Dynamics of Tectonic Plates

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Abstract

We suggest a model that describes a mutual dynamics of tectonic plates. It is a sort of stick-slip dynamics modeled by a Markov random process. The process defines the dynamics on a micro-level. A macro-level is obtained by a scaling limit which leads to a system of integro-differential equations which determines a kind of mean field systems. The conditions when the Gutenberg-Richter empirical law holds are presented at the mean field level. Those conditions are rather universal and independent of the features of the resistant forces.

1 Introduction.

The tectonic plate construction of the earth lithosphere including the plate motions is generally accepted and recently well described (for example, see [1, 4]). Moreover, the tectonic plates of other planets of the solar system, such as Mars and Venus, are a subject of investigations as well [3]. Tectonic plate motion as a cause of earthquakes is also a widely held view. However, the mechanism of the earthquake emergence in a course of the plate motions is subject of intensive investigations in Geosciences as well as in Mathematics and Physics (see [5, 6, 7, 8, 9]).

Of the different models dealing with the emergence of earthquake the most popular is related to *stick-slip* motion. It suggests (see, for example, Brace and Byerlee [10]) that earthquakes must be the result of a stick-slip frictional instability rather than caused by fracture appearance and propagation. It considers the earthquake as the result of a sudden slip along a pre-existing fault on plate surface and the "stick" as the interseismic period of strain accumulation. Benioff-Wadati subduction zones and large well studied active fault zones such as San Andreas, seem to exhibit sudden slip events followed by "silent" slip and renewal of the contact points creating a stick behavior.

Concerned with this model some authors related this mechanism to laboratory experiments consistent with the Ruina-Dietrich "rate and state-variable friction law" [11], [12]. Another model was proposed in [5] (Burridge-Knopoff model), where experimental and theoretical results are discussed. The model in [5] is a one-dimensional chain of massive blocks tied by springs and situated on a rough unmoving surface. The heading block is pulled with a constant speed, followed by the succession of the remnant blocks subjected to the same stick-slip behavior.

Proposed model. Here we suggest a stochastic model for the stick-slip dynamics. The model describes a stochastic dynamics on the micro-level. It is represented as a stochastic dynamics of a set of contact points on surfaces. A combination of three types of change forms the dynamics of the plate: 1) a deterministic motion causing the deformation of the contact points and an increase of strain; 2) an appearance of new contacts that changes the dynamics; 3) break-up of the existent contacts which causes a jump-wise change of the dynamics velocity. Some of the physical phenomena are here described by a number of parameters. One of the parameters is the rate of contact destructions. We show that these parameters are related to the Gutenberg-Richter law (see [18]). For the elastic resistance a linear dependence of the force on a value of the contact deformation is here assumed. A relatively similar model defined and numerically analyzed is discussed in [14].

The micro-level model defined by a Markov piece-wise deterministic random process is a base for a macro-level model represented by a system of integro-differential equations. This rather universal model describing the friction between different matter plates is here applied to describe tectonic plate motions. The mean field method used in this paper was also applied in [17].

The paper is organized as follows: In Section 2 we informally describe our model. Sections 3 and 4 contain the rigorous definitions and the basic results including the condition where the Gutenberg-Richter law is satisfied. This condition requires an inverse deformation asymptotic law for the breaks of the contact points. Section 5 contains some rigorous descriptions.

2 Preliminary descriptions

This section shortly describes the proposed model and gives some explanations about the obtained results. The model of plate dynamics is based on a well known idea of a stick-slip behaviour sliding over each other surfaces with friction. We describe our model on a micro- and macro-levels. The micro-level is defined by a Markov stochastic process, the macro-level is obtained by a scaling of this process. The scaled system is already not stochastic and presented by a intero-differential system of equations.

Micro-level. Assume a plate dynamics on a solid unmoving substrate with $\widehat{\Lambda}$ as an area of possible contacts between the plate and the substrate. The plate is subjected to the action of a constant force F (a *moving force*). The moving force causes a motion of the plate in the direction of F. The contacting surfaces are not smooth. There are sets of *asperities* on the surfaces. In fact, there exist two sets of the asperities, one on the plate and another on the substrate. The asperities of both plate and solid substrate can be contacted, what creates a resistance force due to deformations of the contacting asperities. The contacts are the main obstacles creating the resistance of the plate motion. In further explanations we consider only a set Λ which is the set of the asperities on the plate. We assume a fixed density of the asperities which does not change in the course of the dynamics. This assumption is indirectly presented by a constant $\bar{c}_b$ (see (4.4)). The resistance to the plate dynamics is created by some of the plate asperities which come to contact with some asperities of the substrate. The asperity in contact is called *contacting point* or *contact element*. The set of the contacting points we denote by Ω and its elements by ω. The set Ω is a subset of Λ. The set of contacting points changes in the course of the dynamics.

The plate dynamics is modelled by a piecewise deterministic Markov process in which a state space is composed by values of *deformations* of the contacting points during the dynamics. The values of the resistance forces depend on the values of the deformations. In the case of elastic behavior of the asperities, the resistance force is proportional to the deformation value. We want to emphasize that not only purely elastic cases (see Section 3) are being considered. The deformations increase in the course of the dynamics while the number of the contacting point does not change. We also assume that the law of the plate motion and therefore of the deformation changes are defined by Aristotle's equations (or Aristotelian mechanics). The choice of the dynamical equations is justified by an assumption that the plate is immersed in a partially viscous environment due to the physico-mechanical conditions to which the rocks are subjected at great depths, which creates an additional resistance force suppressing the acceleration of the plate motion. Therefore despite the large mass of the plate we neglect its acceleration what also has been reinstated from the measures of the plate deformation through time.

This described part of the dynamics reflects only a deterministic aspect of the corresponding Markov process. We shall return to the discussion of our choice of use the Aristotelian mechanics further.

In the course of the plate dynamics any contact asperity can be destroyed. It causes an abrupt momentaneous decrease in the number of the contact points. Consequently the resistance force created by this contact element disappears abruptly and the velocity of the plate is enlarged by a jump. We consider the abrupt increase of the velocity as a possible cause of a quake shock. A rate $c_u(x)$ of the contact point destruction may depend on the value of the corresponding asperity deformation x. A very important case which we study is when $c_u(x) = \frac{c}{x}$ for some constant $c > 0$ and large x. The meaning of the shape of this function is rather strange: the greater the deformation the smaller the probability of failure (see (4.15)).

Another jump-wise behaviour of the Markov process happens when a new contact element comes into being which means that an asperity of the plate meets an asperity of the substrate. The rate (or intensity) of the emergence of new contact point in our model depends on the plate velocity. It is clear that the intensity is equal to 0 if the plate is not moving. We adopt the case when the emergence intensity is proportional to the velocity $\bar{c}_b v(t)$ (see Section 4). In the moment of emergence the new contact does not create any resistance force. In the course of the dynamics, the deformation of the new contacting point increases. A path of the Markov process at the birth of the new contact point is continuous, but the derivative of the path is discontinuous (see Fig. 1).

About the Aristotelian mechanics applied here we remark that in the considered case the most essential difference between Newtonian and Aristotelian dynamics arises at the jump moment of the plate velocity. No jump will occur under Newtonian dynamics. Instead, the velocity will have a large acceleration on a small time interval.

The Markov process depicted above describes the tectonic plate motion on a microlevel in which the asperities are minimal elements involved in the dynamics. Recall that states of the Markov process are sets of the deformation values of the contacting points.

Macro-level. Our main analysis (including Gutenberg-Richter law derivation) is done in the case of a macro-level where a big number of the asperities are considered. In fact we take limits to infinity of the asperity number by doing some scaling of the model parameters (see details in Section 4). In the scaled version (on the macro-level) we have a deterministic dynamics instead of the stochastic process. Now we consider a density of the contacting points over the deformations as a state of the scaled system. Let $\rho(x,t)$ be the density at the moment t. Roughly speaking $\rho(x,t)$ is a scaled "number" of the contacting points having their deformations equal to x at time t. We obtain a system of "integro-differential" equations for the deterministic dynamics of $\rho(x,t)$. The system is presented in (4.1) and (4.2). We shall stress again that $\rho(x,t)$ is the distribution of the contacting asperities over their deformations and it is not a distribution over the physical contacting surface.

The system (4.1), (4.2) is not linear. A general solution of the stationary system equations (4.3) is in (4.4). We are interested in what must be a form of the destruction (death) intensity $c_u(x)$, when the stationary solution $\rho(x)$ satisfies the Gutenberg-Richter law. The differential equation (4.14) on the function $c_u(x)$ gives the answer. All solutions of (4.14) are in (4.15), and among them the only physically meaningful is in (4.16) which has its shape $c_u(x) = \frac{c}{x}$, where $c > 0$. The explanation of such strange shape of $c_u(x)$ presumably can be as follows. The asperity which has already a large deformation can withstand a greater deformation. We call this result *inverse deformation law* (see Proposition 4.1).

We investigate in Section 4.2 one of non-stationary cases, namely when $c_u(x) \equiv c$ is a constant.

3 The model description. Finite volume.

We consider a plate moving on a solid unmoving substrate with contact area Λ. The plate is subjected to the action of a constant force F in some fixed direction (a *moving force*). The moving force causes a motion of the plate in the direction of F. Asperities of both plate and solid substrate can be contacted, what creates a resistance force due to deformation of the contacting asperities. The set of *contact points* we denote by Ω and its elements by ω. The set Ω is a finite subset of Λ.

Deterministic part

In the course of the plate motion, any contacting asperity is being deformed until its failure. Here we will introduce a plastic component in the deformation mechanism.

Any contact point $\omega \in \Omega$ creates a resistant force depending on magnitude of the asperity deformation. In a general setting the resistant force is an expression $\varkappa \min\{x_\omega, x_\omega^\alpha\}$, where α is a constant from $[0, 1]$. When $\alpha = 1$ then the resistance is purely elastic, and $\varkappa$ is Hooke's constant. The sum of all resistance forces over all contact points gives the total resistance force. Therefore the resultant force acting on the plate is

$$G = \Big[F - \varkappa \sum_{\omega \in \Omega} \min\{x_\omega, x_\omega^\alpha\}\Big]_+, \tag{3.1}$$

where $[A]_+ = \max\{A, 0\}$. This means that the resistance force cannot be greater than the moving force.

The force G causes a deterministic motion of the plate. We assume that the plate is a monolithic hard object any point of which is moving with the same velocity v. The next our assumption implies that the dynamics of the plate under the force action follows the so called Aristotle mechanics. It means that the velocity v of the motion is proportional to the acting force,

$$v = \gamma G, \tag{3.2}$$

in contrast to Newton mechanics where the acceleration is proportional to the acting force. Here γ is constant. This assumption is due to the fact that the moving plate is immersed in a viscous medium and therefore inertia, despite the large mass of the plate, has little influence on the nature of the motion.

If during a time interval $[t_1, t_2]$ the set Ω is not changed (a new contact does not appear and any existing contact does not disappear) then we have a deterministic dynamics on $[t_1, t_2]$ evolving according to the equation

$$\frac{\mathrm{d}x_\omega(t)}{\mathrm{d}t} = v(t) = \gamma\Big[F - \varkappa \sum_{\omega' \in \Omega} z_{\omega'}^{(\alpha)}(t)\Big]_+ \tag{3.3}$$

with an initial velocity value $v(t_1)$, (see (3.1) and (3.2)). Here we denote $z_\omega^{(\alpha)} = \min\{x_\omega, x_\omega^\alpha\}$.

In the case of the elastic resistance forces ($\alpha = 1$) , if

$$v(t_1) = \gamma\Big[F - \varkappa \sum_{\omega' \in \Omega} x_{\omega'}(t_1)\Big]_+ = \gamma\Big(F - \varkappa \sum_{\omega' \in \Omega} x_{\omega'}(t_1)\Big) > 0 \tag{3.4}$$

then a solution of (3.3) on $[t_1, t_2]$ is

$$v(t) = v(t_1) e^{-\varkappa\gamma|\Omega|(t-t_1)} \tag{3.5}$$

for $t \in [t_1, t_2]$, where $|\Omega|$ is a number of the points (the contacts) in the set Ω (see rigorous evaluations in the section 5.2).

This dynamics determines the motion of the plate when the set of contact points is fixed.

Stochastic part

The deterministic dynamics is interrupted by random events of two kinds: either some contact from Ω disappears or a new contact appears. In both cases the number of the contacts is changed from $n = |\Omega|$ to either $n - 1$ or $n + 1$. Next we describe the dynamics of the appearance and disappearance (birth and death) of the contact points. These dynamics are random and of Markov type.

Assume that there were no any random events on the interval $[t_1, t_2]$, and at t_2 a new contact appears. Its deformation is equal to 0 at the moment t_2. Let $\Delta_\Omega = \{x_\omega\}$ be a set of all contact point deformations. The contact set Ω is changed to a set Ω' at the moment t_2, and the new deformation set is $\Delta_{\Omega'} = \Delta_\Omega \cup \{0\} = \{0, x_\omega : \omega \in \Omega\}$.

It is assumed that the birth of new contact points at t_2 depends on the velocity $v(t_2) = \gamma[F - \varkappa \sum_{\omega' \in \Omega} z_{\omega'}^{(\alpha)}(t_2)]_+$ of the plate. This dependence is determined by a birth rate $c_b(v(t)) > 0$. The function c_b should reflect physical properties of the plate such as a fractal dimension of the asperities and many others. It is clear that new contacts do not appear if the velocity v is zero, hence $c_b(0) = 0$. A natural choice of c_b is $c_b(v) = \bar{c}_b v$, linear dependence on the velocity, where $\bar{c}_b > 0$.

Because the new contact deformation is 0 then

$$\sum_{\omega' \in \Omega} z_{\omega'}^{(\alpha)}(t_2) = \sum_{\omega' \in \Omega'} z_{\omega'}^{(\alpha)}(t_2).$$

Thus the velocity $v(t)$ is continuous at $t = t_2$. However the velocity derivative $\frac{\mathrm{d}v(t)}{\mathrm{d}t}\Big|_{t=t_2}$ at t_2 is discontinuous; see Figure 1 of a typical path of the velocity. The left derivative at t_2 is greater than the right one at the same time t_2. In the case of elastic resistance forces:

$$\begin{aligned}
\lim_{t\uparrow t_2} \frac{\mathrm{d}v(t)}{\mathrm{d}t} &= -\gamma\varkappa n v(t_1) e^{-\gamma\varkappa n(t_2-t_1)} \\
&> \lim_{t\downarrow t_2} \frac{\mathrm{d}v(t)}{\mathrm{d}t} = -\gamma\varkappa(n+1)v(t_1)e^{-\gamma\varkappa n(t_2-t_1)},
\end{aligned}$$

where $n = |\Omega|$.

The disappearance (death) of a contact ω from Ω is determined by the rate $c_u \equiv c_u(x_\omega)$, it may depend on the deformation x_ω. If at the moment t_2 the contact ω disappears then the velocity has discontinuity at t_2. The velocity increases abruptly by the value $\gamma\varkappa z_\omega^{(\alpha)}$. We shall assume that the velocity is continuous from the right at t_2, that is

$$\lim_{t\downarrow t_2} v(t) = v(t_2).$$

Complete view

We describe now a complete evolution of the plate velocity by the dynamic described above. The time is split into the intervals $[0, \infty] = \bigcup [t_i, t_{i+1}]$ such that during every interval $[t_i, t_{i+1}]$ the plate is moving deterministically according to (3.3) and (3.5) substituting t_1 by t_i. The set of the points $R = \{t_i\}$ is the moments of random events: either a new contact appears or one of the existing contacts disappears. The set of the random moments R is splitted by the set R_b of the appearing contacts and the set R_u of the disappearing contacts, $R = R_b \cup R_u$.

Any disappearance of the contact releases an energy which switches to an energy of seismic waves in the plates. The amount of released energy depends on the deformation value x_ω of the disappeared contact. We assume that the seismic wave energy is proportional to the primitive of the resistance force (the potential). For large deformation x it is proportional to $x^{1+\alpha}$. If x_ω is large then the oscillation amplitude may be large which can be observed as an earthquake.

A typical random path of the velocity is presented on Figure 1, which is typical for the stick-slip process.

4 Scaling limit. Infinite volume

In this section we propose an analytical approach which allows to study some properties of the defined model. The idea is to consider a very large number of the asperities. It means that we consider a limit of the size of the contact area Λ going to infinity. In the limit we obtain infinite number of asperities and then a distribution of the asperity deformations is described by a density function $\rho(x,t)$, $\rho(x,t)\mathrm{d}x$ has a meaning of a number of the asperities having their deformation in the interval $(x, x+\mathrm{d}x)$ at the moment t. To obtain a reasonable model in this limit we have to change the values of the parameters determining the plate motion model. Some of the parameters must depend on the size of the contact surface Λ. Namely, the birth intensity $c_b' = \bar{c}_b\Lambda$ and the acting force $F' = F\Lambda$, are proportional to the size Λ. The death parameter we take without changes $c_u' = c_u$. Further we omit the sign $'$.

A system of equations describing a behavior of the density $\rho(x,t)$ and the velocity $v(t)$ derived from balance considerations is

$$\frac{\partial\rho(x,t)}{\partial t} + v(t)\frac{\partial\rho(x,t)}{\partial x} = \bar{c}_b v(t)\delta_0(x) - c_u(x)\rho(x,t), \tag{4.1}$$

and

$$v(t) = \gamma\left(F - \varkappa\int_{-\infty}^{+\infty} z^{(\alpha)}(x)\rho(x,t)dx\right)_+, \tag{4.2}$$

where $z^{(\alpha)}(x) = \min\{x, x^\alpha\}$. Here we define the function $\rho(x,t)$ on all line $x \in \mathbb{R}$ such that it is equal to 0 when $x < 0$.

In the two next subsections we study some properties of the solutions of (4.1), (4.2).

4.1 Stationary state and the Gutenberg-Richter law

In this subsection we study some properties of (4.1) in a stationary regime, when $\rho(x,t)$ does not depend on time t. The equation (4.1) is then

$$v\frac{\mathrm{d}\rho}{\mathrm{d}x}(x) = \bar{c}_b v\delta(x) - c_u(x)\rho(x), \tag{4.3}$$

(cf (4.32)). Because of the stationarity the velocity v of the plate does not depend on time t being constant in (4.3). The solution of (4.3) connects $\rho(x)$ and v as the following

$$\rho(x) = \begin{cases} \bar{c}_b \exp\{-\frac{1}{v}\int_0^x c_u(y)dy\}, & \text{when } x > 0, \\ 0, & \text{when } x < 0. \end{cases} \tag{4.4}$$

The density at $x = 0$ is $\rho(0) = \bar{c}_b$. For $v = 0$ the stationary regime is trivial, $c_u(x)\rho(x) = 0$.

When an asperity subjected to a deformation size x disrupts then it releases the energy

$$e(x) = \mu x^{1+\alpha} \tag{4.5}$$

for large x, where μ is a constant of proportionality.

The Gutenberg-Richter law. The Gutenber-Richter law is observed, in practice, on a restricted area of energy values $[e_1, e_2]$. Therefore we study the intensity destruction (death) function $c_u(x)$ on the corresponding deformation interval $[x_1, x_2]$, where $x_i = e_i^{\frac{1}{1+\alpha}}$, if we put $\mu = 1$ on (4.5). It is assumed that $x_1 > 1$. Let $n(e)\mathrm{d}e$ be a distribution of a number of the destructions along the energy e axis, and let $m(x)\mathrm{d}x$ be a distribution of number of the destructions arising from the asperities destroyed at a size x of their deformations. The Gutenberg-Richter law claims that

$$n(e)\mathrm{d}e \propto \frac{1}{e^w}\mathrm{d}e, \tag{4.6}$$

where $w > 1$. Observations show that w lies in the range $1.7 - 2.1$ (see [18]). The relation (4.6) is equivalent to the popular Gutenberg-Richter relation in terms of magnitude M (see [2], [18]): $\log n(M) = a + bM$, where parameter b is related to w by the equation $b = 3(w-1)/2$.

Changing variables defined by (4.5) we obtain

$$n(e(x))\mathrm{d}e(x) \propto n(e(x))x^\alpha \mathrm{d}x \propto \frac{1}{e^w(x)}x^\alpha \mathrm{d}x \tag{4.7}$$

The distribution $m(x)\mathrm{d}x$ can be expressed by the density $\rho(x)$ of the deformations and the intensity $c_u(x)$ of the destruction as

$$m(x)\mathrm{d}x \propto c_u(x)\rho(x)\mathrm{d}x. \tag{4.8}$$

Remark next that

$$m(x)\mathrm{d}x = n(e(x))x^{\alpha}\mathrm{d}x. \tag{4.9}$$

It follows now from (4.7) that

$$c_u(x)\rho(x)\mathrm{d}x \propto \frac{1}{e^w(x)}x^{\alpha}\mathrm{d}x = \frac{x^{\alpha}}{x^{w(1+\alpha)}}\mathrm{d}x. \tag{4.10}$$

The left hand side of this relation is the frequency (density) of the demolished contacts achieved with deformation values at $\mathrm{d}x$.

The equation (4.10) shows a mutual behavior of $c_u(x)$ and $\rho(x)$ for large x:

$$\rho(x) \propto \frac{1}{x^p c_u(x)}, \tag{4.11}$$

where $p = w(1+\alpha) - \alpha$. The next analysis uses the function $c_u(x)$ on the interval $[0, x_2]$. The function $c_u(x)$ on $x \in [0, x_1]$ is assumed arbitrary continuous. The relation

$$\bar{c}_b \exp\left\{-\frac{1}{v}\int_0^x c_u(y)\mathrm{d}y\right\} \propto \frac{1}{x^p c_u(x)} \tag{4.12}$$

for $x \in [x_1, x_2]$ follows from (4.4). That is

$$\bar{c}_b \exp\left\{-\frac{1}{v}\int_0^x c_u(y)\mathrm{d}y\right\} = \frac{A}{x^p c_u(x)} \tag{4.13}$$

for some proportionality constant $A > 0$. Taking logarithm and derivative of (4.13) we obtain the differential equation

$$\frac{1}{v}c_u(x) = \frac{p}{x} + \frac{c_u'(x)}{c_u(x)} \tag{4.14}$$

on the interval $[x_1, x_2]$, where $c_u'(x)$ means the derivative of $c_u(x)$. We assume an initial value $c_u(x_1) > 0$ is given.

The rate of the disruptions corresponding to Gutenberg-Richter law

Assume that the Gutenberg-Richter law is satisfied on $[x_1, x_2]$*. Under this assumption the general solution of (4.14) is*

$$c_u(x) = \frac{(p-1)v}{x + Bx^p}, \tag{4.15}$$

where $B \geq 0$.

The fact that (4.15) is the solution of (4.14) can be verified directly.

The relation (4.15) shows that the disrupting asperity rate on $[x_1, x_2]$ is decreasing: *greater the asperity deformation* x *less a probability to be disrupted*. On other hand this correspondence means: greater the asperity deformation slower the density $\rho(x)$ decreasing. In the case $B > 0$ the density $\rho(x)$ can not be done as small as possible since $\lim_{x\to\infty} \rho(x) > 0$ (see (4.11) and (4.15). Such ρ has no physical sense, thus we consider the case $B = 0$.

Proposition 4.1 (The inverse deformation law). *If Gutenberg-Richter law is satisfied then a physically meaningful case is when* $B = 0$ *and the disruption rate* c_u *is decreasing hyperbolically*

$$c_u(x) = \frac{(p-1)v}{x} \tag{4.16}$$

Assume that $c_u(x)$ is defined on $[0, \infty)$ such that (4.16) holds on $[x_1, x_2]$ the following equations define v and $\rho(x)$:

$$v = \gamma \left(F - \varkappa \int_{-\infty}^{+\infty} z^{(\alpha)}(x) \rho(x) dx \right)_+, \tag{4.17}$$

$$\rho(x) = \bar{c}_b \exp\Big\{-\frac{1}{v} \int_0^x c_u(y) dy\Big\}, \text{ for } x > 0. \tag{4.18}$$

If solutions of (4.17) and (4.18) are such that $v > 0$ then

$$\rho(x) = \begin{cases} \bar{c}_b \exp\left\{-\frac{1}{v}\int_0^x c_u(y)\mathrm{d}y\right\} & \text{if } x < x_1, \\ \bar{c}_b \exp\left\{-\frac{1}{v}\int_0^{x_1} c_u(y)\mathrm{d}y\right\} \left(\frac{x_1}{x}\right)^{p-1} & \text{if } x_1 \leq x < x_2, \\ \bar{c}_b \exp\left\{-\frac{1}{v}\int_0^{x_1} c_u(y)\mathrm{d}y\right\} \left(\frac{x_1}{x_2}\right)^{p-1} \exp\left\{-\frac{1}{v}\int_{x_2}^x c_u(y)\mathrm{d}y\right\} & \text{if } x_2 \leq x. \end{cases} \tag{4.19}$$

and

$$
\begin{aligned}
v = \ & \gamma \left(F - \varkappa \bar{c}_b \left[\int_0^{x_1} z^{(\alpha)} \exp\left\{ -\frac{1}{v} \int_0^x c_u(y)\mathrm{d}y \right\} \mathrm{d}x \ + \right.\right. \\
& \frac{x_1^{p-1}}{p-\alpha} \exp\left\{ -\frac{1}{v} \int_0^{x_1} c_u(y)\mathrm{d}y \right\} \left[x_1^{\alpha-p} - x_2^{\alpha-p} \right] + \\
& \left.\left. \left(\frac{x_1}{x_2} \right)^p \int_{x_2}^{\infty} x^{\alpha} \exp\left\{ -\frac{1}{v} \int_{x_2}^x c_u(y)\mathrm{d}y \right\} \mathrm{d}x \right] \right).
\end{aligned}
$$

Recall that $p = w(1+\alpha) - \alpha$

4.1.1 Examples

Example 1. Assume that $x_1 = 1$ and the constants $F,\ \gamma,\ \varkappa,\ \bar{c}_b,\ f(\alpha)$ and x_2 are such that

$$
\widetilde{v} = \gamma \left(F - \varkappa \bar{c}_b \left[\frac{1}{1+\alpha} + \frac{1}{p-\alpha} \left[1 - \frac{1}{x_2^{\alpha-p}} \right] \right] \right) > 0. \tag{4.20}
$$

We consider the following rate of the contact destruction

$$
c_u(x) = \begin{cases} 0, & \text{if } x \le 1, \\ \frac{\widetilde{v}(p-1)}{x}, & \text{if } 1 < x \le x_2 \\ \infty, & \text{if } x > x_2. \end{cases} \tag{4.21}
$$

Then the solutions of (4.17) and (4.18) are $v = \widetilde{v}$ and

$$
\rho(x) = \begin{cases} \bar{c}_b & \text{if } x < 1, \\ \bar{c}_b \left(\frac{1}{x}\right)^{p-1} & \text{if } 1 \le x < x_2, \\ 0 & \text{if } x_2 \le x. \end{cases} \tag{4.22}
$$

Example 2. Assume that $x_1 = 1$ and $\alpha = 1$. Let the constants $F,\ \gamma,\ \varkappa,\ \bar{c}_b$ and x_2 be such that there exists a constant $\widetilde{a} > 0$ such that

$$
F > \widetilde{r} = \bar{c}_b \varkappa \left[\frac{1}{\widetilde{a}^2} \left(1 - (1+\widetilde{a})e^{-\widetilde{a}}\right) + \frac{e^{-\widetilde{a}}}{p-1} \left(1 - x_2^{1-p}\right) \right]. \tag{4.23}
$$

Let

$$
c_u(x) = \begin{cases} \widetilde{a}\gamma(F - \widetilde{r}), & \text{if } x < 1 \\ \frac{(p-1)\gamma(F-\widetilde{r})}{x}, & \text{if } 1 \le x < x_2, \\ \infty, & \text{if } x_2 \le x \end{cases} \tag{4.24}
$$

then the solutions of (4.17) and (4.18) are

$$v = \gamma(F - \widetilde{r})$$

$$\rho(x) = \begin{cases} e^{-\widetilde{a}x}, & \text{if } x < 1, \\ e^{-\widetilde{a}} \left(\frac{1}{x}\right)^{p-1}, & \text{if } 1 \le x < x_2, \\ 0, & \text{if } x_2 \le x. \end{cases}$$

For the considered case $p = 2w - 1$.

There existence of the constant $\widetilde{a}$ follows from a limit

$$\lim_{a\to\infty} \frac{1}{a^2}\big(1 - (1+a)e^{-a}\big) + \frac{e^{-a}}{p-1}\big(1 - x_2^{1-p}\big) = \infty \tag{4.25}$$

4.2 Non-stationarity

Finding a solution of (4.1) and (4.2) in general case is rather a difficult problem. We describe a non-stationary behavior of the system in a particular case when the disruption intensity, $c_u(x) \equiv \bar{c}_u$ does not depend on the deformation value x and the resistance is elastic. This model was studied in [15] on the micro-level. Here we obtain the same results on the macro-level.

We reduce the study of (4.1) and (4.2) to a dynamical system which can be completely investigated. For formal considerations see in Section 5

4.2.1 A dynamical system

Define

$$N(t) = \int \rho(x,t)dx \quad \text{and} \quad M(t) = \int x\rho(x,t)dx.$$

Then $v(t) = \gamma\big(\bar{F} - \kappa M(t)\big)_+$ (see (4.2)).

We assume and check later that $\rho(x) \to 0$ and $x\rho(x) \to 0$, when $x \to \infty$.

Integrating (4.1) over x we obtain

$$\frac{dN(t)}{dt} = \bar{c}_b v(t) - \bar{c}_u N(t) \tag{4.26}$$

Multiplying (4.1) by x and integrating, we obtain

$$\frac{dM(t)}{dt} + v(t)\int x\frac{\partial\rho(x,t)}{\partial x}dx = -\bar{c}_u M(t), \tag{4.27}$$

because of $\int x\delta(x)\mathrm{d}x = 0$. Evaluating the integral in the left-hand side of (4.27) by parts we obtain the following system of equations on the plane (N, M):

$$\left\{ \begin{array}{rcl} dN/dt & = & \bar{c}_b v - \bar{c}_u N \\ dM/dt & = & vN - \bar{c}_u M \end{array} \right. \tag{4.28}$$

describing a dynamical system in the plane.

In the quarter-plane (M, N), $M > 0$, $N > 0$, there exists a unique point (M_0, N_0) which is a fixed point of (4.28). It means that (M_0, N_0) is a solution of (4.28) at the assumption that $\frac{\mathrm{d}M}{\mathrm{d}t} = \frac{\mathrm{d}N}{\mathrm{d}t} = 0$. This solution is

$$\begin{aligned} M_0 &= \frac{F}{\kappa} - \frac{F}{2\gamma\kappa a}\left[\sqrt{1+4\gamma a} - 1\right] \\ N_0 &= \frac{\bar{c}_b}{\bar{c}_u}\frac{F}{2a}\left[\sqrt{1+4\gamma a} - 1\right], \end{aligned} \tag{4.29}$$

where

$$a = \gamma\kappa F \frac{\bar{c}_b}{\bar{c}_u^2}. \tag{4.30}$$

The fixed point (M_0, N_0) is stable, that is the dynamic (4.28) is such that a point $(M(t), N(t))$ is attracted to (M_0, N_0) if $(M(t), N(t))$ is in a neighborhood of (M_0, N_0). There are two different ways how a path $(M(t), N(t))$ is moving to (M_0, N_0). The type of the ways depends on a value of a. This behavior is investigated by a linearization of the non-linear equations (4.28) (see Section 5.3). At the fixed point, the density $\rho(x, t)$ does not depend on time and is equal to

$$\rho(x) = \bar{c}_b \exp\left\{-\frac{\bar{c}_u}{v_0}x\right\}, \tag{4.31}$$

being a solution of the stationary version

$$\frac{\partial\rho(x)}{\partial x} = \bar{c}_b v_0 \delta(x) - \bar{c}_u \rho(x) \tag{4.32}$$

of (4.1). The velocity v_0 at the fixed point is a solution of the cubic equation

$$v = \gamma F - \gamma\kappa\frac{\bar{c}_b}{\bar{c}_b^2}v^3,$$

which has an unique positive solution between $v = 0$ and $v = \sqrt[3]{\frac{F\bar{c}_u^2}{\varkappa\bar{c}_b}}$ (see (4.2)).

Remark that in the stationary regime ρ is exponentially decreasing (see (4.31)). Therefore Gutenberg-Richter law is not satisfied for a constant intensity $\bar{c}_u$ of the disruptions.

5 Mathematical tools

In this section we present some mathematical justifications of the facts described in the previous sections.

5.1 Markov process

The model description, see Section 3, shows that the stochastic dynamics of the plate is a Markov process which we define hereby.

Remark that the model does not care about the positions of the points ω in the area Λ, but it essentially depends on the displacements x_ω of any contact point ω (see (3.1) and (3.3)). Thus a state of the moving plate is described by a set

$$\mathbf{x} := \{x_\omega\}_{\omega\in\Omega} \subset \mathbb{R}_+ = [0, \infty). \tag{5.1}$$

of all displacements of the contacts Ω.

It is clear then that we consider an one-dimensional model.

Further we will omit the index ω. Let X be a set of all finite configurations $X = \{\mathbf{x} \subset \mathbb{R}_+,\ |\mathbf{x}| < \infty\}$, where $|\mathbf{x}|$ means a number of points in $\mathbf{x}$.

As was said in the section 3 the random events in the dynamics are separated by a period of the deterministic motion during which configuration $\mathbf{x}$ moves into the positive direction as a rigid rod. The deterministic motion is described by the relations (3.1) – (3.5). Recall that in the elastic case

$$v(t) = v(0)e^{-\gamma\varkappa nt}, \tag{5.2}$$

where $n = |\mathbf{x}|$ (see (3.5)). The formula (5.2) defines the plate dynamics when there are no random events on the interval $[0, t]$. It means that the number the contact points (which is equal to the element number in the set $\mathbf{x}$) is not changed on this interval and equal to n.

Further, a point $x \in \mathbf{x}$ of any configuration $\mathbf{x}$ we will call also *particle*.

The stochastic Markov dynamic

There exist two kinds of the perturbations of the smooth deterministic dynamics: by a birth of a new particle or by a death of an existent particle.

The born particles are always localized at $0 \in \mathbb{R}_+$ at the moment of their appearance. Thus it does not create immediately a resistant force. The birth intensity we denoted by $c_b = c_b(v(t))$. The intensity $c_u = c_u(x(t))$ of the death of any particle may depend on the size $x(t)$ of the particle deformation.

Remark that the velocity $v(t)$ in the stationary state is always positive except the case $c_u(x) \equiv 0$. Any displacement $x \in \mathbf{x}$ cannot exceed the value $\left(\frac{F}{\varkappa}\right)^{\frac{1}{\alpha}}$.

The dynamics of the process is described by the following infinitesimal operator formalism. First, we describe a set of Markov process states.
The configuration set (the set of states) we define as

$$\begin{aligned} \mathcal{X} &= \bigcup_{n=0}^{\infty} \left\{ \{n\} \times \left[0, \frac{F}{\kappa}\right]^n \right\} \qquad (5.3) \\ &= \left\{ (n, x_1, ..., x_n) :\, n \in \mathbb{N},\ \mathbf{x} = (x_1, ..., x_n) \in \left[0, \left(\frac{F}{\varkappa}\right)^{\frac{1}{\alpha}}\right]^n \right\} \end{aligned}$$

Every state $(n, x_1, ..., x_n)$ means that there are n contact points (the particles) and $\mathbf{x} = (x_1, ..., x_n)$ describes the deformations of the contacts.
Infinitesimal generator. Let $\mathbf{H} = \{f = (f_n)\}$ be a set of continuous functions on $\mathcal{X}$, i.e. every $f_n : \{n\} \times \left[0, \frac{F}{\varkappa}\right]^n \to \mathbb{R}$ is continuous. We shall omit the index n if it does not lead to misunderstanding and write $f(n, x_1, ..., x_n)$ instead $f_n(n, x_1, ..., x_n)$. The infinitesimal operator L of the Markov process defined on $\mathbf{H}$ is

$$\begin{aligned} Lf(n, \mathbf{x}) = \ & v \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \qquad (5.4) \\ & + c_b(v)[f(n+1, x_1, ..., x_n, 0) - f(n, x_1, ..., x_n)] \\ & + \sum_{j=1}^{n} c_u(x_j)[f(n-1, x_1, ..., \widehat{x}_j, ..., x_n) - f(n, x_1, ..., x_n)], \end{aligned}$$

where $\widehat{x}_j$ means that the variable x_j is not presented in the list of variables, and we recall that $v(n, x_1, ..., x_n) = \gamma[F - \varkappa \sum_{i=1}^{n} x_i]_+$. The first term on the right of (5.4) corresponds to the deterministic plate motion between the random events. The second term reflects the birth event and the third term reflects the death event.

The Markov process defined by the operator L is a piece-wise deterministic process (see [16]).

The scaling limit of this Markov process is a deterministic process from the section 4 describing by the system (4.1), (4.2) that follows from the general theory (see [19])

5.2 Solution (3.5) of equation (3.3)

Finding the solution of (3.3) on the interval $[t_1, t_2]$ introduce $X(t) = \sum_{\omega \in \Omega} x_\omega$. It follows from the elastic version of (3.3) that

$$nv(t) = \frac{\mathrm{d}X(t)}{\mathrm{d}t} = n\gamma \left(F - \varkappa X(t)\right), \tag{5.5}$$

where $n = |\Omega|$ is the number of the contacts. The general solution (5.5) is

$$X(t) = \frac{F}{\varkappa} + Ce^{-n\gamma\varkappa(t-t_1)}, \tag{5.6}$$

where C must be defined from $X(t_1)$, that is $C = X(t_1) - \frac{F}{\varkappa}$. However if only the velocity $v(t_1)$ is known then $C = -\frac{v(t_1)}{\varkappa\gamma}$. Now (3.5) follows from (5.6).

5.3 A dynamical system

The stationary point (4.29) can be of two types: stable focus or stable node. Essential role in the following plays the combination of the parameters

$$a = \gamma\kappa\bar{F}\frac{\bar{c}_b}{\bar{c}_u^2}, \tag{5.7}$$

which we call an *order parameter*.

Theorem 5.1. *There exists in $\mathbb{R}_+^2$ an unique solution (M_0, N_0) of the equations $dM/dt = dN/dt = 0$ ((4.28)):*

$$\begin{aligned} M_0 &= \frac{\bar{F}}{\kappa} - \frac{\bar{F}}{2\gamma\varkappa a}\left[\sqrt{1+4\gamma a} - 1\right] \\ N_0 &= \frac{\bar{c}_b}{\bar{c}_u}\frac{\bar{F}}{2a}\left[\sqrt{1+4\gamma a} - 1\right] \end{aligned} \tag{5.8}$$

The linearized at (M_0, N_0) equations (4.28) are

$$\begin{aligned} dM/dt &= -(\bar{c}_u + \varkappa N_0)(M - M_0) + (\bar{F} - \varkappa M_0)(N - N_0) \\ dN/dt &= -\varkappa \bar{c}_b (M - M_0) - \bar{c}_u (N - N_0) \end{aligned} \tag{5.9}$$

The determinant of the matrix

$$\mathcal{M} = \begin{pmatrix} -(\bar{c}_u + \varkappa N_0) & \bar{F} - \varkappa M_0 \\ -\varkappa \bar{c}_b & -\bar{c}_u \end{pmatrix}$$

is positive. Therefore the trace (the sum of the eigenvalues) of M is negative.

If the order parameter $a < 20$ then the eigenvalues are complex and the point (M_0, N_0) is the stable focus, if $a > 20$ the both eigenvalues are negative then the point (M_0, N_0) is the stable node. The constant a is defined by (5.7) (see Figure 2 and 3).

6 Acknowledgement

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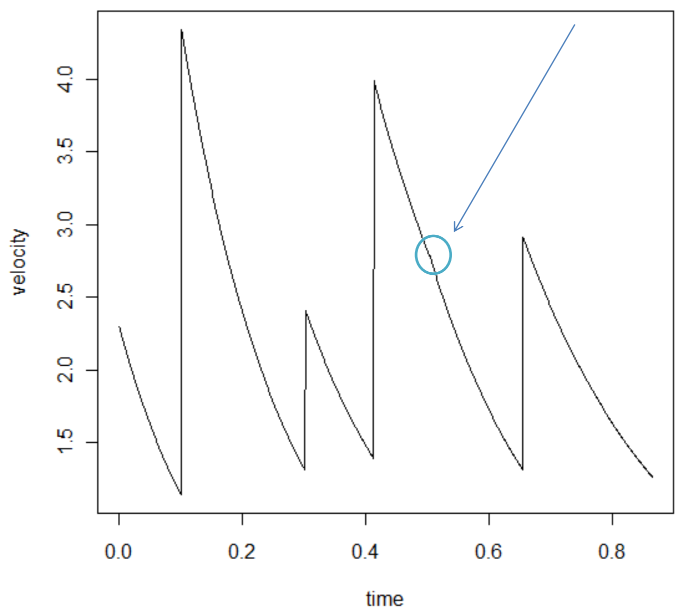

Figure 1: A typical path of the velocity $v(t)$ is a piece-wise continuous function between two nearest deaths of the contacts. The velocity has jumps at moments of the death of particle. The derivative of the velocity increases abruptly at the moment of the birth of a particle (blue circle).

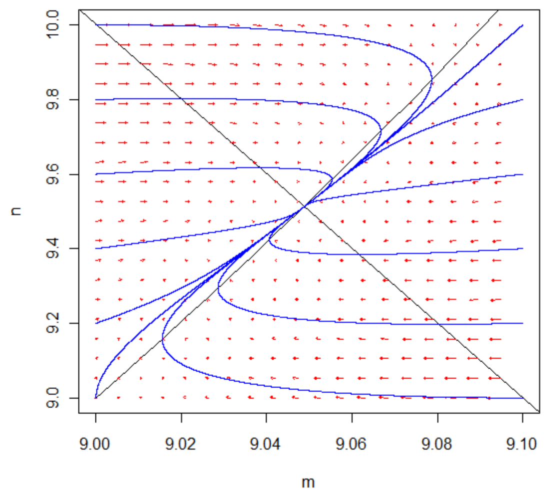

Figure 2: The integral curves of the field (4.28), $a < 20$.

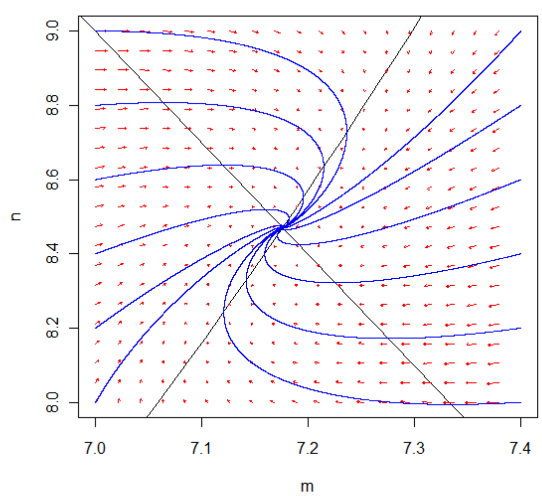

Figure 3: The integral curves of the field (4.28), the case $a > 20$.