List 3. Monte Carlo Method I. Integration.

In general the Monte Carlo Method can be defined in the following way. Let I be some value that we are interested to estimate (for example I can be some definite integral). The method consists on a finding a random variable X, or a function of random variable $h(X)$ such that $I = \mathbb{E}\big(h(X)\big)$. Let X_i be an infinite sequence of i.i.d. with the same distribution of X. Then the value I is estimated by the average

$$\hat{I}_n := \frac{1}{n}\sum_{i=1}^{n} h(X_i) \to I \text{ a.s. as } n \to \infty,$$

where the convergence is guaranteed by the strong low of large numbers. Taking this in mind in order to resolve the exercises.

1. Using Monte Carlo method find the value of integral of the semi-circle $I_1 = \int_{-1}^{1}\sqrt{1-x^2}dx$. The true value of the integral is $\pi/2$.

(1) (2 points) Construct two estimators by MC methods: the classical MC method, m_1, and MC method with partial integration m_2 (use the triangle with vertices $(-1,0),(1,0)$ and $(1,0)$ as the first approximation of the semi-cirle). Provide a random variable X and function $h(X)$ for two methods.

(2) (2 points) Calculate explicitly the variances of random variables used in the two method. Then, using these calculations, find the sample size for the both MC methods described in previous item, such that the absolute error does not exceed 0.01 with probability 95%.

(3) (1 points) Explain the using of "geometrical" (accept-reject) method in order to calculate the above integral.

2. Choose some (positive) function of two variables $f(x,y)$ of your interest (it can be likelihood function, or some other abstract function).

(1) (1 point) On some bounded area, say $Q = \{x \in [a,b], y \in [c,d]\}$, plot the function;

(2) (2 point) Calculate its integral on the same bounded area Q numerically using "classical" Monte Carlo method. What instrumental random variable you will use? Construct the 95% confidence interval (probably you can not calculate explicitly the variance, use its classical estimator s_n^2) and plot all for different increasing values of sample size, for example, $n = 100, 200, \ldots, 4900, 5000$.

(3) (2 points) Let $g(x,y) \equiv c$ some constant function, $x, y \in Q$ (choose the value explicitly, a number, observing the values of your function $f(x,y)$ and thinking on the partial integration estimator). Construct the partial integration estimator. For different values of sample size, compare the variances of the partial integration estimator and the "classical" one constructed in previous item.