

Metric aspects of the dynamics of covering maps of the circle

E. Colli¹ M. L. do Nascimento² E. Vargas³

¹Departamento de Matemática Aplicada
Universidade de São Paulo

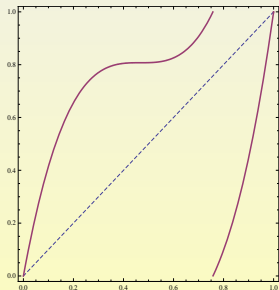
²Centro de Ciências Exatas e Naturais
Universidade Federal do Pará

³Departamento de Matemática
Universidade de São Paulo

OUTLINE

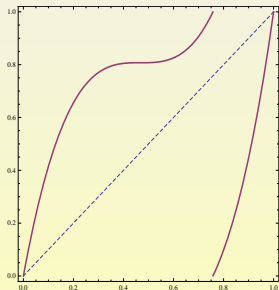
- 1 INTRODUCTION
 - Set up
 - Topological aspects
 - Metric aspects
- 2 MAIN RESULT
 - Statement
 - Strong recurrence
 - History
- 3 TOOLS AND IDEAS
 - Difference equation I
 - Difference equation II

CRITICAL COVERING MAPS ON S^1



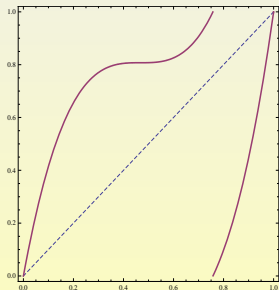
- Class C^1 , degree $d > 2$, critical point c of order $k > 1$.
- Class C^2 and Negative Schwarzian on $S^1 \setminus \{c\}$.
- There is no wandering interval.

CRITICAL COVERING MAPS ON S^1



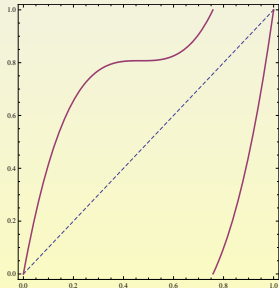
- Class C^1 , degree $d \geq 2$, critical point c of order $\ell > 1$;
- Class C^2 and Negative Schwarzian on $S^1 \setminus \{c\}$;
- There is no wandering interval.

CRITICAL COVERING MAPS ON S^1



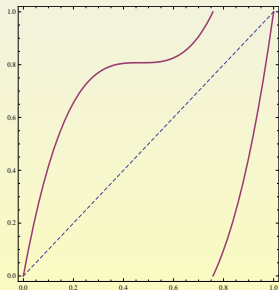
- Class C^1 , degree $d \geq 2$, critical point c of order $\ell > 1$;
- Class C^3 and Negative Schwarzian on $S^1 \setminus \{c\}$;
- There is no wandering interval.

CRITICAL COVERING MAPS ON S^1



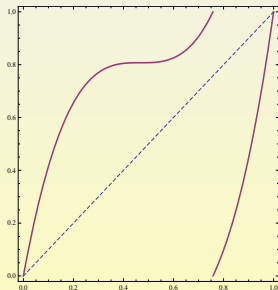
- Class C^1 , degree $d \geq 2$, critical point c of order $\ell > 1$;
 - Class C^3 and Negative Schwarzian on $S^1 \setminus \{c\}$;
 - There is no wandering interval.

CRITICAL COVERING MAPS ON S^1



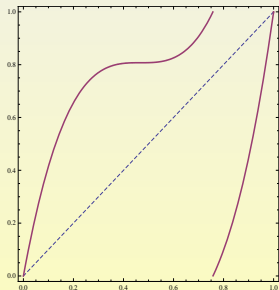
- Class C^1 , degree $d \geq 2$, critical point c of order $\ell > 1$;
- Class C^3 and Negative Schwarzian on $S^1 \setminus \{c\}$;
- There is no wandering interval.

CRITICAL COVERING MAPS ON \mathbb{S}^1



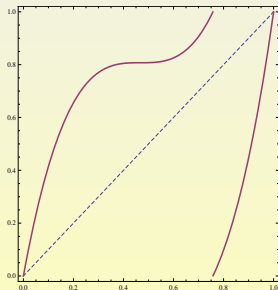
- Class C^1 , degree $d \geq 2$, critical point c of order $\ell > 1$;
- Class C^3 and Negative Schwarzian on $\mathbb{S}^1 \setminus \{c\}$;
- There is no wandering interval.

CRITICAL COVERING MAPS ON \mathbb{S}^1



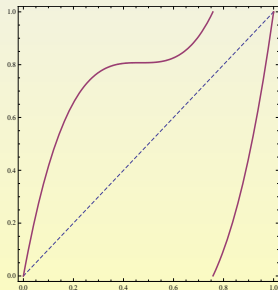
- Class C^1 , degree $d \geq 2$, critical point c of order $\ell > 1$;
- Class C^3 and Negative Schwarzian on $\mathbb{S}^1 \setminus \{c\}$;
- There is no wandering interval.

CRITICAL COVERING MAPS ON \mathbb{S}^1



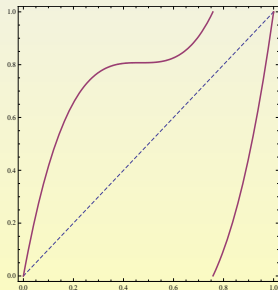
- Class C^1 , degree $d \geq 2$, critical point c of order $\ell > 1$;
- Class C^3 and Negative Schwarzian on $\mathbb{S}^1 \setminus \{c\}$;
- There is no wandering interval.

CRITICAL COVERING MAPS ON \mathbb{S}^1



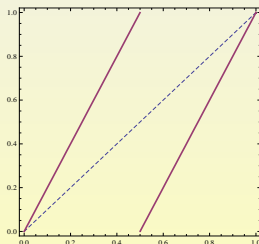
- Class C^1 , degree $d \geq 2$, critical point c of order $\ell > 1$;
- Class C^3 and Negative Schwarzian on $\mathbb{S}^1 \setminus \{c\}$;
- There is no wandering interval.

CRITICAL COVERING MAPS ON \mathbb{S}^1



- Class C^1 , degree $d \geq 2$, critical point c of order $\ell > 1$;
- Class C^3 and Negative Schwarzian on $\mathbb{S}^1 \setminus \{c\}$;
- There is no wandering interval.

THE MAP $\mathcal{R}(z) = 2z \text{ MOD } 1$



QUESTIONS

- Regularity of conjugacy;
- Invariant measures
- Attractors;
- Decay of geometry;
- Renormalization;
- Growth of $Df^n(c')$.

QUESTIONS

- Regularity of conjugacy;
- Invariant measures
- Attractors;
- Decay of geometry;
- Renormalization;
- Growth of $Df^n(c')$.

QUESTIONS

- Regularity of conjugacy;
- Invariant measures
- Attractors;
 - Decay of geometry;
 - Renormalization;
 - Growth of $Df^n(c')$.

QUESTIONS

- Regularity of conjugacy;
- Invariant measures
- Attractors;
- Decay of geometry;
- Renormalization;
- Growth of $Df^n(c^f)$.

QUESTIONS

- Regularity of conjugacy;
- Invariant measures
- Attractors;
- Decay of geometry;
- Renormalization;
- Growth of $Df^n(c^f)$.

QUESTIONS

- Regularity of conjugacy;
- Invariant measures
- Attractors;
- Decay of geometry;
- Renormalization;
- Growth of $Df^n(c^f)$.

GROWTH OF DERIVATIVES

THEOREM

Let f be as above, $\mu^+ := \frac{1+\sqrt{1+4\ell}}{2\ell}$ and $\varphi := \frac{1+\sqrt{5}}{2}$. If in addition f has the Fibonacci combinatorics then:

① If $1 < \ell < 2$ then $\frac{\log \log Df^{sn}(c^f)}{n} \rightarrow \log \mu^+ > 0$.

If $1 < \ell < 2$ then $\frac{\log \log Df^{sn}(c^f)}{\log n} \rightarrow \frac{\log \mu^+}{\log \varphi} < 1$.

GROWTH OF DERIVATIVES

THEOREM

Let f be as above, $\mu^+ := \frac{1+\sqrt{1+4\ell}}{2\ell}$ and $\varphi := \frac{1+\sqrt{5}}{2}$. If in addition f has the Fibonacci combinatorics then:

1 If $1 < \ell < 2$ then $\frac{\log \log Df^{sn}(c^f)}{n} \rightarrow \log \mu^+ > 0$.

GROWTH OF DERIVATIVES

THEOREM

Let f be as above, $\mu^+ := \frac{1+\sqrt{1+4\ell}}{2\ell}$ and $\varphi := \frac{1+\sqrt{5}}{2}$. If in addition f has the Fibonacci combinatorics then:

1 If $1 < \ell < 2$ then $\frac{\log \log Df^{sn}(c^f)}{n} \rightarrow \log \mu^+ > 0$.

2 If $1 < \ell < 2$ then $\frac{\log \log Df^k(c^f)}{\log k} \rightarrow \frac{\log \mu^+}{\log \varphi} < 1$.

3 If $\ell = 2$ then $Df^k(c^f) \leq \exp(\log k) \leq K$.

GROWTH OF DERIVATIVES

THEOREM

Let f be as above, $\mu^+ := \frac{1+\sqrt{1+4\ell}}{2\ell}$ and $\varphi := \frac{1+\sqrt{5}}{2}$. If in addition f has the Fibonacci combinatorics then:

1 If $1 < \ell < 2$ then $\frac{\log \log Df^{sn}(c^f)}{n} \rightarrow \log \mu^+ > 0$.

2 If $1 < \ell < 2$ then $\frac{\log \log Df^k(c^f)}{\log k} \rightarrow \frac{\log \mu^+}{\log \varphi} < 1$.

GROWTH OF DERIVATIVES

THEOREM

Let f be as above, $\mu^+ := \frac{1+\sqrt{1+4\ell}}{2\ell}$ and $\varphi := \frac{1+\sqrt{5}}{2}$. If in addition f has the Fibonacci combinatorics then:

1 If $1 < \ell < 2$ then $\frac{\log \log Df^{sn}(c^f)}{n} \rightarrow \log \mu^+ > 0$.

2 If $1 < \ell < 2$ then $\frac{\log \log Df^k(c^f)}{\log k} \rightarrow \frac{\log \mu^+}{\log \varphi} < 1$.

3 If $\ell = 2$ then $0 < K \leq \frac{\log Df^{sn}(c^f)}{n} \leq K^{-1}$.

4 If $\ell > 2$ then $Df^n(c^f)$ is bounded.

GROWTH OF DERIVATIVES

THEOREM

Let f be as above, $\mu^+ := \frac{1+\sqrt{1+4\ell}}{2\ell}$ and $\varphi := \frac{1+\sqrt{5}}{2}$. If in addition f has the Fibonacci combinatorics then:

1 If $1 < \ell < 2$ then $\frac{\log \log Df^{sn}(c^f)}{n} \rightarrow \log \mu^+ > 0$.

2 If $1 < \ell < 2$ then $\frac{\log \log Df^k(c^f)}{\log k} \rightarrow \frac{\log \mu^+}{\log \varphi} < 1$.

3 If $\ell = 2$ then $0 < K \leq \frac{\log Df^{sn}(c^f)}{n} \leq K^{-1}$.

GROWTH OF DERIVATIVES

THEOREM

Let f be as above, $\mu^+ := \frac{1+\sqrt{1+4\ell}}{2\ell}$ and $\varphi := \frac{1+\sqrt{5}}{2}$. If in addition f has the Fibonacci combinatorics then:

- 1 If $1 < \ell < 2$ then $\frac{\log \log Df^{sn}(c^f)}{n} \rightarrow \log \mu^+ > 0$.
- 2 If $1 < \ell < 2$ then $\frac{\log \log Df^k(c^f)}{\log k} \rightarrow \frac{\log \mu^+}{\log \varphi} < 1$.
- 3 If $\ell = 2$ then $0 < K \leq \frac{\log Df^{sn}(c^f)}{n} \leq K^{-1}$.

4 If $\ell > 2$ then $Df^{sn}(c^f)$ is bounded.

GROWTH OF DERIVATIVES

THEOREM

Let f be as above, $\mu^+ := \frac{1+\sqrt{1+4\ell}}{2\ell}$ and $\varphi := \frac{1+\sqrt{5}}{2}$. If in addition f has the Fibonacci combinatorics then:

- 1 If $1 < \ell < 2$ then $\frac{\log \log Df^{S_n}(c^f)}{n} \rightarrow \log \mu^+ > 0$.
- 2 If $1 < \ell < 2$ then $\frac{\log \log Df^k(c^f)}{\log k} \rightarrow \frac{\log \mu^+}{\log \varphi} < 1$.
- 3 If $\ell = 2$ then $0 < K \leq \frac{\log Df^{S_n}(c^f)}{n} \leq K^{-1}$.
- 4 If $\ell > 2$ then $Df^{S_n}(c^f)$ is bounded.

GROWTH OF DERIVATIVES

THEOREM

Let f be as above, $\mu^+ := \frac{1+\sqrt{1+4\ell}}{2\ell}$ and $\varphi := \frac{1+\sqrt{5}}{2}$. If in addition f has the Fibonacci combinatorics then:

- 1 If $1 < \ell < 2$ then $\frac{\log \log Df^{s_n}(c^f)}{n} \rightarrow \log \mu^+ > 0$.
- 2 If $1 < \ell < 2$ then $\frac{\log \log Df^k(c^f)}{\log k} \rightarrow \frac{\log \mu^+}{\log \varphi} < 1$.
- 3 If $\ell = 2$ then $0 < K \leq \frac{\log Df^{s_n}(c^f)}{n} \leq K^{-1}$.
- 4 If $\ell > 2$ then $Df^{s_n}(c^f)$ is bounded.

5 If $1 < \ell \leq 2$ then $\omega(c)$ (which is a minimal Cantor set) has Hausdorff dimension zero.

GROWTH OF DERIVATIVES

THEOREM

Let f be as above, $\mu^+ := \frac{1+\sqrt{1+4\ell}}{2\ell}$ and $\varphi := \frac{1+\sqrt{5}}{2}$. If in addition f has the Fibonacci combinatorics then:

- 1 If $1 < \ell < 2$ then $\frac{\log \log Df^{s_n}(c^f)}{n} \rightarrow \log \mu^+ > 0$.
- 2 If $1 < \ell < 2$ then $\frac{\log \log Df^k(c^f)}{\log k} \rightarrow \frac{\log \mu^+}{\log \varphi} < 1$.
- 3 If $\ell = 2$ then $0 < K \leq \frac{\log Df^{s_n}(c^f)}{n} \leq K^{-1}$.
- 4 If $\ell > 2$ then $Df^{s_n}(c^f)$ is bounded.
- 5 If $1 < \ell \leq 2$ then $\omega(c)$ (which is a minimal Cantor set) has Hausdorff dimension zero.

GROWTH OF DERIVATIVES

THEOREM

Let f be as above, $\mu^+ := \frac{1+\sqrt{1+4\ell}}{2\ell}$ and $\varphi := \frac{1+\sqrt{5}}{2}$. If in addition f has the Fibonacci combinatorics then:

- 1 If $1 < \ell < 2$ then $\frac{\log \log Df^{s_n}(c^f)}{n} \rightarrow \log \mu^+ > 0$.
- 2 If $1 < \ell < 2$ then $\frac{\log \log Df^k(c^f)}{\log k} \rightarrow \frac{\log \mu^+}{\log \varphi} < 1$.
- 3 If $\ell = 2$ then $0 < K \leq \frac{\log Df^{s_n}(c^f)}{n} \leq K^{-1}$.
- 4 If $\ell > 2$ then $Df^{s_n}(c^f)$ is bounded.
- 5 If $1 < \ell \leq 2$ then $\omega(c)$ (which is a minimal Cantor set) has Hausdorff dimension zero.

GROWTH OF DERIVATIVES

THEOREM

Let f be as above, $\mu^+ := \frac{1+\sqrt{1+4\ell}}{2\ell}$ and $\varphi := \frac{1+\sqrt{5}}{2}$. If in addition f has the Fibonacci combinatorics then:

- 1 If $1 < \ell < 2$ then $\frac{\log \log Df^{s_n}(c^f)}{n} \rightarrow \log \mu^+ > 0$.
- 2 If $1 < \ell < 2$ then $\frac{\log \log Df^k(c^f)}{\log k} \rightarrow \frac{\log \mu^+}{\log \varphi} < 1$.
- 3 If $\ell = 2$ then $0 < K \leq \frac{\log Df^{s_n}(c^f)}{n} \leq K^{-1}$.
- 4 If $\ell > 2$ then $Df^{s_n}(c^f)$ is bounded.
- 5 If $1 < \ell \leq 2$ then $\omega(c)$ (which is a minimal Cantor set) has Hausdorff dimension zero.

THE FIBONACCI COMBINATORICS

- c is recurrent;
- Set $I_0 := (0, 1)$ and $I_n \ni c$, a domain of the first return map ϕ_n to I_{n-1} ($\phi_n : I_n \rightarrow I_{n-1}$);
- if $\phi_n(c) = F^{-1}(c)$ then (ϕ_n) is the Fibonacci sequence $1, 2, 3, 5, 8, \dots$

THE FIBONACCI COMBINATORICS

- c is recurrent;
- Set $I_0 := (0, 1)$ and $I_n \ni c$, a domain of the first return map ϕ_n to I_{n-1} ($\phi_n: I_n \rightarrow I_{n-1}$);
- If $\phi_n(c) = I^{\delta_n}(c)$ then (δ_n) is the Fibonacci sequence $(1, 2, 3, 5, 8, \dots)$.

THE FIBONACCI COMBINATORICS

- c is recurrent;
- Set $I_0 := (0, 1)$ and $I_n \ni c$, a domain of the first return map ϕ_n to I_{n-1} ($\phi_n : I_n \rightarrow I_{n-1}$);
- If $\phi_n(c) = F^n(c)$ then (s_n) is the Fibonacci sequence $1, 2, 3, 5, 8, \dots$.

THE FIBONACCI COMBINATORICS

- c is recurrent;
- Set $I_0 := (0, 1)$ and $I_n \ni c$, a domain of the first return map ϕ_n to I_{n-1} ($\phi_n : I_n \rightarrow I_{n-1}$);
- If $\phi_n(c) = f^{s_n}(c)$ then $\{s_n\}$ is the Fibonacci sequence $1, 2, 3, 5, 8, \dots$.

THE FIBONACCI COMBINATORICS

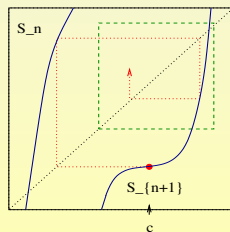
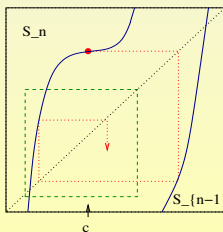
- c is recurrent;
- Set $I_0 := (0, 1)$ and $I_n \ni c$, a domain of the first return map ϕ_n to I_{n-1} ($\phi_n : I_n \rightarrow I_{n-1}$);
- If $\phi_n(c) = f^{s_n}(c)$ then s_n is the Fibonacci sequence $1, 2, 3, 5, 8, \dots$

THE FIBONACCI COMBINATORICS

- c is recurrent;
- Set $I_0 := (0, 1)$ and $I_n \ni c$, a domain of the first return map ϕ_n to I_{n-1} ($\phi_n : I_n \rightarrow I_{n-1}$);
- If $\phi_n(c) = f^{S_n}(c)$ then S_n is the Fibonacci sequence $1, 2, 3, 5, 8, \dots$

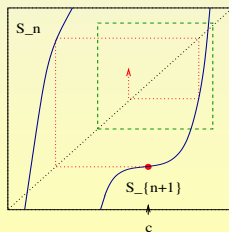
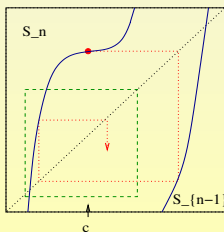
THE FIBONACCI COMBINATORICS

- c is recurrent;
- Set $I_0 := (0, 1)$ and $I_n \ni c$, a domain of the first return map ϕ_n to I_{n-1} ($\phi_n : I_n \rightarrow I_{n-1}$);
- If $\phi_n(c) = f^{S_n}(c)$ then S_n is the Fibonacci sequence $1, 2, 3, 5, 8, \dots$

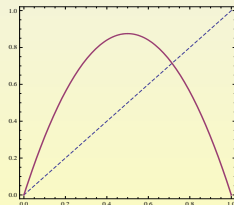
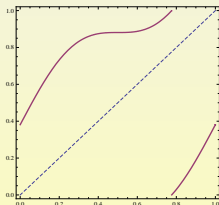
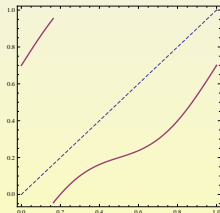


THE FIBONACCI COMBINATORICS

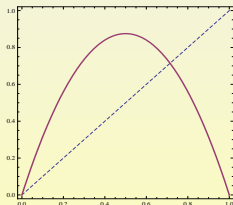
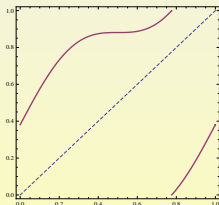
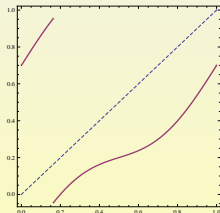
- c is recurrent;
- Set $I_0 := (0, 1)$ and $I_n \ni c$, a domain of the first return map ϕ_n to I_{n-1} ($\phi_n : I_n \rightarrow I_{n-1}$);
- If $\phi_n(c) = f^{S_n}(c)$ then S_n is the Fibonacci sequence $1, 2, 3, 5, 8, \dots$



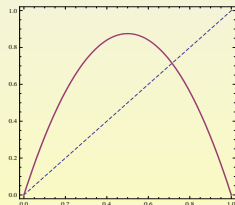
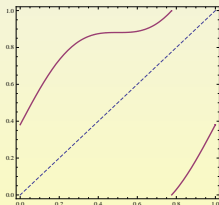
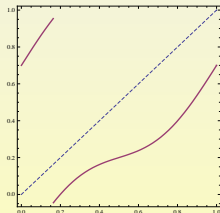
OTHER DYNAMICS



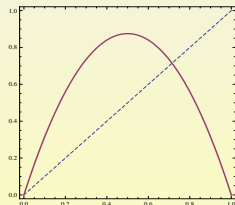
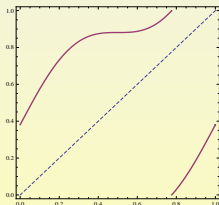
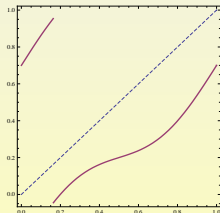
OTHER DYNAMICS



OTHER DYNAMICS

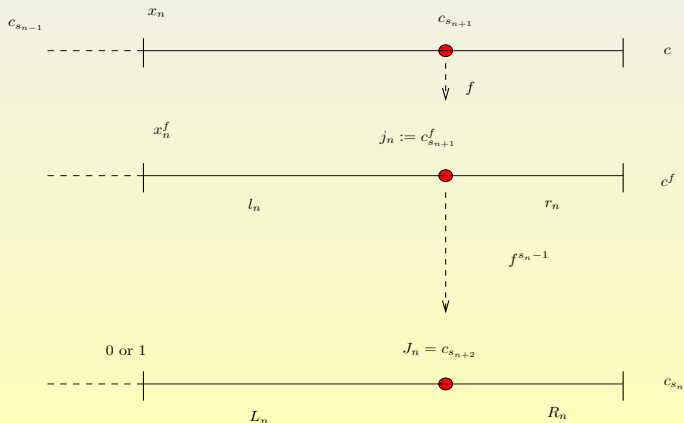


OTHER DYNAMICS



CROSS RATIO

$$Df^{s_n-1}(c_{s_{n+1}}^f) = \frac{|J_n|}{|j_n|} \geq \frac{|L_n||R_n|}{|L_n \cup R_n|} \frac{|l_n \cup r_n|}{|l_n||r_n|}$$



DIFFERENCE INEQUATION I

- $$Df^{S_n-1}(c_{S_{n+1}}^f) = \frac{|J_n|}{|j_n|} \geq \frac{|L_n||R_n|}{|L_n \cup R_n|} \frac{|l_n \cup r_n|}{|l_n||r_n|}$$

$$Df^{S_n-1}(c_{S_{n+1}}^f) \geq \frac{|L_n|}{|L_n \cup R_n|} \frac{d_n - d_{n+2}}{d_{n+1}'} \frac{d_{n-1}'}{d_{n-1}' - d_{n+1}'},$$

where $d_n := |c_{S_n} - c|$.

- For the next slide $\lambda_n := \frac{d_n}{d_{n+1}'} \geq \lambda > 1$.

DIFFERENCE INEQUATION I

- $$Df^{s_n-1}(c_{s_{n+1}}^f) = \frac{|J_n|}{|j_n|} \geq \frac{|L_n||R_n|}{|L_n \cup R_n|} \frac{|l_n \cup r_n|}{|l_n||r_n|}$$

- $$Df^{s_n-1}(c_{s_{n+1}}^f) \geq \frac{|L_n|}{|L_n \cup R_n|} \frac{d_n - d_{n+2}}{d_{n+1}'} \frac{d_{n-1}'}{d_{n-1}' - d_{n+1}'},$$

where $d_n := |c_{s_n} - c|$.

- For the next slide $\lambda_n := \frac{d_n}{d_{n+2}'} \geq \lambda > 1$.

DIFFERENCE INEQUATION I



$$Df^{S_n-1}(c_{S_{n+1}}^f) = \frac{|J_n|}{|j_n|} \geq \frac{|L_n||R_n|}{|L_n \cup R_n|} \frac{|l_n \cup r_n|}{|l_n||r_n|}$$



$$Df^{S_n-1}(c_{S_{n+1}}^f) \geq \frac{|L_n|}{|L_n \cup R_n|} \frac{d_n - d_{n+2}}{d_{n+1}^f} \frac{d_{n-1}^f}{d_{n-1}^f - d_{n+1}^f},$$

where $d_n := |c_{S_n} - c|$.

- For the next slide $\lambda_n := \frac{d_n}{d_{n+1}} \geq \lambda > 1$.

DIFFERENCE INEQUALITY I



$$Df^{S_n-1}(c_{S_{n+1}}^f) = \frac{|J_n|}{|j_n|} \geq \frac{|L_n||R_n|}{|L_n \cup R_n|} \frac{|l_n \cup r_n|}{|l_n||r_n|}$$



$$Df^{S_n-1}(c_{S_{n+1}}^f) \geq \frac{|L_n|}{|L_n \cup R_n|} \frac{d_n - d_{n+2}}{d_{n+1}^f} \frac{d_{n-1}^f}{d_{n-1}^f - d_{n+1}^f},$$

where $d_n := |c_{S_n} - c|$.

- For the next slide $\lambda_n := \frac{d_n}{d_{n+2}} \geq \lambda > 1$.

DIFFERENCE INEQUATION I



$$Df^{s_n-1}(c_{s_{n+1}}^f) = \frac{|J_n|}{|j_n|} \geq \frac{|L_n||R_n|}{|L_n \cup R_n|} \frac{|l_n \cup r_n|}{|l_n||r_n|}$$



$$Df^{s_n-1}(c_{s_{n+1}}^f) \geq \frac{|L_n|}{|L_n \cup R_n|} \frac{d_n - d_{n+2}}{d_{n+1}^f} \frac{d_{n-1}^f}{d_{n-1}^f - d_{n+1}^f},$$

where $d_n := |c_{s_n} - c|$.

- For the next slide $\lambda_n := \frac{d_n}{d_{n+2}} \geq \lambda > 1$.

DIFFERENCE INEQUATION I



$$\frac{(Df^{s_{n+1}}(c^f))^2}{Df^{s_n}(c^f)Df^{s_{n-1}}(c^f)} \geq K_n \ell^2 \lambda_{n-1}^{2-\ell} \frac{q_{n-1}}{q_n} \frac{1 - \lambda_n^{-1}}{1 - \lambda_{n-2}^{-\ell}} \frac{1 - \lambda_{n-1}^{-1}}{1 - \lambda_{n-1}^{-\ell}}$$

$$\left[\frac{Df^{s_{n+1}}(c^f)}{1 - \lambda_n^{-1}} \right]^2 \left[\frac{Df^{s_n}(c^f)}{1 - \lambda_{n-1}^{-1}} \right]^{-1} \left[\frac{Df^{s_{n-1}}(c^f)}{1 - \lambda_{n-2}^{-1}} \right]^{-1} \geq \frac{Q_{n-1}}{Q_n} \sigma_n$$

where

$$\sigma_n := K_n \ell^2 \lambda_{n-1}^{2-\ell} \frac{1 - \lambda_{n-1}^{-1}}{1 - \lambda_{n-1}^{-\ell}} \frac{1 - \lambda_{n-2}^{-1}}{1 - \lambda_{n-2}^{-\ell}} \geq \sigma > 1$$

DIFFERENCE INEQUATION I



$$\frac{(Df^{s_{n+1}}(c^f))^2}{Df^{s_n}(c^f)Df^{s_{n-1}}(c^f)} \geq K_n \ell^2 \lambda_{n-1}^{2-\ell} \frac{q_{n-1}}{q_n} \frac{1 - \lambda_n^{-1}}{1 - \lambda_{n-2}^{-\ell}} \frac{1 - \lambda_{n-1}^{-1}}{1 - \lambda_{n-1}^{-\ell}}$$



$$\left[\frac{Df^{s_{n+1}}(c^f)}{1 - \lambda_n^{-1}} \right]^2 \left[\frac{Df^{s_n}(c^f)}{1 - \lambda_{n-1}^{-1}} \right]^{-1} \left[\frac{Df^{s_{n-1}}(c^f)}{1 - \lambda_{n-2}^{-1}} \right]^{-1} \geq \frac{Q_{n-1}}{Q_n} \sigma_n$$

where

$$\sigma_n := K_n \ell^2 \lambda_{n-1}^{2-\ell} \frac{1 - \lambda_n^{-1}}{1 - \lambda_{n-1}^{-\ell}} \frac{1 - \lambda_{n-2}^{-1}}{1 - \lambda_{n-2}^{-\ell}} \geq \sigma > 1$$

DIFFERENCE INEQUATION I



$$\frac{(Df^{s_{n+1}}(c^f))^2}{Df^{s_n}(c^f)Df^{s_{n-1}}(c^f)} \geq K_n \ell^2 \lambda_{n-1}^{2-\ell} \frac{q_{n-1}}{q_n} \frac{1 - \lambda_n^{-1}}{1 - \lambda_{n-2}^{-\ell}} \frac{1 - \lambda_{n-1}^{-1}}{1 - \lambda_{n-1}^{-\ell}}$$



$$\left[\frac{Df^{s_{n+1}}(c^f)}{1 - \lambda_n^{-1}} \right]^2 \left[\frac{Df^{s_n}(c^f)}{1 - \lambda_{n-1}^{-1}} \right]^{-1} \left[\frac{Df^{s_{n-1}}(c^f)}{1 - \lambda_{n-2}^{-1}} \right]^{-1} \geq \frac{Q_{n-1}}{Q_n} \sigma_n$$

where

$$\sigma_n := K_n \ell^2 \lambda_{n-1}^{2-\ell} \frac{1 - \lambda_{n-1}^{-1}}{1 - \lambda_{n-1}^{-\ell}} \frac{1 - \lambda_{n-2}^{-1}}{1 - \lambda_{n-2}^{-\ell}} \geq \sigma > 1$$

EXPONENTIAL GROWTH

- $X_n := \log \left[\frac{Df^{s_n}(c^f)}{1 - \lambda_{n-1}^{-1}} \right] \geq Y_n$, where

$$Y_{n+1} - \frac{1}{2} Y_n - \frac{1}{2} Y_{n-1} = \frac{1}{2} \log Q_{n-1} - \frac{1}{2} \log Q_n + \frac{1}{2} \log \sigma_n$$

- $\liminf \frac{Y_n}{n} \geq \frac{1}{3} \liminf \log \sigma_n \geq \varepsilon > 0$

EXPONENTIAL GROWTH

- $X_n := \log \left[\frac{Df^{s_n}(c^f)}{1-\lambda_{n-1}^{-1}} \right] \geq Y_n$, where

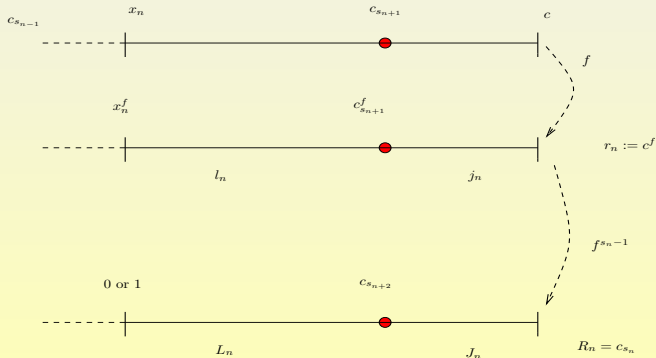
$$Y_{n+1} - \frac{1}{2} Y_n - \frac{1}{2} Y_{n-1} = \frac{1}{2} \log Q_{n-1} - \frac{1}{2} \log Q_n + \frac{1}{2} \log \sigma_n$$



$$\liminf \frac{Y_n}{n} \geq \frac{1}{3} \liminf \log \sigma_n \geq \varepsilon > 0$$

DIFFERENCE INEQUALITY II

$$Df^{s_n-1}(c^f) = \frac{|R_n|}{|r_n|} \leq \frac{(|L_n \cup J_n|)|J_n|}{|L_n|} \frac{|I_n|}{(|I_n \cup j_n|)|j_n|}$$



DIFFERENCE INEQUATION II

- $$C^{-1} \lambda_{n-1}^{2-\ell} \leq \frac{(Df^{s_{n+1}}(c^f))^2}{Df^{s_n}(c^f) Df^{s_{n-1}}(c^f)} \leq C \lambda_{n-1}^{2-\ell}.$$

$$C_1^{-1} \lambda_{n-1}^{\ell} \leq Df^{s_n}(c^f) Df^{s_{n-1}}(c^f) \leq C_1 \lambda_{n-1}^{\ell}$$

- $X_n := \log Df^{s_n}(c^f)$ satisfies

$$-\Delta \leq X_{n+1} - \frac{1}{2}(X_{n-1} + X_n) \leq \Delta$$

DIFFERENCE INEQUALITY II

- $$C^{-1} \lambda_{n-1}^{2-\ell} \leq \frac{(Df^{S_{n+1}}(c^f))^2}{Df^{S_n}(c^f) Df^{S_{n-1}}(c^f)} \leq C \lambda_{n-1}^{2-\ell}.$$

- $$C_1^{-1} \lambda_{n-1}^\ell \leq Df^{S_n}(c^f) Df^{S_{n-1}}(c^f) \leq C_1 \lambda_{n-1}^\ell$$

- $X_n := \log Df^{S_n}(c^f)$ satisfies

$$-\Delta \leq X_{n+1} - \frac{1}{\ell}(X_{n-1} + X_n) \leq \Delta$$

DIFFERENCE INEQUALITY II



$$C^{-1} \lambda_{n-1}^{2-\ell} \leq \frac{(Df^{S_{n+1}}(c^f))^2}{Df^{S_n}(c^f) Df^{S_{n-1}}(c^f)} \leq C \lambda_{n-1}^{2-\ell}.$$



$$C_1^{-1} \lambda_{n-1}^\ell \leq Df^{S_n}(c^f) Df^{S_{n-1}}(c^f) \leq C_1 \lambda_{n-1}^\ell$$

- $X_n := \log Df^{S_n}(c^f)$ satisfies

$$-\Delta \leq X_{n+1} - \frac{1}{2}(X_{n-1} + X_n) \leq \Delta$$

DIFFERENCE INEQUATION II



$$C^{-1} \lambda_{n-1}^{2-\ell} \leq \frac{(Df^{s_{n+1}}(c^f))^2}{Df^{s_n}(c^f) Df^{s_{n-1}}(c^f)} \leq C \lambda_{n-1}^{2-\ell}.$$



$$C_1^{-1} \lambda_{n-1}^\ell \leq Df^{s_n}(c^f) Df^{s_{n-1}}(c^f) \leq C_1 \lambda_{n-1}^\ell$$

- $X_n := \log Df^{s_n}(c^f)$ satisfies

$$-\Delta \leq X_{n+1} - \frac{1}{\ell}(X_{n-1} + X_n) \leq \Delta$$

DIFFERENCE INEQUATION II



$$C^{-1} \lambda_{n-1}^{2-\ell} \leq \frac{(Df^{s_{n+1}}(c^f))^2}{Df^{s_n}(c^f) Df^{s_{n-1}}(c^f)} \leq C \lambda_{n-1}^{2-\ell}.$$



$$C_1^{-1} \lambda_{n-1}^\ell \leq Df^{s_n}(c^f) Df^{s_{n-1}}(c^f) \leq C_1 \lambda_{n-1}^\ell$$

- $X_n := \log Df^{s_n}(c^f)$ satisfies

$$-\Delta \leq X_{n+1} - \frac{1}{\ell}(X_{n-1} + X_n) \leq \Delta$$

SUPEREXPONENTIAL GROWTH

- $Z_{n+1} = \frac{1}{\ell}(Z_n + Z_{n-1}) - \Delta$

- $T(x, y) := (y, \frac{1}{\ell}(x + y) - \Delta)$;

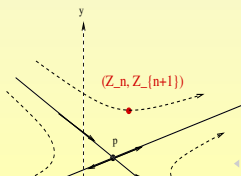
- $(Z_n, Z_{n+1}) = T(Z_{n-1}, Z_n)$

- $(Z_n, Z_{n+1}) = T^n(Z_0, Z_1)$

- $D T = \begin{pmatrix} 0 & 1 \\ \frac{1}{\ell} & \frac{1}{\ell} \end{pmatrix}$ eigenvalues: $\lambda_{\pm} = \frac{1 \pm \sqrt{1 + 4\ell\Delta}}{2\ell}$

corresponding eigenvectors: $(1, \mu_{\pm})$.

- $X_n := \log D T^n(c^0) \geq Z_n$



SUPEREXPONENTIAL GROWTH

- $Z_{n+1} = \frac{1}{\ell}(Z_n + Z_{n-1}) - \Delta$

- $T(x, y) := (y, \frac{1}{\ell}(x + y) - \Delta)$;

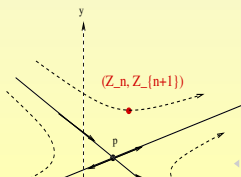
- $(Z_n, Z_{n+1}) = T(Z_{n-1}, Z_n)$

- Fixed point

- $p := \begin{pmatrix} \Delta & \Delta \\ \frac{2}{\ell} - 1 & \frac{2}{\ell} - 1 \end{pmatrix}$, eigenvalues: $\mu_{\pm} = \frac{1 \pm \sqrt{1 + 4\ell\Delta}}{2\ell}$

- corresponding eigenvectors: $(1, \mu_{\pm})$.

- $X_n := \log Df^{2n}(c^{\ell}) \geq Z_n$



SUPEREXPONENTIAL GROWTH

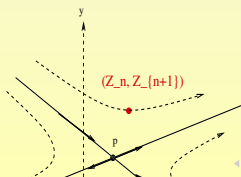
- $Z_{n+1} = \frac{1}{\ell}(Z_n + Z_{n-1}) - \Delta$
- $T(x, y) := (y, \frac{1}{\ell}(x + y) - \Delta)$;
 $(Z_n, Z_{n+1}) = T(Z_{n-1}, Z_n)$

• Fixed point

$$p := \left(\frac{\Delta}{\frac{\ell}{2} - 1}, \frac{\Delta}{\frac{\ell}{2} - 1} \right), \quad \text{eigenvalues: } \mu_{\pm} = \frac{1 \pm \sqrt{1 + 4\ell}}{2\ell}$$

• corresponding eigenvectors: $(1, \mu_{\pm})$.

- $X_n := \log DF^n(c^0) \geq Z_n$



SUPEREXPONENTIAL GROWTH

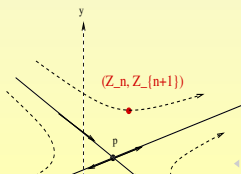
- $Z_{n+1} = \frac{1}{\ell}(Z_n + Z_{n-1}) - \Delta$
- $T(x, y) := (y, \frac{1}{\ell}(x + y) - \Delta)$;
 $(Z_n, Z_{n+1}) = T(Z_{n-1}, Z_n)$

• Fixed point

$$p := \left(\frac{\Delta}{\frac{2}{\ell} - 1}, \frac{\Delta}{\frac{2}{\ell} - 1} \right), \quad \text{eigenvalues: } \mu_{\pm} = \frac{1 \pm \sqrt{1 + 4\ell}}{2\ell}$$

• corresponding eigenvectors: $(1, \mu_{\pm})$.

- $X_n := \log Df^{2n}(c^{\ell}) \geq Z_n$



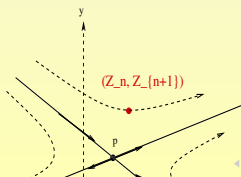
SUPEREXPONENTIAL GROWTH

- $Z_{n+1} = \frac{1}{\ell}(Z_n + Z_{n-1}) - \Delta$
- $T(x, y) := (y, \frac{1}{\ell}(x + y) - \Delta)$;
 $(Z_n, Z_{n+1}) = T(Z_{n-1}, Z_n)$
- Fixed point

$$p := \left(\frac{\Delta}{\frac{2}{\ell} - 1}, \frac{\Delta}{\frac{2}{\ell} - 1} \right), \quad \text{eigenvalues: } \mu_{\pm} = \frac{1 \pm \sqrt{1 + 4\ell}}{2\ell}$$

corresponding eigenvectors: $(1, \mu_{\pm})$.

$$\bullet X_n := \log DF^n(c^*) \geq Z_n$$



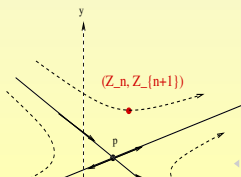
SUPEREXPONENTIAL GROWTH

- $Z_{n+1} = \frac{1}{\ell}(Z_n + Z_{n-1}) - \Delta$
- $T(x, y) := (y, \frac{1}{\ell}(x + y) - \Delta)$;
 $(Z_n, Z_{n+1}) = T(Z_{n-1}, Z_n)$
- Fixed point

$$p := \left(\frac{\Delta}{\frac{2}{\ell} - 1}, \frac{\Delta}{\frac{2}{\ell} - 1} \right), \quad \text{eigenvalues: } \mu_{\pm} = \frac{1 \pm \sqrt{1 + 4\ell}}{2\ell}$$

corresponding eigenvectors: $(1, \mu_{\pm})$.

- $X_n := \log Df^{2n}(c^{\ell}) \geq Z_n$



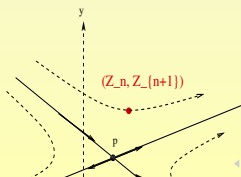
SUPEREXPONENTIAL GROWTH

- $Z_{n+1} = \frac{1}{\ell}(Z_n + Z_{n-1}) - \Delta$
- $T(x, y) := (y, \frac{1}{\ell}(x + y) - \Delta)$;
 $(Z_n, Z_{n+1}) = T(Z_{n-1}, Z_n)$
- Fixed point

$$p := \left(\frac{\Delta}{\frac{2}{\ell} - 1}, \frac{\Delta}{\frac{2}{\ell} - 1} \right), \quad \text{eigenvalues: } \mu_{\pm} = \frac{1 \pm \sqrt{1 + 4\ell}}{2\ell}$$

corresponding eigenvectors: $(1, \mu_{\pm})$.

- $X_n := \log Df^{S_n}(c^f) \geq Z_n$



THANKS