3
R2	F3	A5	B4
R3	F0	A3	B0
R3	F1	A4	B1
R3	F2	A5	B2
R3	F3	A6	B3.

In general, the rows of the 0-1 matrix for the N queens problem are

$$\mathrm{R}i \quad \mathrm{F}j \quad \mathrm{A}(i+j) \quad \mathrm{B}(N-1-i+j)$$

for $0 \le i, j < N$. (Here Ri and Fj represent ranks and files of a chessboard; Ak and Bℓ represent diagonals and reverse diagonals. The secondary columns A0, A$(2N-2)$, B0, and B$(2N-2)$ each arise in only one row of the matrix so they can be omitted.)

When we apply algorithm DLX to this generalized cover problem, it behaves quite differently from the traditional algorithms for the N queens problem, because it branches sometimes on different ways to occupy a rank of the chessboard and sometimes on different ways to occupy a file. Furthermore, we gain efficiency by paying attention to the order in which primary columns of the cover problem are considered when those columns all have the same S value (the same branching factor): It is better to place queens near the middle of the board first, because central positions rule out more possibilities for later placements.

Consider, for example, the eight queens problem. Figure 16(a) shows an empty board, with 8 possible ways to occupy each rank and each file. Suppose we decide to place a queen in R4 and F7, as shown in Figure 16(b). Then there are five ways to cover F4; after choosing R5 and F4, Figure 16(c), there are four ways to cover R3, and so on. At each stage we choose the most constrained rank or file, using the "organ pipe ordering"

R4 F4 R3 F3 R5 F5 R2 F2 R6 F6 R1 F1 R7 F7 R0 F0

to break ties. Placing a queen in R2 and F3 after Figure 16(d) makes it impossible to cover F2, so backtracking will occur even though only four queens have been tentatively placed.

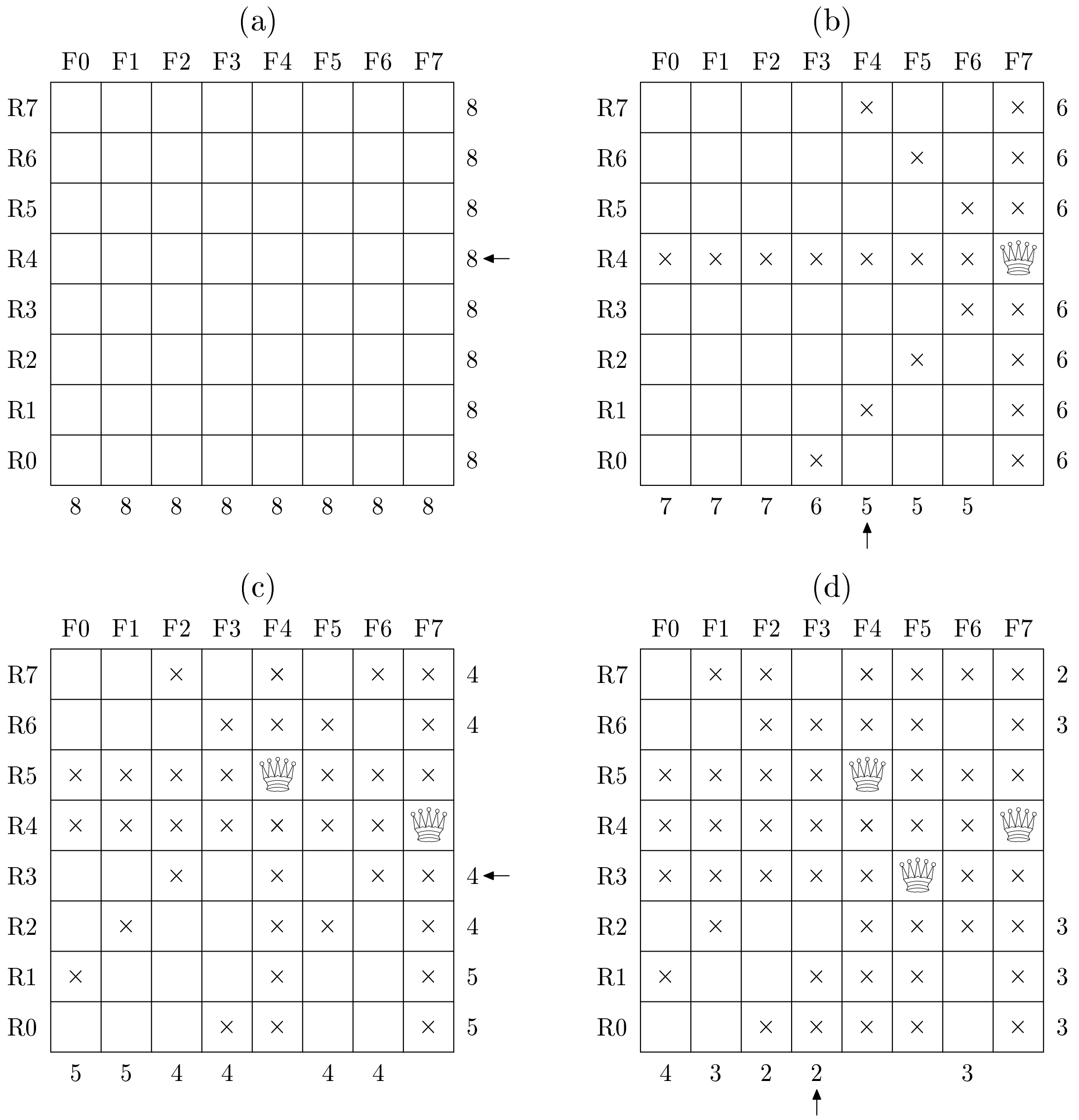


FIGURE 16. Solving the 8 queens problem by treating ranks and files symmetrically.

The order in which header nodes are linked together at the start of algorithm DLX can have a significant effect on the running time. For example, experiments on the 16 queens problem show that the search tree has 312,512,659 nodes and requires 5,801,583,789 updates, if the ordering R0 R1 ... R15 F0 F1 ... F15 is used, while the organ-pipe ordering R8 F8 R7 F7 R9 F9 ... R0 F0 requires only about 54% as many updates. On the other hand, the order in which individual elements of a row or column are linked together has no effect on the algorithm's total running time.

Here are some statistics observed when algorithm DLX solved small cases of the N queens problem using organ-pipe order, without reducing the number of solutions by taking symmetries of the board into account:

N	Solutions	Nodes	Updates	R-Nodes	R-Updates
1	1	2	3	2	3
2	0	3	19	3	19
3	0	4	56	6	70
4	2	13	183	15	207
5	10	46	572	50	626
6	4	93	1,497	115	1,765
7	40	334	5,066	376	5,516
8	92	1,049	16,680	1,223	18,849
9	352	3,440	54,818	4,640	71,746
10	724	11,578	198,264	16,471	269,605
11	2,680	45,393	783,140	67,706	1,123,572
12	14,200	211,716	3,594,752	312,729	5,173,071
13	73,712	1,046,319	17,463,157	1,589,968	26,071,148
14	365,596	5,474,542	91,497,926	8,497,727	139,174,307
15	2,279,184	31,214,675	513,013,152	49,404,260	800,756,888
16	14,772,512	193,032,021	3,134,588,055	308,130,093	4,952,973,201
17	95,815,104	1,242,589,512	20,010,116,070	2,015,702,907	32,248,234,866
18	666,090,624	8,567,992,237	141,356,060,389	13,955,353,609	221,993,811,321

Here "R-nodes" and "R-Updates" refer to the results when we consider only R0, R1, ..., R$(N-1)$ to be primary columns that need to be covered; columns Fj are secondary. In this case the algorithm reduces to the usual procedure in which branching occurs only on ranks of the chessboard. The advantage of mixing rows with columns becomes evident as N increases, but I'm not sure whether the ratio of R-Updates to Updates will be unbounded or approach a limit as N goes to infinity.

I should point out that special methods are known for counting the number of solutions to the N queens problem without actually generating the queen placements [33].

Concluding remarks. Algorithm DLX, which uses dancing links to implement the "natural" algorithm for exact cover problems, is an effective way to enumerate all solutions to such problems. On small cases it is nearly as fast as algorithms that have been tuned to solve particular classes of problems, like pentomino packing or the N queens problem, where geometric structure can be exploited. On large cases it appears to run even faster than those special-purpose algorithms, because of its ordering heuristic. And as computers get faster and faster, we are of course tackling larger and larger cases all the time.

In this paper I have used the exact cover problem to illustrate the versatility of dancing links, but I could have chosen many other backtrack applications in which the same ideas apply. For example, the approach works nicely with the Waltz filtering algorithm [36]; perhaps this fact has subliminally influenced my choice of names. I recently used dancing links together with a dictionary of about 600 common three-letter words of English to find word squares such as

```
ATE   BED   OHM   PEA   TWO
WIN   OAR   RUE   URN   ION
LED   WRY   BET   BAY   TEE
```

in which each row, column, and diagonal is a word; about 60 million updates produced all solutions. I believe that a terpsichorean technique is significantly better than the alternative of copying the current state at every level, as considered in the pioneering paper by Haralick and Elliott on constraint satisfaction problems [19]. Certainly the use of (1) and (2) is simple, useful, and fun.

"What a dance / do they do / Lordy, I am tellin' you!" [2]

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Historical notes. (1) Although the IAS computer was popularly known in Princeton as the "MANIAC," that title properly belonged only to a similar but different series of computers built at Los Alamos. (See [27].) (2) George Jelliss [23] discovered that the great puzzle masters H. D. Benjamin and T. R. Dawson experimented with the concept of polysticks already in 1946–1948. However, they apparently did not publish any of their work. (3) My names for the tetrasticks are slightly different from those originally proposed by Barwell [3]: I prefer to use the letters J, R, and U for the pieces he called U, J, and C, respectively.

Program notes. The implementation of algorithm DLX that I used when preparing this paper is file `dance.w` on webpage `http://www-cs-faculty.stanford.edu/~knuth/programs.html`. See also the related files `polyominoes.w`, `polyiamonds.w`, `polysticks.w`, and `queens.w`.

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Addendum. During November, 1999, Alfred Wassermann of Universität Bayreuth succeeded in covering the Aztec diamond of Figure 15 with one-sided tetrasticks, using a cluster of workstations running algorithm DLX. The 107 possible solutions, which are quite beautiful, have been posted at `http://did.mat.uni-bayreuth.de/wassermann/`. He subsequently enumerated the 10,440,433 solutions to the 9×10 one-sided pentomino problem; many of these turn out to be more symmetric than the one in Figure 10.