Recordar: Teorema da função implícita

$\Omega \subset \mathbb{R}^{N+M}_{(x,y)}$, $f:\Omega \to \mathbb{R}^N$, de classe C^1. $(a,b)\in\Omega$, $f(a,b)=0$, $A := f'(a,b) \in L(\mathbb{R}^{N+M},\mathbb{R}^N)$.

$A_x \in L(\mathbb{R}^N)$, $A_y = L(\mathbb{R}^M,\mathbb{R}^N)$. Impomos $A_x \in GL(\mathbb{R}^N)$. Conclusão:

$\exists U \subset \Omega$ aberto, $(a,b)\in U$, $W \subset \mathbb{R}^M$, aberto, $b \in W$, $g : W \to \mathbb{R}^N$ de classe C^1 t.q. $g(b)=a$

$$\{(x,y)\in U : f(x,y)=0\} = \{(g(y),y) : y \in W\}$$

Além disto, $g'(b) = -A_x^{-1} A_y$.

<u>Exemplo</u>: $N=2, M=3$ $f:\mathbb{R}^5 \to \mathbb{R}^2$, $f=(f_1,f_2)$, $f\in C^1$.

$$f_1(x_1,x_2,y_1,y_2,y_3) = 2e^{x_1} + x_2y_1 - 4y_2 + 3 = 0$$

$$f_2(x_1,x_2,y_1,y_2,y_3) = x_2\cos x_1 - 6x_1 + 2y_1 - y_3 = 0$$

$a = (0,1),\ b = (3,2,7)\ :\ f(a,b) = 0$

Objetivo: resolver o sistema acima em (x_1, x_2).

$$f'(x,y) = \begin{bmatrix} 2e^{x_1} & y_1 & x_2 & -4 & 0 \\ -x_2 \operatorname{sen} x_1 - 6 & \cos x_1 & 2 & 0 & -1 \end{bmatrix}$$

$$f'(a,b) = \left(\begin{bmatrix} 2 & 3 \\ -6 & 1 \end{bmatrix} \begin{bmatrix} 1 & -4 & 0 \\ 2 & 0 & -1 \end{bmatrix} \right)$$

$$\quad\ \ \| \qquad\qquad\quad A_x \qquad\qquad\qquad A_y$$

$$\quad\ \ A$$

$$A(h_1, h_2, k_1, k_2, k_3) = A_x(h_1, h_2) + A_y(k_1, k_2, k_3)$$

$A_x \in GL(\mathbb{R}^2)$?

$$\det \begin{bmatrix} 2 & 3 \\ -6 & 1 \end{bmatrix} = 20$$

Conclusão: $\exists\, U \subset \mathbb{R}^5$, $(0,1,3,2,7) \in U$, $\exists\, W \subset \mathbb{R}^3$ aberto, $(3,2,7) \in W$ e $g: W \to \mathbb{R}^2$, de classe C^1, $g(3,2,7) = (0,1)$, tal que nosso sistema pode ser resolvido na forma $x_1 = g_1(y_1,y_2,y_3)$, $x_2 = g_2(y_1,y_2,y_3)$ em U. Temos

$$g'(3,2,7) = -A_x^{-1} A_y \; : \; ?$$

$$A_x^{-1} = \frac{1}{20}\begin{bmatrix} 1 & -3 \\ 6 & 2 \end{bmatrix}$$

$$-A_x^{-1} A_y = \frac{-1}{20}\begin{bmatrix} 1 & -3 \\ 6 & 2 \end{bmatrix}\begin{bmatrix} 1 & -4 & 0 \\ 2 & 0 & -1 \end{bmatrix}$$

$$= \frac{-1}{20}\begin{bmatrix} -5 & -4 & 3 \\ 10 & -24 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{4} & \frac{1}{5} & -\frac{3}{20} \\ -\frac{1}{2} & \frac{6}{5} & \frac{1}{10} \end{bmatrix}$$

$$x_1 = g_1(y_1, y_2, y_3), \quad x_2 = g_2(y_1, y_2, y_3)$$

$$0 = g_1(3,2,7), \quad 1 = g_2(3,2,7)$$

$$\frac{\partial g_1}{\partial y_1}(3,2,7) = \frac{1}{4}; \quad \frac{\partial g_1}{\partial y_2}(3,2,7) = \frac{1}{5}, \text{ etc...}$$

$$\frac{\partial g_2}{\partial y_1}(3,2,7) = -\frac{1}{2}, \ldots \quad \blacksquare$$

<u>Exercício</u>: Seja $f: \mathbb{R} \to \mathbb{R}$ contínua, $f(t) > 0$ para todo t. Suponha que

$\int_0^1 f(t)\,dt = 3$. Mostre que existem

$\delta > 0$ e $g :]-\delta, \delta[\longrightarrow \mathbb{R}$ de classe C^1

tal que $\int_x^{g(x)} f(t)\,dt = 2 \quad \forall x \in]-\delta, \delta[$.

Solução: Seja $F(x,y) \doteq \int_x^y f(t)\,dt$

$$\left.\begin{array}{l} \dfrac{\partial F}{\partial x}(x,y) = -f(x) \\[2ex] \dfrac{\partial F}{\partial y}(x,y) = f(y) \end{array}\right] \longrightarrow \text{contínuas em } \mathbb{R}^2$$

$\therefore F$ é C^1 em $\mathbb{R}^2$.

$$F(x,y) - 2 = 0$$

$(0, y_0) \in \mathbb{R}^2$ t.q. $F(0, y_0) - 2 = 0$

Considere $h(T) = \int_0^T f(t)\,dt$, $0 \leq T \leq 1$

$h(0)=0,\ h(1)=3 \underset{\text{T.V.I.}}{\Longrightarrow} \exists y_0,\ 0<y_0<1.$

tal $\int_0^{y_0} f(t)\,dt = 2.$

Assim $\begin{cases} F(0,y_0)=2 \\ \dfrac{\partial F}{\partial y}(0,y_0) = f(y_0) > 0 \end{cases}$

$$F'(0,y_0) = \begin{bmatrix} \dfrac{\partial F}{\partial x}(0,y_0) & \underbrace{\dfrac{\partial F}{\partial y}(0,y_0)}_{\det \neq 0} \end{bmatrix}$$

$$= \begin{bmatrix} -f(0) & f(y_0) \end{bmatrix}$$

$$\begin{matrix} \shortparallel \\ 0 \end{matrix}$$

Pelo teorema da função implícita

$\exists \delta>0,\ g:]-\delta,\delta[\longrightarrow \mathbb{R},\ g(0)=y_0$

$$\int_x^{g(x)} f(t)\,dt = 2.$$

Exemplo:

$\Omega \subset \mathbb{R}^N$, $f: \Omega \to \mathbb{R}$ de classe C^1.

Seja $x_0 \in \Omega$ t.q. $\begin{cases} f(x_0) = 0 \\ \dfrac{\partial f}{\partial x_N}(x_0) \neq 0 \end{cases}$

Escrevemos $x \in \mathbb{R}^N$ como

$$x = (x', x_N), \quad x' \in \mathbb{R}^{N-1}$$

$$x_0 = (x_0', x_{N0})$$

$$f'(x_0) \text{ `='} \left[\frac{\partial f}{\partial x_1}(x_0) \ \dots \ \underbrace{\frac{\partial f}{\partial x_N}(x_0)}_{\det \neq 0} \right].$$

Pelo teorema da função implícita: $\begin{cases} \exists\, U \subset \Omega, \ x_0 \in U \\ \exists\, W \subset \mathbb{R}^{N-1}, \ x_0' \in W \\ \exists\, g: W \to \mathbb{R} \text{ de classe } C^1, \end{cases}$

com $g(x_0') = x_{0N}$ e

$$\{x \in U : f(x) = 0\} = \{(x', g(x')) : x' \in W\}$$