Metric aspects of the dynamics of covering maps of the circle

E. Colli[1] M. L. do Nascimento[2] E. Vargas[3]

[1]Departamento de Matemática Aplicada
Universidade de São Paulo

[2]Centro de Ciências Exatas e Naturais
Universidade Federal do Pará

[3]Departamento de Matemática
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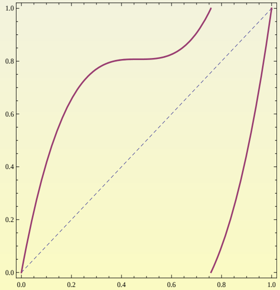
OUTLINE

CRITICAL COVERING MAPS ON $\mathbb{S}^1$



- Class C^1, degree $d \geq 2$, critical point c of order $\ell > 1$;
- Class C^3 and Negative Schwarzian on $\mathbb{S}^1 \setminus \{c\}$;
- There is no wandering interval.

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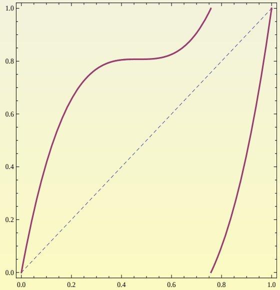

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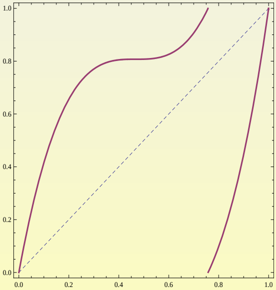

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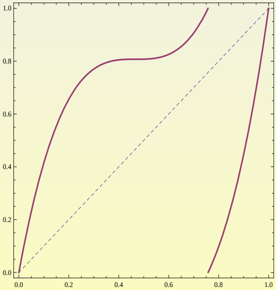

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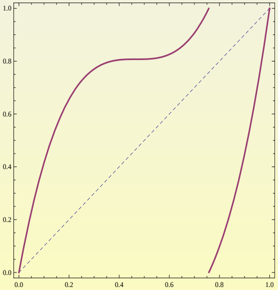

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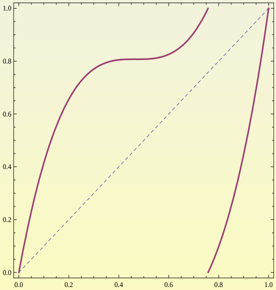

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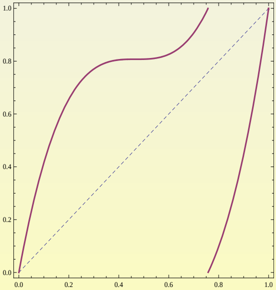

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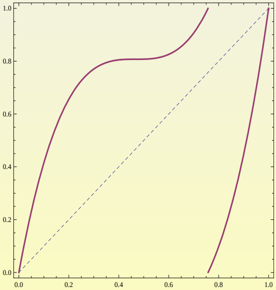

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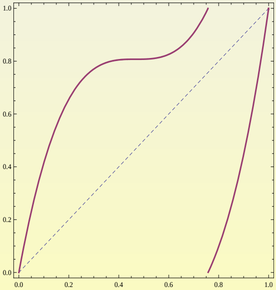

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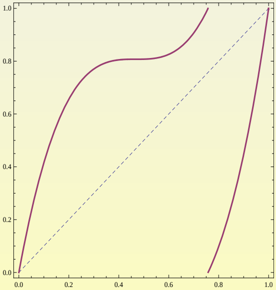

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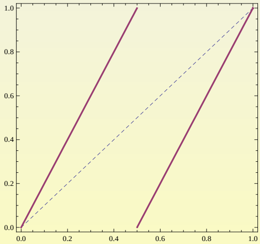
The map $\mathcal{R}(z) = 2z \bmod 1$



Questions

- Regularity of conjugacy;
- Invariant measures
- Attractors;
- Decay of geometry;
- Renormalization;
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GROWTH OF DERIVATIVES

THEOREM

Let f be as above, $\mu^+ := \frac{1+\sqrt{1+4\ell}}{2\ell}$ *and* $\varphi := \frac{1+\sqrt{5}}{2}$. *If in addition f has the Fibonacci combinatorics then:*

1. *If* $1 < \ell < 2$ *then* $\frac{\log\log Df^{S_n}(c^f)}{n} \to \log \mu^+ > 0.$
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THE FIBONACCI COMBINATORICS

- c is recurrent;
- Set $I_0 := (0, 1)$ and $I_n \ni c$, a domain of the first return map ϕ_n to I_{n-1} ($\phi_n : I_n \to I_{n-1}$);
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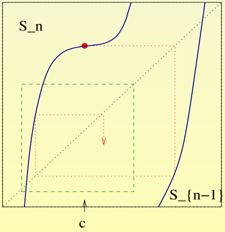

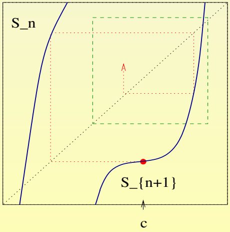

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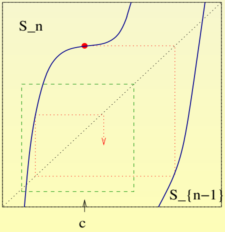
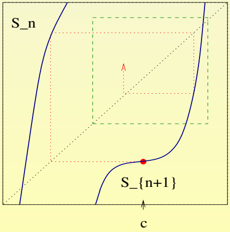
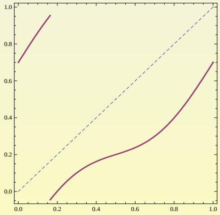
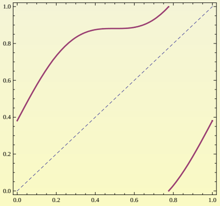
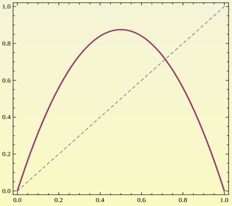




Other dynamics







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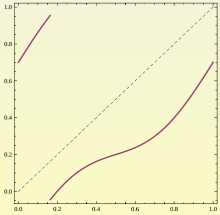

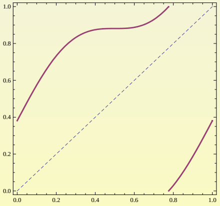

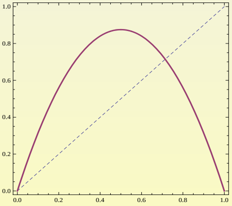

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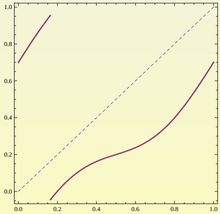

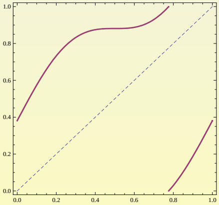

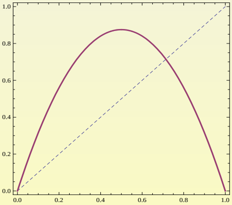

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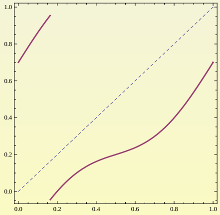
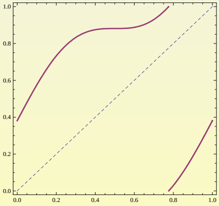
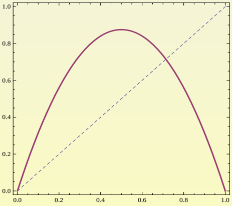
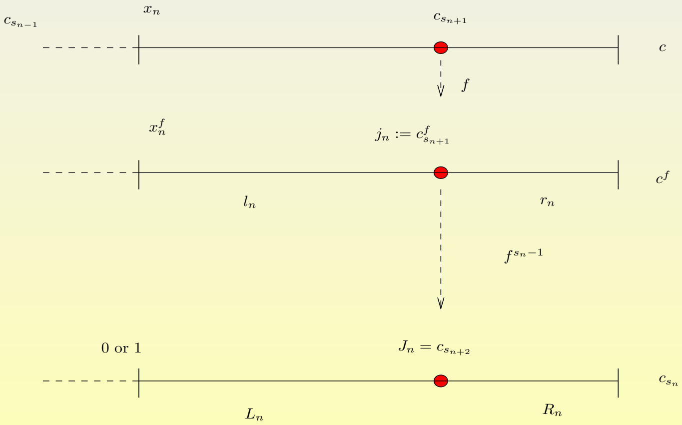






CROSS RATIO

$$Df^{s_n-1}(c^f_{s_{n+1}}) = \frac{|J_n|}{|j_n|} \geq \frac{|L_n||R_n|}{|L_n \cup R_n|} \frac{|l_n \cup r_n|}{|l_n||r_n|}$$



Difference inequation I

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- $$Df^{s_n-1}(c^f_{s_{n+1}}) \geq \frac{|L_n|}{|L_n \cup R_n|} \frac{d_n - d_{n+2}}{d^f_{n+1}} \frac{d^f_{n-1}}{d^f_{n-1} - d^f_{n+1}},$$

  where $d_n := |c_{s_n} - c|$.

- For the next slide $\lambda_n := \frac{d_n}{d_{n+2}} \geq \lambda > 1$.

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- For the next slide $\lambda_n := \frac{d_n}{d_{n+2}} \geq \lambda > 1$.

DIFFERENCE INEQUATION I

- $$\frac{\left(Df^{s_{n+1}}(c^f)\right)^2}{Df^{s_n}(c^f)\,Df^{s_{n-1}}(c^f)} \geq K_n\,\ell^2\,\lambda_{n-1}^{2-\ell}\,\frac{q_{n-1}}{q_n}\,\frac{1-\lambda_n^{-1}}{1-\lambda_{n-2}^{-\ell}}\,\frac{1-\lambda_{n-1}^{-1}}{1-\lambda_{n-1}^{-\ell}}$$

- $$\left[\frac{Df^{s_{n+1}}(c^f)}{1-\lambda_n^{-1}}\right]^2\left[\frac{Df^{s_n}(c^f)}{1-\lambda_{n-1}^{-1}}\right]^{-1}\left[\frac{Df^{s_{n-1}}(c^f)}{1-\lambda_{n-2}^{-1}}\right]^{-1} \geq \frac{Q_{n-1}}{Q_n}\sigma_n$$

  where

  $$\sigma_n := K_n\ell^2\lambda_{n-1}^{2-\ell}\,\frac{1-\lambda_{n-1}^{-1}}{1-\lambda_{n-1}^{-\ell}}\,\frac{1-\lambda_{n-2}^{-1}}{1-\lambda_{n-2}^{-\ell}} \geq \sigma > 1$$

Difference inequation I

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- $$\left[\frac{Df^{s_{n+1}}(c^f)}{1-\lambda_n^{-1}}\right]^2 \left[\frac{Df^{s_n}(c^f)}{1-\lambda_{n-1}^{-1}}\right]^{-1} \left[\frac{Df^{s_{n-1}}(c^f)}{1-\lambda_{n-2}^{-1}}\right]^{-1} \geq \frac{Q_{n-1}}{Q_n}\,\sigma_n$$

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Exponential growth

- $X_n := \log \left[\frac{Df^{s_n}(c^f)}{1-\lambda_{n-1}^{-1}} \right] \geq Y_n$, where

$$Y_{n+1} - \frac{1}{2} Y_n - \frac{1}{2} Y_{n-1} = \frac{1}{2} \log Q_{n-1} - \frac{1}{2} \log Q_n + \frac{1}{2} \log \sigma_n$$

- $$\liminf \frac{Y_n}{n} \geq \frac{1}{3} \liminf \log \sigma_n \geq \varepsilon > 0$$

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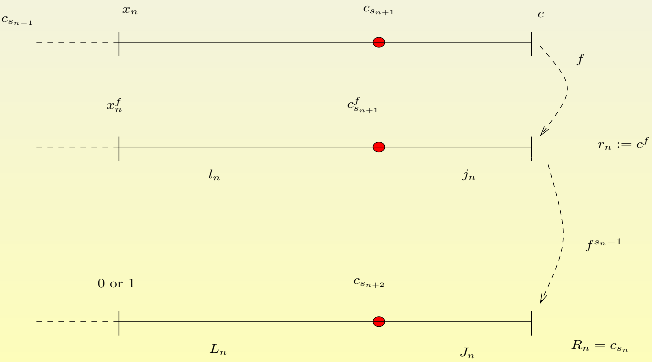
DIFFERENCE INEQUATION II

$$Df^{s_n-1}(c^f) = \frac{|R_n|}{|r_n|} \leq \frac{(|L_n \cup J_n|)|J_n|}{|L_n|} \frac{|l_n|}{(|l_n \cup j_n|)|j_n|}$$



DIFFERENCE INEQUATION II

- $$C^{-1}\lambda_{n-1}^{2-\ell} \le \frac{\left(Df^{s_{n+1}}(c^f)\right)^2}{Df^{s_n}(c^f)\,Df^{s_{n-1}}(c^f)} \le C\lambda_{n-1}^{2-\ell}\,.$$

- $$C_1^{-1}\lambda_{n-1}^{\ell} \le Df^{s_n}(c^f)\,Df^{s_{n-1}}(c^f) \le C_1\lambda_{n-1}^{\ell}$$

- $X_n := \log Df^{s_n}(c^f)$ satisfies

$$-\Delta \le X_{n+1} - \frac{1}{\ell}(X_{n-1}+X_n) \le \Delta$$

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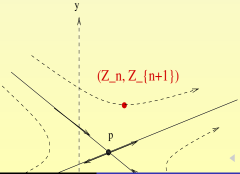
Superexponential growth

- $Z_{n+1} = \frac{1}{\ell}(Z_n + Z_{n-1}) - \Delta$
- $T(x, y) := (y, \frac{1}{\ell}(x + y) - \Delta)$; $(Z_n, Z_{n+1}) = T(Z_{n-1}, Z_n)$
- Fixed point

$$p := \left(\frac{\Delta}{\frac{2}{\ell} - 1}, \frac{\Delta}{\frac{2}{\ell} - 1} \right), \quad \text{eigenvalues:} \quad \mu_\pm = \frac{1 \pm \sqrt{1 + 4\ell}}{2\ell}$$

corresponding eigenvectors: $(1, \mu_\pm)$.

- $X_n := \log Df^{S_n}(c^f) \geq Z_n$



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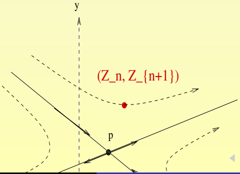

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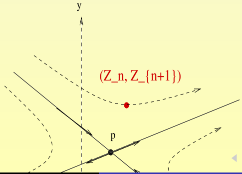

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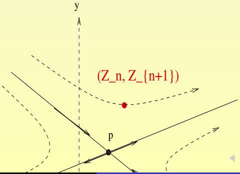

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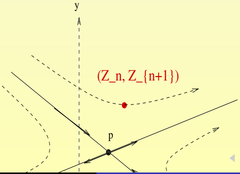

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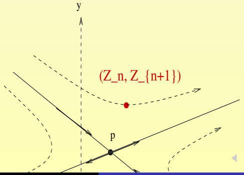

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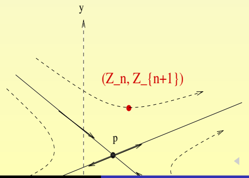

THANKS