List 3. Monte Carlo Method I. Integration.

In general the Monte Carlo Method can be defined in the following way. Let I be some value that we are interested to estimate (for example I can be some definite integral). The method consists on a finding a random variable X, or a function of random variable $h(X)$ such that $I = \mathbb{E}\big(h(X)\big)$. Let X_i be an infinite sequence of i.i.d. with the same distribution of X. Then the value I is estimated by the average

$$\hat{I}_n := \frac{1}{n}\sum_{i=1}^{n} h(X_i) \to I \text{ a.s. as } n \to \infty,$$

where the convergence is guaranteed by the strong low of large numbers. Taking this in mind resolve the exercises.

1. Using Monte Carlo method find the value of integral $I_1 = \int_0^1 \sqrt{1+x^2}dx$.

(1) Construct two estimators by MC methods: the classical MC method, m_1, and MC method with partial integration m_2 (use the first two term of Taylor expansion). Provide a random variable X and function $h(X)$ for two methods.
(2) Compare the variances of random variables used in the two method. For the both MC methods described in previous item find the sample size such that the absolute error does not exceed 0.001 with probability 95%.
(3) Simulate 100 points and provide the two numerical estimations for the integral with their respective 95% confidence intervals.
(4) Consider MC geometric method (consider an integral as an area (volume) under a function). Construct the estimator of I by geometric method. Calculate the variance of random variable used in this method.

2. Using Monte Carlo method find the value of integral $I_2 = \int_0^1 cos(x)dx$.

(1) Construct two estimators by MC methods: the classical MC method, m_1, and MC method with partial integration m_2 based on the function $y = -0.4x+1$. Provide a random variable X and function $h(X)$ for two methods.
(2) Compare the variances of random variables used in two methods.
(3) Simulate 100 points and provide the two numerical estimations for the integral with their respective 95% confidence intervals.

3. Provide the Monte Carlo simulation algorithm in order to estimate the integral

$$I_3 = \frac{1}{(2\pi)^5}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} e^{-(x_1^2+\cdots+x_{10}^2)/2}\sqrt{1+\frac{\sin^2(x_1\ldots x_{10}}{4}}dx_1\ldots dx_{10}.$$

Find the 95% confidence intervals, based on 1000 simulation points and on 100000 simulation points.