

# Notes about percolation on causal triangulations

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**Abstract.** In this work we study bond percolation on random causal triangulation. We show that the phase transition is non-trivial and we compute a lower and an upper bound for the critical value. Obviously, the critical value depends strongly on the nature of the underlying graph, but the critical value is shown to be constant a.s. for the random causal triangulation ensemble.

## 1 Introduction

In the study of two dimensional quantum gravity models of discrete random surfaces appear as a powerful technique....

Percolation is the fundamental stochastic model for spatial disorder. We consider bond percolation on random causal triangulations. Detailed accounts of the basic theory on  $\mathbb{L}^d = (\mathbb{Z}^d, \mathbb{E}^d)$  in  $d \geq 2$  dimensions may be found in [Grimmett, G. R. \(1999\)](#) and [Bollobas, B. and Riordan, O. \(2006\)](#). Percolation comes in two forms, *bond* and *site*, and we concentrate here on the bond model....

## 2 Causal Triangulations Ensemble

Here we start with definition of *rooted causal (or Lorentzian) triangulations* of the cylinder  $\mathbb{C} = S \times [0, \infty]$ , where  $S$  is a unite circle. We maintain definitions and denotations of [Krikun, M. and Yambartsev, A. \(2012\)](#). Consider a connected graph  $G$  with countable set of vertices embedded into the cylinder  $\mathbb{C}$ . Any connected component of  $\mathbb{C} \setminus G$  is called a *face*. Let the size of a face be the number of edges incident to it, with the convention that an edge incident to the same face on both sides counts for two. We then call a face with size 3 (or 3-sided face) a *triangle*.

The graph  $G$  defines an infinite causal (or Lorentzian) triangulation  $\mathbf{t}$  of if (i) all vertices lie in circles  $S \times \{j\}, j \in \mathbb{N} \cup \{0\} = \{0, 1, \dots\}$ ; (ii) each face is triangle; (iii) each face of  $\mathbf{t}$  belongs to some strip  $S \times [j, j+1], j = 0, 1, \dots$ ,

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and has all vertices and exactly one edge on the boundary  $(S \times \{j\}) \cup (S \times \{j+1\})$  of the strip  $S \times [j, j+1]$ ; and (iv) the number of edges on  $S \times \{j\}$  is positive and finite for any  $j = 0, 1, \dots$ . See Figure 4 for an example of Lorenzian triangulation.

We note that two vertices of a triangle on a same circle, say  $S \times \{j\}$ , may coincide (in this case the corresponding edge stretches over the whole circle  $S \times \{j\}$ , i.e. is a loop).

The root in a triangulation  $\mathbf{t}$  consists of a triangle  $\Delta$  of  $\mathbf{t}$ , called the *root face*, with the anti-clockwise ordering on its vertices  $(o, x, y)$ , where  $o$  and  $x$  lie in  $S \times \{0\}$  (can coincide) and  $y$  belongs to  $S \times \{1\}$ . Vertex  $o$  is the *root vertex* or simply *root*. The edge  $(o, x)$  belong to  $S \times \{0\}$ .

Two rooted triangulations, say  $\mathbf{t}$  and  $\mathbf{t}'$ , are equivalent if  $\mathbf{t}$  and  $\mathbf{t}'$  are embeddings  $i_{\mathbf{t}}, i_{\mathbf{t}'}$  of the same graph  $G$  and there exists a self-homeomorphism  $h : \mathbb{C} \rightarrow \mathbb{C}$  such that  $hi_{\mathbf{t}} = i_{\mathbf{t}'}$ . We suppose that the the homeomorphism  $h$  transforms each slice  $S \times \{j\}, j \in \mathbb{N}$  to itself and preserves the root:  $h$  sends the root of  $\mathbf{t}$  to the root of  $\mathbf{t}'$ .

The equivalence class of embedded rooted causal (Lorentzian) triangulations we call causal triangulation.

In the same way we also can define a causal triangulation of a cylinder  $\mathbb{C}_N = S \times [0, N]$ .

Let  $\mathbb{LT}_N$  and  $\mathbb{LT}_\infty$  be the sets of all causal triangulations with the supports  $\mathbb{C}_N = S \times [0, N]$  and  $\mathbb{C} = S \times [0, \infty)$ , respectively. The number of edges on the upper boundary  $S \times \{N\}$  is not fixed. We introduce a Gibbs measure on the set  $\mathbb{LT}_N$  as

$$P_{N,\mu}(\mathbf{t}) = \frac{1}{Z_N(\mu)} e^{-\mu F_N(\mathbf{t})}, \quad (2.1)$$

where  $F_N(\mathbf{t})$  is the number of triangles in the firsts  $N$  strips of the triangulation  $\mathbf{t}$ , and  $Z_N(\mu)$  is the partition function.

The measure on the set of infinite triangulations  $\mathbb{LT}_\infty$  is defined by the weak limit

$$P_\mu := \lim_{n \rightarrow \infty} P_{N,\mu}.$$

It was shown in [Malyshev, V. et. al. \(2001\)](#) that this limit exists for all  $\mu \geq \mu_c := \ln 2$ , and provided some properties of the causal triangulations under the limit measure  $P_\mu$ . The probability space  $(\mathbb{LT}_\infty, \mathcal{F}, P_\mu)$  we will call a *causal triangulations ensemble*, for any  $\mu \geq \ln 2$ .

**Theorem 2.1.** *For any  $n \geq 0$  let  $k_n = k_n(\mathbf{t})$  be the number of the vertices at the  $n$ -th level (on slice  $S \times \{n\}$ ) in a triangulation  $\mathbf{t}$ .*

a) *For  $\mu > \ln 2$ , the sequence  $\{k_n\}$  is a positive recurrent Markov chain*

with respect to the limit measure  $P_\mu$ , with invariant measure

$$\pi = \{\pi(n) = (1 - \Lambda)^2 n \Lambda^{n-1} : n \in \mathbb{N}\},$$

where  $\Lambda(\mu) = \left[ \frac{1 - \sqrt{1 - 4e^{-2\mu}}}{2e^{-\mu}} \right]^2$ . In addition, the transition of the Markov chain are given by

$$P(n, n') = \frac{n'}{n} \Lambda^{n'-n-1} \binom{n+n'-1}{n-1} e^{-\mu(n+n')}. \quad (2.2)$$

b) For  $\mu = \ln 2$  the sequence  $\{k_n\}$  is distributed as the branching process  $\xi_n$  with a geometric offspring distribution with parameter  $1/2$ , conditioned to non-extinction at infinity, and we obtain

$$P_{\mu_c}(k_n = m) = \lim_{N \rightarrow \infty} Pr(\xi_n = m | \xi_N > 0) = \frac{mn^{m-1}}{(n+1)^{m+1}} \quad (2.3)$$

### 3 Percolation on Causal Triangulation. Main Results

For any causal triangulation  $\mathbf{t} \in \mathbb{L}\mathbb{T}_\infty$ , we define a bond percolation on  $\mathbf{t}$  with parameter  $p \in (0, 1)$ . Let us denote the resulting probability measure on  $\mathbf{t}$  by  $\mathbb{P}_p^{(\mathbf{t})}$ , and  $\mathbb{E}_p^{(\mathbf{t})}$  denotes expectation w.r.t.  $\mathbb{P}_p^{(\mathbf{t})}$ . We define the *percolation function*  $p \rightarrow \theta(p)$  by

$$\theta^{(\mathbf{t})}(p) = \mathbb{P}_p^{(\mathbf{t})}(|C_o| = \infty) \quad (3.1)$$

where  $v$  is an arbitrary vertex, and  $C_o$  is percolation cluster contained the root  $o$  of the triangulation  $\mathbf{t}$ .

If  $\theta^{(\mathbf{t})}(p) = 0$ , then the probability that the root  $o$  is inside of an infinite connected component is 0, therefore it also means that no infinite connected component exists a.s. On the other hand, if  $\theta^{(\mathbf{t})}(p) > 0$  then the proportion of the vertices in infinite connected components is equals to  $\theta^{(\mathbf{t})}(p)$ , which is positive, and we say that the system *percolates*. We define the *critical value* on the triangulation  $\mathbf{t}$  by

$$p_c(\mathbf{t}) = \inf\{p : \theta^{(\mathbf{t})}(p) > 0\}. \quad (3.2)$$

For percolation on causal triangulations it is natural to ask whether the critical value is non-trivial (different from both 0 and 1), and whether it depends on the triangulation  $\mathbf{t}$  sampled from distribution  $P_\mu$ .

In the following sections we show that the critical value obeys a zero-one law and is constant  $P_\mu$ -a.s. for any  $\mu \geq \ln 2$ , and we show that the critical value is non-trivial only in the case  $\mu = \ln 2$   $P_\mu$ -a.s. These results are summarized in Theorem 3.1 below.

**Theorem 3.1.** *For the considered percolation model on causal triangulations the following statements hold.*

1. *The critical value  $p_c(\mathbf{t})$  is constant  $P_\mu$ -a.s.,*
2. *The critical value satisfy the following relation*

$$p_c(\mathbf{t}) = \begin{cases} 1 & \text{if } \mu > \ln 2, \\ 0 < p_c(\mathbf{t}) < 1 & \text{if } \mu = \ln 2. \end{cases} \quad P_\mu - a.s. \quad (3.3)$$

3. *If  $\mu = \mu_c$ , then  $\frac{1}{\sqrt{21}} < p_c(\mathbf{t}) \leq \frac{1}{2}$ ,  $P_{\mu_c}$ -a.s.*

#### 4 Absence of infinite cluster for subcritical causal triangulations

In this section we prove the first and second statement of Theorem 3.1 for the subcritical random causal triangulations ensemble  $(\mathbb{LT}_\infty, \mathcal{F}, P_\mu)$ , i.e.,  $\mu > \ln 2$ .

According Theorem 2.1 the sequence  $\{k_n\}_{n \in \mathbb{N}}$  defines the Markov chain on the probability space  $(\mathbb{LT}_\infty, \mathcal{F}, P_\mu)$ , where  $k_n = k_n(\mathbf{t})$  is the number of vertices of the triangulation  $\mathbf{t}$  on slice  $S \times \{n\}$ . Let  $X_1$  be the *first passage* time to state 1 (space contraction) defined by

$$X_1(\mathbf{t}) = \inf\{n > 0 : k_n(\mathbf{t}) = 1 \text{ and } k_{n+1}(\mathbf{t}) = 1\}$$

where  $\inf \emptyset = \infty$ . We now define inductively the  $r$ th passage time  $X_r$  to state 1 by

$$X_{r+1}(\mathbf{t}) = \inf\{n \geq X_r(\mathbf{t}) + 2 : k_n(\mathbf{t}) = 1 \text{ and } k_{n+1}(\mathbf{t}) = 1\},$$

for  $r = 0, 1, 2, \dots$ . By Theorem 2.1, for  $\mu > \ln 2$ , the sequence  $\{k_n\}_{n \in \mathbb{N}}$  is the positive recurrent Markov chain with the measure  $P_\mu$ , thus  $\lim_{r \rightarrow \infty} X_r(\mathbf{t}) = +\infty$   $P_\mu$ -a.s.

For each  $N \in \mathbb{N}$ , let  $T_N$  be the number of contractions up to time  $N$ , which can be written in terms of indicator functions as

$$T_N = \sum_{k=1}^{\infty} \mathbf{1}_{\{X_k \leq N\}}.$$

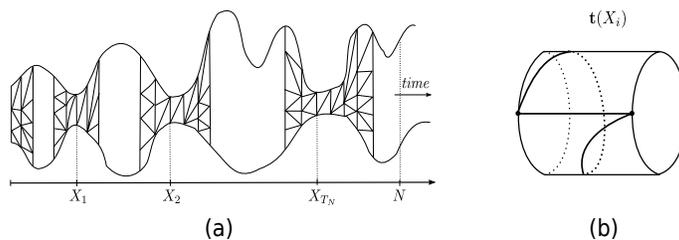
By recurrence of the Markov chain  $\{k_n\}$ , we have the following result.

**Lemma 4.1.** *For any  $\mu > \ln 2$  and for all  $N \geq 1$  the number of contractions is finite  $P_\mu$ -a.s., i.e.  $T_N(\mathbf{t}) < \infty$ ,  $P_\mu$ -a.s. Moreover, the following limit holds true:*

$$\lim_{N \rightarrow \infty} T_N = \infty \quad P_\mu - a.s.$$

Furthermore, by ergodic theorem for Markov chains, we have that

$$\frac{T_N(\mathbf{t})}{N} \rightarrow \frac{(1-\Lambda)^2}{\Lambda} e^{-2\mu} \text{ as } N \rightarrow \infty \quad P_\mu - a.s.$$



**Figure 1** Typical path in the case  $\mu > \ln 2$ .

Each causal triangulation  $\mathbf{t}$  from  $\text{LT}_\infty$  is identified as a consistent sequence

$$\mathbf{t} = (\mathbf{t}(0), \mathbf{t}(1), \dots, \mathbf{t}(N), \dots),$$

where  $\mathbf{t}(i)$  is a causal triangulation of the strip  $S \times [i, i+1]$ . The property of consistency means that for each pair  $(\mathbf{t}(i), \mathbf{t}(i+1))$  every side of a triangle from  $\mathbf{t}(i)$  lying in  $S \times \{i+1\}$  serves as a side of a triangle from  $\mathbf{t}(i+1)$ , and vice versa.

Denote  $\partial_N := S \times \{N\}$ . Let  $\{0 \leftrightarrow \partial_N\}$  be the event that there exists an open path, in the classical bound percolation sense, joining the root vertex in the first strip to some vertex in  $\partial_N$ .

**Lemma 4.2.** For any  $\mu > \ln 2$  and  $p \in [0, 1)$

$$\mathbb{P}_p^{(\mathbf{t})}(0 \leftrightarrow \partial_N) \leq e^{-(1-p)^2 T_N(\mathbf{t})} \quad P_\mu - a.s. \quad (4.1)$$

**Proof.** Denote by  $X_1, \dots, X_{T_N}$  the contraction times of triangulation  $\mathbf{t}$  up to time  $N$  (see Figure 1). We say that a strip  $\mathbf{t}(i)$  of the triangulation  $\mathbf{t}$  is open if there exist at least one open edge connecting  $S \times \{i\}$  with  $S \times \{i+1\}$ . Thus, we obtain that following relation

$$\begin{aligned} \mathbb{P}_p^{(\mathbf{t})}(0 \leftrightarrow \partial_N) &\leq \mathbb{P}_p^{(\mathbf{t})}(\mathbf{t}(X_1) \text{ is open}, \dots, \mathbf{t}(X_{T_N}) \text{ is open}) \\ &= \prod_{i=1}^{T_N} \mathbb{P}_p^{(\mathbf{t})}(\mathbf{t}(X_i) \text{ is open}) = (1 - (1-p)^2)^{T_N}. \end{aligned}$$

Using the inequality  $1 - a \leq e^{-a}$ , when  $a \in [0, 1)$ , we obtain

$$\mathbb{P}_p^{(\mathbf{t})}(0 \leftrightarrow \partial_N) \leq e^{-(1-p)^2 T_N}.$$

□

Using Lemma 4.1 and Lemma 4.2, and letting  $N \rightarrow \infty$  in (4.1), we obtain the following Lemma.

**Lemma 4.3.** *If  $\mu > \ln 2$ , then for all  $p \in [0, 1)$*

$$\mathbb{P}_p^{(\mathbf{t})}(0 \leftrightarrow \infty) = 0 \quad P_\mu - a.s. \quad (4.2)$$

Lemma 4.3 implies that the critical value for percolation model on causal triangulations, when  $\mu > \ln 2$ , is  $p_c = 1$ . This prove Theorem 3.1 in the subcritical case.

## 5 Phase transition for percolation model in the critical case

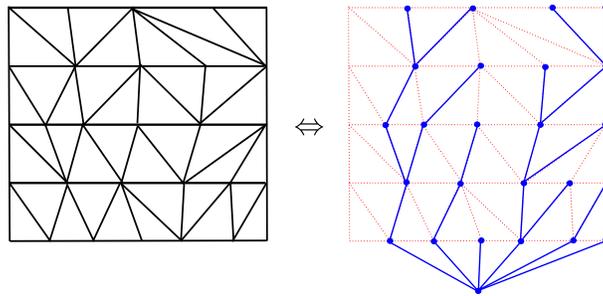
In this section we prove the third statement of Theorem 3.1.

### 5.1 Two-dimensional CDT and Galton-Watson trees

Bijection between causal triangulations and planar trees was established in Malyshev, V. et. al. (2001), see Figure 4. This bijection permit to obtain a tree parametrization of infinite causal triangulation (see also Durhuus, et al. (2010)).

Below we briefly sketch this bijection, which also serves as a way to simulate random causal triangulations.

Given a triangulation  $\mathbf{t} \in \mathbb{LT}_n$ , define the subgraph  $\tau \subset \mathbf{t}$  by taking, for each vertex  $v \in \mathbf{t}$ , the leftmost edge going from  $v$  downwards (see Figure 4). The obtained graph is a spanning forest of  $\mathbf{t}$ , and connecting all vertices on the circle  $S \times \{0\}$  we obtain a tree  $\tau$ . Moreover,  $\mathbf{t}$  can be reconstructed knowing  $\tau$ . We call  $\tau$  the *tree parametrization* of  $\mathbf{t}$ . Denote by  $\eta$  this bijection.



**Figure 2** *Tree parametrization.*

According to this bijection the measure  $P_{\mu_c}$  on infinite triangulations will create a measure  $\rho_\infty$  on the set of infinite trees. In Malyshev, V. et. al.

(2001) it was proved that that the measure  $\rho_\infty$  corresponds to the critical Galton-Watson process with offspring distribution  $\mathbf{p} = (p_k = 1/2^{k+1}, k = 0, 1, \dots)$  conditioned to non-extinction at infinity. Moreover, (see, for example, Durhuus, B. (2006)) an infinite tree generated by this process belongs to the set of so-called *single spine trees*:

- (i) it contains a single infinite path,  $\{v_0, v_1, \dots\}$ , starting at the root vertex  $v_0 = o$ ; this path is called a *a spine*;
- (ii) at each vertex  $v_i$  of the spine a pair of finite trees  $(L_i, R_i)$  is attached, one of each side of the spine;
- (iii) the pairs  $(L_i, R_i)$  are i.i.d. each distributed by critical Galton-Watson with offspring distribution  $\mathbf{p}$ .

This representation helps prove that the critical probability is constant almost sure according to the measure  $P_{\mu_c}$ .

Note here that the same construction works for any critical Walton-Watson process: see Lyons, R. et al. (1995) and Geiger, J. (1999).

## 5.2 Critical value is Constant $P_{\mu_c}$ a.s.

Consider the critical value  $p_c(\mathbf{t})$  of the bond percolation model as a function of  $\mathbf{t}$  defined on the space  $(\mathbb{L}\mathbb{T}_\infty, \mathcal{F}, P_\mu)$ . In the above sections we have shown that  $p_c(\mathbf{t})$   $P_{\mu_c}$ -a.s. causal triangulation  $\mathbf{t}$ .

**Lemma 5.1.** *Let  $G, G'$  be two infinite, locally finite graphs that differ only by a finite subgraph, i.e. there exist two finite subgraphs  $H \subset G, H' \subset G'$ , such that  $G \setminus H$  is isomorphic to  $G' \setminus H'$ . Then  $p_c(G) = p_c(G')$ .*

**Proof.** Let  $\varphi : G \setminus H \rightarrow G' \setminus H'$  be an isomorphism between  $G \setminus H$  and  $G' \setminus H'$ . For each configuration  $w \in \{0, 1\}^{E_{G \setminus H}}$  we define the configuration  $w' \in \{0, 1\}^{E_{G' \setminus H'}}$  as follows: for any  $e' \in G' \setminus H'$  we define  $w'(e') = w(\varphi^{-1}(e'))$ . Note that such defined correspondance  $w \in \{0, 1\}^{E_{G \setminus H}} \rightarrow w' \in \{0, 1\}^{E_{G' \setminus H'}}$  is a bijection. Thus,

$$\mathbb{P}_p^{G' \setminus H'}(w' : w'(\varphi(e)) = 1) = \mathbb{P}_p^{G \setminus H}(w : w(e) = 1) = p.$$

Therefore

$$\varphi(\{w : |C_{v_0}(w)| = \infty\}) = \{w' : |C_{\varphi(v_0)}(w')| = \infty\},$$

and

$$\mathbb{P}_p^{G \setminus H}(w : |C_{v_0}(w)| = \infty) = \mathbb{P}_p^{G' \setminus H'}(w' : |C_{\varphi(v_0)}(w')| = \infty).$$

Since the events  $\{w : |C_{v_0}(w)| = \infty\}$  and  $\{w' : |C_{\varphi(v_0)}(w')| = \infty\}$  are independent of finite subsets, we complete the proof.  $\square$

The critical probability  $p_c \equiv p_c(\mathbf{t})$  is a function of  $\mathbf{t}$ . Given a causal triangulation  $\mathbf{t}$ , let us consider its Galton-Watson tree parametrization  $(L_i(\tau), R_i(\tau))$ , where  $\tau = \eta(\mathbf{t})$ . Let

$$\mathcal{S}_\pi(\mathbf{t}) = (L_{\pi(i)}(\mathbf{t}), R_{\pi(i)}(\mathbf{t})),$$

where  $\pi : \mathbb{N} \rightarrow \mathbb{N}$  is a bijection such that  $\pi(n) = n$  for all but finitely many  $n$ . Define the set of finite permutations of  $\mathbf{t}$  by  $\mathcal{S}(\mathbf{t}) = \cup_\pi \mathcal{S}_\pi(\mathbf{t})$ . The set  $\mathcal{S}(\mathbf{t})$  is symmetric for all  $\mathbf{t} \in \mathbb{LT}_\infty$ , i.e. invariant under finite permutations. Thus, by Hewitt-Savage zero-one law we obtain the following lemma.

**Lemma 5.2.** *Let  $\mathbf{t}$  be a random causal triangulation. Then*

$$P_{\mu_c}(\mathcal{S}(\mathbf{t})) = 1 \quad P_{\mu_c} - a.s. \mathbf{t}.$$

*Proof of Theorem 3.1.* By Lemma 5.2 there exists a subset  $\mathcal{A} \subset \mathbb{LT}_\infty$  such that  $P_{\mu_c}(\mathcal{S}(\mathbf{t})) = 1$  for any  $\mathbf{t} \in \mathcal{A}$ . We define the set  $\mathcal{S} = \bigcap_{\mathbf{t} \in \mathcal{A}} \mathcal{S}(\mathbf{t})$ . Note that  $P_{\mu_c}(\mathcal{S}) = 1$ , and if  $\tau_1, \tau_2 \in \mathcal{P}$  there exists a permutation  $\pi$  such that  $\mathbf{t}_1 = \pi \mathbf{t}_2$ , i.e.,  $\mathbf{t}_1$  and  $\mathbf{t}_2$  only differ by a finite subgraph. Therefore, by Lemma 5.1 we conclude that the function  $p_c : \mathbb{LT}_\infty \rightarrow [0, \infty)$  is constant  $P_{\mu_c}$  a.s. This completes the proof of the part 1 of Theorem 3.1.  $\square$

### 5.3 The critical value is non-trivial

We prove first that  $p_c(\mathbf{t}) > 0$   $P_{\mu_c}$ -a.s.  $\mathbf{t}$ . We shall show that  $E_{\mu_c}(\theta^{(\cdot)}(p)) = 0$  whenever  $p$  is sufficiently close to 0. Let  $\sigma^{(\mathbf{t})}(n)$  be the number of paths of  $\mathbf{t}$  of the length  $n$  beginning at the root, and let  $N^{(\mathbf{t})}(n)$  be the number of such paths which are open. Any such path is open with probability  $p^n$ , so that

$$\mathbb{E}_p(N^{(\mathbf{t})}(n)) = p^n \sigma^{(\mathbf{t})}(n).$$

Now, if the root belongs to an infinite open cluster then there exists open paths of all lengths beginning at the root, so that

$$\begin{aligned} \theta^{(\mathbf{t})}(p) &\leq \mathbb{P}_p(N^{(\mathbf{t})}(n) \geq 1) \\ &\leq \mathbb{E}_p(N^{(\mathbf{t})}(n)) = p^n \sigma^{(\mathbf{t})}(n). \end{aligned}$$

Let  $X \sim \left(\frac{1}{2^{n+1}}\right)_{n \geq 0}$ ,  $Y \sim \left(\frac{n}{2^{n+1}}\right)_{n \geq 1}$  be two random variables.

Let  $\gamma \in \sigma^{(\mathbf{t})}(n)$  and let  $v_0, v_1, \dots, v_{n+1}$  denote its vertices. The degree  $d_i$  of the vertex  $v_i$  is a random variable, and has as distribution  $X$  or  $Y$ . Thus,  $\sigma^{(\mathbf{t})}(n) \leq \prod_{i=0}^{n-1} (2d_i + 1)$ . Therefore

$$E_{\mu_c}(\theta^{(\cdot)}(p)) \leq p^n 21^{n/2}.$$

Thus, we obtain that  $\theta^{(\mathbf{t})}(p) = 0$   $P_{\mu_c}$ -a.s.  $\mathbf{t}$  if  $p < 1/\sqrt{21}$ .

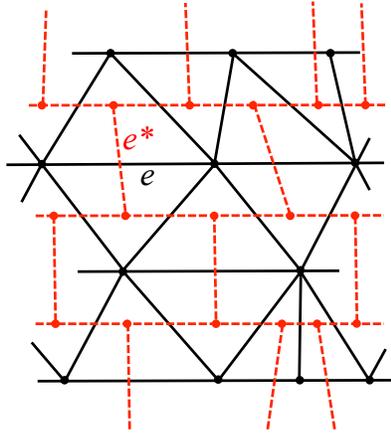
We also show that  $p_c(\mathbf{t}) < 1$ , and we use an approach which is commonly called a *Peierls argument*. We shall show that  $\theta^{(\cdot)}(p) > 0$  if  $p$  is sufficiently close to 1,  $P_{\mu_c}$ -a.s. causal triangulation  $\mathbf{t}$ .

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For any causal triangulation  $\mathbf{t}$  let  $\gamma_\infty \equiv \gamma_\infty(\mathbf{t}) = (v_0, v_1, \dots)$  be the spine path of the tree parametrization for  $\mathbf{t}$  starting at the root  $v_0$ , such that  $v_i \in S \times \{i\}$ .



**Figure 3** A dual graph (red) for a causal triangulation (black).

For any finite cluster  $C_o$  of open edges contained the root there exists a closed circuit (or contour) consisted of edges on dual graph. Let  $C^{(\mathbf{t})}(n)$  be the set of closed circuits (set of contours) of the length  $n$  in  $\mathbf{t}^*$ , separating  $v_0$  from the infinite part of the graph. Then

$$\begin{aligned} \mathbb{P}_p(|C_0| < \infty) &\leq \sum_{\gamma} \mathbb{P}_p(\gamma \text{ is closed in } \mathbf{t}^*) \\ &\leq \sum_{n \geq 1} |C^{(\mathbf{t})}(n)|(1-p)^n. \end{aligned}$$

**Lemma 5.3.** *Let  $\mathbf{t}$  be a random causal triangulation, and let  $v_0$  be the root vertex of  $\mathbf{t}$ . There exists  $p_0 < 1$  such that for any  $p \in (p_0, 1)$*

$$E_{\mu_c} \sum_{n \geq 1} |C^{(\mathbf{t})}(n)|(1-p)^n < 1. \tag{5.1}$$

**Proof.** Let  $C_{R,n}^{(\mathbf{t})} \subset C^{(\mathbf{t})}(n)$  be the set of contours of length  $n$  which surround  $v_0$  and intersect  $\gamma_\infty$  at height  $R$ . Note that any such contour does not exist from the strip  $[R-n, R+n]$ . Let also  $S_{R,n}$  be the number of particles in the tree parametrization of  $\mathbf{t}$  at height  $R-n$  which have nonempty offspring in the generation located at height  $R+n$ . Since every contour of  $C^{(\mathbf{t})}(n)$ , in order to surround  $v_0$ , must cross each subtree starting of each vertex of  $S_{R,n}$ , we have

$$\{S_{R,n} > n\} \Rightarrow \{\gamma \in C^{(\mathbf{t})}(n)\} = \emptyset.$$

In the other hand

$$|C_{R,n}^{(\mathbf{t})}| \leq 2^n,$$

since the contours  $C_{R,n}^{(\mathbf{t})}$  live on the dual graph  $\mathbf{t}^*$ , which has all vertices of degree 3, thus there are at most  $2^n$  self-avoiding path with a fixed starting point (which is in our case the intersection with  $\gamma_\infty$ ). In addition, note that  $C^{(\mathbf{t})}(n) = \cup_{R \geq 2} C_{R,n}^{(\mathbf{t})}$ . Utilizing the decomposition

$$E_{\mu_c} |C_{R,n}^{(\mathbf{t})}| = E_{\mu_c} [|C_{R,n}^{(\mathbf{t})}| \mid S_{R,n} > n] + E_{\mu_c} [|C_{R,n}^{(\mathbf{t})}| \mid S_{R,n} \leq n],$$

we have the following inequality

$$E_{\mu_c} \sum_{n \geq 1} |C^{(\mathbf{t})}(n)| (1-p)^n \leq \sum_{n \geq 1} (1-p)^n 2^n \sum_{R \geq 2} P_{\mu_c}(S_{R,n} \leq n). \quad (5.2)$$

Now, let us estimate  $P_{\mu_c}(S_{R,n} \leq n)$ . For this we write this probability as follow

$$P_{\mu_c}(S_{R,n} \leq n) = \sum_{k \geq 1} P_{\mu_c}(S_{R,n} \leq n \mid k_{R-n} = k) P_{\mu_c}(k_{R-n} = k).$$

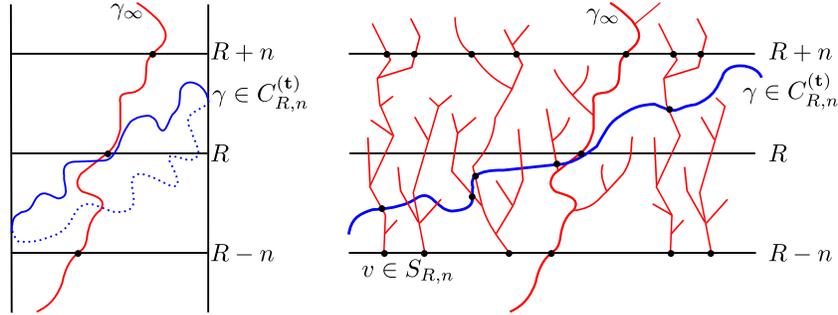
Note that, if  $k \leq n$ , then  $P_{\mu_c}(S_{R,n} \leq n \mid k_{R-n} = k) = 1$ . Thus, we write the probability  $P_{\mu_c}(S_{R,n} \leq n)$  as

$$P_{\mu_c}(S_{R,n} \leq n) = P_{\mu_c}(k_{R-n} \leq n) + \sum_{k > n} P_{\mu_c}(S_{R,n} \leq n \mid k_{R-n} = k) P_{\mu_c}(k_{R-n} = k). \quad (5.3)$$

Utilizing the distribution of  $k_n$ , respect to the Gibbs measure  $P_{\mu_c}$ , the first sum above satisfies the inequality

$$P_{\mu_c}(k_{R-n} \leq n) \leq \left( \frac{n}{R-n} \right)^2. \quad (5.4)$$

The second sum in (5.3) can be estimated by estimating the probability



**Figure 4** Construction of the set  $S_{R,n}$  in the proof of the Lemma 5.3

$P_{\mu_c}(S_{R,n} \leq n | k_{R-n} = k)$ . For this, note that the probability for a particle of the branching process  $\xi$  to survive up to time  $2n$  is equals  $1/(1+2n)$ , thus conditionally on  $\{k_{R-n} = k\}$ ,  $S_{R,n}$  is stochastically minorated by a binomial distribution with parameters  $(k_{R-n} = k, \frac{1}{1+2n})$ . Utilizing Hoeffding's inequality for binomial distribution we obtain

$$P_{\mu_c}(S_{R,n} \leq n | k_{R-n} = k) < e^{-2\frac{(k/(2n+1)-n)^2}{k}}. \quad (5.5)$$

Employing the upper bound (5.5), we obtain

$$P_{\mu_c}(S_{R,n} \leq n) \leq \left(\frac{n}{R-n}\right)^2 + \frac{1}{(R-n)^2} \sum_{k>n} k e^{-2\frac{(k/(2n+1)-n)^2}{k}}.$$

The above sum over  $k > n$  is bounded by some absolute constant  $A_1$ , thus we obtain the following upper bound for the probability

$$P_{\mu_c}(S_{R,n} \leq n) \leq \left(\frac{n}{R-n}\right)^2 + \frac{A_1}{(R-n)^2}.$$

Utilizing this upper bound, we prove that there exist constants  $A_2, A_3$  such that

$$E_{\mu_c} \sum_{n \geq 1} |C^{(t)}(n)|(1-p)^n \leq \sum_{n \geq 1} (2(1-p))^n (A_2 n^2 + A_3). \quad (5.6)$$

Now, we define the function  $\varphi(p) = \sum_{n \geq 1} (2(1-p))^n (A_2 n^2 + A_3)$ . This function is analytic and decreasing on  $(1/2, 1]$ , and  $\varphi(1) = 0$ . Therefore there exist  $p_0 \in (1/2, 1)$  such that  $\varphi(p) < 1$  if  $p \in (p_0, 1]$ . This conclude the proof of the Lemma.  $\square$

Finally, in order to prove that  $p_c(\mathbf{t}) < 1$   $P_{\mu_c}$ -a.s.  $\mathbf{t}$ , we utilize the identities  $\mathbb{P}_p^{(\mathbf{t})}(|C_0| = \infty) = 1 - \mathbb{P}_p^{(\mathbf{t})}(|C_0| < \infty)$ , and

$$\mathbb{P}_p^{(\mathbf{t})}(|C_0| = \infty) \geq 1 - \sum_{n \geq 1} |C^{(\mathbf{t})}(n)|(1-p)^n.$$

Employing Lemma 5.3, we prove that  $E_{\mu_c}(\mathbb{P}_p^{(\mathbf{t})}(|C_0| = \infty)) > 0$  for  $p \in (p_0, 1]$ , with  $p_0 < 1$ . This complete the proof of the part 2 of the Theorem 3.1 in the critical case.

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