

COALESCENCE AND MINIMAL SPANNING TREES OF IRREGULAR GRAPHS

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ABSTRACT. This paper concentrates on breaching the gap between the Smoluchowski coagulation equations for Marcus-Lushnikov processes and the theory of random graphs. It is known that in many cases the cluster dynamics of a random graph process can be replicated with the corresponding coalescent process. The cluster dynamics of a coalescent process (without merger history) is reflected in a auxiliary process called the Marcus-Lushnikov process. The merger dynamics of the Marcus-Lushnikov processes will correspond to a greedy algorithm for finding the minimal spanning tree in the respective random graph process. This observation allows one to express the limiting mean length of a minimal spanning tree in terms of the solutions of the Smoluchowski coagulation equations that represent the hydrodynamic limit of the Marcus-Lushnikov process corresponding to the random graph process.

We concentrate on finding the limiting mean length of a minimal spanning tree on an irregular graph. Specifically, an Erdős-Rényi random graph process on the bipartite graph $K_{\alpha[n],\beta[n]}$ is considered with $\alpha[n] = \alpha n + o(n)$ and $\beta[n] = \beta n + o(n)$. There, the following expression for the limiting mean length of the minimal spanning tree is derived via the Smoluchowski coagulation equations of the Marcus-Lushnikov processes with multidimensional weight vectors:

$$\lim_{n \rightarrow \infty} E[L_n] = \gamma + \frac{1}{\gamma} + \sum_{i_1 \geq 1; i_2 \geq 1} \frac{(i_1 + i_2 - 1)!}{i_1! i_2!} \frac{\gamma^{i_1} i_1^{i_2 - 1} i_2^{i_1 - 1}}{(i_1 + \gamma i_2)^{i_1 + i_2}},$$

where $\gamma = \frac{\alpha}{\beta}$. This is a completely new formula for the case of an irregular bipartite graph $\gamma \neq 1$. In the case of $\gamma = 1$, the above series adds up to

$$\lim_{n \rightarrow \infty} E[L_n] = 2\zeta(3)$$

as derived in Frieze and McDiarmid [14] for a regular bipartite graph. A generalization of the approach is considered in the discussion section.

1. INTRODUCTION

We begin with the following quote from Aldous [1]: *It turns out that there is a large scientific literature relevant to the Marcus-Lushnikov process, mostly focusing on its deterministic approximation. Curiously, this literature has been largely ignored by random graph theorists.* The broader goal of this paper is in breaching the gap between the theory

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of the Smoluchowski coagulation equations for the Marcus-Lushnikov processes and the random graph theory. This paper concentrates on extending the connection between coalescent processes and random graph processes, e.g. Erdős-Rényi random graph evolution as described in Subsection 2.1. In particular, deriving a formula for the limiting length of the minimal spanning tree in a random graph process in terms of the solutions of the Smoluchowski coagulation equations for the corresponding coalescent process.

In this paper we will concentrate on analyzing the length of the minimal spanning tree as the prime example that demonstrates the usefulness of the Marcus-Lushnikov processes and the coalescence theory in general for answering questions about random graphs. We recall that the asymptotic limit for the mean length of a minimal spanning tree on K_n was derived in Frieze [13], $\lim_{n \rightarrow \infty} E[L_n] = \zeta(3) = \sum_{k=1}^{\infty} \frac{1}{k^3}$. The mean length of a minimal spanning tree on the complete bipartite graph $K_{n,n}$ was shown to be $\lim_{n \rightarrow \infty} E[L_n] = 2\zeta(3)$. See [14]. In Beveridge et al [5], the minimal spanning tree problem was addressed for d -regular graphs. In this paper we will concentrate on finding $\lim_{n \rightarrow \infty} E[L_n]$ for irregular bipartite graphs by means of developing a connection between the coalescence theory and the random graph theory.

We observe that in many cases the cluster dynamics of a random graph process can be replicated with the corresponding coalescent process. For example, the Erdős-Rényi random graph process on K_n can be tied to the n -particle multiplicative coalescent, and there are many other, more elaborate examples. The cluster dynamics of a coalescent process (without merger history) is reflected in a auxiliary process called the Marcus-Lushnikov process. The merger dynamics of such coalescent processes corresponds to a greedy algorithm for finding the minimal spanning tree in the respective random graph process. This observation allows us to express the limiting mean length of a minimal spanning tree in terms of the solutions of the Smoluchowski coagulation equations that represent the hydrodynamic limit of the Marcus-Lushnikov process corresponding to the random graph process. At the end of Subsection 2.3 a general approach for finding $\lim_{n \rightarrow \infty} E[L_n]$ via the Smoluchowski coagulation equations is proposed. As a particular application of the proposed general approach, we find the asymptotics for the mean length of a minimal spanning tree on the complete bipartite graph $K_{\alpha n, \beta n}$ in Section 3. Observe that for $\alpha \neq \beta$, the asymmetric complete bipartite graph $K_{\alpha n, \beta n}$ is an irregular graph.

2. ERDŐS-RÉNYI PROCESS ON K_n AND MULTIPLICATIVE COALESCENT

2.1. Erdős-Rényi random graph. Recall that Erdős-Rényi random graph is a model on a complete graph of n vertices, K_n , where each edge e of $\binom{n}{2}$ edges there is an associated uniform random variable U_e over $[0, 1]$. The random variables $\{U_e\}_e$ are assumed to be independent. For the “time” parameter $p \in [0, 1]$, an edge e is considered “open” if $U_e \leq p$. Erdős-Rényi random graph $G(n, p)$ will consist of all n vertices and all open edges at time p . The number of open edges is a binomial random variable with parameters $\binom{n}{2}$ and p ,

and mean value $\binom{n}{2}p \sim \frac{pn^2}{2}$. As we increase p , more and more edges open up, new clusters are created, and cluster merges occur. Thus Erdős-Rényi random graph model can be viewed as a dynamical model that describes an evolution of a random graph [7].

If we condition on the number of edges in $G(n, p)$, the graph structure will no longer depend on p . Let $\xi_{n, N}$ be the number of components in an Erdős-Rényi random graph with n vertices and N edges. For $t > 0$, letting $N \sim \frac{tn}{2}$, Theorem 6 in [7] by P. Erdős and A. Rényi states that

$$(1) \quad \frac{E[\xi_{n, N}]}{n} = \frac{1}{t} \sum_{k=1}^{\infty} \frac{k^{k-2}(te^{-t})^k}{k!} + \mathcal{R}_t,$$

where the error term is

$$\mathcal{R}_t = \begin{cases} O\left(\frac{1}{n}\right) & \text{if } 0 < t < 1 \\ O\left(\frac{\log n}{n}\right) & \text{if } t = 1 \\ o(1) & \text{if } t > 1 \end{cases}.$$

There $\varphi(t) = \frac{1}{t} \sum_{k=1}^{\infty} \frac{k^{k-2}(te^{-t})^k}{k!}$ reaches its maximum at $t = 1$, and $\varphi(1) = \sum_{k=1}^{\infty} \frac{k^{k-2}e^{-k}}{k!} = \frac{1}{2}$.

Let for $t > 0$,

$$(2) \quad x(t) := \min\{x > 0 : xe^{-x} = te^{-t}\}.$$

In other words $x(t)$ is the unique $x \in (0, 1]$ such that $xe^{-x} = te^{-t}$. Obviously, $x(t) = t$ for $0 < t \leq 1$. It was pointed out by P. Erdős and A. Rényi that $\varphi(t)$ in the equation (1) can be represented via $x(t)$ as follows,

$$\varphi(t) = \frac{x(t) - \frac{x^2(t)}{2}}{t}.$$

Observe that here, since we are letting $N \sim \frac{tn}{2}$, parameter t is essentially equivalent to np . So t is a scaled time parameter.

2.2. Multiplicative Coalescent. A general finite coalescent process begins with n singletons (clusters of mass one). The cluster formation is governed by a symmetric collision rate kernel $K(i, j) = K(j, i) > 0$. Specifically, a pair of clusters with masses (weights) i and j coalesces at the rate $K(i, j)/n$, independently of the other pairs, to form a new cluster of mass $i + j$. The process continues until there is a single cluster of mass n . See [24, 2, 4, 3, 10] and references therein.

Formally, for a given n consider the space $\mathcal{P}_{[n]}$ of partitions of $[n] = \{1, 2, \dots, n\}$. Let $\Pi_0^{(n)}$ be the initial partition in singletons, and $\Pi_t^{(n)}$ ($t \geq 0$) be a strong Markov process such that $\Pi_t^{(n)}$ transitions from partition $\pi \in \mathcal{P}_{[n]}$ to $\pi' \in \mathcal{P}_{[n]}$ with rate $K(i, j)/n$ provided that partition π' is obtained from partition π by merging two clusters of π of weights i and j . If $K(i, j) \equiv 1$ for all positive integer masses i and j , the process $\Pi_t^{(n)}$ is known as Kingman's n -coalescent process. If $K(i, j) = i + j$ the process is called n -particle additive coalescent.

Finally, if $K(i, j) = ij$ the process is called n -particle multiplicative coalescent. The so called Marcus-Lushnikov process

$$\mathbf{ML}_n(t) = \left(\zeta_{1,n}(t), \zeta_{2,n}(t), \dots, \zeta_{n,n}(t), 0, 0, \dots \right)$$

is an auxiliary process to the corresponding coalescent process that keeps track of the numbers of clusters in each weight category. Here $\zeta_{k,n}(t)$ denotes the number of clusters of weight k at time $t \geq 0$. See [20] and [19] for the original papers by Marcus and Lushnikov. The latter work considered the gelation phenomenon emerging in some of the Marcus-Lushnikov processes. The Marcus-Lushnikov process does not differentiate between the clusters of the same weight, and therefore does not keep track of the merger history of the n -particle coalescent process.

Let the number of vertices in a connected component of a random graph be referred to as a weight of the cluster (or cluster size). Consider the Marcus-Lushnikov process $\mathbf{ML}_n(t)$ corresponding to the multiplicative coalescent process of n particles. Since the coalescent process begins with n singletons, $\mathbf{ML}_n(0) = (n, 0, 0, \dots)$. By construction, the process $\mathbf{ML}_n(t)$ describes cluster size dynamics of the Erdős-Rényi random graph process $G(n, p)$ with $p = 1 - e^{-t/n}$. Here the scaled time parameter in the Erdős-Rényi process is $np = n(1 - e^{-t/n}) \sim t$. Thus the time scale is consistent with the one used in [7] by P. Erdős and A. Rényi. This Marcus-Lushnikov process keeps track of the history of cluster mergers and cluster sizes, but not of individual clusters' history. Let $\zeta_{k,n}(t)$ be the number of clusters of mass k in a multiplicative coalescent process of n particles at time t . The deterministic dynamics of the limiting fractions $\zeta_k(t) = \lim_{n \rightarrow \infty} \frac{\zeta_{k,n}(t)}{n}$ is described by the Smoluchowski coagulation equations [2, 23, 26] as follows

$$(3) \quad \frac{d}{dt} \zeta_k = -k\zeta_k \sum_{j=1}^{\infty} j\zeta_j + \frac{1}{2} \sum_{j=1}^{k-1} j(k-j)\zeta_j\zeta_{k-j} \quad (k = 1, 2, \dots) \quad \text{with } \zeta_k(0) = \delta_{1,k}.$$

We begin with $\sum_{j=1}^{\infty} j\zeta_j(0) = 1$, and following McLeod [21], we have

$$\begin{aligned} \frac{d}{dt} \sum_{j=1}^{\infty} j\zeta_j &= \sum_{j=1}^{\infty} j \frac{d}{dt} \zeta_j = - \sum_{i,j=1}^{\infty} ij^2\zeta_j\zeta_i + \frac{1}{2} \sum_{j=1}^{\infty} \sum_{i=1}^{j-1} (i + (j-i))i(j-i)\zeta_i\zeta_{j-i} \\ &= - \sum_{i,j=1}^{\infty} ij^2\zeta_j\zeta_i + \frac{1}{2} \sum_{i,j=1}^{\infty} (i+j)ij\zeta_i\zeta_j = 0 \end{aligned}$$

provided convergence of $\sum_{j=1}^{\infty} j^2\zeta_j(t)$. Thus there exists a time $T_{gel} \in [0, \infty]$, defined as

$$T_{gel} := \sup \left\{ t > 0 : \sum_{j=1}^{\infty} j^2\zeta_j(t) \text{ converges} \right\},$$

such that the following *conservation of mass* formula is satisfied up to time T_{gel} ,

$$\sum_{j=1}^{\infty} j\zeta_j(t) = 1.$$

Time $T_{gel} > 0$, if finite, is called the *gelation time*. The kernel function $K(\cdot, \cdot)$ for which such $T_{gel} < \infty$ is called the *gelling kernel*. It is well known [21] that the multiplicative kernel $K(i, j) = ij$ is a gelation kernel with the gelling time $T_{gel} = 1$. Indeed, as it was done in [21], the Smoluchowski coagulation equations reduce to

$$(4) \quad \frac{d}{dt}\zeta_k = -k\zeta_k + \frac{1}{2} \sum_{j=1}^{k-1} j(k-j)\zeta_j\zeta_{k-j} \quad (k = 1, 2, \dots) \quad \text{with } \zeta_k(0) = \delta_{1,k}$$

which can be explicitly solved:

$$(5) \quad \zeta_k(t) = \frac{k^{k-2}t^{k-1}}{k!} e^{-kt} \quad \text{for } t \geq 0.$$

The above reduced Smoluchowski system (4) is also known as the Flory's coagulation system of equations (named after Flory [12]). Notice, that for the solution (5) of system (4),

$$\begin{cases} \sum_{k=1}^{\infty} k\zeta_k(t) = 1 & \text{if } t \leq T_{gel} \\ \sum_{k=1}^{\infty} k\zeta_k(t) < 1 & \text{if } t > T_{gel}. \end{cases}$$

The phenomenon of losing total mass after a certain finite time T_{gel} is called *gelation*. It is an important phenomenon that was studied extensively in the coagulations equations literature. See [1, 2, 28, 17, 19].

The hydrodynamic limit $\lim_{n \rightarrow \infty} \frac{\zeta_{k,n}(t)}{n} = \zeta_k(t)$ is proven in formula (41) of Subsection 4.2 of this paper, where for a fixed time $T > 0$, it is shown that

$$\lim_{n \rightarrow \infty} \sup_{s \in [0, T]} |n^{-1}\zeta_{k,n}(s) - \zeta_k(s)| = 0 \quad \text{a.s.}$$

for each $k \geq 1$. Also, for a given integer $K > 0$, we have

$$\lim_{n \rightarrow \infty} \sup_{s \in [0, T]} \left| \sum_{k=1}^K n^{-1}\zeta_{k,n}(s) - \sum_{k=1}^K \zeta_k(s) \right| = 0 \quad \text{a.s.}$$

Recall the function $x(t)$ defined in (2). It was observed in [21] that since $\sum_{k=1}^{\infty} \frac{k^{k-1}(te^{-t})^k}{k!} = x(t)$, the first moment of ζ_k ,

$$\sum_{k=1}^{\infty} k\zeta_k(t) = \frac{1}{t} \sum_{k=1}^{\infty} \frac{k^{k-1}(te^{-t})^k}{k!} = \frac{x(t)}{t}.$$

Thus, $\sum_{k=1}^{\infty} k\zeta_k(t) = 1$ if and only if $t \leq 1$, and $\sum_{k=1}^{\infty} k\zeta_k(t) < 1$ for $t > 1$. Therefore $T_{gel} = 1$.

It is important to observe that we are interested in the solution $\zeta_k(t)$ (as in (5)) of the reduced system (4) of Smoluchowski coagulation equations as the hydrodynamic limit of $\frac{\zeta_{k,n}(t)}{n}$ over the whole time interval $[0, \infty)$. The reason for considering $t \in [0, \infty)$ is based on mass conservation property in the Marcus-Lushnikov processes for all $t \geq 0$:

$$\sum_{k=1}^{\infty} k \frac{\zeta_{k,n}(t)}{n} = 1.$$

While for the solution of (3), the mass is conserved only until T_{gel} , and $\sum_{k=1}^{\infty} k\zeta_k(t) < 1$ for $t > T_{gel} = 1$. As we know, in the Erdős-Rényi process, T_{gel} corresponds to a time after which a single giant component emerges, and continues to absorb components of smaller size. The giant cluster dynamics is unobserved in (3), while the reduced Smoluchowski system (4) captures its influence on the dynamics of the smaller size clusters.

Indeed, in [7], P. Erdős and A. Rényi showed that the cycles are rare for a given fixed $t > 0$, and the clusters of size k at time t consist mainly of *isolated trees of order k* . Specifically, if τ_k denotes the number of isolated trees of order k , Theorem 4b in [7] asserts that

$$(6) \quad \lim_{n \rightarrow \infty} \frac{kE[\tau_k]}{n} = \frac{k^{k-2}t^{k-1}}{k!} e^{-kt} = \zeta_k(t)$$

and

$$(7) \quad \lim_{n \rightarrow \infty} \frac{\sum_{k=1}^{\infty} kE[\tau_k]}{n} = \lim_{n \rightarrow \infty} \frac{\sum_{k=1}^n kE[\tau_k]}{n} = \frac{x(t)}{t}.$$

Moreover, Theorem 9b in [7] proves the emergence of one giant component after $T_{gel} = 1$. There, if we let $\gamma_n(t)$ denote the size of the greatest component at time t , then

$$\lim_{n \rightarrow \infty} \frac{\gamma_n(t)}{n} = 1 - \frac{x(t)}{t} \quad \text{in probability.}$$

So the dynamics of $g(t) := 1 - \sum_{k=1}^{\infty} k\zeta_k(t) = 1 - \frac{x(t)}{t}$ represents the asymptotic size of the giant component.

2.3. The length of the minimal spanning tree in K_n . Recall that in the construction of the Erdős-Rényi random graph model, each edge e of the complete graph K_n had a random variable U_e associated with it. Here we considered U_e to be uniform over $[0, 1]$. However, in general, various types of probability distributions are considered in the extensive literature on the topic. Now, thinking of U_e as the length of the edge e , one can construct a minimal spanning tree on K_n . Let random variable L_n denote the length of such minimal spanning tree. The asymptotic limit of the mean value of L_n was considered in Frieze [13]. There, the results (6) and (7) from P. Erdős and A. Rényi [7] are used in proving the following limit

$$(8) \quad \lim_{n \rightarrow \infty} E[L_n] = \int_0^{\infty} \frac{x(t)}{t} dt = \sum_{k=1}^{\infty} \int_0^{\infty} \frac{k^{k-2} t^{k-1}}{k!} e^{-kt} dt = \zeta(3),$$

where $\zeta(3) = \sum_{k=1}^{\infty} \frac{1}{k^3} = 1.202\dots$ is the value of the Riemann zeta function at 3.

Consider a coalescent process with a kernel $K(x, y)$ for which $T_{gel} < \infty$ has been proved. See [1, 17]. Then for a corresponding random graph model, we use the following S. Janson's formula [16]

$$(9) \quad \lim_{n \rightarrow \infty} E[L_n] = \lim_{n \rightarrow \infty} \int_0^1 E[\kappa(G(n, p))] dp - 1,$$

where $\kappa(G(n, p))$ is the number of components in the Erdős-Rényi random graph $G(n, p)$, and prove the following statement.

Theorem 2.1. *The average length of the minimal spanning tree of the corresponding graph is*

$$(10) \quad \lim_{n \rightarrow \infty} E[L_n] = \sum_{k=1}^{\infty} \int_0^{\infty} \zeta_k(t) dt,$$

where $\zeta_k(t)$ are the solutions (5) of the corresponding system of Smoluchowski coagulation equations (4).

Observe that the above equation (10) is consistent with result (8) of Frieze [13].

We already observed that the Marcus Lushnikov process $\mathbf{ML}_n(t)$ corresponding to the multiplicative coalescent process that begins with n singletons is equivalent to the cluster

size dynamics in the process $G(n, 1 - e^{-t/n})$. Here

$$\begin{aligned}
\lim_{n \rightarrow \infty} E[L_n] &= \lim_{n \rightarrow \infty} \int_0^1 E[\kappa(G(n, p))] dp - 1 = \lim_{n \rightarrow \infty} \int_0^\infty \frac{1}{n} E[\kappa(G(n, 1 - e^{-t/n}))] e^{-t/n} dt - 1 \\
&= \lim_{n \rightarrow \infty} \int_0^\infty \sum_{k=1}^\infty \frac{1}{n} E[\kappa^{er}(k, n, 1 - e^{-t/n})] e^{-t/n} dt - 1 \\
&= \lim_{n \rightarrow \infty} \int_0^\infty \sum_{k=1}^\infty \frac{E[\zeta_{k,n}(t)]}{n} e^{-t/n} dt - 1,
\end{aligned}$$

where $\kappa^{er}(k, n, p)$ is the number of components of size k in $G(n, p)$ and $p = 1 - e^{-t/n}$. Therefore, one could informally calculate the limit as follows:

$$\begin{aligned}
\lim_{n \rightarrow \infty} E[L_n] &= \sum_{k=1}^\infty \int_0^\infty \zeta_k(t) dt + \lim_{n \rightarrow \infty} \int_{T_{gel}}^\infty \frac{1}{n} e^{-t/n} dt - 1 \\
(11) \qquad &= \sum_{k=1}^\infty \int_0^\infty \zeta_k(t) dt + \lim_{n \rightarrow \infty} e^{-T_{gel}/n} - 1 = \sum_{k=1}^\infty \int_0^\infty \zeta_k(t) dt,
\end{aligned}$$

Here $\int_{T_{gel}}^\infty \frac{1}{n} e^{-t/n} dt$ represents the emergence of one giant component at time $T_{gel} = 1$.

Beveridge et al [5] extended S. Janson's formula (9) to all connected graphs with i.i.d. uniform $[0, 1]$ edge lengths:

$$E[L_n] = \int_0^1 E[\kappa(G(n, p))] dp - 1.$$

This allows us to state our general objective as follows. Consider a Marcus-Lushnikov processes equivalent to the cluster size dynamics in a general random graph model. The solutions $\zeta_{\mathbf{k}}(t)$ for the corresponding reduced Smoluchowski coagulation equations are considered with \mathbf{k} in a certain index space, e.g. $\mathbf{k} \in \mathbb{Z}_+^2 \setminus \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$ in the model analyzed in Section 3. In case of a gelling kernel, the following generalization of formula (10) is proposed as a method for computing the mean length of the minimal spanning tree:

$$\lim_{n \rightarrow \infty} E[L_n] = \sum_{\mathbf{k}} \int_0^\infty \zeta_{\mathbf{k}}(t) dt.$$

2.4. Proof of Theorem 2.1. Here we give a rigorous proof of the approach in formula (11). Note that unlike the original proof in Frieze [13], our proof will not rely on knowing the distribution of sizes and the geometry of clusters in the Erdős-Rényi process as provided in [7]. Nor will it require knowing anything about large clusters or the emergence of a unique giant component at time $T_{gel} = 1$. All that we use is the weak convergence results of Kurtz [9, 18] that we applied to the Marcus-Lushnikov processes in Section 4.

Proof. Observe that

$$(12) \quad \lim_{t \rightarrow \infty} \sum_{k=1}^{\infty} k \zeta_k(t) = \lim_{t \rightarrow \infty} \sum_{k=1}^{\infty} \frac{k^{k-1} t^{k-1}}{k!} e^{-kt} = \lim_{t \rightarrow \infty} \frac{x(t)}{t} = 0.$$

Thus, for any given $\epsilon \in (0, 1/4)$, we can fix $T \gg T_{gel}$ so large that

$$(13) \quad \sum_{k=1}^{\infty} k \zeta_k(T) \leq \frac{\epsilon}{2}.$$

Notice that the above inequality (13) ties T to ϵ .

Fix integer $K > 0$. Since we know from Section 4 that $\lim_{n \rightarrow \infty} \frac{\zeta_{k,n}(t)}{n} = \zeta_k(t)$ *a.s.* over the interval $[0, T]$ for all $k \in \{1, 2, \dots, K\}$, the probability of the complement of the event

$$(14) \quad Q_{K,T,n}^{\epsilon} := \left\{ \sum_{k=1}^K \frac{k \zeta_{k,n}(T)}{n} \leq \epsilon \right\}$$

is decreasing to zero as $n \rightarrow \infty$. Moreover,

$$q_{K,T}^{\epsilon}(n) := P(\overline{Q_{K,T,n}^{\epsilon}}) = o(n^{-1})$$

by equation (47) in Subsection 4.4.

We will split $\int_0^\infty \sum_{k=1}^\infty \frac{E[\zeta_{k,n}(t)]}{n} e^{-t/n} dt$ as follows.

$$\begin{aligned}
\int_0^\infty \sum_{k=1}^\infty \frac{E[\zeta_{k,n}(t)]}{n} e^{-t/n} dt &= \int_0^T \sum_{k=1}^K \frac{E[\zeta_{k,n}(t)]}{n} e^{-t/n} dt && \text{(Term I)} \\
&+ \int_0^T \sum_{k=K+1}^\infty \frac{E[\zeta_{k,n}(t)]}{n} e^{-t/n} dt && \text{(Term II)} \\
&+ (1 - q_{K,T}^\epsilon(n)) \int_T^\infty \sum_{k=1}^K \frac{E[\zeta_{k,n}(t) \mid Q_{K,T,n}^\epsilon]}{n} e^{-t/n} dt && \text{(Term III)} \\
&+ (1 - q_{K,T}^\epsilon(n)) \int_T^\infty \sum_{k=K+1}^\infty \frac{E[\zeta_{k,n}(t) \mid Q_{K,T,n}^\epsilon]}{n} e^{-t/n} dt && \text{(Term IV)} \\
(15) \quad &+ q_{K,T}^\epsilon(n) \int_T^\infty \sum_{k=1}^\infty \frac{E[\zeta_{k,n}(t) \mid \overline{Q_{K,T,n}^\epsilon}]}{n} e^{-t/n} dt && \text{(Term V)}
\end{aligned}$$

Next, we estimate the terms **I-V** in (15).

Term I. As it is proven in Section 4, $\lim_{n \rightarrow \infty} \frac{\zeta_{k,n}(t)}{n} = \zeta_k(t)$ *a.s.* on $[0, T]$ for all $k = 1, 2, \dots, K$. Therefore,

$$\lim_{n \rightarrow \infty} \int_0^T \sum_{k=1}^K \frac{E[\zeta_{k,n}(t)]}{n} e^{-t/n} dt = \sum_{k=1}^K \int_0^T \zeta_k(t) dt.$$

Term II. Observe that,

$$\sum_{k=K+1}^\infty \frac{\zeta_{k,n}(t)}{n} \leq \frac{1}{Kn} \sum_{k=K+1}^\infty k \zeta_{k,n}(t) = \frac{1}{K} \left(1 - \sum_{k=1}^K \frac{k \zeta_{k,n}(t)}{n} \right) \leq \frac{1}{K}.$$

Thus,

$$\int_0^T \sum_{k=K+1}^\infty \frac{E[\zeta_{k,n}(t)]}{n} e^{-t/n} dt = O\left(\frac{T}{K}\right)$$

regardless of the value of $n > 0$.

Term III. Recall that in the theory of Marcus-Lushnikov processes the *gel* is the set of all “large” clusters. By analogy, we define the K -*gel* to be the collection of all clusters of mass bigger than K . Let $M_{K \text{ gel}}(t)$ denote the total mass of all clusters in the K -*gel* at time $t \geq 0$.

Now, conditioning on the event $Q_{K,T,n}^\epsilon$, the mass of the K -gel is $M_{K\text{-gel}}(t) \geq (1 - \epsilon)n$ for all $t \geq T$. Thus each cluster not in K -gel will be gravitating toward the K -gel with the rate of at least $\frac{M_{K\text{-gel}}(t)}{n} \geq 1 - \epsilon$. Consider a cluster that was not in K -gel at time T . Let $T + L$ be the time it becomes a part of the K -gel. Then, its contribution to the integral $\int_T^\infty \sum_{k=1}^K \frac{E[\zeta_{k,n}(t) \mid Q_{K,T,n}^\epsilon]}{n} e^{-t/n} dt$ is at most

$$\begin{aligned} \int_T^\infty \frac{E[\mathbf{1}_{[T,T+L]}(t) \mid Q_{K,T,n}^\epsilon]}{n} e^{-t/n} dt &\leq \frac{\int_T^\infty E[\mathbf{1}_{[T,T+L]}(t) \mid Q_{K,T,n}^\epsilon] dt}{n} e^{-T/n} \\ &= \frac{E[L \mid Q_{K,T,n}^\epsilon]}{n} e^{-T/n} \leq \frac{1}{(1 - \epsilon)n}, \end{aligned}$$

where

$$\mathbf{1}_A = \begin{cases} 1 & \text{if } t \in A \\ 0 & \text{if } t \notin A \end{cases}.$$

The number of clusters not in K -gel at time $t \geq T$ is

$$\sum_{k=1}^K \zeta_{k,n}(t) \leq \sum_{k=1}^K k \zeta_{k,n}(t) \leq \epsilon n.$$

Therefore,

$$\int_T^\infty \sum_{k=1}^K \frac{E[\zeta_{k,n}(t) \mid Q_{K,T,n}^\epsilon]}{n} e^{-t/n} dt \leq \frac{\epsilon n}{(1 - \epsilon)n} = \frac{\epsilon}{1 - \epsilon} < 2\epsilon.$$

Term IV. We let $\mathcal{C} = \{C_1, C_2, C_3, \dots, C_M\}$ denote the set of all clusters that ever exceeded mass K in the whole history of the process $\{\mathbf{ML}_n(t)\}_{t \in [0, \infty)}$. There are less than n/K such clusters, i.e. $M < n/K$. For each C_i , the emergence time a_i is the time when a pair of clusters of mass not exceeding K mergers into a new cluster C_i of mass greater than K . We enumerate these clusters in the order they emerge.

Let $M_i(t)$ denote the mass of cluster C_i at time t . Consider a pair of clusters, C_i and C_j , coexisting in the K -gel at time t , each of mass smaller than $n/2$. We split their merger rate into two by saying that C_i absorbs C_j with rate $\frac{1}{2n} M_i(t) M_j(t)$, and C_j absorbs C_i with rate $\frac{1}{2n} M_i(t) M_j(t)$. In other words, C_i and C_j merge with rate $\frac{1}{n} M_i(t) M_j(t)$, and which one of the two clusters absorbs the other is decided with a toss of an independent fair coin.

There is a finite stopping time

$$t^* = \min\{t \geq 0 : \exists C_i \in \mathcal{C} \text{ with } M_i(t) \geq n/2\}$$

when a cluster C_{i^*} has its mass $M_{i^*}(t^*) \geq n/2$. After t^* , the rules of interactions of cluster C_{i^*} with the other clusters in \mathcal{C} change as follows. For $t > t^*$, C_{i^*} absorbs C_j with rate $\frac{1}{n}M_{i^*}(t)M_j(t)$, while C_{i^*} itself cannot be absorbed by any other cluster in \mathcal{C} .

Let b_i denote the time when cluster C_i is absorbed by another cluster in collection \mathcal{C} . Naturally, there will be only one survivor C_{i^*} with $b_{i^*} = \infty$. Let $J_i = [a_i, b_i) \cap [T, \infty)$ denote the lifespan of cluster C_i . Note that a cluster C_i from the set \mathcal{C} existing at time $t \in [a_i, b_i)$ is absorbed into one of the clusters in the K -gel with the total instantaneous rate of

$$\lambda_i(t) \geq \frac{1}{2n}M_i(t)(M_{K\text{-gel}}(t) - M_i(t)).$$

Conditioning on the event $Q_{K,T,n}^\epsilon$ defined in (14), we have that if $M_i(t) < n/2$ for $t \in J_i$, then the rate of absorption of C_i into the K -gel is

$$\lambda_i(t) \geq \frac{1}{2n}M_i(t) \left((1 - \epsilon)n - \frac{1}{2}n \right) \geq \frac{1}{2n}M_i(t) \left(\frac{3}{4}n - \frac{1}{2}n \right) \geq \frac{1}{8}M_i(t) > \frac{K}{8}.$$

Next,

$$(16) \quad \int_T^\infty \sum_{k=K+1}^\infty \frac{E[\zeta_{k,n}(t) \mid Q_{K,T,n}^\epsilon]}{n} e^{-t/n} dt = \int_T^\infty \frac{1}{n} e^{-t/n} dt + \mathcal{E}$$

where $\int_T^\infty \frac{1}{n} e^{-t/n} dt$ is due to the event $Q_{K,T,n}^\epsilon$ which guarantees the existence of at least one component from \mathcal{C} in the K -gel for all $t \in [T, \infty)$ and the second term \mathcal{E} is responsible for all the times $t \geq T$ when the number of clusters in the K -gel is greater than one. The term \mathcal{E} is bounded as follows

$$\mathcal{E} \leq \int_T^\infty \frac{E \left[\sum_{i: i \neq i^*} \mathbf{1}_{J_i}(t) \mid Q_{K,T,n}^\epsilon \right]}{n} e^{-t/n} dt.$$

Now, each cluster C_i is gravitating towards the rest of the K -gel with the rate of at least $K/8$. Thus, for each $i \neq i^*$,

$$\int_T^\infty \frac{E[\mathbf{1}_{J_i}(t) \mid Q_{K,T,n}^\epsilon]}{n} e^{-t/n} dt \leq \frac{E[|J_i| \mid Q_{K,T,n}^\epsilon]}{n} e^{-\frac{T}{n}} \leq \frac{8}{nK}.$$

Hence, since the cardinality of set \mathcal{C} is $M < n/K$,

$$\mathcal{E} < \frac{n}{K} \cdot \frac{8}{nK} = \frac{8}{K^2},$$

and from (16), we obtain

$$\int_T^\infty \sum_{k=K+1}^\infty \frac{E[\zeta_{k,n}(t) \mid Q_{K,T,n}^\epsilon]}{n} e^{-t/n} dt = 1 + O(K^{-2}) + O\left(\frac{T}{n}\right) \quad \text{as } n \rightarrow \infty,$$

where the term $O(K^{-2})$ does not depend on the value of $n > 0$.

Term V. Here

$$q_{K,T}^\epsilon(n) \int_T^\infty \sum_{k=1}^\infty \frac{E[\zeta_{k,n}(t) \mid \overline{Q_{K,T,n}^\epsilon}]}{n} e^{-t/n} dt \leq nq_{K,T}^\epsilon(n) \int_T^\infty \frac{1}{n} e^{-t/n} dt \leq nq_{K,T}^\epsilon(n) = o(1).$$

Finally, by putting together the analysis in **Terms I-V** in the equation (15), we obtain for a given fixed $\epsilon \in (0, 1/4)$, sufficiently large fixed $T \gg T_{gel}$ satisfying (13), and arbitrarily large K ,

$$(17) \quad \int_0^\infty \sum_{k=1}^\infty \frac{E[\zeta_{k,n}(t)]}{n} e^{-t/n} dt = \sum_{k=1}^K \int_0^T \zeta_k(t) dt + 1 + O\left(\frac{T}{K}\right) + O(K^{-2}) + O(\epsilon) + O\left(\frac{T}{n}\right) + o(1),$$

which, when we increase n to infinity will yield

$$\begin{aligned} \limsup_{n \rightarrow \infty} \left| \int_0^\infty \sum_{k=1}^\infty \frac{E[\zeta_{k,n}(t)]}{n} e^{-t/n} dt - \sum_{k=1}^\infty \int_0^\infty \zeta_k(t) dt - 1 \right| &= \sum_{k=K+1}^\infty \int_0^T \zeta_k(t) dt + \sum_{k=1}^\infty \int_T^\infty \zeta_k(t) dt \\ &\quad + O\left(\frac{T}{K}\right) + O(K^{-2}) + O(\epsilon). \end{aligned}$$

Consequently, taking $\limsup_{K \rightarrow \infty}$, we obtain

$$\limsup_{n \rightarrow \infty} \left| \int_0^\infty \sum_{k=1}^\infty \frac{E[\zeta_{k,n}(t)]}{n} e^{-t/n} dt - \sum_{k=1}^\infty \int_0^\infty \zeta_k(t) dt - 1 \right| = \sum_{k=1}^\infty \int_T^\infty \zeta_k(t) dt + O(\epsilon).$$

Finally, formula (13) guarantees that decreasing ϵ down to zero will propel T to $+\infty$, and

$$\lim_{n \rightarrow \infty} \int_0^\infty \sum_{k=1}^\infty \frac{E[\zeta_{k,n}(t)]}{n} e^{-t/n} dt = \sum_{k=1}^\infty \int_0^\infty \zeta_k(t) dt + 1.$$

Thus we confirmed formula (10) for the case of the multiplicative coalescent process. \square

2.5. Gelation. Here, we would like to summarize the main results regarding the gelation phenomenon. Consider a general system of Smoluchowski coagulation equations with a positive symmetric kernel $K(i, j) = K(j, i) > 0$,

$$\frac{d}{dt}\zeta_j = -\zeta_j \sum_{j=1}^{\infty} K(i, j)\zeta_i + \frac{1}{2} \sum_{i=1}^{j-1} K(i, j-i)\zeta_i\zeta_{j-i} \quad (k = 1, 2, \dots) \quad \text{with } \zeta_j(0) = \delta_{1,j}.$$

Then, following [21], we use the above Smoluchowski coagulation equations to obtain

$$\begin{aligned} \frac{d}{dt} \sum_{j=1}^{\infty} j\zeta_j &= \sum_{j=1}^{\infty} j \frac{d}{dt} \zeta_j = - \sum_{i,j=1}^{\infty} jK(i, j)\zeta_j\zeta_i + \frac{1}{2} \sum_{j=1}^{\infty} \sum_{i=1}^{j-1} (i + (j - i))K(i, j - i)\zeta_i\zeta_{j-i} \\ &= - \sum_{i,j=1}^{\infty} jK(i, j)\zeta_j\zeta_i + \frac{1}{2} \sum_{i,j=1}^{\infty} (i + j)K(i, j)\zeta_i\zeta_j = 0 \end{aligned}$$

provided convergence of $\sum_{i,j=1}^{\infty} jK(i, j)\zeta_j\zeta_i$. Therefore, letting

$$T_{gel} := \sup \left\{ t > 0 : \sum_{i,j=1}^{\infty} jK(i, j)\zeta_j(t)\zeta_i(t) \text{ converges} \right\},$$

we have $\frac{d}{dt} \sum_{j=1}^{\infty} j\zeta_j = 0$ for $t \in [0, T_{gel})$, which in turn implies $\sum_{j=1}^{\infty} j\zeta_j(t) = 1$ for $t \in [0, T_{gel})$.

The question whether $T_{gel} < \infty$ is the question of whether gelation phenomenon occurs in a given system of Smoluchowski equations. The first mathematical proof of gelation was produced in McLeod [21] for the multiplicative kernel. Historically, that happened around the time when the formation of a giant cluster was proved by P. Erdős and A. Rényi [7]. The overlap in mathematical formulas obtained in the two papers, [21] and [7], representing the two different branches of mathematics is quite remarkable. The work of finding a mathematically solid proof of gelation phenomenon for other conjectured gelling kernels began fifteen years later with a work by Lushnikov [19]. It continued with publications of Ziff [29], Ernst et al. [8], van Dongen and Ernst [28], and many other mathematicians and mathematical physicists. In Spouge [27], the gelation is demonstrated numerically for the general bilinear kernel $K(i, j) = A + B(i + j) + Cij$. While in Aldous [1], the gelation is proved for $K(i, j) = \frac{2(ij)^\gamma}{(i+j)^\gamma - i^\gamma - j^\gamma}$, where $\gamma \in (1, 2)$. There $\gamma = 2$ would correspond to the multiplicative kernel for which, as we know, gelatin also occurs. Jeon [17] proved that complete and instantaneous gelation occurs if $K(i, j) \geq ij\psi(i, j)$, where $\psi(i, j)$ is a function increasing in both variables, i and j , such that $\sum_{j=1}^{\infty} \frac{1}{j\psi(i, j)} < \infty$ for all i . This includes $K(i, j) = (ij)^\alpha$, $\alpha > 1$, as a primary example.

3. ERDŐS-RÉNYI PROCESS ON $K_{\alpha n, \beta n}$ AND CROSS-MULTIPLICATIVE COALESCENT

For $\alpha, \beta > 0$, consider two integer valued functions, $\alpha[n] = \alpha n + o(n)$ and $\beta[n] = \beta n + o(n)$. Consider an Erdős-Rényi random graph process on the bipartite graph $K_{\alpha[n], \beta[n]}$ with $\alpha[n]$ vertices on the left side and $\beta[n]$ vertices on the right side. There each edge e of $\sim \alpha\beta n^2$ edges has an associated random variable U_e . The random variables U_e are assumed to be independent and uniform over $[0, 1]$. For the time parameter $p \in [0, 1]$, an edge e is considered open if $U_e \leq p$. Erdős-Rényi random graph $G(n, p)$ will consist of all n vertices and all open edges at time p .

Next, consider a coalescent process corresponding to an Erdős-Rényi random graph process on $K_{\alpha[n], \beta[n]}$. Specifically, let each cluster connecting i_1 vertices on the left side of the bipartite graph with i_2 vertices on the right side of the bipartite graph be assigned a two-dimensional weight vector $\mathbf{i} = \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}$. There $i_1, i_2 \geq 0$ and $i_1 + i_2 > 0$. Define the

coalescence kernel as follows. For any pair of clusters with weight vectors $\mathbf{i} = \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}$ and

$\mathbf{j} = \begin{bmatrix} j_1 \\ j_2 \end{bmatrix}$, let

$$(18) \quad K(\mathbf{i}, \mathbf{j}) := i_1 j_2 + i_2 j_1.$$

The coalescent process begins with $\alpha[n] + \beta[n]$ singletons, of which $\alpha[n]$ singletons are of weight $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and the other $\beta[n]$ singletons are of weight $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. There, each pair of clusters of respective weights \mathbf{i} and \mathbf{j} would coalesce into a cluster of weight $\mathbf{i} + \mathbf{j}$ with rate $K(\mathbf{i}, \mathbf{j})/n$. The last merger will create a cluster of weight $\begin{bmatrix} \alpha[n] \\ \beta[n] \end{bmatrix}$. We will call this a **cross-multiplicative coalescent process**.

3.1. Smoluchowski coagulation equations. Consider the Marcus-Lushnikov process $\mathbf{ML}_n(t)$ that keeps track of cluster counts in the above defined cross-multiplicative coalescent process that begins with the $\alpha[n] + \beta[n]$ singletons of the two types, $\alpha[n]$ of weight $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\beta[n]$ of weight $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Specifically, let $\zeta_{i_1, i_2}^{[n]}(t)$ denote the number of the components in the cross-multiplicative coalescent process of weight $\mathbf{i} = \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}$ at time t . Then $\mathbf{ML}_n(t)$ is the infinite-dimensional process with coordinates $\zeta_{i_1, i_2}^{[n]}(t)$, i.e.

$$\mathbf{ML}_n(t) = \left(\zeta_{i_1, i_2}^{[n]}(t) \right)_{i_1, i_2}.$$

The Smoluchowski coagulation equations for the Marcus-Lushnikov process $\mathbf{ML}_n(t)$ with bipartite multiplicative kernel are written as follows:

$$\frac{d}{dt}\zeta_{i_1, i_2}(t) = -\zeta_{i_1, i_2}(t) \sum_{j_1, j_2} (i_1 j_2 + i_2 j_1) \zeta_{j_1, j_2}(t) + \frac{1}{2} \sum_{\substack{\ell_1, k_1: \ell_1 + k_1 = i_1, \\ \ell_2, k_2: \ell_2 + k_2 = i_2}} (\ell_1 k_2 + \ell_2 k_1) \zeta_{\ell_1, \ell_2}(t) \zeta_{k_1, k_2}(t) \quad (19)$$

with the initial conditions $\zeta_{1,0}(0) = \alpha$ and $\zeta_{0,1}(0) = \beta$.

A reduced system of differential equation corresponding to the above Smoluchowski coagulation equations (19) will be given in (20). It will take into account the mass conservation property of the above Marcus-Lushnikov process $\mathbf{ML}_n(t)$, and therefore will represent the smaller cluster dynamics over the whole time interval $[0, \infty)$.

First, we notice that here the initial total mass is $\sum_{i_1, i_2} (i_1 + i_2) \zeta_{i_1, i_2}(0) = \alpha + \beta$. Moreover, the initial total ‘left mass’ is $\sum_{i_1, i_2} i_1 \zeta_{i_1, i_2}(0) = \alpha$ and the initial total ‘right mass’ is $\sum_{i_1, i_2} i_2 \zeta_{i_1, i_2}(0) = \beta$.

Next, we consider the rate of change for the total left mass and the total right mass, and use (19) to obtain

$$\begin{aligned} \frac{d}{dt} \sum_{i_1, i_2} i_1 \zeta_{i_1, i_2}(t) &= - \sum_{i_1, i_2, j_1, j_2} i_1 (i_1 j_2 + i_2 j_1) \zeta_{i_1, i_2}(t) \zeta_{j_1, j_2}(t) \\ &\quad + \frac{1}{2} \sum_{\ell_1, k_1, \ell_2, k_2} (\ell_1 + k_1) (\ell_1 k_2 + \ell_2 k_1) \zeta_{\ell_1, \ell_2}(t) \zeta_{k_1, k_2}(t) = 0 \end{aligned}$$

and

$$\begin{aligned} \frac{d}{dt} \sum_{i_1, i_2} i_2 \zeta_{i_1, i_2}(t) &= - \sum_{i_1, i_2, j_1, j_2} i_2 (i_1 j_2 + i_2 j_1) \zeta_{i_1, i_2}(t) \zeta_{j_1, j_2}(t) \\ &\quad + \frac{1}{2} \sum_{\ell_1, k_1, \ell_2, k_2} (\ell_2 + k_2) (\ell_1 k_2 + \ell_2 k_1) \zeta_{\ell_1, \ell_2}(t) \zeta_{k_1, k_2}(t) = 0 \end{aligned}$$

whenever $\sum_{i_1, i_2} (i_1 + i_2)^2 \zeta_{i_1, i_2}(t)$ converges. Thus in order to establish whether the kernel defined in (18) is a gelling kernel, we need to consider whether

$$T_{gel} := \sup \left\{ t > 0 : \sum_{i_1, i_2} (i_1 + i_2)^2 \zeta_{i_1, i_2}(t) \text{ converges} \right\}$$

is finite.

Here, for $t < T_{gel}$, $\sum_{j_1, j_2} j_1 \zeta_{j_1, j_2}(t) = \alpha$ and $\sum_{j_1, j_2} j_2 \zeta_{j_1, j_2}(t) = \beta$. Therefore, for any i_1 and i_2 , $\sum_{j_1, j_2} (i_1 j_2 + i_2 j_1) \zeta_{j_1, j_2}(t) = \beta i_1 + \alpha i_2$. Thus we can consider the following reduced Smoluchowski coagulation equations:

$$(20) \quad \frac{d}{dt} \zeta_{i_1, i_2}(t) = -(\beta i_1 + \alpha i_2) \zeta_{i_1, i_2}(t) + \frac{1}{2} \sum_{\substack{\ell_1, k_1: \ell_1 + k_1 = i_1, \\ \ell_2, k_2: \ell_2 + k_2 = i_2}} (\ell_1 k_2 + \ell_2 k_1) \zeta_{\ell_1, \ell_2}(t) \zeta_{k_1, k_2}(t)$$

with the initial conditions $\zeta_{1,0}(0) = \alpha$ and $\zeta_{0,1}(0) = \beta$. Once again, the solutions of Smoluchowski coagulation system (19) and the above reduced Smoluchowski coagulation system (20) will match up until T_{gel} . Consecutively, the solution (24) of the reduced Smoluchowski system of equations (20) will be used in Subsection 3.3 in establishing the finiteness of the gelation time and for finding its value, T_{gel} .

The following hydrodynamic limit is proven in the equation (43) of Subsection 4.3. Fix a pair of positive integers K_1 and K_2 , and a real $T > 0$. Then,

$$\lim_{n \rightarrow \infty} \sup_{s \in [0, T]} \left| n^{-1} \zeta_{i_1, i_2}^{[n]}(s) - \zeta_{i_1, i_2}(s) \right| = 0 \quad \text{a.s.}$$

for all $i_1, i_2 \geq 1$. Consequently,

$$\lim_{n \rightarrow \infty} \sup_{s \in [0, T]} \left| n^{-1} \sum_{\substack{1 \leq i_1 \leq K_1 \\ 1 \leq i_2 \leq K_2}} \zeta_{i_1, i_2}^{[n]}(s) - \sum_{\substack{1 \leq i_1 \leq K_1 \\ 1 \leq i_2 \leq K_2}} \zeta_{i_1, i_2}(s) \right| = 0 \quad \text{a.s.}$$

3.2. Solution $\zeta_{i_1, i_2}(t)$ of (20). Observe that as it was the case with the multiplicative coalescent, we will consider the above reduced system of differential equations (20) over the whole time interval $[0, \infty)$ because the mass conservation property

$$\sum_{i_1, i_2} i_1 \zeta_{i_1, i_2}(t) = \alpha \quad \text{and} \quad \sum_{i_1, i_2} i_2 \zeta_{i_1, i_2}(t) = \beta$$

in the Marcus-Lushnikov process $\mathbf{ML}_n(t)$ holds for all $t \in [0, \infty)$. Thus, while the solutions to (19) and (20) are identical over $[0, T_{gel})$, the reduced system (20) continues to reflect the dynamics of the smaller clusters even after T_{gel} .

Next, we want to find the solution $\zeta_{i_1, i_2}(t)$ of reduced system (20) for all $t \geq 0$. Here we observe that $\zeta_{1,0}(t) = \alpha e^{-\beta t}$ and $\zeta_{0,1}(t) = \beta e^{-\alpha t}$, and extend the approach of McLeod [21] by considering the solutions of the following form

$$(21) \quad \zeta_{i_1, i_2}(t) = \alpha^{i_1} \beta^{i_2} S_{i_1, i_2} e^{-(\beta i_1 + \alpha i_2)t} t^{i_1 + i_2 - 1}$$

and plugging them into the equation (20). After cancelations, we arrive with the following recursion

$$(22) \quad (i_1 + i_2 - 1)S_{i_1, i_2} = \frac{1}{2} \sum_{\substack{\ell_1, k_1: \ell_1 + k_1 = i_1, \\ \ell_2, k_2: \ell_2 + k_2 = i_2}} (\ell_1 k_2 + \ell_2 k_1) S_{\ell_1, \ell_2} S_{k_1, k_2}$$

with the initial conditions $S_{i,0} = S_{0,i} = \delta_{1,i}$. Here, $S_{i_1, i_2} = S_{i_2, i_1}$.

Here, we managed to solve the problem of finding S_{i_1, i_2} combinatorially.

Proposition 3.1. *The system of equations (22) with the initial conditions $S_{i,0} = S_{0,i} = \delta_{1,i}$ has the following unique solution*

$$S_{i_1, i_2} = \frac{i_1^{i_2-1} i_2^{i_1-1}}{i_1! i_2!}.$$

Proof. In Theorem 1.1(3) of [15], F. Huang and B. Liu generalize the Abel's binomial theorem as follows:

$$(23) \quad \sum_{k_1=0}^{i_1} \sum_{k_2=0}^{i_2} \binom{i_1}{k_1} \binom{i_2}{k_2} (v + zi_1 - zk_1)^{k_2-1} (-z(i_1 - k_1))^{i_2-k_2} (-zk_2)^{k_1} (u + zk_2)^{i_1-k_1-1} \\ = \frac{[uv - i_1 i_2 z^2] u^{i_1-1} v^{i_2-1}}{(v + i_1 z)(u + i_2 z)}$$

Then, we use (23) with $z = -1$ to confirm our candidate solution satisfies (22) by plugging it into the right hand side of (22) as follows.

$$\begin{aligned}
& \frac{1}{2} \sum_{\substack{\ell_1, k_1: \ell_1+k_1=i_1, \\ \ell_2, k_2: \ell_2+k_2=i_2}} (\ell_1 k_2 + \ell_2 k_1) S_{\ell_1, \ell_2} S_{k_1, k_2} = \sum_{\substack{\ell_1, k_1: \ell_1+k_1=i_1, \\ \ell_2, k_2: \ell_2+k_2=i_2}} \ell_1 k_2 S_{\ell_1, \ell_2} S_{k_1, k_2} \\
&= \sum_{\substack{\ell_1, k_1: \ell_1+k_1=i_1, \\ \ell_2, k_2: \ell_2+k_2=i_2, \\ (k_1, k_2), (\ell_1, \ell_2) \neq (0, 0)}} \frac{\ell_1^{\ell_2} \ell_2^{\ell_1-1} k_1^{k_2-1} k_2^{k_1}}{\ell_1! \ell_2! k_1! k_2!} \\
&= \frac{1}{i_1! i_2!} \sum_{\substack{k_1: 0 \leq k_1 \leq i_1, \\ k_2: 0 \leq k_2 \leq i_2, \\ (k_1, k_2) \neq (0, 0), (i_1, i_2)}} \binom{i_1}{k_1} \binom{i_2}{k_2} k_1^{k_2-1} (i_1 - k_1)^{i_2-k_2} k_2^{k_1} (i_2 - k_2)^{i_1-k_1-1} \\
&= \frac{1}{i_1! i_2!} \lim_{\substack{v \rightarrow i_1, \\ u \rightarrow i_2}} \left\{ \sum_{k_1=0}^{i_1} \sum_{k_2=0}^{i_2} \binom{i_1}{k_1} \binom{i_2}{k_2} (v - i_1 + k_1)^{k_2-1} (i_1 - k_1)^{i_2-k_2} k_2^{k_1} (u - k_2)^{i_1-k_1-1} \right. \\
&\quad \left. - \frac{i_1^{i_2} u^{i_1-1}}{v - i_1} - \frac{i_2^{i_1} v^{i_2-1}}{u - i_2} \right\} \\
&= \frac{1}{i_1! i_2!} \lim_{\substack{v \rightarrow i_1, \\ u \rightarrow i_2}} \left\{ \frac{[uv - i_1 i_2] u^{i_1-1} v^{i_2-1}}{(v - i_1)(u - i_2)} - \frac{i_1^{i_2} u^{i_1-1}}{v - i_1} - \frac{i_2^{i_1} v^{i_2-1}}{u - i_2} \right\} \\
&= \frac{1}{i_1! i_2!} \lim_{\substack{v \rightarrow i_1, \\ u \rightarrow i_2}} \left\{ \frac{i_1 v^{i_2-1} u^{i_1-1}}{v - i_1} + \frac{u^{i_1} v^{i_2-1}}{u - i_2} - \frac{i_1^{i_2} u^{i_1-1}}{v - i_1} - \frac{i_2^{i_1} v^{i_2-1}}{u - i_2} \right\}
\end{aligned}$$

Hence,

$$\begin{aligned}
\frac{1}{2} \sum_{\substack{\ell_1, k_1: \ell_1+k_1=i_1, \\ \ell_2, k_2: \ell_2+k_2=i_2}} (\ell_1 k_2 + \ell_2 k_1) S_{\ell_1, \ell_2} S_{k_1, k_2} &= \frac{1}{i_1! i_2!} \lim_{\substack{v \rightarrow i_1, \\ u \rightarrow i_2}} \left\{ i_1 u^{i_1-1} \frac{v^{i_2-1} - i_1^{i_2-1}}{v - i_1} + v^{i_2-1} \frac{u^{i_1} - i_2^{i_1}}{u - i_2} \right\} \\
&= \frac{1}{i_1! i_2!} \left((i_2 - 1) \cdot i_1^{i_2-1} i_2^{i_1-1} + i_1 \cdot i_1^{i_2-1} i_2^{i_1-1} \right) \\
&= (i_1 + i_2 - 1) \frac{i_1^{i_2-1} i_2^{i_1-1}}{i_1! i_2!} \\
&= (i_1 + i_2 - 1) S_{i_1, i_2}
\end{aligned}$$

thus completing the proof. \square

The solution of equations (20) follows from (21) and Proposition 3.1,

$$(24) \quad \zeta_{i_1, i_2}(t) = \frac{i_1^{i_2-1} i_2^{i_1-1} \alpha^{i_1} \beta^{i_2}}{i_1! i_2!} e^{-(\beta i_1 + \alpha i_2) t} t^{i_1+i_2-1}.$$

3.3. Gelation in $K_{\alpha n, \beta n}$. Next, we prove the finiteness of the gelation time

$$T_{gel} := \sup \left\{ t > 0 : \sum_{i_1, i_2} (i_1 + i_2)^2 \zeta_{i_1, i_2}(t) \text{ converges} \right\}.$$

From equation (24) we know that T_{gel} is the radius of convergence for the series

$$\sum_{i_1, i_2} (i_1 + i_2)^2 \zeta_{i_1, i_2}(t) = \sum_{i_1, i_2} (i_1 + i_2)^2 \frac{i_1^{i_2-1} i_2^{i_1-1} \alpha^{i_1} \beta^{i_2}}{i_1! i_2!} e^{-(\beta i_1 + \alpha i_2)t} t^{i_1 + i_2 - 1}.$$

Here, by Stirling's approximation, T_{gel} solves $1 - (\alpha \wedge \beta)t + \ln((\alpha \vee \beta)t) = 0$.

We also notice that the mass of the system in (20) is conserved until T_{gel} , after which it begins to dissipate, i.e.

$$\begin{cases} \sum_{i_1, i_2} (i_1 + i_2) \zeta_{i_1, i_2}(t) = \alpha + \beta & \text{if } t \leq T_{gel} \\ \sum_{i_1, i_2} (i_1 + i_2) \zeta_{i_1, i_2}(t) < \alpha + \beta & \text{if } t > T_{gel}. \end{cases}$$

3.4. The length of the minimal spanning tree on $K_{\alpha n, \beta n}$ via $\zeta_{i_1, i_2}(t)$. Once again, let us consider U_e to be the length of the edge e . Then one can construct a minimal spanning tree on $K_{\alpha n, \beta n}$. Let random variable L_n denote the length of such minimal spanning tree. We want to represent the asymptotic limit of the mean value of L_n via $\zeta_{i_1, i_2}(t)$.

For a random graph process $G(n, p)$ over $K_{\alpha n, \beta n}$, Lemma 1 in Beveridge et al [5] implies

$$(25) \quad E[L_n] = \int_0^1 E[\kappa(G(n, p))] dp - 1$$

provided the latter limit exists, where $\kappa(G(n, p))$ is the number of components in the random graph process $G(n, p)$ at time p . This will be used in establishing the following theorem that will be proved in Subsection 3.5.

Theorem 3.2. *The limiting mean length of the minimal spanning tree is*

$$(26) \quad \lim_{n \rightarrow \infty} E[L_n] = \sum_{i_1, i_2} \int_0^{\infty} \zeta_{i_1, i_2}(t) dt.$$

Observe that if we plug-in the solutions (21) of the reduced system of Smoluchowski coagulation equations (20) into the right hand side of (26), we get

$$\begin{aligned}
\sum_{i_1, i_2}^{\infty} \int_0^{\infty} \zeta_{i_1, i_2}(t) d(t) &= \frac{\alpha}{\beta} + \frac{\beta}{\alpha} + \sum_{i_1 \geq 1; i_2 \geq 1} \alpha^{i_1} \beta^{i_2} S_{i_1, i_2} \int_0^{\infty} t^{i_1+i_2-1} e^{-(\beta i_1 + \alpha i_2)t} dt \\
&= \frac{\alpha}{\beta} + \frac{\beta}{\alpha} + \sum_{i_1 \geq 1; i_2 \geq 1} \frac{\alpha^{i_1} \beta^{i_2} S_{i_1, i_2}}{(\beta i_1 + \alpha i_2)^{i_1+i_2}} (i_1 + i_2 - 1)! \\
(27) \quad &= \gamma + \frac{1}{\gamma} + \sum_{i_1 \geq 1; i_2 \geq 1} \frac{\gamma^{i_1} S_{i_1, i_2}}{(i_1 + \gamma i_2)^{i_1+i_2}} (i_1 + i_2 - 1)!
\end{aligned}$$

with $\gamma = \frac{\alpha}{\beta}$.

Next, by combining Proposition 3.1 with (27) we obtained the following main theorem.

Theorem 3.3. *The limiting mean length of the minimal spanning tree is*

$$\lim_{n \rightarrow \infty} E[L_n] = \gamma + \frac{1}{\gamma} + \sum_{i_1 \geq 1; i_2 \geq 1} \frac{(i_1 + i_2 - 1)!}{i_1! i_2!} \frac{\gamma^{i_1} i_1^{i_2-1} i_2^{i_1-1}}{(i_1 + \gamma i_2)^{i_1+i_2}}.$$

Theorem 3.3 is consistent with [14], where it was shown that for $\alpha = \beta$, $\lim_{n \rightarrow \infty} E[L_n] = 2\zeta(3)$. Indeed, we have the following Corollary reproducing the results in [14]. Observe however that for $\alpha \neq \beta$ the bipartite graph is irregular and the results in Frieze and McDiarmid [14] no longer apply.

Corollary 3.4. *If $\gamma = 1$,*

$$\lim_{n \rightarrow \infty} E[L_n] = 2\zeta(3).$$

Proof. Abel's binomial theorem [25] states that

$$\sum_{i=0}^n \binom{n}{i} (w + n - i)^{n-i-1} (z + i)^i = w^{-1} (z + w + n)^n.$$

Plugging-in $z = w = -n$, we obtain

$$\sum_{i=0}^n \binom{n}{i} i^{n-i-1} (n - i)^i = n^{n-1}$$

and therefore,

$$\sum_{i_1, i_2: i_1+i_2=n} i_1 S_{i_1, i_2} = \sum_{i_1, i_2: i_1+i_2=n} i_1 \frac{i_1^{i_2-1} i_2^{i_1-1}}{i_1! i_2!} = \frac{n^{n-1}}{n!}.$$

Hence,

$$n \cdot \sum_{i_1, i_2: i_1+i_2=n} S_{i_1, i_2} = \sum_{i_1, i_2: i_1+i_2=n} (i_1 + i_2) S_{i_1, i_2} = 2 \sum_{i_1, i_2: i_1+i_2=n} i_1 S_{i_1, i_2} = 2 \frac{n^{n-1}}{n!}$$

and

$$\sum_{i_1, i_2: i_1+i_2=n} S_{i_1, i_2} = 2 \frac{n^{n-2}}{n!}.$$

Plugging the above into (27) with $\gamma = 1$, we obtain

$$\begin{aligned} \lim_{n \rightarrow \infty} E[L_n] &= 2 + \sum_{i_1 \geq 1: i_2 \geq 1} \frac{S_{i_1, i_2}}{(i_1 + i_2)^{i_1+i_2}} (i_1 + i_2 - 1)! \\ &= 2 + \sum_{n=2}^{\infty} \left(\sum_{i_1, i_2: i_1+i_2=n} \frac{S_{i_1, i_2}}{n^n} \right) (n-1)! \\ &= 2 + \sum_{n=2}^{\infty} 2 \frac{n^{n-2}}{n!} \cdot \frac{1}{n^n} (n-1)! \\ (28) \quad &= 2 + \sum_{n=2}^{\infty} \frac{2}{n^3} = 2\zeta(3). \end{aligned}$$

Thus confirming the results in [14]. □

3.5. Proof of Theorem 3.2. Let us give a rigorous proof of Theorem 3.2. Here, we will follow the strategy used for proving Theorem 2.1 in Subsection 2.4.

Proof. Observe that

$$(29) \quad \lim_{t \rightarrow \infty} \sum_{i_1, i_2} i_1 \zeta_{i_1, i_2}(t) = 0 \quad \text{and} \quad \lim_{t \rightarrow \infty} \sum_{i_1, i_2} i_2 \zeta_{i_1, i_2}(t) = 0.$$

Indeed, by plugging in $\zeta_{i_1, i_2}(t)$ as in (24), we obtain

$$\frac{d}{dt} \sum_{i_1, i_2} i_1 \zeta_{i_1, i_2}(t) = \sum_{i_1, i_2} i_1 \zeta_{i_1, i_2}(t) \left(\frac{i_1 + i_2 - 1}{t} - (\beta i_1 + \alpha i_2) \right) \leq -\frac{\alpha \wedge \beta}{2} \sum_{i_1, i_2} i_1 \zeta_{i_1, i_2}(t)$$

for $t > \frac{1}{\alpha \wedge \beta}$. Thus, $\sum_{i_1, i_2} i_1 \zeta_{i_1, i_2}(t)$, and similarly $\sum_{i_1, i_2} i_2 \zeta_{i_1, i_2}(t)$, would decrease to zero exponentially fast when $t > \frac{1}{\alpha \wedge \beta}$.

Now, having established (29), for any given $\epsilon \in (0, 1/4)$, we can fix $T \gg T_{gel}$ so large that

$$(30) \quad \sum_{i_1, i_2} i_1 \zeta_{i_1, i_2}(t) \leq \frac{\alpha \epsilon}{2} \quad \text{and} \quad \sum_{i_1, i_2} i_2 \zeta_{i_1, i_2}(t) \leq \frac{\beta \epsilon}{2}.$$

Notice that the above inequalities (30) ties T to ϵ .

Fix integers $K_1 > 0$ and $K_2 > 0$, and let $R := R(K_1, K_2) = \{1, 2, \dots, K_1\} \times \{1, 2, \dots, K_2\}$.

Since we know from Section 4 that $\lim_{n \rightarrow \infty} \frac{\zeta_{i_1, i_2}^{[n]}(t)}{n} = \zeta_{i_1, i_2}(t)$ *a.s.* over the interval $[0, T]$ for

all $\mathbf{i} = \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} \in R$, the probability of the complement of the event

$$(31) \quad Q_{R,T,n}^\epsilon := \left\{ \sum_{\mathbf{i} \in R} i_1 \frac{\zeta_{i_1, i_2}^{[n]}(T)}{n} \leq \frac{3}{4} \alpha \epsilon \quad \text{and} \quad \sum_{\mathbf{i} \in R} i_2 \frac{\zeta_{i_1, i_2}^{[n]}(T)}{n} \leq \frac{3}{4} \beta \epsilon \right\}$$

is decreasing to zero as $n \rightarrow \infty$. Moreover,

$$q_{R,T}^\epsilon(n) := P(\overline{Q_{R,T,n}^\epsilon}) = o(n^{-1})$$

by equation (47) in Subsection 4.4.

We know from (25) that

$$\lim_{n \rightarrow \infty} E[L_n] = \lim_{n \rightarrow \infty} \int_0^1 E[\kappa(G(n, p))] dp - 1 = \lim_{n \rightarrow \infty} \int_0^\infty \sum_{i_1, i_2} \frac{E[\zeta_{i_1, i_2}^{[n]}(t)]}{n} e^{-t/n} dt - 1$$

provided the latter limit exists.

We will split $\int_0^\infty \sum_{i_1, i_2} \frac{E[\zeta_{i_1, i_2}^{[n]}(t)]}{n} e^{-t/n} dt$ as follows.

$$\begin{aligned} \int_0^\infty \sum_{i_1, i_2} \frac{E[\zeta_{i_1, i_2}^{[n]}(t)]}{n} e^{-t/n} dt &= \int_0^T \sum_{\mathbf{i} \in R} \frac{E[\zeta_{i_1, i_2}^{[n]}(t)]}{n} e^{-t/n} dt && \text{(Term I)} \\ &+ \int_0^T \sum_{\mathbf{i} \notin R} \frac{E[\zeta_{i_1, i_2}^{[n]}(t)]}{n} e^{-t/n} dt && \text{(Term II)} \\ &+ (1 - q_{R,T}^\epsilon(n)) \int_T^\infty \sum_{\mathbf{i} \in R} \frac{E[\zeta_{i_1, i_2}^{[n]}(t) \mid Q_{R,T,n}^\epsilon]}{n} e^{-t/n} dt && \text{(Term III)} \\ &+ (1 - q_{R,T}^\epsilon(n)) \int_T^\infty \sum_{\mathbf{i} \notin R} \frac{E[\zeta_{i_1, i_2}^{[n]}(t) \mid Q_{R,T,n}^\epsilon]}{n} e^{-t/n} dt && \text{(Term IV)} \\ (32) \quad &+ q_{R,T}^\epsilon(n) \int_T^\infty \sum_{i_1, i_2} \frac{E[\zeta_{i_1, i_2}^{[n]}(t) \mid \overline{Q_{R,T,n}^\epsilon}]}{n} e^{-t/n} dt && \text{(Term V)} \end{aligned}$$

Next, we estimate the terms **I-V** in (32).

Term I. As it is proven in Section 4, $\lim_{n \rightarrow \infty} \frac{\zeta_{i_1, i_2}^{[n]}(t)}{n} = \zeta_{i_1, i_2}(t)$ *a.s.* on $[0, T]$ for all $\mathbf{i} = \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} \in R$. Therefore,

$$\lim_{n \rightarrow \infty} \int_0^T \sum_{\mathbf{i} \in R} \frac{E[\zeta_{i_1, i_2}^{[n]}(t)]}{n} e^{-t/n} dt = \sum_{\mathbf{i} \in R} \int_0^T \zeta_{i_1, i_2}(t) dt.$$

Term II. Observe that,

$$\begin{aligned} \sum_{\mathbf{i} \notin R} \frac{\zeta_{i_1, i_2}^{[n]}(t)}{n} &\leq \frac{1}{n} \sum_{i_1 > K_1} \sum_{i_2} \zeta_{i_1, i_2}^{[n]}(t) + \frac{1}{n} \sum_{i_1} \sum_{i_2 > K_2} \zeta_{i_1, i_2}^{[n]}(t) \\ &\leq \frac{1}{K_1 n} \sum_{i_1 > K_1} \sum_{i_2} i_1 \zeta_{i_1, i_2}^{[n]}(t) + \frac{1}{n K_2} \sum_{i_1} \sum_{i_2 > K_2} i_2 \zeta_{i_1, i_2}^{[n]}(t) \\ &\leq \frac{\alpha[n]}{K_1 n} + \frac{\beta[n]}{n K_2} \leq 2 \frac{\alpha}{K_1} + 2 \frac{\beta}{K_2} \end{aligned}$$

for all n large enough. Thus,

$$\int_0^T \sum_{\mathbf{i} \notin R} \frac{E[\zeta_{i_1, i_2}^{[n]}(t)]}{n} e^{-t/n} dt = O\left(\frac{T}{K_1}\right) + O\left(\frac{T}{K_2}\right).$$

Term III. We define the R -gel to be the collection of all clusters whose mass vector is not in R . Let

$$(33) \quad M_{R\text{gel}}(t) = \begin{bmatrix} m_1(t) \\ m_2(t) \end{bmatrix}$$

denote the total mass vector of all clusters in the R -gel at time $t \geq 0$.

Now, conditioning on the event $Q_{R, T, n}^\epsilon$, we have $m_1(t) \geq \alpha(1 - \epsilon)n$ and $m_2(t) \geq \beta(1 - \epsilon)n$ for all $t \geq T$, and n large enough. Thus each cluster in R will be gravitating toward the R -gel with the rate of at least $(\alpha \wedge \beta)(1 - \epsilon)$. Consider a cluster in R at time T . Let $T + L$ be the time it becomes a part of the R -gel. Then, its contribution to the integral $\int_T^\infty \sum_{\mathbf{i} \in R} \frac{E[\zeta_{i_1, i_2}^{[n]}(t) \mid Q_{R, T, n}^\epsilon]}{n} e^{-t/n} dt$ is at most

$$\int_T^\infty \frac{E[\mathbf{1}_{[T, T+L]}(t) \mid Q_{R, T, n}^\epsilon]}{n} e^{-t/n} dt \leq \frac{E[L \mid Q_{R, T, n}^\epsilon]}{n} e^{-T/n} \leq \frac{1}{(\alpha \wedge \beta)(1 - \epsilon)n}.$$

The number of clusters in R at time $t \geq T$ is

$$\sum_{\mathbf{i} \in R} \zeta_{i_1, i_2}^{[n]}(t) \leq \sum_{\mathbf{i} \in R} (i_1 + i_2) \zeta_{i_1, i_2}^{[n]}(t) \leq (\alpha + \beta) \epsilon n.$$

Therefore,

$$\int_T^\infty \sum_{\mathbf{i} \in R} \frac{E[\zeta_{i_1, i_2}^{[n]}(t) \mid Q_{R, T, n}^\epsilon]}{n} e^{-t/n} dt \leq \frac{(\alpha + \beta) \epsilon n}{(\alpha \wedge \beta)(1 - \epsilon)n} = \frac{2\epsilon}{1 - \epsilon} < 3\epsilon.$$

Term IV. We let $\mathcal{C} = \{C_1, C_2, C_3, \dots, C_M\}$ denote the set of all clusters whose mass vectors ever exceeded K_1 in the first coordinate and/or ever exceeded K_2 in the second coordinate in the history of the process $\mathbf{ML}_n(t)$, i.e. all clusters that were ever a part of R -gel. The number of clusters in \mathcal{C} is less than $\alpha[n]/K_1 + \beta[n]/K_2$. For each C_i , the emergence time a_i is the time of a merger of a pair of clusters in R , resulting in appearance of a new cluster C_i in R -gel. We enumerate these clusters in the order they emerge.

Let $M_i(t) = \begin{bmatrix} m_{1,i}(t) \\ m_{2,i}(t) \end{bmatrix}$ denote the mass vector of cluster C_i at time t . Consider a pair of clusters, C_i and C_j , coexisting in the R -gel at time t , such that $m_{1,i}, m_{1,j} < \alpha n/2$ and $m_{2,i}, m_{2,j} < \beta n/2$. We split their merger rate into two by saying that C_i absorbs C_j with rate $\frac{1}{2n}(m_{1,i}(t)m_{2,j}(t) + m_{2,i}(t)m_{1,j}(t))$, and C_j absorbs C_i with rate $\frac{1}{2n}(m_{1,i}(t)m_{2,j}(t) + m_{2,i}(t)m_{1,j}(t))$.

There is a finite stopping time

$$t^* = \min\{t \geq 0 : \exists C_i \in \mathcal{C} \text{ with } m_{1,i}(t) \geq \alpha n/2 \text{ or } m_{2,i}(t) \geq \beta n/2\}$$

when a cluster C_{i^*} has its mass vector satisfying either $m_{1,i^*}(t^*) \geq \alpha n/2$ or $m_{2,i^*}(t^*) \geq \beta n/2$. After time t^* the rules of interactions of cluster C_{i^*} with the other clusters in \mathcal{C} change as follows. For $t > t^*$, C_{i^*} absorbs C_j with rate $\frac{1}{n}(m_{1,i^*}(t)m_{2,j}(t) + m_{2,i^*}(t)m_{1,j}(t))$, while C_{i^*} itself cannot be absorbed by any other cluster in \mathcal{C} .

Let b_i denote the time when cluster C_i is absorbed by another cluster in collection \mathcal{C} . Naturally, there will be only one survivor C_{i^*} with $b_{i^*} = \infty$. Let $J_i = [a_i, b_i) \cap [T, \infty)$ denote the lifespan of cluster C_i . Note that a cluster C_i from the collection \mathcal{C} existing at time $t \in [a_i, b_i)$ is absorbed into one of the clusters in the R -gel with the total instantaneous rate of

$$\lambda_i(t) \geq \frac{1}{2n} \left(m_{1,i}(t)(m_2(t) - m_{2,i}(t)) + m_{2,i}(t)(m_1(t) - m_{1,i}(t)) \right),$$

where $m_1(t)$ and $m_2(t)$ are as defined in (33). Conditioning on the event $Q_{R, T, n}^\epsilon$ defined in (31), we have that if $m_{1,i}(t) < \alpha n/2$ and $m_{2,i}(t) < \beta n/2$ for $t \in J_i$, then the rate of

absorption of C_i into the R -gel is

$$\begin{aligned} \lambda_i(t) &\geq \frac{1}{2n} m_{1,i}(t) \beta \left((1-\epsilon)n - \frac{1}{2}n \right) + \frac{1}{2n} m_{2,i}(t) \alpha \left((1-\epsilon)n - \frac{1}{2}n \right) \\ &\geq \frac{1}{2n} m_{1,i}(t) \beta \left(\frac{3}{4}n - \frac{1}{2}n \right) + \frac{1}{2n} m_{2,i}(t) \alpha \left(\frac{3}{4}n - \frac{1}{2}n \right) \\ &\geq \frac{m_{1,i}(t) \beta + m_{2,i}(t) \alpha}{8} > \frac{K_1 \beta + K_2 \alpha}{8}. \end{aligned}$$

Next,

$$(34) \quad \int_T^\infty \sum_{i \notin R} \frac{E[\zeta_{i_1, i_2}^{[n]}(t) \mid Q_{R,T,n}^\epsilon]}{n} e^{-t/n} dt = \int_T^\infty \frac{1}{n} e^{-t/n} dt + \mathcal{E}$$

where $\int_T^\infty \frac{1}{n} e^{-t/n} dt$ is due to the event $Q_{R,T,n}^\epsilon$ which guarantees the existence of at least one component from \mathcal{C} in the R -gel for all $t \in [T, \infty)$ and the second term \mathcal{E} is responsible for all the times $t \geq T$ when the number of clusters in the R -gel is greater than one. The term \mathcal{E} is bounded as follows

$$\mathcal{E} \leq \int_T^\infty \frac{E \left[\sum_{i: i \neq i^*} \mathbf{1}_{J_i}(t) \mid Q_{R,T,n}^\epsilon \right]}{n} e^{-t/n} dt.$$

Now, each cluster C_i is gravitating towards the rest of the R -gel with the rate of at least $\frac{K_1 \beta + K_2 \alpha}{8}$. Thus, for each $i \neq i^*$,

$$\int_T^\infty \frac{E[\mathbf{1}_{J_i}(t) \mid Q_{R,T,n}^\epsilon]}{n} e^{-t/n} dt \leq \frac{E[|J_i| \mid Q_{R,T,n}^\epsilon]}{n} e^{-\frac{T}{n}} \leq \frac{8}{n(K_1 \beta + K_2 \alpha)}.$$

Hence, since the cardinality of set \mathcal{C} is $M < \alpha[n]/K_1 + \beta[n]/K_2$,

$$\mathcal{E} < (\alpha[n]/K_1 + \beta[n]/K_2) \cdot \frac{8}{n(K_1 \beta + K_2 \alpha)} = \frac{8(\alpha/K_1 + \beta/K_2)}{K_1 \beta + K_2 \alpha} + o(1),$$

and from (16), we obtain

$$\int_T^\infty \sum_{i \notin R} \frac{E[\zeta_{i_1, i_2}^{[n]}(t) \mid Q_{R,T,n}^\epsilon]}{n} e^{-t/n} dt = 1 + O(K_1^{-2}) + O(K_2^{-2}) + O\left(\frac{T}{n}\right) + o(1) \quad \text{as } n \rightarrow \infty.$$

Term V. Here

$$\begin{aligned} q_{R,T}^\epsilon(n) \int_T^\infty \sum_{i_1, i_2} \frac{E[\zeta_{i_1, i_2}^{[n]}(t) \mid \overline{Q_{R,T,n}^\epsilon}]}{n} e^{-t/n} dt &\leq q_{R,T}^\epsilon(n) \int_T^\infty \frac{\alpha[n] + \beta[n]}{n} e^{-t/n} dt \\ &\leq (\alpha[n] + \beta[n]) q_{R,T}^\epsilon(n) = o(1) \end{aligned}$$

as $q_{R,T}^\epsilon(n) = o(n^{-1})$.

Finally, by putting together the analysis in **Terms I-V** in the equation (32), we obtain for a given fixed $\epsilon \in (0, 1/4)$, sufficiently large fixed $T \gg T_{gel}$ satisfying (30), and arbitrarily large K_1 and K_2 ,

$$\begin{aligned} \int_0^\infty \sum_{i_1, i_2} \frac{E[\zeta_{i_1, i_2}^{[n]}(t)]}{n} e^{-t/n} dt &= \sum_{\mathbf{i} \in R(K_1, K_2)} \int_0^T \zeta_{i_1, i_2}(t) dt + 1 + O\left(\frac{T}{K_1}\right) + O\left(\frac{T}{K_2}\right) \\ (35) \qquad \qquad \qquad &+ O(K_1^{-2}) + O(K_2^{-2}) + O(\epsilon) + O\left(\frac{T}{n}\right) + o(1), \end{aligned}$$

which when we increase n to infinity will yield

$$\lim_{n \rightarrow \infty} \int_0^\infty \sum_{i_1, i_2} \frac{E[\zeta_{i_1, i_2}^{[n]}(t)]}{n} e^{-t/n} dt = \sum_{i_1, i_2} \int_0^\infty \zeta_{i_1, i_2}(t) dt + 1.$$

□

4. HYDRODYNAMIC LIMITS FOR MARCUS-LUSHNIKOV PROCESSES

In [9] a certain class of Markov processes, called *density dependent population process*, was considered. These are jump Markov processes which depend on some parameter which can be interpreted depending on context of the model. For example, it could be the population size. Coalescence processes can be considered as a case of density dependent population processes when we choose the total population mass n as a parameter for the processes. Specifically, assuming the process starts with n clusters of unit mass each (aka singletons). In Chapter 11 of Ethier and Kurtz [9], the law of large numbers and the central limit theorems were established for such class of processes, as $n \rightarrow \infty$.

4.1. Density dependent population processes. We first formulate the framework for the convergence result of Kurtz as stated in Theorem 2.1 in Chapter 11 of [9] (Theorem 8.1 in [18]). There, the *density dependent population processes* are defined as continuous time Markov processes with state spaces in \mathbb{Z}^d , and transition intensities represented as follows

$$(36) \qquad q^{(n)}(k, k + \ell) = n \left[\beta_\ell \left(\frac{k}{n} \right) + O\left(\frac{1}{n} \right) \right],$$

where $\ell, k \in \mathbb{Z}^d$, and β_ℓ is a given collection of rate functions.

In Section 5.1 of [2], Aldous observes that the results from Chapter 11 of Ethier and Kurtz [9] can be used to prove the weak convergence of a Marcus-Lushnikov process to the solutions of Smoluchowski system of equations in the case when the Marcus-Lushnikov process can be formulated as a finite dimensional density dependent population process. Specifically, the Marcus-Lushnikov processes corresponding to the multiplicative and Kingman coalescent with the monodisperse initial conditions (n singletons) can be represented as finite dimensional density dependent population processes defined above.

Define $F(x) = \sum_{\ell} \beta_{\ell}(x)$. Then, Theorem 2.1 in Chapter 11 of [9] (Theorem 8.1 in [18]) states the following law of large numbers. Let $\hat{X}_n(t)$ be the Markov process with the intensities $q^{(n)}(k, k + \ell)$ given in (36), and let $X_n(t) = n^{-1} \hat{X}_n(t)$. Finally, let $|x| = \sqrt{\sum x_i^2}$ denote the Euclidean norm in \mathbb{R}^d .

Theorem 4.1. *Suppose for all compact $\mathcal{K} \subset \mathbb{R}^d$,*

$$\sum_{\ell} |\ell| \sup_{x \in \mathcal{K}} \beta_{\ell}(\bar{x}) < \infty,$$

and there exists $M_{\mathcal{K}} > 0$ such that

$$(37) \quad |F(x) - F(y)| \leq M_{\mathcal{K}} |x - y|, \quad \text{for all } x, y \in \mathcal{K}.$$

Suppose $\lim_{n \rightarrow \infty} X_n(0) = x_0$, and $X(t)$ satisfies

$$(38) \quad X(t) = X(0) + \int_0^t F(X(s)) ds,$$

for all $T \geq 0$. Then

$$(39) \quad \lim_{n \rightarrow \infty} \sup_{s \in [0, T]} |X_n(s) - X(s)| = 0 \quad \text{a.s.}$$

4.2. Hydordynamic limit for multiplicative coalescent process. Consider a multiplicative coalescent process with kernel $K(i, j) = ij$. Recall that in the definition of a coalescent process given in Subsection 2.2, a pair of clusters with masses i and j coalesces at the rate $K(i, j)/n$. Consider the corresponding Marcus-Lushnikov process

$$\mathbf{ML}_n(t) = \left(\zeta_{1,n}(t), \zeta_{2,n}(t), \dots, \zeta_{n,n}(t), 0, 0, \dots \right)$$

that keeps track for the numbers of clusters in each weight category. There, the initial conditions will be $\mathbf{ML}_n(0) = (n, 0, 0, \dots) = ne_1$, where e_i denotes the i -th coordinate vector.

Next, for a fixed positive integer K , let $\hat{X}_n(t)$ be the restriction of process $\mathbf{ML}_n(t)$ to the first K dimensions, i.e.

$$\hat{X}_n(t) = \left(\zeta_{1,n}(t), \zeta_{2,n}(t), \dots, \zeta_{K,n}(t) \right)$$

with the initial conditions $\hat{X}_n(0) = ne_1$. Apparently, $\hat{X}_n(t)$ is itself a (finite dimensional) Markov process with the following transition rates of $\hat{X}_n(t)$ stated as in (36). Let $x = (x_1, x_2, \dots, x_K)$. Then, for any positive $1 \leq i < j \leq K$, the change vector $\ell = -e_i - e_j + e_{i+j}\mathbf{1}_{i+j \leq K}$ corresponding to a merger of clusters of respective sizes i and j would be assigned the rate

$$q^{(n)}(x, x + \ell) = nijx_i x_j = n\beta_\ell(x),$$

where $\beta_\ell(x) = ijx_i x_j$.

For a given $1 \leq i \leq K$, the change vector $\ell = -2e_i + e_{2i}\mathbf{1}_{2i \leq K}$ corresponding to a merger of a pair of clusters of size i will be assigned the rate

$$q^{(n)}(x, x + \ell) = n \left[\frac{i^2 x_i^2}{2} - \frac{i^2 x_i}{2n} \right] = n \left[\beta_\ell(x) + O\left(\frac{1}{n}\right) \right],$$

where $\beta_\ell(x) = i^2 \frac{x_i^2}{2}$.

For a given $1 \leq i \leq K$, the change vector $\ell = -e_i$ corresponding to a cluster of mass i merging with a cluster of mass greater than K will be assigned the rate

$$q^{(n)}(x, x + \ell) = nix_i \left[1 - \sum_{j=1}^K jx_j \right] = n\beta_\ell(x),$$

where $\beta_\ell(x) = ix_i \left(1 - \sum_{j=1}^K jx_j \right)$.

Then, by Theorem 4.1, $X_n(t) = n^{-1}\hat{X}_n(t)$ converges to $X(t)$ as in (39), where $X(t)$ satisfies (38) with

$$\begin{aligned} F(x) &:= \sum_{\ell} \beta_\ell(x) = \sum_{ij: 1 \leq i < j \leq K} ijx_i x_j [-e_i - e_j + e_{i+j}\mathbf{1}_{i+j \leq K}] \\ &\quad + \frac{1}{2} \sum_{i=1}^K i^2 x_i^2 [-2e_i + e_{2i}\mathbf{1}_{2i \leq K}] - \sum_{i=1}^K ix_i \left(1 - \sum_{j=1}^K jx_j \right) e_i \\ (40) \quad &= \sum_{i=1}^K \left(-ix_i + \frac{1}{2} \sum_{\substack{1 \leq i_1, i_2 \leq K \\ i_1 + i_2 = i}} i_1 i_2 x_{i_1} x_{i_2} \right) e_i. \end{aligned}$$

Here, $F(x)$ is naturally satisfying the Lipschitz continuity conditions (37), and the initial conditions $X(0) = X_n(0) = e_1$.

Observe that the system of equations (38) with $F(x)$ as in (40) will yield the reduced system of Smoluckowski coagulation equations (4) also known as the Flory's coagulation system [12]. Thus, for a given integer $K > 0$ and a fixed real $T > 0$,

$$(41) \quad \lim_{n \rightarrow \infty} \sup_{s \in [0, T]} |n^{-1} \zeta_{k,n}(s) - \zeta_k(s)| = 0 \quad \text{a.s.}$$

for $k = 1, 2, \dots, K$.

Note that the above limit no longer requires a fixed K for each individual k in (41). However, we will mainly need the following limit in our calculations,

$$\lim_{n \rightarrow \infty} \sup_{s \in [0, T]} \left| \sum_{k=1}^K n^{-1} \zeta_{k,n}(s) - \sum_{k=1}^K \zeta_k(s) \right| = 0 \quad \text{a.s.}$$

4.3. Hydordynamic limit for cross-multiplicative coalescent processes. Fix integers $K_1 > 0$ and $K_2 > 0$, and let $R := R(K_1, K_2) = \{1, 2, \dots, K_1\} \times \{1, 2, \dots, K_2\}$. Let e_i be the standard basis vectors in $\mathbb{R}^{K_1 K_2}$, enumerated by $\mathbf{i} = \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} \in R$. Consider a restriction of a Marcus-Lushnikov processes with the cross-multiplicative kernel $\zeta_{i_1, i_2}(t)$ to $\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} \in R$. Let

$$\hat{X}_n(t) = \left\{ \zeta_{i_1, i_2}^{[n]}(t) \right\}_{\mathbf{i} \in R}$$

with the initial conditions $\hat{X}_n(0) = \alpha[n]e_{0'} + \beta[n]e_{0''}$, where $0' = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $0'' = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

We observe the following transition rates of $\hat{X}_n(t)$ stated as in (36). Let $x = \sum_{\mathbf{i} \in R} x_i e_i$. Then, for any \mathbf{i} and \mathbf{j} in R , the change vector $\ell = -e_i - e_j + \mathbf{1}_{\{\mathbf{i}+\mathbf{j} \in R\}} e_{\mathbf{i}+\mathbf{j}}$ corresponding to a merger of clusters of respective weights \mathbf{i} and \mathbf{j} would be assigned the rate

$$q^{(n)}(x, x + \ell) = n(i_1 j_2 + i_2 j_1) x_i x_j = n \beta_\ell(x),$$

where $\beta_\ell(x) = (i_1 j_2 + i_2 j_1) x_i x_j$.

For a given $\mathbf{i} \in R$, the change vector $\ell = -e_i$ corresponding to the merger of clusters whose weight vector is \mathbf{i} with clusters whose weight vectors are not in R will be assigned the rate

$$q^{(n)}(x, x + \ell) = n \left[i_1 x_i \left(\beta - \sum_{\mathbf{j} \in R} j_2 x_j \right) + i_2 x_i \left(\alpha - \sum_{\mathbf{j} \in R} j_1 x_j \right) \right] = n \beta_\ell(x),$$

where $\beta_\ell(x) = i_1 x_i \left(\beta - \sum_{\mathbf{j} \in R} j_2 x_j \right) + i_2 x_i \left(\alpha - \sum_{\mathbf{j} \in R} j_1 x_j \right)$.

Thus, by Theorem 4.1, $X_n(t)$ converges to $X(t)$ as in (39), where $X(t)$ satisfies (38) with

$$\begin{aligned}
(42) \quad F(x) &:= \sum_{\ell} \ell \beta_{\ell}(x) = \frac{1}{2} \sum_{\mathbf{i}, \mathbf{j} \in R} [-e_{\mathbf{i}} - e_{\mathbf{j}} + \mathbf{1}_{\{\mathbf{i}+\mathbf{j} \in R\}} e_{\mathbf{i}+\mathbf{j}}] (i_1 j_2 + i_2 j_1) x_{\mathbf{i}} x_{\mathbf{j}} \\
&\quad - \sum_{\mathbf{i} \in R} e_{\mathbf{i}} i_1 x_{\mathbf{i}} \left(\beta - \sum_{\mathbf{j} \in R} j_2 x_{\mathbf{j}} \right) - \sum_{\mathbf{i} \in R} e_{\mathbf{i}} i_2 x_{\mathbf{i}} \left(\alpha - \sum_{\mathbf{j} \in R} j_1 x_{\mathbf{j}} \right) \\
&= \sum_{\mathbf{i} \in R} e_{\mathbf{i}} \left(-(\beta i_1 + \alpha i_2) x_{\mathbf{i}} + \frac{1}{2} \sum_{\ell, \mathbf{k}: \ell+\mathbf{k}=\mathbf{i}} (\ell_1 k_2 + \ell_2 k_1) x_{\ell} x_{\mathbf{k}} \right)
\end{aligned}$$

for a fixed $T > 0$. The system of equations (38) with $F(x)$ given in (42) will yield the reduced system of Smoluckowski coagulation equations (20). So, for a fixed a pair of positive integers K_1 and K_2 , and a fixed real number $T > 0$,

$$(43) \quad \lim_{n \rightarrow \infty} \sup_{s \in [0, T]} \left| n^{-1} \zeta_{i_1, i_2}^{[n]}(s) - \zeta_{i_1, i_2}(s) \right| = 0 \quad \text{a.s.}$$

for all $\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} \in R$. Therefore,

$$\lim_{n \rightarrow \infty} \sup_{s \in [0, T]} \left| n^{-1} \sum_{\substack{1 \leq i_1 \leq K_1 \\ 1 \leq i_2 \leq K_2}} \zeta_{i_1, i_2}^{[n]}(s) - \sum_{\substack{1 \leq i_1 \leq K_1 \\ 1 \leq i_2 \leq K_2}} \zeta_{i_1, i_2}(s) \right| = 0 \quad \text{a.s.}$$

4.4. Central Limit Theorem and related results. The usefulness of the framework set in [9, 18] for proving weak convergence is that the law of large numbers Theorem 4.1 is enhanced with the corresponding central limit theorem (see Theorem 4.2 below) and the large deviation theory [11]. The following central limit theorem is derived in Theorem 8.2 in [18] (and Theorem 2.3 in Chapter 11 of [9]).

Theorem 4.2. *Suppose for all compact $\mathcal{K} \subset \mathbb{R}^d$,*

$$(44) \quad \sum_{\ell} |\ell|^2 \sup_{x \in \mathcal{K}} \beta_{\ell}(x) < \infty$$

and that the β_{ℓ} and ∂F are continuous. Suppose X_n and X are as in Theorem 4.1, and suppose $V_n = \sqrt{n}(X_n - X)$ is such that $\lim_{n \rightarrow \infty} V_n(0) = V(0)$, where $V(0)$ is a constant. Then V_n converges in distribution to V , which is the solution of

$$(45) \quad V(t) = V(0) + U(t) + \int_0^t \partial F(X(s)) V(s) ds,$$

where $U(t)$ is a Gaussian process and $\partial F(X(s)) = (\partial_j F_i(X(s)))_{i,j}$.

The proof of Theorem 4.2 is based on representing $V_n(t)$ as follows. Let Y_ℓ be independent Poisson processes with rate one. Then,

$$(46) \quad V_n(t) = V_n(0) + U_n(t) + \int_0^t \sqrt{n}(F(X_n(s)) - F(X(s)))ds,$$

where

$$U_n(t) = \sum_{\ell} \ell W_{\ell}^{(n)} \left(\int_0^t \beta_{\ell}(X_n(s))ds \right),$$

$W_{\ell}^{(n)}(u) = n^{-1/2} \hat{Y}_{\ell}(nu)$, and $\hat{Y}_{\ell}(u) := Y_{\ell}(u) - u$ are centralized Poisson processes.

Next, we will use formula (46) in order to derive an upper bound (47) on probability $P(|X_n(T) - X(T)| \geq \delta)$. Let us consider a simple case of a density dependent population process on \mathbb{R}^d for which the following three conditions are satisfied.

- $V_n(0) = 0$.
- There are finitely many vectors $\ell \in \mathbb{R}^d$.
- Both $X_n(t)$ and $X(t)$ live on a compact set \mathcal{K} .

Notice that the above conditions are satisfied for the Marcus-Lushnikov processes considered here, with the general bilinear kernel as in Subsection 4.2 and with the cross-multiplicative kernel as in Subsection 4.3. Specifically, for a given $m > 0$, let

$$\mathcal{K}_m = \left\{ x \in \mathbb{R}_+^d : \sum_i x_i \leq m \right\}.$$

Then, in Subsection 4.2, $X_n(t), X(t) \in \mathcal{K}_2$, and in Subsection 4.3, $X_n(t), X(t) \in \mathcal{K}_m$, where $m > \alpha + \beta$.

Proposition 4.3. *Assuming the above conditions are satisfied together with the Lipschitz continuity conditions (37), we have*

$$(47) \quad P(|X_n(T) - X(T)| \geq \delta) = O(n^{-2}).$$

Proof. Here,

$$\sqrt{n}|F(X_n(s)) - F(X(s))| \leq \sqrt{n}M_{\mathcal{K}}|X_n(s) - X(s)| = M_{\mathcal{K}}|V_n(s)|$$

and for a fixed $T > 0$ and any $t \leq T$,

$$|U_n(t)| \leq \varepsilon_n(T) := \sum_{\ell} |\ell| \max \left\{ \left| W_{\ell}^{(n)}(s) \right| : s \in [0, T \sup_{x \in \mathcal{K}} \beta_{\ell}(x)] \right\}.$$

Hence, for a fixed $T > 0$, equation (46) implies the following inequality,

$$|V_n(t)| \leq \varepsilon_n(T) + M_{\mathcal{K}} \int_0^t |V_n(s)|ds \quad \text{for all } t \in [0, T].$$

Then, by Grönwall's inequality (see Appendix 5 in [9]),

$$(48) \quad |V_n(t)| \leq \varepsilon_n(T)e^{M_{\mathcal{K}}t}.$$

In particular, we use equation (48) together with Markov inequality to obtain the following simple bound for any $\delta > 0$,

$$(49) \quad P(|X_n(T) - X(T)| \geq \delta) \leq \frac{V_n^4(T)}{n^2\delta^4} \leq \frac{E[\varepsilon_n^4(T)]e^{4M\kappa T}}{n^2\delta^4}.$$

Here, for any fixed real $S > 0$, integer $r > 0$, and any real $\lambda > 0$, we have by Doob's martingale inequality,

$$P\left(\max_{s \in [0, S]} |W_\ell^{(n)}(s)|^r \geq \lambda\right) = P\left(\max_{s \in [0, S]} |W_\ell^{(n)}(s)| \geq \lambda^{1/r}\right) \leq \frac{E\left[\left(W_\ell^{(n)}(S)\right)^{2+2r}\right]}{\lambda^{2+2/r}}$$

as $|W_\ell^{(n)}(s)|$ is a non-negative sub-martingale. Therefore,

$$E\left[|W_\ell^{(n)}(s)|^r\right] \leq 1 + \int_1^\infty P\left(\max_{s \in [0, S]} |W_\ell^{(n)}(s)|^r \geq \lambda\right) d\lambda \leq 1 + (1 + 2/r)E\left[\left(W_\ell^{(n)}(S)\right)^{2+2r}\right],$$

where by the classical central limit theorem,

$$\lim_{n \rightarrow \infty} E\left[\left(W_\ell^{(n)}(S)\right)^{2+2r}\right] = S^{1+r}E[Z^{2+2r}], \quad Z - \text{standard normal r.v.}$$

Thus, $E[\varepsilon_n^4(T)] = O(1)$, and (47) follows from (49). \square

5. DISCUSSION: GENERALIZATIONS AND OPEN PROBLEMS.

As the natural next step we see finding the limit mean length of the minimal spanning tree of random graph processes on irregular multipartite graphs via the corresponding Marcus-Lushnikov processes with multidimensional weight vectors. Note that the theory presented here extends to other irregular graphs beyond multipartite graphs.

The gelation of bipartite Marcus-Lushnikov processes and other coalescent processes with multidimensional weight vectors has not been studied before. This on its own provides an interesting problem. We will look for a generalizing the existing gelation results [27, 1, 17] for the Smoluchowski coagulation equations of the Marcus-Lushnikov processes with multidimensional weight vectors.

One of the issues facing the use of coalescent processes in genetics as models of genetic drift viewed backwards in time is that of genetic recombination. There, distinct gene loci would follow different pathways of ancestry, resulting in different gene genealogies. As a biological application, it will be useful to consider a coalescent process with multidimensional weight vectors as a means of addressing the issue of genetic recombination, and possibly, the issue of biological compatibility.

Finally, analyzing the convergence rates in the hydrodynamic limits, we could obtain a central limit theorem for L_n on $K_{\alpha n, \beta n}$ similar to the central limit theorem for L_n on K_n proved in Jensen [16]. Specifically, we hope to apply Theorem 4.2 in the analysis. Also, similarly to [6], we could examine the second and third order terms in $E[L_n]$.

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