A fórmula de Taylor de ordem 2

$\Omega \subset \mathbb{R}^N$ aberto, $f \in C^2(\Omega)$, $x_0 \in \Omega$; $B_\delta(x_0) \subset \Omega$, $\delta > 0$

Para $h \in B_\delta(0)$ definimos

$$R(h) := f(x_0+h) - f(x_0) - \sum_{j=1}^{N} \frac{\partial f}{\partial x_j}(x_0) h_j - \frac{1}{2} \sum_{j,k=1}^{N} \frac{\partial^2 f}{\partial x_j \partial x_k}(x_0) h_j h_k$$

Proposição: $\dfrac{R(h)}{|h|^2} \xrightarrow[h \to 0]{} 0$

Dem. Já provamos que

$$R(0) = \frac{\partial R}{\partial h_j}(0) = \frac{\partial^2 R}{\partial h_j \partial h_k}(0) = 0, \quad j,k = 1, \dots, N$$

Fixemos $j \in \{1, \dots, N\}$

$$\left\| \left(\frac{\partial R}{\partial h_j} \right)'(h) \right\| = \left\{ \sum_{k=1}^{N} \frac{\partial^2 R}{\partial h_k \partial h_j}(h)^2 \right\}^{1/2}$$

Seja $\varepsilon > 0$. Existe $0 < \delta_1 < \delta$:

$$|h| < \delta_1 \Rightarrow \left| \frac{\partial^2 R}{\partial h_k \partial h_j}(h) \right| < \varepsilon$$

Logo $\left\| \left(\frac{\partial R}{\partial h_j}\right)'(h) \right\| \leq \sqrt{N}\varepsilon, \quad |h| < \delta_1$

Pela desigualdade do valor médio,

$$\left| \frac{\partial R}{\partial h_j}(h) - \frac{\partial R}{\partial h_j}(0) \right| \leq \sqrt{N}\varepsilon |h - 0|,$$

se $|h| < \delta_1$

ie, $\left| \frac{\partial R}{\partial h_j}(h) \right| \leq \sqrt{N}\varepsilon |h|, \quad |h| \leq \delta_1$

Repetindo o argumento

$$\|R'(h)\| = |\vec{\nabla} R(h)| =$$

$$\left\{ \sum_{j=1}^{N} \frac{\partial R}{\partial h_j}(h)^2 \right\}^{1/2}$$

$$\leq \left\{ \sum_{j=1}^{N} N\varepsilon^2 |h|^2 \right\}^{1/2}$$

$= N\varepsilon|h|, \quad |h|<\delta_1$

Aplico a desigualdade do valor médio:

$$|R(h)-\underbrace{R(0)}_{=0}| \leq N\varepsilon|h|\,|h-0|$$
$$= N\varepsilon|h|^2$$

$$\Rightarrow |R(h)| \leq N\varepsilon|h|^2, \quad |h|<\delta_1.$$

ie, $$\frac{|R(h)|}{|h|^2} \leq N\varepsilon, \quad |h|<\delta_1$$

ie, $$\frac{|R(h)|}{|h|^2} \xrightarrow[h\to 0]{} 0 \quad \blacksquare$$

Transformações simétricas

Def. Seja $A \in L(\mathbb{R}^N)$. Dizemos que A é simétrica se

$$(Ax)\cdot y = x\cdot(Ay),\ \forall x,y \in \mathbb{R}^N$$

Seja $\{e_1, \dots, e_N\}$ a base canônica de $\mathbb{R}^N$. Se A é simétrica

$$\underbrace{(Ae_j)\cdot e_i}_{\doteq a_{ij}} = e_j \cdot Ae_i = \underbrace{Ae_i \cdot e_j}_{a_{ji}}$$

matriz de A com relação à base canônica $\{e_1, \dots, e_N\}$: A

$$a_{ij} = a_{ji}$$

ou

$$A = {}^tA$$

<u>Lema</u>: A é simétrica $\Longleftrightarrow A = A^t$

<u>Dem</u>. $\Longrightarrow$ já provado

$\Longleftarrow$ Se $x, y \in \mathbb{R}^N$, $x = \sum_{j=1}^{N} x_j e_j$

$y = \sum_{j=1}^{N} y_j e_j$ então

$$(Ax) \cdot y = \left(\sum_{j=1}^{N} x_j A e_j\right) \cdot \left(\sum_{k=1}^{N} y_k e_k\right)$$

$$= \sum_{j,k=1}^{N} x_j y_k (A e_j \cdot e_k)$$

$$= \sum_{j,k=1}^{N} x_j y_k a_{kj}$$

$$\overset{\text{hipótese}}{=} \sum_{j,k=1}^{N} x_j y_k a_{jk}$$

$$= \sum_{j,k=1}^{N} x_j y_k (Ae_k \cdot e_j)$$

$$= \dots = x \cdot (Ay) \quad ▨$$

<u>Teorema</u> espectral: Seja $A \in L(\mathbb{R}^N)$. Então existem $\{u_1, \dots, u_N\}$ base ortonormal de $\mathbb{R}^N$, isto é,

$$u_i \cdot u_j = \delta_{ij} \quad \forall\, i, j = 1, \dots, N,$$

e $\lambda_1, \dots, \lambda_N \in \mathbb{R}$ tais que

$$A u_j = \lambda_j u_j \quad , \; j = 1, \dots, N.$$

<u>Obs</u>: A matriz de A com relação à base $\{u_1, \dots, u_N\}$, $\{\hat{a}_{ij}\}$, satisfaz:

$$A u_j = \sum_{i=1}^{N} \tilde{a}_{ij} u_i$$

Tomo o produto escalar por u_k em ambos os lados desta igualdade. Obtemos

$$(A u_j) \cdot u_k = \left(\sum_{i=1}^{N} \tilde{a}_{ij} u_i \right) \cdot u_k$$

$$= \sum_{i=1}^{N} \tilde{a}_{ij} (u_i \cdot u_k)$$

$$= \tilde{a}_{kj}$$

$$= (\lambda_j u_j) \cdot u_k$$

$$= \lambda_j (u_j \cdot u_k) = \lambda_j \delta_{jk}$$

$$\tilde{a}_{kj} = \lambda_j \delta_{jk} \;, \text{ ie,}$$

$$\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_N \end{bmatrix}$$

Note ainda que, se $x, y \in \mathbb{R}^N$,

$$x = \sum_{j=1}^{N} x_j u_j$$

$$y = \sum_{j=1}^{N} y_j u_j$$

então

$$(Ax) \cdot y = \left(\sum_{j=1}^{N} x_j A u_j , \sum_{k=1}^{N} y_k u_k \right)$$

$$= \sum_{j,k=1}^{N} x_i y_k (Au_j) \cdot u_k =$$

$$= \sum_{j,k=1}^{N} x_j y_k \lambda_j (u_j \cdot u_k)$$

ie,

$$\boxed{(Ax) \cdot y = \sum_{j=1}^{N} \lambda_j x_j y_j}$$

$$(Ax) \cdot x = \lambda_1 x_1^2 + \dots + \lambda_N x_N^2$$

<u>Def</u>. Seja $A \in L(\mathbb{R}^N)$ simétrica.

Dizemos que A é <u>positiva</u> (e escreveremos $A \geqslant 0$) se

$$(Ax) \cdot x \geqslant 0, \quad \forall x \in \mathbb{R}^N$$

Dizemos que A é <u>definida positiva</u> (e escreveremos $A > 0$) se

$(Ax).x > 0$ se $x \neq 0$.

<u>Observação</u>: Se A é definida positiva então $\exists c > 0$:

$$(Ax).x \geq c|x|^2, \forall x \in \mathbb{R}^N$$

De fato $\min_{|y|=1} (Ay).y = c > 0$

[$y \longmapsto (Ay).y$ é contínua e $\{y \in \mathbb{R}^N : |y| = 1\}$ é compacto]

Se $x \in \mathbb{R}^N$, $x \neq 0$ então

$$\left| \frac{x}{|x|} \right| = 1$$

e portanto $\left(A\left(\frac{x}{|x|}\right)\right) \cdot \frac{x}{|x|} \geq c$

$$\frac{1}{|x|^2}(Ax) \cdot x \geq c \quad \boxed{///}$$

Pelo teorema espectral, se $x = \sum_{j=1}^{N} x_j u_j$

então $(Ax) \cdot x = \lambda_1 x_1^2 + \dots + \lambda_N x_N^2$

Assim $\left[\begin{array}{l} A \geq 0 \iff \lambda_j \geq 0, \forall j = 1..N \\ A > 0 \iff \lambda_j > 0, \forall j = 1,..,N \end{array}\right.$

$$\Downarrow$$

$$c = \min\{\lambda_1, \dots, \lambda_N\}$$

Lembrar: no teorema espectral $\{\lambda_1, \dots, \lambda_N\}$ são as raízes de

$$\det(\lambda I - A) = 0$$

como antes