List 8 IBI 5081 – Optimization III. MCMC.

For all next problems describe Metropolis-Hasting algorithm providing pseudo-code (without numerical implementation):

(1) (see, Ross "Simulations", Ch.10, Ex.3) Let S be the set of all $n \times n$ matrices A whose elements are either 0 or 1. (Thus, there are 2^{n^2} matrices in S.) The pair of elements $a_{i,j}$ and $a_{r,s}$ of the matrix A are said to be neigbours if $|r-i|+|s-j| = 1$. (Thus, for instance, the neighbors of $a_{2,2}$ are $a_{1,2}, a_{2,1}, a_{2,3}$ and $a_{3,2}$.) Let $\mathcal{N}$ denote all the pairs of neighboring elements of A. The "Ising energy" of the matrix A is defined by

$$H(A) = -\sum_{\mathcal{N}} a_{i,j} a_{r,s}$$

where the sum is over all the pairs of neighboring elements.

Provide an algorithm for sampling randomly a matrix A according to the probability mass function

$$P(A) \propto \exp(-\lambda H(A)), \text{ or } P(A) = \frac{\exp(-\lambda H(A))}{\sum_{A \in S} \exp(-\lambda H(A))}, \quad A \in S,$$

where λ is a specified positive constant. *Hint*: Let the matrices A and B be neighbors of $A - B$ has only one nonzero elements.

(2) Consider the subset G of symmetric matrices from the set S with 0's on diagonal, $G \subset S$. The set G represents the set of adjacency matrices for simple (without self-loops) undirected graphs. Let $A = (a_{i,j}, i, j \in V) \in G$, where $V = \{1, 2, \ldots, n\}$. The graph with corresponds to the adjacency matrix A defined by the set of vertices, V, and two distinct vertices $i, j \in V$ are connected by an edge iff the corresponding matrix element $a_{i,j} = 1$. On the set of simple undirected graphs G we define the following probability measure: for any $g \in G$

$$P(g) \propto \text{ the number of edges on } g.$$

Provide a sampling algorithm form the measure.

(3) (from internet: http://web.csulb.edu/ tebert/teaching/fall17/552/ assignments/mcmc/mcmc.pdf, Ex.7)

Consider the state space $X = \{0, 1\}^n$ of binary strings having length n. Define an auxiliary Markov chain: let $p(y|x) = 1/n$ if y differs from x in exactly one bit, and $p(y|x) = 0$ otherwise. Suppose we desire an equilibrium distribution π for which $\pi(x)$ is proportional to the number of ones that occur in vector x. For example, in the long run a random walk should visit a string having five 1s five times as often as it visits a string having only a single 1. Provide a general formula for $A(x, y)$ that would be used if we were to obtain the desired equilibrium distribution using the Hastings-Metropolis algorithm.