Category numbers of posets and topological spaces

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Abstract

We define the concept of category numbers of posets and topological spaces and show some relations between some of them.

1 Introduction

In this section we discuss the definitions of category numbers for both topological spaces and forcing posets. We expect the reader is familiarized with topological spaces, with the Baire category theorem for complete metric spaces and compact Hausdorff spaces, and with the basics of either Martin’s axiom or forcing. We will not discuss forcing in this document, we will just deal with filters in partial orders and dense open subsets in them.

1.1 Topological Spaces

The Baire Category Theorem is a very famous theorem that states that the intersection of a countable family of dense open subsets in a complete metric space or of a compact Hausdorff topological space is nonempty (in fact, dense). Proofs may be found in most general topology books, such as [3] or [1].

With this theorem in mind, a natural question arises: given a topological space \( X \), what is the least cardinal \( \kappa \) of a family of dense open subsets of \( X \) whose intersection is empty? Such family does not exist for every topological space. The following proposition shows a necessary and sufficient condition.

Proposition 1.1. Let \( X \) be a topological space. Then \( X \) has a nonempty collection of dense open subsets whose intersection is empty if and only if \( \text{int} \, \text{cl} \{ x \} = \emptyset \) for every \( x \in X \).

Proof. Let \( D = \{ D \subseteq X : D \text{ is open and dense} \} \). It suffices to see that \( \bigcap D = \{ a \in X : \text{int} \, \text{cl} \{ a \} \neq \emptyset \} \). Fix \( a \in X \).

Suppose \( \text{int} \, \text{cl} \{ a \} = \emptyset \). Then \( X \setminus \text{cl} \{ a \} \) is a dense open set, so \( a \notin \bigcap D \).

Now suppose \( \text{int} \, \text{cl} \{ a \} \neq \emptyset \). We show that \( a \) is in every dense open subset \( D \). Since \( D \) is dense, \( D \cap \text{int} \, \text{cl} \{ a \} \neq \emptyset \), so \( D \cap \text{cl} \{ a \} \neq \emptyset \). Since \( D \) is open, \( D \cap \{ a \} \neq \emptyset \), so \( a \in D \).

Notice that every \( T_1 \) space with no isolated points satisfies the condition of the previous proposition. Moreover, if a space has isolated points, it fails to satisfy the condition. It is not enough for a space to have no isolated points and be \( T_0 \), since if we let \( X \) be an infinite set, \( x \in X \) and give it the topology generated by the cofinite subsets that have \( x \) as an element, then \( X \) will have no isolated points, \( X \) will be \( T_0 \) and \( \{ x \} \) will be dense.

Following [2], we define the Baire category number of a topological space as follows:

Definition 1.2. Suppose \( X \) is a topological space such that \( \text{int} \, \text{cl} \{ x \} = \emptyset \) for every \( x \in X \). Then \( m(X) \) is the least cardinal \( \kappa \) for which there exists a collection \( D \) of dense open sets of cardinality \( \kappa \) such that \( \bigcap D = \emptyset \).

Notice that by fixing, for each \( x \), a dense open set \( D_x \) such that \( x \notin D_x \), we obtain a collection of at most \( |X| \) dense open sets with empty intersection. So \( m(X) \leq |X| \).

When we talk about the Baire Category Theorem, we usually refer to classes of spaces, such as the class of completely metrizable spaces, or the class of compact Hausdorff spaces. Therefore it is useful to define the Baire category number of a class of spaces, which indicates what is the least cardinal \( \kappa \) for which there exists a space in the class with a collection of dense open sets of size \( \kappa \) and empty intersection.

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**Definition 1.3.** Suppose \( K \) is a class of topological spaces with at least one member \( X \) with no points \( x \) such that \( \text{cl}\{x\} \neq \emptyset \). Then we define

\[
m(K) = \min\{\kappa : \exists X \in K \exists D \text{ nonempty collection of dense open subsets of } D \text{ such that } |D| = \kappa \text{ and } \bigcap D = \emptyset\}.
\]

Now we state a basic fact about the Baire category number. Notice that if \( A \) is a dense open subset of \( X \) and \( D \subseteq X \) is dense, then \( D \cap A \) is a dense open subspace of the open set \( D \). So if \( X \) has a collection of dense open subsets of \( X \) with empty intersection, so does \( D \).

**Lemma 1.4.** Let \( X \) be a topological space which has a collection of dense open subsets with empty intersection, and let \( D \subseteq X \) be dense. Then \( m(D) \leq m(X) \). Moreover, if \( D \) is the intersection of less than \( m(X) \) open sets, then the equality holds.

**Proof.** Consider \( A \) a collection of dense open subsets of \( X \) such that \( \bigcap A = \emptyset \) and \( |A| = m(X) \). Since \( D \) is dense, it follows that \( \{D \cap A : A \in A\} \) is a collection of dense open subsets of \( D \). Now, \( \bigcap_{A \in A} D \cap A = D \cap \bigcap A = \emptyset \). Thus, \( m(D) \leq |A| = m(X) \).

Now suppose \( D = \bigcap D \), where \( D \) is a nonempty family of less than \( m(X) \) open sets of \( X \). Since they all contain \( D \), they all are dense. Let \( A \) be a collection of less than \( m(X) \) open dense sets of \( D \). For each \( A \in A \), let \( U_A \) be an open subset of \( X \) such that \( U_A \cap D = A \). Each \( U_A \) is an open dense set of \( X \). Then \( \bigcap_{A \in A} U_A \cap \bigcap D = \bigcap_{A \in A} U_A \cap D = \bigcap_{A \in A} A \) is not empty. This concludes that \( m(D) \geq m(X) \). \( \square \)

### 1.2 Partial Orders

Preorders play an important role in infinitary combinatorics and forcing. In what follows, we review all the definitions we will be using. We expect the reader to be familiar with most of the definitions below. We leave some straightforward verifications to the reader.

**Definition 1.5.** A preorder is a pair \((P, \leq)\) such that \( \leq \) is a reflexive and transitive relation on \( P \).

- If \((P, \leq)\) is a preorder, we say that \( p, q \in P \) are comparable if \( p \leq q \) or \( q \leq p \). Otherwise they are called incomparable.

- If \((P, \leq)\) is a preorder, we define \( p \approx q \) iff \( p \leq q \) and \( q \leq p \). When \( \leq \) is clear by the context, we just write \( \approx \) instead of \( \approx \). The reader may verify that \( \approx \) is an equivalence relation.

- If \((P, \leq)\) is a preorder, the induced strict preorder is the transitive and irreflexive relation \( < \) given by \( p < q \) iff \( p \leq q \) and \( p \not\approx q \).

- A partial order is a preorder \((P, \leq)\) which is also antisymmetric, that is, for all \( p, q \in P \), if \( p \leq q \) and \( q \leq p \) then \( p = q \). In other words, \( p \approx q \) implies \( p = q \).

- A greatest (least) element of a preorder \((P, \leq)\) is an element \( p \in P \) such that for all \( q \in P \), \( q \leq p \) \((p \leq q)\). The reader may verify that in the case \((P, \leq)\) is a partial order, it has at most one greatest and one least element, and in this case we denote then by \( \max P \) and \( \min P \).

- A forcing partial order, (or forcing preorder), is a triple \((P, \leq, \mathbb{1})\), where \((P, \leq)\) is a partial order (or preorder) with greatest element \( \mathbb{1} \).

- A maximal element of a preorder \((P, \leq)\) is an element \( p \in P \) such that for all \( q \in P \), if \( q \geq p \) then \( q \approx p \) (in other words, there is no \( q \in P \) such that \( q > p \)). A minimal element of a preorder \((P, \leq)\) is an element \( p \in P \) such that for all \( q \in P \), if \( q \leq p \) then \( q \approx p \) (in other words, there is no \( q \in P \) such that \( q < p \)).

- Let \((P, \leq)\) be a preorder. Two elements \( a, b \) of \( P \) are said to be compatible if there exists \( c \in P \) such that \( c \leq a, b \). If they are not compatible, they are said to be incompatible and we write \( a \perp b \). Otherwise, we write \( a \not\perp b \).

- Given a preorder \((P, \leq)\) we define the cone \( V_p = \{q \in P : q \leq p\} \). Notice that these sets form a basis for a topology in \( P \), and that \( \{V_p\} \) is a local one-set basis for the point \( p \) in this topology. This topology is, in general, not even \( T_0 \). For partial orders, it is \( T_0 \) but in general not \( T_1 \).
• Given a preorder \((P, \leq)\), and \(A \subseteq P\), we define \(A^\uparrow = \{ p \in P : \exists a \in A : a \leq p\}\) and \(A^\downarrow = \{ p \in P : \exists a \in A : a \geq p\}\). If \(A = A^\uparrow\), we say \(A\) is closed upwards. If \(A = A^\downarrow\), we say \(A\) is closed downwards or open. Notice that \(A\) is open in this sense if and only if it is open in the topology defined above. Given \(p \in P\), we denote \(\{p\}^\uparrow = p^\uparrow\) and \(\{p\}^\downarrow = p^\downarrow\).

• Given a preorder \((P, \leq)\), we say that \(D \subseteq P\) is dense iff for every \(p \in P\) there exists \(d \in D\) such that \(d \leq p\). Notice that this agrees with density in the topology defined above.

• Given a preorder \((P, \leq)\), we say that \(G \subseteq P\) is a filter iff it is nonempty, closed upwards and such that for every \(a, b \in G\) there exists \(c \in G\) such that \(c \leq a, b\).

• Given a preorder \((P, \leq)\), we say that \(p \in P\) is an atom if every two elements below \(p\) are compatible.

• Given a preorder \((P, \leq)\), we say that \(A \subseteq P\) is a chain iff every two elements are comparable. We say it is an antichain if every two elements of \(A\) are incompatible. Notice that by Zorn’s Lemma, every chain and every antichain can be extended to a maximal chain or antichain.

In theorem statements, when it is clear from the context that a set \(P\) has an order relation \(\leq\) and no confusion may arise, we just write \(P\) instead of \((P, \leq)\).

In the previous subsection we defined \(m(X)\) for a topological space \(X\). Now we define the cardinal invariant \(m(P)\) for a preorder \(P\).

**Definition 1.6.** Let \(P\) be a preorder with no atoms. Then \(m(P)\) is the least cardinal \(\kappa\) for which there exists a family of \(\kappa\) dense open sets \(D\) such that no filter \(G \subseteq P\) intersects every element of \(D\).

The invariant \(m(P)\) really only makes sense if \(P\) has no atoms.

**Proposition 1.7.** Let \(P\) be a preorder. Then the following are equivalent:

1. \(P\) has an atom,

2. There exists a filter \(G \subseteq P\) that intersects all dense open subsets of \(P\).

**Proof.** Suppose \(P\) has an atom \(a\). Let \(G = \{ p \in P : a\) is compatible with \(p\}\}. \(G\) is clearly closed upwards. If \(p, q \in G\), then there exists \(p', q' \in P\) such that \(p' \leq a, p\) and \(q' \leq a, q\). Since \(a\) is an atom, \(p', q'\) are compatible, so there exists \(r\) such that \(r \leq q' \leq q\) and \(r \leq p' \leq p\). Moreover, \(r \leq a\), so \(r \in G\). This proves that \(G\) is a filter.

Now we show that \(G\) intersects every dense open subset of \(P\). Let \(D\) be a dense open subset of \(P\). There exists \(d \in D\) such that \(d \leq a\). But then \(d \in G\) by the definition of \(G\) and we are done.

Now suppose \(P\) has no atom. Then given a filter \(G, P \setminus G\) is a dense open set:

- It is open: given \(p \in P \setminus G\) and \(q \leq p\), we have that \(q \in P \setminus G\), since if we had that \(q \in G\), it would imply that \(p \in G\).

- It is dense: given \(p \in P\), there exists two incompatible elements \(q, q'\) below \(p\). Since every two elements of \(G\) are compatible, \(G\) can have at most one of the elements \(q, q'\). So one of them is in \(P \setminus G\).

Thus, let \(G\) be the collection of all filters in \(P\) and consider \(D = \{ P \setminus G : G \in G\}\). \(\Box\)

Differently to what happens to topological spaces, the cardinal \(m(P)\) is always uncountable when it is defined.

**Proposition 1.8.** Let \(P\) be a preorder with no atoms. Then \(m(P) \geq \omega_1\).

**Proof.** Suppose \(\{D_n : n \in \omega\}\) is a countable collection of dense open subsets of \(P\) (which can be finite by repeating sets if necessary). Recursively, by density we define a sequence \(p_0 \geq p_1 \geq p_2 \geq \ldots\) such that \(p_n \in D_n\) for every \(n \in \omega\). It is easy to verify that \(G = \{ p \in P : \exists n \in \omega, p_n \leq p\}\) is a filter. Clearly, \(G \cap D_n\) for every \(n \in \omega\). \(\Box\)

As what we did for topological spaces, it is also useful to define \(m(P)\) when \(P\) is a class of preorders.

**Definition 1.9.** Suppose \(P\) is a class of preorders with at least one preorder with no atoms. Then we define

\[
m(P) = \min\{m(P) : P \in P\text{ and }P\text{ has no atoms}\}.
\]

There are many variations of these concepts, such as considering only partial orders, or only orders with greatest elements, or dense sets instead of restricting ourselves to dense open sets. These restrictions and distinctions are sometimes useful theoretically, and we dedicate the rest of this subsection to showing the relations these variations have.

The following proposition is straightforward and the proof is left to the reader.
Proposition 1.10. Let \((P, \leq)\) be a preorder. Then:

1. Let \(G \subseteq P\) be a filter and \(D \subseteq P\) be any set. Then \(G \cap D \neq \emptyset\) iff \(G \cap D^\uparrow \neq \emptyset\).
2. Let \(A \subseteq P\) be a maximal antichain. Then \(A^\uparrow\) is open and dense.
3. Every dense subset of \(P\) contains a maximal antichain.
4. \(m(P)\) is the least cardinality for which exists a collection of dense subsets of \(P\) with no filter intersecting all of them.
5. \(m(P)\) is the least cardinality for which exists a collection of maximal antichains of \(P\) with no filter intersecting all of them.
6. Let \(D\) be a dense subset of \(P\), \(A \subseteq P\) be a dense open set and \(G \subseteq P\) be a filter. Then \(G \cap D\) is a filter in \(D\), \(A \cap D\) is a dense open subset in \(D\) and \(G \cap D\) intersects \(A \cap D\) if and only if \(G\) intersects \(A\).
7. Let \(D\) be a dense subset of \(P\), \(A \subseteq D\) be a dense open subset of \(D\) and \(G \subseteq D\) be a filter in \(D\). Then \(G^\uparrow\) is a filter in \(P\), \(A^\uparrow\) is a dense open subset of \(P\) and \(G^\uparrow\) intersects \(A^\uparrow\) iff \(G\) intersects \(A\).
8. Let \(D\) be a dense subset of \(P\). Then \(P\) has no atoms if, and only if \(D\) has no atoms, and in this case, \(m(P) = m(D)\).

As a corollary of the last item, when we add a maximum element to a preorder, it doesn’t change its Baire invariant.

Corollary 1.11. Let \((P, \leq)\) be a preorder. Fix \(\mathbf{1} \notin P\) and let \(Q = P \cup \{\mathbf{1}\}\). Extend \(\leq\) by defining \(\leq = \leq \cup \{(p, \mathbf{1}) : p \in P\}\), so that \(\mathbf{1}\) is the greatest element of \(Q\) and the order relation in \(P\) is preserved. Then \(P\) is a dense subset of \((Q, \leq)\), \(P\) has atoms iff \(Q\) has atoms and, in the case they have no atoms, \(m(P) = m(Q)\). Moreover, if \(P\) is a partial order, so is \(Q\).

Now we show how to associate a preorder to a partial order with the same category invariant. We also leave the proof to the reader.

Proposition 1.12. Let \((P, \leq)\) be a preorder. Let \(Q = P/\approx\) and \(i : P \to Q\) the quotient map from \(P\) onto \(Q\). Define \(i(p) \leq i(q)\) iff \(p \leq q\) for every \(p, q \in P\). Then:

1. The relation \(\leq\) is well defined and \((Q, \leq)\) is a partial order.
2. \(p\) is a greatest element of \(P\) iff \(i(p) = \text{max} Q\).
3. If \(G \subseteq P\) is a filter, \(A \subseteq P\) an open set and \(D \subseteq P\) a dense set, so are (resp) \(i[G]\), \(i[A]\) and \(i[D]\).
4. If \(G \subseteq P\) is a filter and \(D \subseteq P\) is a dense open set, then \(i[G] \cap i[D] \neq \emptyset\) iff \(G \cap D \neq \emptyset\).
5. If \(H \subseteq Q\) is a filter, \(B \subseteq Q\) an open set and \(E \subseteq Q\) a dense set, so are (resp) \(i^{-1}[H]\), \(i^{-1}[B]\) and \(i^{-1}[E]\).
6. If \(H \subseteq Q\) is a filter and \(E \subseteq Q\) is a dense open set, then \(i^{-1}[H] \cap i^{-1}[E] \neq \emptyset\) iff \(H \cap E \neq \emptyset\).
7. For every \(p, q \in P\), \(p \perp q\) iff \(i(p) \perp i(q)\).
8. For every \(p \in P\), \(p\) is an atom iff \(i(p)\) is an atom.
9. \(P\) has atoms iff \(Q\) has atoms, and in case they have no atoms, \(m(P) = m(Q)\).

2  Countable orders and completely metrizable separable spaces

2.1 Introduction

The cardinal known as covering of category is defined as \(m(K)\), where \(K\) is the class of separable completely metrizable topological spaces.

The cardinal \(m_{\text{emb}}\) is the cardinal \(m(P)\), where \(P\) is the class of all countable preorders.

The aim of this section is to show that there two cardinals are the same, and that they are equal to \(m(\mathbb{R})\).
The Baire space is defined as the space $\omega^\omega$, where $\omega$ is discrete and the standard product topology is used. The cones $V_s = \{ f \in \omega^\omega : s \subseteq f \}$ where $s \in \omega^<\omega$ are a base for this topology.

It is well known that the Baire Space is homeomorphic to the set of the irrationals $\mathbb{I} = \mathbb{R} \setminus \mathbb{Q}$ with the standard topology (e.g. this is left as an exercise with a very generous hint in [3]). In fact, the space $\mathbb{I}$ is topologically characterized for being a completely metrizable, separable, nowhere locally compact\(^1\), zero dimensional space, which is also a well known fact. This will be shown in the following subsection.

2.2 Characterizing the irrationals

In this section, we show that the Baire space $\omega^\omega$ is homeomorphic to every completely metrizable, nowhere locally compact, zero dimensional space. Notice that the set of irrational numbers satisfies these properties (it is completely metrizable since it is a $G_\delta$ subset of $\mathbb{R}$, so in particular every space satisfying these properties is homeomorphic to $\mathbb{I}$).

We start with a lemma.

**Lemma 2.1.** Let $(X, d)$ be a separable, zero dimensional, non compact metric space and $r > 0$ be a real number. Then $X$ can be written as a union of infinitely many countable nonempty pairwise disjoint clopen subsets of diameter at most $r$.

**Proof.** Fix an open cover $U_0$ of $X$ with no finite subcover. Now fix an open cover $U_1$ refining $U_0$ of sets of diameter at most $r$. Now fix an open cover $U_2$ refining $U_1$ of clopen sets. Since $X$ is Lindelöf, fix a countable subcover $U$ of $U_2$. Then $U$ is a countable set of clopen sets of diameter at most $r$ whose union is $X$.

Now enumerate $U = \{ U_n : n \in \omega \}$ and define $V_n = U_n \setminus \bigcup_{m < n} U_m$. Then $\mathcal{V} = \{ V_n : n \in \omega \} \setminus \{ \emptyset \}$ is a countable collection of nonempty clopen pairwise disjoint subsets of $X$ of diameter $\leq r$ whose union is $X$. Since $\mathcal{V}$ refines $U_0$, it cannot be finite.

Now we use this lemma to prove our characterization.

**Theorem 2.2.** Every separable, zero dimensional, nowhere locally compact, completely metrizable topological space is homeomorphic to $\omega^\omega$.

**Proof.** Let $d$ be a complete metric for the topological space $X$. We define a family $(U_s : s \in \omega^<\omega)$ recursively in the length of $s$ satisfying, for every $s \in \omega^<\omega$

1. $U_s$ is a nonempty clopen subset of $X$,
2. $U_\emptyset = X$ and $\bigcup_{n \in \omega} U_{s^{<}(n)} = U_s$,
3. $U_{s^{<}(n)} \cap U_{s^{<}(m)} = \emptyset$ for all $n,m \in \omega$ with $n \neq m$,
4. If $s \neq \emptyset$, $U_s$ has diameter at most $\frac{1}{|s|}$ (with respect to $d$).

To show that such a family exists, we first define $U_\emptyset = X$. Now suppose $U_s$ has been defined for all $s \in \omega^<k$. We fix $s \in \omega^<k$ and define $(U_{s^{<}(n)} : n \in \omega)$ by applying Lemma 2.1 to the metric space $(U_s, d)$ and $r = \frac{1}{|s| + 1}$. $U_s$ is not compact since it is nonempty open and $X$ is nowhere locally compact.

Now we let $f : \omega^\omega \rightarrow X$ be such that for each $a \in \omega^\omega$, $f(a)$ is the unique point of $\bigcap_{n \in \omega} U_{a|n}$. This point exists and is unique since we are considering a shrinking intersection of a sequence of closed sets whose diameters converges to 0 in a complete metric space. Now we verify $f$ is an homeomorphism.

$f$ is injective: if $a, b$ are distinct points of $\omega^\omega$, let $n$ be the first natural number such that $a(n) \neq b(n)$. Then by (3), $U_{a|(n+1)}$ and $U_{b|(n+1)}$ are disjoint, so by the definition of $f$, $f(a) \neq f(b)$.

$f$ is onto: fix $x \in X$. Recursively, we define $a : \omega \rightarrow \omega$ such that $x \in U_{a|n}$. Clearly, $x \in U_\emptyset$. Suppose $a|k$ is defined and call it $s$. Then $x \in U_s$. By (2), there exists $n$ such that $x \in U_{s^{<}(n)}$, define $a(k) = n$. Now we have that $x \in \bigcap_{n \in \omega} U_{a|n}$, so $f(a) = x$.

$\{U_s : s \in \omega^\omega\}$ is a basis for $X$: let $x \in X$ be given and an open set $G$ such that $x \in G$. Since $f$ is surjective, let $a \in \omega^\omega$ be such that $x = f(a)$. Since $G$ is open, there exists $r > 0$ such that the ball $B$ of radius $r$ and center $x$ is contained in $G$. Fix an integer $k > 0$ such that $\frac{1}{k} < r$. Then by (4), since $x \in U_{a|k}$, it follows that $U_{a|k} \subseteq B \subseteq G$.

\(^1\)*A nowhere locally compact space is a space such that every compact set has empty interior.*
f is continuous and open: it suffices to show that f bijects to basis. So given s ∈ ω<ω we show that f[V_s] = U_s. Fix s. ⊆: if a ∈ V_s, then s ⊆ a, so by definition f(a) ∈ U_s. Therefore f[V_s] ⊆ U_s.

≥: fix x ∈ U_s. There exists a such that f(a) = x. We claim that s ⊆ a, which completes the proof. If not, let n be the least integer such that s(n) ≠ a(n). Let t = s|n. Then x = f(a) ∈ U_{a(n+1)} = U_{a\smallsetminus(s(n))} by the definition of f and f(a) = x ∈ U_{s(n+1)} = U_{a\smallsetminus(s(n))}, which contains U_s by (2). This implies that x = f(a) ∈ U_{a\smallsetminus(s(n))} ∩ U_{a\smallsetminus(s(n))}, which contradicts (3).

2.3 Determining m_{ctbl}

ω<ω has the natural structure of a forcing partial order: we order it my reverse inclusion (we say that s ≤ t iff t ⊆ s). With this order, its greatest element is the empty sequence, ∅. It is easy to verify that this order has no atoms.

In this section, we show that every countable partial order with no atoms contains a dense copy of ω<ω, which, by the last bullet point of proposition 1.10, implies that m_{ctbl} is m(ω<ω).

Proposition 2.3. Let P be a preorder with no atoms. Then for every p ∈ P there exists an infinite antichain contained in p^<.

Proof. We define two sequences (p_n : n ∈ ω) and (q_n : n ∈ ω) satisfying, for every n ∈ ω:

1. p_0, q_0 ≤ p,
2. p_{n+1}, q_{n+1} ≤ p_n,
3. p_n ⊥ q_n.

In order to do this, we use the fact that p is not an atom to obtain p_0, q_0 satisfying 1. and 3. If we have constructed the sequences until step n, we use the fact that p_n is not an atom to obtain p_{n+1}, q_{n+1} satisfying 2. and 3.

Then the family {q_n : n ∈ ω} must be an infinite antichain: given n < m, we have q_n ⊥ q_m; if we had that r ≤ q_n, q_m, then we would have r ≤ q_n, p_{m-1} and p_{m-1} ≤ p_n, so q_n and p_n would be compatible, a contradiction. □

Proposition 2.4. Let (P, ≤, 1) be a countable forcing partial order with no atoms. Then P contains a dense copy of ω<ω, that is, there exists Q ⊆ P such that Q is dense in P and Q is isomorphic to ω<ω.

Proof. Enumerate P = {q_n : n ∈ ω}. We define a function f : ω<ω → P recursively on the length of s ∈ ω<ω satisfying:

1. f(∅) = 1,
2. f[{s\smallsetminus(n) : n ∈ ω}] is a maximal antichain in f(s)^f for every s ∈ ω<ω, and f(s\smallsetminus(n)) ≠ f(s\smallsetminus(m)) if n ≠ m,
3. For every n ∈ ω there exists s ∈ ω<n+1 such that f(s) ≤ q_n.
4. f[ω^n] is a maximal antichain in P for every n ∈ ω and f[ω^n] is injective,
5. f(s\smallsetminus(n)) < f(s) for every s ∈ ω, n ∈ ω.

To do this, we define f(∅) = 1. Now suppose we have defined f for every s ∈ ω≤n. Since f[ω^n] is a maximal antichain, there exists s ∈ ω^n such that f(s) ⊧ q_n. Fix r such that r ≤ f(s), q_n. By the previous proposition, there exists an infinite antichain below r which may be expanded to a maximal antichain \{f(s\smallsetminus(n)) : n ∈ ω\} in f(s)^f. For the other s ∈ ω^n, just let \{f(s\smallsetminus(n)) : n ∈ ω\} be an infinite maximal antichain in f(s)^f (we require that f to be injective in these domains).

1., 2. and 3. are clearly satisfied. To see f[ω^n+1] is a maximal antichain, let p ∈ P be given. Since f[ω^n] is a maximal antichain, there exists s ∈ ω^n and p_0 ∈ P such that p_0 ≤ p, f(s). Since \{f(s\smallsetminus(n)) : n ∈ ω\} is a maximal antichain in f(s)^f, there exists p_1 ∈ P and n ∈ ω such that p_1 ≤ f(s\smallsetminus(n)), p_0. Since p_0 ≤ p, it follows that f(s\smallsetminus(n)) ⊧ p. This proves half of 4. Now we see that if s, t ∈ ω<n+1, then f(s) ≠ f(t). If s|n = t|n, they were chosen to be incompatible. If not, they cannot be compatible (and cannot be equal) since they are below the two incompatible elements s|n, t|n.

5. This follows from 2. The equality cannot hold since the only antichain containing f(s) below f(s) is \{f(s)\}, which is not infinite.

Let Q be the range of f. 3. implies Q is dense in P. Now we must see that f : ω<ω → Q is an isomorphism.
Claim 1: if \( s, t \in \omega^n \) and \( s < t \), then \( f(s) < f(t) \).

Suppose \( s < t \). Let \( m \) be the length of \( t \) and \( n \) the length of \( s \). By 5., for every \( i \) such that \( m \leq i < n \), it follows that \( f(s(i + 1)) < f(s(i)) \), so by transitivity of \( < \) and finite induction, \( f(s) < f(t) \).

Claim 2: if \( s, t \in \omega^m \) and \( f(s) = f(t) \), then \( s = t \).

Suppose not. Then \( f(s) = f(t) \) and \( s \neq t \). \( s \) and \( t \) cannot have the same length since it contradicts 4. so one of them has greater length, say, \( s \). Let \( n \) be the length of \( t \). We cannot have that \( s < t \) or by Claim 1 we would have \( f(s) < f(t) \), so \( t \) and \( s/n \) are distinct elements in \( f[\omega^n] \), which implies by 4. that \( f(s) = f(t) \perp f(s|n) \). But Claim 1 implies \( f(s) < f(s|n) \), a contradiction.

Claim 3: if \( s, t \in \omega^m \) and \( f(s) \leq f(t) \), then \( s \leq t \).

Suppose not. Then either \( t < s \) or there exists \( n \) such that \( t|n \neq s|n \). In the first case, Claim 1 implies that \( f(t) < f(s) \), a contradiction. In the second case \( f(s) \leq f(t|n), f(s|n) \), which contradicts the fact that \( f[\omega^n] \) is an antichain and that \( f \) is injective.

Corollary 2.5. If \( P \) is a countable preorder with no atoms, then \( m(P) = m(\omega^{<\omega}) \). In particular, \( m_{ctbl} = m(\omega^{<\omega}) \)

Proof. Let \( P \) be a countable preorder with no atoms. By Proposition 1.12, we have that \( Q = P/ \approx \) is a partial order with no atoms such that \( m(P) = m(Q) \). Since \( i : P \to Q \) is onto, we have that \( Q \) is also countable. By adjoining a greatest element to \( Q \) as in Corollary 1.11, we obtain a countable forcing partial order \( R \) with no atoms such that \( m(R) = m(Q) \). Finally, the Proposition above shows that \( R \) contains a dense copy of \( \omega^{<\omega} \), and thus, by item 10 of Proposition 1.10, it follows that \( m(R) = m(\omega^{<\omega}) \).

2.4 Determining the category number of the completely metrizable separable spaces

In this section, we show that every completely metrizable separable space with no isolated points has a \( G_\delta \) dense copy of the irrationals. This implies that, being \( K \) the class of the completely metrizable separable spaces, then \( m(K) = m(\omega^\omega) \).

Proposition 2.6. Let \( X \) be a completely metrizable space with no isolated points. Then there exists a \( G_\delta \) dense set \( Y \subseteq X \) which is homeomorphic to \( \omega^\omega \).

Proof. Let \( d \) be a bounded complete metric for \( X \). Let \( B \) be a countable basis for \( X \) composed of open balls with \( X \in B \). Given \( U, V \in B \), define \( V \subseteq U \) iff \( \text{cl}V \subseteq U \) and the diameter of \( V \) is less than half of the diameter of \( U \). Then \( B \) is a countable forcing partial order with maximum element \( X \). Since \( X \) has no isolated points, \( B \) has no atoms.

Let \( f : \omega^{<\omega} \to B \) be an injective function such that for all \( s, t \in \omega^{<\omega} \), \( s < t \) iff \( f(s) < f(t) \) with \( f[\omega^{<\omega}] \) dense in \( B \). Notice that this implies that \( s, t \) are incompatible iff \( f(s), f(t) \) are incompatible in \( B \) (that is, if \( f(s) \cap f(t) = \emptyset \)).

Given \( a \in \omega^\omega \), since \( (X, d) \) is a complete metric space, there exists a unique \( \phi(a) \in \bigcap_{n \in \omega} f(a|n) = \bigcap_{n \in \omega} \text{cl} f(a|n) \).

\( \phi \) is injective: given \( a, b \) distinct elements of \( \omega^\omega \), let \( n \) be such that \( a|n \neq b|n \). Then \( \phi(a) \in f(a|n), \phi(b) \in f(b|n) \) and \( \phi(a|n) \cap \phi(b|n) = \emptyset \) since \( a|n \) and \( b|n \) are incompatible. So \( \phi(a) \neq \phi(b) \).

The range of \( \phi \) is dense: given a nonempty open set \( G \subseteq X \), there exists \( A \in B \) such that \( A \subseteq G \). Since the range of \( f \) is dense in \( B \), there exists \( s \in \omega^{<\omega} \) such that \( f(s) \subseteq A \). Then if \( a \in \omega^\omega \) contains \( s \), it follows that \( \phi(a) \in f(s) \subseteq G \).

The range of \( \phi \) is a \( G_\delta \) set: we show that \( \phi[\omega^{\omega}] = \bigcap_{n \in \omega} \bigcup f[\omega^n] \).

\( \subseteq \): given \( a \in \omega^\omega \), \( \phi(a) \in \bigcap_{n \in \omega} f(a|n) \subseteq \bigcap_{n \in \omega} \bigcup f[\omega^n] \).

\( \supseteq \): given \( x \in \bigcap_{n \in \omega} \bigcup f[\omega^n] \), for each \( n \in \omega \) there exists \( s_n \in \omega^n \) such that \( x \in f(s_n) \). We claim that if \( n < m \) then \( s_n \equiv s_m \). Since dom \( s_n = n < m = \text{dom} s_m \), if \( s_n \not\equiv s_m \), then \( s_n \) and \( s_m \) are incompatible, therefore \( f(s_n) \) and \( f(s_m) \) are disjoint, a contradiction. Let \( a \in \omega^\omega \) be given by \( a = \bigcup_{n \in \omega} s_n \). Then for every \( n \in \omega \), \( a|n = s_n \), so \( f(a) \in \bigcap_{n \in \omega} f(a|n) \). But \( x \) is also in this intersection. Since the diameter of \( f(a|n) \) converges to 0, this implies that \( x = f(a) \).

\( \{f(s) \cap \phi[\omega^n] : s \in \omega^{<\omega}\} \) is a basis for \( \phi[\omega^n] \). In fact, we show that given \( a \), \( \{f(a|n) : n \in \omega\} \) is a local basis for \( \phi(a) \) in \( X \). For let \( a \in \omega^\omega \) and \( U \subseteq X \) be an open set containing \( \phi(a) \). Since the diameters of \( f(a|n) \) converges to 0, there exists \( n \) such that \( f(a|n) \) has diameter smaller than \( d(f(a), X \setminus U) \). So \( \phi(a) \in f(a|n) \subseteq U \).
\( f \) is open and continuous: it suffices to see that given \( s \in S \), \( \phi[V_s] = f(s) \cap \phi[\omega^\omega] \).

\[ \subseteq: \text{let } a \in V_s \text{ be given. We must see that } \phi(a) \in f(s). \text{ Since } a \in V_s, \text{ we have } s \subseteq a, \text{ therefore by definition } \phi(a) \in f(s). \]

\[ \supseteq: \text{let } a \in \omega^\omega \text{ be such that } \phi(a) \in f(s). \text{ We must see that } a \in V_s, \text{ that is, } s \subseteq a. \text{ If not, let } n \text{ be the length of } s. \text{ Then } s \text{ an } a/n \text{ are incompatible, so } f(s) \cap f(a/n) = \emptyset. \text{ However, } \phi(a) \text{ is in this intersection, a contradiction.} \]

\[ \square \]

**Corollary 2.7.** If \( X \) is a separable and completely metrizable space without isolated points, then \( m(X) = m(\omega^\omega) \).

In particular, \( m(K) = m(\omega^\omega) \).

**Proof.** By the above theorem, there exists \( Y \subseteq X \) such that \( Y \) is dense, \( G_\delta \) and homeomorphic to \( \omega^\omega \). First, being a \( G_\delta \) implies that \( Y \) is also completely metrizable and therefore has Baire category number equal to the same Baire category number, in particular, all equal to \( m(\omega^\omega) \).

We have also shown in Theorem 2.4 that every countable forcing partial order without atoms has a dense copy of \( \omega^{<\omega} \), and therefore, by item 8 of Proposition 1.10, they all have the same Baire category number as \( \omega^{<\omega} \).

We pointed out in Corollary 2.5 that every countable preorder without atoms has Baire category number equal to some countable forcing partial order without atoms, and therefore has Baire category number equal to \( m(\omega^{<\omega}) \).

Finally, a direct argument shows that the cardinals \( m(\omega^\omega) \) and \( m(\omega^{<\omega}) \) are the same.

**Proposition 2.8.** \( m(\omega^\omega) = m(\omega^{<\omega}) \).

**Proof.** Suppose \( D \) is a collection of dense open subsets of \( \omega^\omega \) of cardinality less than \( m(\omega^{<\omega}) \).

Notice that for each \( D \in D \), \( \hat{D} = \{ s \in \omega^{<\omega} : V_s \subseteq D \} \) is an dense open subset of \( \omega^{<\omega} \). Now \( \hat{D} = \{ \hat{D} : D \in D \} \cup \{ s \in \omega^{<\omega} : \text{dom } s \geq n \cap n \in \omega \} \) is a collection of less than \( m(\omega^{<\omega}) \) dense open sets, so there exists a filter \( G \) intersecting all of them. \( G \) is a collection of compatible functions. Since \( G \) intersects \( \{ s \in \omega^{<\omega} : \text{dom } s \geq n \} \) for each \( n \in \omega \), it follows that \( g = \bigcup G \in \omega^\omega \).

We claim \( g \in \bigcap D \): given \( D \in D \), \( G \) intersects \( \hat{D} \), so there exists \( s \in G \) such that \( V_s \subseteq D \). Since \( s \subseteq g \), \( g \in V_s \subseteq D \) and we are done. This concludes \( m(\omega^\omega) \geq m(\omega^{<\omega}) \).

Now suppose \( D \) is a collection of dense open subsets of \( \omega^{<\omega} \) of cardinality less than \( m(\omega^\omega) \). For each \( D \in D \), let \( \tilde{D} = \{ a \in \omega^\omega : \exists n \in \omega : a[n] \in D \} \). Notice that \( \tilde{D} = \{ \tilde{D} : D \in D \} \) is a collection of less than \( m(\omega^\omega) \) dense open sets of \( \omega^\omega \), so it has an element \( g \) in its intersection. Let \( G = \{ g[n] : n \in \omega \} \). \( G \) is a filter and \( G \) intersects each \( D \in D \): given \( D, g \in D \), which means there exists \( n \in \omega \) such that \( g[n] \in D \). This concludes \( m(\omega^\omega) \leq m(\omega^{<\omega}) \). \( \square \)

**References**

