Exercícios extraídos do livro ”Probability and Statistics” de M. H. DeGroot.

**1.** Suppose that the proportion $\theta$ of defective items in a large manufactured lot is unknown and that the prior distribution of $\theta$ is a uniform distribution on the interval $(0, 1)$. When eight items are selected at random from the lot, it is found that exactly three of them are defective. Determine the posterior distribution of $\theta$.

**2.** Suppose that $X = (X_1, X_2)$ is observed to learn about an unknown parameter $\theta$. Suppose that $X$, given $\theta$, is absolutely continuous with p.d.f $f(\cdot|\theta)$ and that the prior p.d.f. of $\theta$ is $\pi(\cdot)$. Show that the posterior distribution $\pi(\cdot|x_1, x_2)$ is the same regardless of whether it is calculated directly (in one stage) from the whole sample $x$ or it is calculated sequentially (in two stages) by first determining the posterior distribution of $\theta$ given $X_1 = x_1$ (after observing only $X_1 = x_1$), $\pi(\cdot|x_1)$, and then determining a second posterior for $\theta$, after observing $X_2 = x_2$, from $\pi(\cdot|x_1)$.

**3.** Consider again exercise 1 and assume the same prior distribution for $\theta$. Suppose now, however, that instead of selecting a random sample of eight items from the lot, we perform the following experiment: items from the lot are selected at random one by one until exactly three defective items have been found. If we find that we must select a total of eight items in this experiment, what is the posterior distribution of $\theta$ at the end of the experiment?

**4.** Suppose that a single observation $X$ is to be taken from a uniform distribution on the interval $(\theta - \frac{1}{2}, \theta + \frac{1}{2})$, $\theta$ unknown, and that the prior distribution of $\theta$ is a uniform distribution on the interval $(10, 20)$. If the observed value of $X$ is 12, what is the posterior distribution of $\theta$?

**5.** Consider again the conditions of exercise 4 and assume the same prior distribution of $\theta$. Suppose now, however, that six observations are selected at random from the uniform distribution on the interval $(\theta - \frac{1}{2}, \theta + \frac{1}{2})$. Suppose that the observed values are: 11.0 , 11.5 , 11.7 , 11.1 , 10.9 and 11.4. Determine the posterior distribution of $\theta$.

**6.** Consider again the conditions of exercise 1. Suppose that another statistician has observed exactly three defective items among 100 items selected at random from the lot. Suppose also that the posterior distribution the statistician has obtained for $\theta$ is a beta distribution for which the mean is $\frac{2}{51}$ and the variance is $\frac{98}{[(51)^2(103)]}$. Which beta prior distribution had the statistician assigned to $\theta$?

**7.** Let $\theta$ denote the average number of defects per 100 feet of a certain type of magnetic tape. Suppose that the value of $\theta$ is unknown and that the prior distribution of $\theta$ is a gamma distribution with parameters $a = 2$ and $b = 10$. When a 1200-foot roll of this tape is inspected, four defects are found. Determine the posterior distribution of $\theta$.

**8.** Suppose that the heights of the individuals in a certain population have a normal distribution for which the value of the mean $\theta$ is unknown and the standard deviation is 2 inches. Suppose also that the prior distribution of $\theta$ is a normal distribution for which the mean is 68 inches and the standard deviation is 1 inch. If ten people are selected at random from the population and their average height is found to be 69.5 inches, what is the posterior distribution of $\theta$?

**9.** Consider again the problem described in exercise 8.

a) Which interval 1 inch long had the highest prior probability of containing the value of $\theta$?

b) Which interval 1 inch long has the highest posterior probability of containing the value of $\theta$?

c) Find the values of the probabilities in parts a) and b).

**10.** Suppose that a random sample is to be taken from a normal distribution for which the value of the mean $\theta$ is unknown and the standard deviation is 2 and that the prior distribution of $\theta$ is a normal distribution for which the standard deviation is 1. What is the smallest number of observations that must be included in the sample in order to reduce the standard deviation of the posterior distribution of $\theta$ to the value 0.1?

**11.** Suppose that the time in minutes required to serve a customer at a certain facility has an exponential distribution for which the value of the parameter $\theta$ is unknown, and that the prior distribution of $\theta$ is a gamma distribution for which the mean is 0.2 and the standard deviation is 1. If the average time required to serve a random sample of 20 customers is observed to be 3.8 minutes, what is the posterior distribution of $\theta$?

**12.** For a distribution with mean $\mu \neq 0$ and standard deviation $\sigma > 0$, the *coefficient of variation* of the distribution is defined as $\frac{\sigma}{|\mu|}$. Consider again the problem described in exercise 11, and suppose that the coefficient of variation of the prior gamma distribution of $\theta$ is 2. What is the smallest number of customers that must be observed in order to reduce the coefficient of variation of the posterior distribution to 0.1?

**13.** Show that the family of beta distributions is a conjugate family of prior distributions for samples from a negative binomial distribution with a known value of the parameter $r$ and an unknown value of the parameter $\theta$, $0 < \theta < 1$.

**14.** Let $\pi(\theta)$ be a p.d.f. which is defined as follows for constants $a > 0$ and $b > 0$:

$\pi(\theta) \;=\; \frac{b^a}{\Gamma(a)}\theta^{-(a+1)}e^{-\frac{b}{\theta}}\mathbb{I}_{(0,\infty)}(\theta).$

a) Verify that $\pi(\theta)$ is actually a p.d.f..

b) Show that the family of p.d.f.s $\pi(\cdot)$, $a > 0$, $b > 0$, is a conjugate family of prior distributions for samples from a normal distribution with a known value of the mean $\mu_0$ and an unknown value of the variance $\theta$.

**15.** Show that the family of Pareto distributions is a conjugate family of prior distributions for samples from a uniform distribution on the interval $(0, \theta)$, where the value of the endpoint $\theta$ is unknown.

**16.** Suppose that $X_1, ..., X_n$ form a random sample from a beta distribution with parameters $\theta$ and 1, $\theta > 0$ unknown. Suppose also that the prior distribution of $\theta$ is a gamma distribution with parameters $a > 0$ and $b > 0$. Determine the mean and the variance of the posterior distribution of $\theta$.

**17.** Suppose that a regular light bulb, a long-life light bulb, and an extra-long-life light bulb are being tested. The lifetime $X_1$ of the regular bulb has an exponential distribution with mean $\theta$, the lifetime $X_2$ of the long-life bulb has an exponential distribution with mean $2\theta$, and the lifetime $X_3$ of the extra-long-life bulb has an exponential distribution with mean $3\theta$. Let $\psi = \frac{1}{\theta}$ and suppose that the prior distribution of $\psi$ is a gamma distribution with parameters $a$ and $b$. Determine the posterior distribution of $\psi$ given $X_1 = x_1$, $X_2 = x_2$ and $X_3 = x_3$.

**18.** Suppose that each of two statisticians $A$ and $B$ must estimate a certain parameter $\theta$, $\theta > 0$. Statistician $A$ can observe the value of a random variable $X$ which has a gamma distribution with parameters $a = 3$ and $b = \theta$, and statistician $B$ can observe the value of a random variable $Y$ which has a Poisson distribution with mean $2\theta$. Suppose that the value observed by statistician $A$ is $X = 2$ and the value observed by statistician $B$ is $Y = 3$. Show that the likelihood functions determined by these observed values are proportional. Show that if both statisticians use a common prior p.d.f. for $\theta$ then they will obtain a common posterior p.d.f. for $\theta$.

**19.** Suppose that each of two statisticians $A$ and $B$ must estimate a certain parameter $\theta$, $0 < \theta < 1$. Statistician $A$ can observe the value of a random variable $X$ which has a binomial distribution with parameters $n = 10$ and $p = \theta$, and statistician $B$ can observe the value of a random variable $Y$ which has a negative binomial distribution with parameters $r = 4$ and $p = \theta$. Suppose that the value observed by statistician $A$ is $X = 4$ and the value observed by statistician $B$ is $Y = 6$. Show that the likelihood functions determined by these observed values are proportional. Show that if both statisticians use a common prior p.d.f. for $\theta$ then they will obtain a common posterior p.d.f. for $\theta$.

**20.** Suppose that $X_1, ..., X_n$ form a random sample from a uniform distribution on the interval $(\theta, 2\theta)$, $\theta > 0$. Suppose the prior distribution of $\theta$ is a Pareto distribution with parameters $a > 0$ and $b > 0$. What is the posterior distribution of $\theta$ given $X_1 = x_1, ..., X_n = x_n$.

**21.** Consider a Markov Chain with two possible states 0 and 1 and with stationary transition probabilities as given in the following transition matrix $\mathbf{P}$ :

$$\mathbf{P} = \begin{bmatrix} \theta & 1-\theta \\ \frac{3}{4} & \frac{1}{4} \end{bmatrix},$$

where the value of $\theta$ is unknown, $0 < \theta < 1$. Suppose that the initial state of the chain is $X_1 = x_1$ and let $X_2, ..., X_{n+1}$ denote the states of the chain at the next $n$ successive periods. Suppose the prior distribution of $\theta$ is a beta distribution with parameters $a > 0$ and $b > 0$. Determine the posterior distribution of $\theta$ given $X_1 = x_1, X_2 = x_2, ..., X_{n+1} = x_{n+1}$.

**22.** Consider a Markov Chain with two possible states 0 and 1 and with stationary transition probabilities as given in the following transition matrix $\mathbf{P}$ :

$$\mathbf{P} = \begin{bmatrix} \theta & 1-\theta \\ 1-\theta & \theta \end{bmatrix},$$

where the value of $\theta$ is unknown, $0 < \theta < 1$. Suppose that the initial state of the chain is $X_1 = x_1$ and let $X_2, ..., X_{n+1}$ denote the states of the chain at the next $n$ successive periods. Suppose the prior distribution of $\theta$ is a beta distribution with parameters $a > 0$ and $b > 0$. Determine the posterior distribution of $\theta$ given $X_1 = x_1, X_2 = x_2, ..., X_{n+1} = x_{n+1}$.

**23.** Consider a Markov Chain with two possible states 0 and 1 and with stationary transition probabilities as given in the following transition matrix $\mathbf{P}$ :

$$\mathbf{P} = \begin{bmatrix} 1-\theta_1 & \theta_1 \\ 1-\theta_2 & \theta_2 \end{bmatrix},$$

where the values of $\theta_1$ and $\theta_2$ are unknown, $0 < \theta_1, \theta_2 < 1$. Suppose that the initial state of the chain is $X_1 = x_1$ and let $X_2, ..., X_{n+1}$ denote the states of the chain at the next $n$ successive periods. Let $\theta = (\theta_1, \theta_2)$. Suppose the prior distribution of $\theta$ is as follows: $\theta_1$ and $\theta_2$ are independent random variables, with $\theta_i$ being a beta random variable with parameters $a_i > 0$ and $b_i > 0$, $i = 1, 2$. Determine the posterior distribution of $\theta$ given $X_1 = x_1, X_2 = x_2, ..., X_{n+1} = x_{n+1}$.