Mathematics for Economic Analysis

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Introduction

The economic world is a misty region. The first explorers used unaided vision. Mathematics is the lantern by which what was before dimly visible now looms up in firm, bold outlines. The old phantasmagoria[1] *disappear. We see better. We also see further.*
—*Irving Fisher* (1892)

1.1 Why Economists Use Mathematics

Economic activity has been part of human life for thousands of years. The word "economics" itself originates from a classical Greek word meaning "household management." Even before the Greeks there were merchants and traders who exhibited an understanding of some economic phenomena; they knew, for instance, that a poor harvest would increase the price of corn, but that a shortage of gold might result in a decrease in the price of corn. For many centuries, the most basic economic concepts were expressed in simple terms requiring only the use of rudimentary mathematics. Concepts like integers and fractions, together with the operations of addition, subtraction, multiplication, and division, were sufficient to allow traders, merchants, farmers and other economic agents to discuss and debate the economic activities and events that affected their daily lives. These tools were enough to enable merchants to keep accounts and to work out what prices to charge.

[1]"Phantasmagoria" is a term invented in 1802 to describe an exhibition of optical illusions produced by means of a magic lantern.

Even calculations of interest on loans were not very complicated. Arithmetic could perform the tasks that merchants required of it even without the concept of zero and the decimal system of notation. Where a calculating device was required, the abacus was powerful enough.

The science of economics reached a turning point in the eighteenth century with the publication of works such as David Hume's *Political Discourses* (1752), François Quesnay's *Tableau Economique* (1758–1759), and Adam Smith's *The Wealth of Nations* (1776). Economic arguments began to be formalized and developed into theories. This created the need to express increasingly complex ideas and interrelationships in a straightforward manner. By the mid-1800s, some writers were beginning to use mathematics to communicate their theories. Some of the first to do this were economists such as Antoine Cournot (the first writer to define and draw an explicit demand curve, and to use calculus in solving a maximization problem in economics) and Léon Walras (who distinguished himself by writing down and solving the first multiequation model of general equilibrium of supply and demand in all markets simultaneously). They found that many of their ideas could be formulated most effectively by means of mathematical language, including algebraic symbols, simple diagrams, and graphs. Indeed, much more sophisticated economic concepts and increasingly complex economic theories have become possible as mathematical language has been used to express them.

Today, a firm understanding of mathematics is essential for any serious student of economics. Although simple economic arguments relying on only two or three variables can sometimes be made in a clear and convincing fashion without mathematics, if we want to consider many variables and the way they interact, it becomes essential to resort to a mathematical model.

As an example, suppose that some government agency is planning to allow a large amount of new housing to be constructed on some land it controls. What consequences will this have for employment? Initially, new jobs will be created in the construction sector as laborers are hired for the project. Moreover, the construction of new houses requires bricks, cement, reinforcing steel, timber, glass, and other building materials. Employment must also grow in firms that manufacture these materials. These producers in turn require materials from other producers, and so on. In addition to all these production effects, increased employment leads to increased incomes. If these income gains are not entirely neutralized by taxes, then a greater demand for consumer goods results. This in turn leads to an increased need for employment among producers of consumer goods, and again the flow of input requirements expands. At the same time, there are feedbacks in the system; for example, increased incomes also generate more demand for housing. In this manner, both positive and negative changes in one sector are transmitted to other sectors of the economy.

The point of this example is that the economic system is so complex that the final effects are difficult to determine without resorting to more formal mathematical devices such as a "circular-flow model" of the entire economy. An example will be the input–output model presented in Section 12.1.

Mathematical Analysis

The principal topic of this book is an important branch of mathematics called **mathematical analysis**. This includes differential and integral calculus and their extensions. Calculus was developed at the end of the seventeenth century by Newton and Leibniz. Their discoveries completely transformed mathematics, physics, and the engineering sciences, giving them all new life. In similar fashion, the introduction of calculus into economics has radically changed the way in which economists analyze the world around them. Calculus is now employed in many different areas of economics: for example, it is used to study the effects of relative price changes on demand, the effects of a change in the price or availability of an essential input such as oil on the production process, the consequences of population growth for the economy, and the extent to which a tax on energy use might reduce carbon dioxide emissions.

The following episode illustrates how economists can use mathematical analysis to solve practical problems. In February 1953, the Netherlands was struck by a catastrophic flood far more extensive than any previously recorded. The dikes protecting the country were washed away and over 1800 people died. Total damages were estimated at about 7% of national income for that year. A commission was established to determine how to prevent similar disasters in the future. Rebuilding the dikes to ensure 100% security would have cost an astronomical amount, even if it were possible at all. The real problem therefore involved a trade-off between cost and security: higher dikes would obviously cost more, but would reduce both the probability and likely severity of future flooding. So the commission had to try to select the optimal height for the dikes. Some economists on the commission applied *cost-benefit analysis*, a branch of economics that involves the use of mathematical analysis, in order to weigh the relative costs and benefits of different alternatives for rebuilding the dikes. This problem is discussed in more detail in Problem 7 in Section 8.4.

Such trade-offs are central to economics. They lead to optimization problems of a type that is naturally handled by mathematical analysis.

1.2 Scientific Method in the Empirical Sciences

Economics is now generally considered to be one of the *empirical sciences*. These sciences share a common methodology that includes the following as its most important elements:

1. Qualitative and quantitative observations of phenomena, either directly or by carefully designed experiment.
2. Numerical and statistical processing of the observed data.
3. Constructing theoretical models that describe the observed phenomena and explain the relationships between them.

4. Using these theoretical models in order to derive predictions.
5. Correcting and improving models so that they predict better.

Empirical sciences thus rely on processes of *observation*, *modeling*, and *verification*. If an activity is to qualify fully as an empirical science, each of the foregoing points is important. Observations without theory can only give purely descriptive pictures of reality that lack explanatory power. But theory without observation risks losing contact with the reality that it is trying to explain.

Many episodes in the history of science show the danger of error when "pure theory" lacks any foundation in reality. For example, around 350 B.C., Aristotle developed a theory that concluded that a freely falling object travels at a constant speed, and that a heavier object falls more quickly than a lighter one. This was convincingly refuted by Galileo Galilei in the sixteenth century when he demonstrated (partly by dropping objects from the Leaning Tower of Pisa) that, excluding the effects of air friction, the speed at which any object falls is proportional to the time it has fallen, and that the constant of proportionality is the same for all objects, regardless of their weight. Thus, Aristotle's theory was eventually disproved by empirical observation.

A second example comes from the science of astronomy. In the year 1800, Hegel advanced a philosophical argument to show that there could only be seven planets in the solar system. Hegel notwithstanding, an eighth planetary body, the asteroid Ceres, was discovered in January 1801. The eighth principal planet, Neptune, was discovered in 1846, and by 1930 the existence of Pluto was known.[2]

With hindsight, the falseness of these assertions by Aristotle and Hegel appears elementary. In all sciences, however, false assertions are being put forth repeatedly, only to be refuted later. Correcting inaccurate theories is an important part of scientific activity, and the previous examples demonstrate the need to ensure that theoretical models are supported by empirical evidence.

In economics, hypotheses are usually less precise than in the physical sciences, and so less obviously wrong than Aristotle's and Hegel's assertions just discussed. But there are a few old theories that have since become so discredited that few economists now take them seriously. One example is the "Phillips curve," that purported to show how an economy could trade off unemployment against inflation. The idea was that employment might be created through tax cuts and/or increased public expenditure, but at the cost of increased inflation. Conversely, inflation could be reduced by tax increases or expenditure cuts, but at the cost of higher unemployment.

[2]The process of discovery relied on looking at how the motion of other known planets deviated from the orbits predicted by Newton's theory of gravitation. These deviations even suggested where to look for an additional planet that could, according to Newton's theory, account for them. Until recently, scientists were still using Newton's theory to search for a tenth planetary body whose existence they suspected. However, more accurate estimates of the masses of the outer planets now suggest that there are no further planets to find after all.

Unlike Hegel, who could never hope to count all the planets, or Aristotle, who presumably never watched with any care the fall of an object that was dropped from rest, the Phillips curve was in fact based on rather careful empirical observation. In an article published in 1958, A. W. Phillips examined the average yearly rates of wage increases and unemployment for the economy of the United Kingdom over the long period from 1861 to 1957. The plot of those observations formed the Phillips curve, and the inflation–unemployment trade-off was part of conventional economic thinking until the 1970s. Then, however, the decade of simultaneous high inflation and high unemployment (stagnation and inflation, generally abbreviated "stagflation") that many Western economies experienced during the period 1973–1982 produced observations that obviously lay well above the usual Phillips curve. The alleged trade-off became hard to discern.

Just as Aristotle's and Hegel's assertions were revised in the light of suitable evidence, this stagflationary episode caused the theory behind the Phillips curve to be modified. It was suggested that as people learn to live with inflation, they adjust wage and loan contracts to reflect expected rates of inflation. Then the trade-off between unemployment and inflation that seemed to be described by the Phillips curve becomes replaced by a new trade-off between unemployment and the deviation in inflation from its expected rate. Moreover, this expected rate increases as the current rate of inflation rises. So lowering unemployment was thought to lead not simply to increased inflation, but to accelerating inflation that increased each period by more than was expected previously. On the other hand, when high inflation came to be expected, combating it with policies leading to painfully high unemployment would lead only to gradual decreases in inflation, as people's expectations of inflation fall rather slowly. Thus, the original Phillips curve theory has been significantly revised and extended in the light of more recent evidence.

Models and Reality

In the eighteenth century, the philosopher Immanuel Kant considered Euclidean geometry to be an absolutely true description of the physical space we observe through our senses. This conception seemed self-evident and was shared by all those who had reflected upon it. The reason for this agreement was undoubtedly that all the results of this geometry could be derived by way of irrefutable logic from only a few axioms, and that these axioms were regarded as self-evident truths about physical space. The first person to question this point of view was the German mathematician Gauss at the beginning of the 1800s. He insisted that the relationship between physical space and Euclid's model could only be made clear by empirical methods. During the 1820s, the first non-Euclidean geometry was developed—that is, a geometry built upon axioms other than Euclid's. Since that time it has been accepted that only observations can decide which geometric model gives the best description of physical space.

This shows how there can be an important difference between a mathematical model and its possible interpretations in reality. Moreover, it may happen that more than one model is capable of describing a certain phenomenon, such as the relationship between money supply and inflation in the United States or Germany. Indeed, this often seems to be the case in economics. As long as all the models to be considered are internally consistent, the best way to select among competing explanations is usually to see which one gives the best description of reality. But this is often surprisingly difficult, especially in economics.

In addition, we must recognize that a model intended to explain a phenomenon like inflation can never be considered as absolutely true; it is at best only an approximate representation of reality. We can never consider all the factors that influence such a complex phenomenon. If we tried to do so, we would obtain a hopelessly complicated theory. This is true not only for models of physical phenomena, but for all models within the empirical sciences.

These comments are particularly relevant for economic research. Consider once again the effects of allowing new housing to be built. In order to understand the full implications of this, an economist would require an incredible amount of data on millions of consumers, businesses, goods and services, etc. Even if it were available in this kind of detail, the amount of data would swamp the capacities of even the most modern computers. In their attempts to understand the underlying relationships in the economy, economists are therefore forced to use various kinds of aggregate data, among other simplifications. Thus, we must always remember that a model is only able to give an approximate description of reality; the goal of empirical researchers should be to make their models reflect reality as closely and accurately as possible.

1.3 The Use of Symbols in Mathematics

Before beginning to study any subject, it is important that everyone agrees on a common "language" in which to discuss it. Similarly, in the study of mathematics, which is in a sense a "language" of its own, it is important to ensure that we all understand exactly the same thing when we see a given symbol. Some symbols in mathematics nearly always signify the same definite mathematical object. Examples are 3, $\sqrt{2}$, π, and $[0, 1]$, which respectively signify three special numbers and a closed interval. Symbols of this type are called *logical constants*. We also frequently need symbols that can represent **variables.** The objects that a variable is meant to represent are said to make up the **domain of variation**. For example, we use the letter x as a symbol for an arbitrary number when we write

$$x^2 - 16 = (x + 4)(x - 4)$$

In words the expression reads as follows:

> The difference between the square of the number (hereby called x) and 16 is always equal to the product of the two numbers obtained by adding 4 to the number and subtracting 4 from the number x.

The equality $x^2 - 16 = (x+4)(x-4)$ is called an *identity* because it is valid identically for all x. In such cases, we sometimes write $x^2 - 16 \equiv (x+4)(x-4)$, where $\equiv$ is the symbol for an identity.

The equality sign (=) is also used in other ways. For example, we write $A = \pi r^2$ as the formula for the area A of a circle with radius r. In addition, the equality sign is used in equations such as

$$x^2 + x - 12 = 0$$

where x stands as a symbol for the unknown number. If we substitute various numbers for x, we discover that the equality sign is often invalid. In fact, the equation is only true for $x = 3$ and for $x = -4$, and these numbers are therefore called its *solutions*.

Example 1.1

A farmer has 1000 meters of fence wire with which to enclose a rectangle. If one side of the rectangle is x (measured in meters), find the area enclosed when x is chosen to be 150, 250, 350, and for general x. Which value of x do you believe gives the greatest possible area?

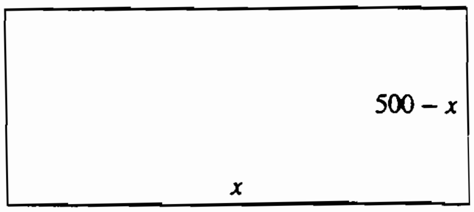
Solution If the other side of the rectangle is y, then $2x + 2y = 1000$. Hence, $x + y = 500$, so that $y = 500 - x$. (See Fig. 1.1.) The area A (in m^2) of this rectangle is, therefore,

$$A = x(500 - x) = 500x - x^2$$

Because both sides must be positive, x must be positive and $500 - x$ must be positive. This means that x must be between 0 and 500 m. The areas when $x = 150$, 250, and 350 are $150 \cdot 350 = 52{,}500$, $250 \cdot 250 = 62{,}500$, and $350 \cdot 150 = 52{,}500$, respectively. Of these, $x = 250$ gives the greatest value. In Problem 7 of Section 3.1 you will be asked to show that $x = 250$ really does give the greatest possible area.

When studying problems requiring several (but not too many) variables, we usually denote these with different letters such as a, b, c, x, y, z, A, B, and so on. Often, we supplement the letters of the Latin alphabet with lowercase and capital

FIGURE 1.1



Greek letters such as α, β, γ, Γ, and Ω. If the number of variables becomes large, we can use subscripts or superscripts to distinguish variables from each other. For example, suppose that we are studying employment in a country that is divided into 100 regions, numbered from 1 to 100. We can then denote employment in region 1 by N_1, employment in region 2 by N_2, and so on. In general, we can define

$$N_i = \text{total employment in region } i, \qquad i = 1, 2, \ldots, 100$$

The suffixes $i = 1, 2, \ldots, 100$ suggest that the index i can be an arbitrary number in the range from 1 to 100. If $N_{59} = 2690$, this means that 2690 people are employed in region 59. If we want to go further and divide the employed into men and women, we could denote the number of women (men) employed in region i by $N_i^{(W)}$ ($N_i^{(M)}$). Then, we would have $N_i^{(W)} + N_i^{(M)} = N_i$, for $i = 1, 2, \ldots, 100$. Note that this notation is actually much clearer than if we were to use 100 different letters to represent the variables N_i—even if we could find 100 different letters from some combination of the Latin, Greek, Cyrillic, and Sanskrit alphabets!

Many students who are used to dealing with algebraic expressions involving only *one* variable (usually x) have difficulties at first in handling expressions involving several variables. For economists, however, the previous example shows how important it is to be able to handle algebraic expressions and equations with many different variables. Here is another example.

Example 1.2

Consider the simple macroeconomic model

$$Y = C + \bar{I}, \qquad C = a + bY \tag{1}$$

where Y is the net national product, C is consumption, and $\bar{I}$ is the total investment, which is treated as fixed.[3] The three letters, $\bar{I}$, a, and b, denote positive numerical constants—for example, $\bar{I} = 100$, $a = 500$, and $b = 0.8$ are possible values of these constants. Rather than thinking of the two models with $\bar{I} = 100$, $C = 500 + 0.8Y$ and with $\bar{I} = 150$, $C = 600 + 0.9Y$ as entirely different, however, it is often more sensible to regard them as two particular instances of the general model [1], where $\bar{I}$, a, and b are unknown, and can vary; they are often called **parameters**. But they should be distinguished from the **variables** C and Y of the model.

After this discussion of constants as parameters of the model, solve [1] for Y.

Solution Substituting $C = a + bY$ from the second equation of [1] for C into the first equation gives

$$Y = a + bY + \bar{I}$$

[3]In economics, we often use a bar over a symbol to indicate that it is fixed.

Now rearrange this equation so that all the terms containing Y are on the left-hand side. This can be done by adding $-bY$ to both sides, thus canceling the bY term on the right-hand side to give

$$Y - bY = a + \bar{I}$$

Notice that the left-hand side is equal to $(1-b)Y$, so $(1-b)Y = a + \bar{I}$. Dividing both sides by $1-b$, so that the coefficient of Y becomes 1, then gives the answer, which is

$$Y = \frac{a}{1-b} + \frac{1}{1-b}\bar{I}$$

This solution is a formula expressing Y in terms of the three parameters $\bar{I}$, a, and b. The formula can be applied to particular values of the constants, such as $\bar{I} = 100$, $a = 500$, $b = 0.8$, to give the right answer in every case. Note the power of this approach: The model is solved only once, and then numerical answers are found simply by substituting appropriate numerical values for the parameters of the model.

Problems

1. **a.** A person buys x_1, x_2, and x_3 units of three goods whose prices per unit are, respectively, p_1, p_2, and p_3. What is the total expenditure?

b. A rental car costs F dollars per day in fixed charges and b dollars per kilometer. How much must a customer pay to drive x kilometers in 1 day?

c. A company has fixed costs of F dollars per year and variable costs of c dollars per unit produced. Find an expression for the total cost per unit (total average cost) incurred by the company if it produces x units in one year.

d. A person has an annual salary of \$$L$ and then receives a raise of $p\%$ followed by a further increase of $q\%$. What is the person's new yearly salary?

e. A square tin plate 18 cm wide is to be made into an open box by cutting out equally sized squares of width x in each corner and then folding over the edges. Find the volume of the resulting box. (Draw a figure.)

2. **a.** Prove that

$$a + \frac{a \cdot p}{100} - \frac{\left(a + \frac{a \cdot p}{100}\right) \cdot p}{100}$$

can be written as

$$a\left[1 - \left(\frac{p}{100}\right)^2\right]$$

b. An item initially costs \$2000 and then its price is increased by 5%. Afterwards the price is lowered by 5%. What is the final price?

c. An item initially costs a dollars and then its price is increased by $p\%$. Afterwards the (new) price is lowered by $p\%$. What is the final price of the item? (After considering this problem, look at the expression in part (a).)

d. What is the result if one first *lowers* a price by $p\%$ and then *increases* it by $p\%$?

3. Solve the following equations for the variables specified:

a. $x = \frac{2}{3}(y-3) + y$ for y **b.** $ax - b = cx + d$ for x

c. $AK\sqrt{L} = Y_0$ for L **d.** $px + qy = m$ for y

e. $\dfrac{\dfrac{1}{1+r} - a}{\dfrac{1}{1+r} + b} = c$ for r **f.** $Y = a(Y - tY - k) + b + I_p + G$ for Y

4. The relationship between a temperature measured in degrees Celsius (or Centigrade) (C) and in Fahrenheit (F) is given by $C = \frac{5}{9}(F - 32)$.

a. Find C when F is 32; find F when $C = 100$.

b. Find a general expression for F in terms of C.

c. One day the temperature in Oslo was $40°F$, while in Los Angeles it was $80°F$. How would you respond to the assertion that it was twice as warm in Los Angeles as in Oslo? (*Hint:* Find the two temperatures in degrees Celsius.)

5. If a rope could be wrapped around the earth's surface at the equator, it would be approximately circular and about 40 million meters long. Suppose we wanted to extend the rope to make it 1 meter above the equator at every point. How many more meters of rope would be needed? (Guess first, and then find the answer by precise calculation. For the formula for the circumference of the circle, see Appendix D.)

Harder Problems

6. Solve the following pair of simultaneous equations for x and y:

$$px + (1-q)y = R \qquad \text{and} \qquad qx + (1-p)y = S$$

7. Consider an equilateral triangle, and let P be an arbitrary point within the triangle. Let h_1, h_2, and h_3 be the shortest distances from P to each of the three sides. Show that the sum $h_1 + h_2 + h_3$ is independent of where point P is placed in the triangle. (*Hint:* Compute the area of the triangle as the sum of three triangles.)

1.4 The Real Number System

God created the integers;
everything else is the work of man.
—L. Kronecker

Real numbers were originally developed in order to measure physical characteristics such as length, temperature, and time. Economists also use real numbers to measure prices, quantities, incomes, tax rates, interest rates, and average costs, among other things. We assume that you have some knowledge of the real number system, but because of its fundamental role, we shall restate its basic properties.

Natural Numbers, Integers, and Rational Numbers

The everyday numbers we use for counting are 1, 2, 3, ... These are called **natural numbers.** Though familiar, such numbers are in reality rather abstract and advanced concepts. Civilization crossed a significant threshold when it grasped the idea that a flock of four sheep and a collection of four stones have something in common, namely "fourness." This idea came to be represented by symbols such as the primitive :: (still used on dominoes or playing cards), the modern 4, and the Roman numeral IV. This notion is grasped again and again as young children develop their mathematical skills.

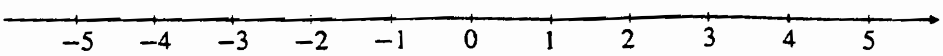
During the early stages of many cultures, day-to-day problems motivated the four basic arithmetic operations of addition, subtraction, multiplication, and division. If we add or multiply two natural numbers, we always obtain another natural number. Moreover, the operations of subtraction and division suggest the desirability of having a number zero $(4-4=0)$, negative numbers $(3-5=-2)$, and fractions $(3 \div 5 = 3/5)$. The numbers $0, \pm 1, \pm 2, \pm 3, \ldots$ are called the **integers**. They can be represented on a **number line** like the one shown in Fig. 1.2.

The **rational numbers** are those like 3/5, that can be written in the form a/b, where a and b are both integers. An integer n is also a rational number, because $n = n/1$. Examples of rational numbers are

$$\frac{1}{2}, \quad \frac{11}{70}, \quad \frac{125}{7}, \quad -\frac{10}{11}, \quad 0 = \frac{0}{1}, \quad -19, \quad -1.26 = -\frac{126}{100}$$

The rational numbers can also be represented on the number line. Imagine that we first mark $1/2$ and all the multiples of $1/2$. Then we mark $1/3$ and all the

FIGURE 1.2 The number line.



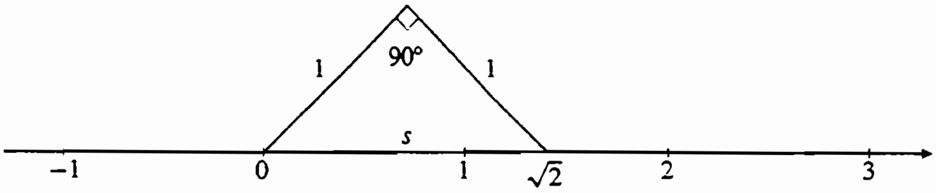


FIGURE 1.3

multiples of 1/3, and so forth. You can be excused for thinking that "finally" there will be no more places left for putting more points on the line. But in fact this is quite wrong. The ancient Greeks already understood that "holes" would remain in the number line even after all the rational numbers had been marked off. This is demonstrated in the construction in Fig. 1.3.

Pythagoras' theorem tells us that $s^2 = 1^2 + 1^2 = 2$, so $s = \sqrt{2}$. It can be shown, however, that there are no integers p and q such that $\sqrt{2} = p/q$. Hence, $\sqrt{2}$ is not a rational number. (Euclid proved this fact in about 300 B.C. See Problem 3 in Section 1.6.)

The rational numbers are therefore insufficient for measuring all possible lengths, let alone areas and volumes. This deficiency can be remedied by extending the concept of numbers to allow for the so-called **irrational numbers**. This extension can be carried out rather naturally by using decimal notation for numbers.

The Decimal System

The way most people write numbers today is called the **decimal system**, or the **base 10 system**. It is a positional system with 10 as the base number. Every natural number can be written using only the symbols, 0, 1, 2, ..., 9, that are called **digits**. You will note that "digit" also means "finger," or "thumb," and that most humans have 10 digits. The positional system defines each combination of digits as a sum of exponents of 10. For example,

$$1984 = 1 \cdot 10^3 + 9 \cdot 10^2 + 8 \cdot 10^1 + 4 \cdot 10^0$$

Each natural number can be uniquely expressed in this manner. With the use of the signs + and −, all integers, positive or negative, can be written in the same way. Decimal points also enable us to express rational numbers other than natural numbers. For example,

$$3.1415 = 3 + 1/10^1 + 4/10^2 + 1/10^3 + 5/10^4$$

Rational numbers that can be written exactly using only a finite number of decimal places are called **finite decimal fractions**.

Each finite decimal fraction is a rational number, but not every rational number can be written as a finite decimal fraction. We also need to allow for **infinite**

decimal fractions such as

$$100/3 = 33.333\ldots$$

where the three dots indicate that the decimal 3 recurs indefinitely.

If the decimal fraction is a rational number, then it will always be **periodic**—that is, after a certain place in the decimal expansion, it either stops or continues to repeat a finite sequence of digits. For example, $11/70 = 0.1\underbrace{571428}\underbrace{571428}5\ldots$.

Real Numbers

The definition of a real number follows from the previous discussion. We define a **real number** as an arbitrary infinite decimal fraction. Hence, a real number is of the form $x = \pm m.\alpha_1\alpha_2\alpha_3\ldots$, where m is an integer, and α_n $(n = 1, 2\ldots)$ is an infinite series of digits, each in the range 0 to 9. We have already identified the periodic decimal fractions with the rational numbers. In addition, there are infinitely many new numbers given by the nonperiodic decimal fractions. These are called **irrational numbers.** Examples include $\sqrt{2}$, $-\sqrt{5}$, π, $2^{\sqrt{2}}$, and $0.121121112111112\ldots$.

It turns out that, in general, it is very difficult to decide whether a given number is rational or irrational. It has been known since the year 1776 that π is irrational and since 1927 that $2^{\sqrt{2}}$ is irrational. However, we still do not know as of 1993 whether $2^{\sqrt{2}} + 3^{\sqrt{3}}$ is irrational or not. One might gain the impression that there are relatively few irrational numbers. In fact, there are (in a certain precise sense) infinitely more irrational numbers than there are rational numbers.

We mentioned earlier that each rational number can be represented by a point on the number line. But not all points on the number line represent rational numbers. It is as if the irrational numbers "close up" the remaining holes on the number line after all the rational numbers have been positioned. Hence, an unbroken and endless straight line with an origin and a positive unit of length is a satisfactory model for the real numbers. We frequently state that there is a *one-to-one correspondence* between the real numbers and the points on a number line.

The rational and irrational numbers are said to be "dense" on the number line. This means that between any two different real numbers, irrespective of how close they are to each other, we can always find both a rational and an irrational number—in fact, we can always find infinitely many of each.

When applied to the real numbers, the four basic arithmetic operations always result in a real number. The only exception is that we cannot divide by 0.

> $\frac{a}{0}$ is not defined for any real number a

This is very important and should not be confused with $0/a = 0$, for all $a \neq 0$. Notice especially that 0/0 is not defined as any real number. For example, if a car requires 60 liters of fuel to go 600 kilometers, then its fuel consumption is 60/600 = 10 liters per 100 kilometers. However, if told that a car uses 0 liters of fuel to go 0 kilometers, we know nothing about its fuel consumption; 0/0 is completely undefined.

Inequalities

In mathematics and especially in economics, inequalities are encountered almost as often as equalities. It is important, therefore, to know and understand the rules for carring out calculations involving inequalities. These are presented in Section A.7 in Appendix A. The following example is of interest in statistics.

Example 1.3

Show that if $a \geq 0$ and $b \geq 0$, then

$$\sqrt{ab} \leq \frac{a+b}{2} \tag{1.1}$$

Solution (You should first test this inequality by choosing some specific numbers, using a calculator if you wish.) To show the given inequality, it is enough to verify that $ab \leq (a+b)^2/4$ because then the square root of the left-hand side cannot exceed the square root of the right-hand side—that is, $\sqrt{ab} \leq \frac{1}{2}(a+b)$. To verify this, it is enough to check that the right-hand side minus the left-hand side is nonnegative. But indeed

$$\frac{(a+b)^2}{4} - ab = \frac{a^2+2ab+b^2-4ab}{4} = \frac{a^2-2ab+b^2}{4} = \frac{(a-b)^2}{4} \geq 0$$

In fact, essentially the same proof can be used to show that $\sqrt{ab} < \frac{1}{2}(a+b)$ unless $a = b$.

The number $\frac{1}{2}(a+b)$ is called the **arithmetic mean** of a and b, and $\sqrt{ab}$ is called the **geometric mean**. What does the inequality in [1.1] state about the different means?

Intervals

If a and b are two numbers on the number line, then we call the set of all numbers that lie between a and b an **interval**. In many situations, it is important to distinguish between the intervals that include their endpoints and the intervals that do not. When $a < b$, there are four different intervals that all have a and b as endpoints, as shown in Table 1.1. Note that the names in the table do not distinguish $[a, b)$ from $(a, b]$. To do so, one could speak of "closed on the left," "open on the right," and so on. Note, too, that an open interval includes neither of its endpoints, but a closed interval includes both of its endpoints. All four intervals, however, have the same length, $b - a$.

We usually illustrate intervals on the number line as in Fig. 1.4, with included endpoints represented by dots, and excluded endpoints at the tips of arrows. The intervals mentioned so far are all *bounded intervals.* We also use the word "interval" to signify certain unbounded sets of numbers. For example, we have

$$[a, \infty) = \text{all numbers } x, \text{ with } x \geq a$$

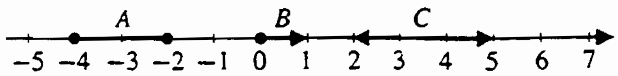
$$(-\infty, b) = \text{all numbers } x, \text{ with } x < b$$

with ∞ as the common symbol for infinity. Note that the symbol ∞ is not a number at all, and therefore the usual arithmetic rules do not apply to it. In $[a, \infty)$, the symbol ∞ is only a handy notation indicating that we are are considering the collection of *all* numbers larger than or equal to a, without any upper limit to the size of the number. From the preceding, it should be readily apparent what we mean

TABLE 1.1

Notation	*Name*	*The interval consists of all x satisfying:*
(a,b)	The **open** interval from a to b.	$a < x < b$
$[a,b]$	The **closed** interval from a to b.	$a \leq x \leq b$
$(a,b]$	The **half-open** interval from a to b.	$a < x \leq b$
$[a,b)$	The **half-open** interval from a to b.	$a \leq x < b$

FIGURE 1.4 $A = [-4, -2]$, $B = [0, 1)$, and $C = (2, 5)$.



by (a, ∞) and $(-\infty, b]$. The collection of all real numbers is sometimes denoted by the symbol $(-\infty, \infty)$.

Absolute Value

Let a be a real number and imagine its position on the number line. The distance between a and 0 is called the **absolute value** of a. If a is positive or 0, then the absolute value is the number a itself; if a is negative, then because distance must be positive, the absolute value is equal to the positive number $-a$.

The **absolute value** of a is denoted by $|a|$, and

$$|a| = \begin{cases} a, & \text{if } a \geq 0 \\ -a, & \text{if } a < 0 \end{cases} \qquad [1.2]$$

For example, $|13| = 13$, $|-5| = -(-5) = 5$, $|-1/2| = 1/2$, and $|0| = 0$.

Note: It is a common fallacy to assume that a must denote a positive number, even if this is not explicitly stated. Similarly, on seeing $-a$, many students are led to believe that this expression is always negative. Observe, however, that the number $-a$ is positive when a itself is negative. For example, if $a = -5$, then $-a = -(-5) = 5$. Nevertheless, it is often a useful convention in economics to define variables so that, as far as possible, their values are positive rather than negative. Where a variable has a definite sign, we shall try to follow this convention.

Example 1.4

(a) Compute $|x - 2|$ for $x = -3$, $x = 0$, and $x = 4$.
(b) Rewrite $|x - 2|$ using (1.2).

Solution

(a) For $x = -3$,

$$|x - 2| = |-3 - 2| = |-5| = 5$$

For $x = 0$,

$$|x - 2| = |0 - 2| = |-2| = 2$$

For $x = 4$,

$$|x - 2| = |4 - 2| = |2| = 2$$

(b) According to [1.2], $|x - 2| = x - 2$ if $x - 2 \geq 0$, that is, $x \geq 2$. However, $|x - 2| = -(x - 2) = 2 - x$ if $x - 2 < 0$, that is, $x < 2$.

Hence,

$$|x-2| = \begin{cases} x-2, & \text{if } x \geq 2 \\ 2-x, & \text{if } x < 2 \end{cases}$$

(Check this answer by trying the values of x tested in part (a).)

Let x_1 and x_2 be two arbitrary numbers. The **distance** between x_1 and x_2 on the number line is equal to $x_1 - x_2$ if $x_1 \geq x_2$, and equal to $-(x_1 - x_2)$ if $x_1 < x_2$. Therefore, we have

$$|x_1 - x_2| = \text{distance between } x_1 \text{ and } x_2 \text{ on the number line} \qquad [1.3]$$

In Fig. 1.5, we have indicated geometrically that the distance between 7 and 2 is 5, whereas the distance between -3 and -5 is equal to 2, because $|-3-(-5)| = |-3+5| = |2| = 2$.

Suppose $|x| = 5$. What values can x have? There are only two possibilities: either $x = 5$ or $x = -5$, because no other numbers have absolute values equal to 5. Generally, if a is greater than or equal to 0, then $|x| = a$ means that $x = a$ or $x = -a$. Because $|x| \geq 0$ for all x, the equation $|x| = a$ has no solution when $a < 0$.

If a is a positive number and $|x| < a$, then the distance from x to 0 is less than a, and so

$$|x| < a \text{ means that } -a < x < a \qquad [1.4]$$

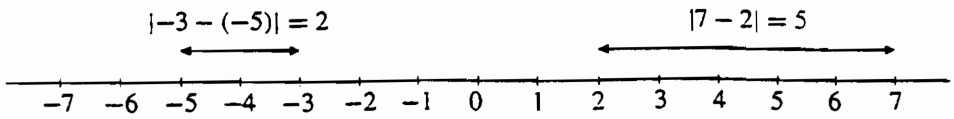
Furthermore, when a is nonnegative; it is clear that

$$|x| \leq a \text{ means that } -a \leq x \leq a \qquad [1.5]$$

Example 1.5

Find all the x such that $|3x - 2| \leq 5$. Check first to see if this inequality is fulfilled for $x = -3$, $x = 0$, $x = 7/3$, and $x = 10$.

FIGURE 1.5 The distance between 7 and 2, and between −3 and −5.



Solution For $x = -3$, $|3x - 2| = |-9 - 2| = 11$; for $x = 0$, we have $|3x - 2| = |-2| = 2$; for $x = 7/3$, $|3x - 2| = |7 - 2| = 5$; and for $x = 10$, $|3x - 2| = |30 - 2| = 28$. Hence, we see that the given inequality is satisfied for $x = 0$ and $x = 7/3$, but not for $x = -3$ or $x = 10$.

From [1.5] we see that $|3x - 2| \le 5$ means $-5 \le 3x - 2 \le 5$. Adding 2 to all three expressions gives $-5 + 2 \le 3x - 2 + 2 \le 5 + 2$, or $-3 \le 3x \le 7$. Dividing by 3 gives $-1 \le x \le 7/3$.

Problems

1. Which of the following numbers is a natural number, an integer, or a rational number?

a. 3.1415926 **b.** $\sqrt{\frac{9}{2} - \frac{1}{2}}$ **c.** $(\sqrt{3} + \sqrt{2})(\sqrt{3} - \sqrt{2})$ **d.** $3\pi - \frac{1}{4}$

2. Which of the following statements are correct?

a. 1984 is a natural number.
b. -5 is to the right of -3 on the number line.
c. -13 is a natural number.
d. There is no natural number that is not rational.
e. 3.1415 is not rational.
f. The sum of two irrational numbers is irrational.

3. For what real numbers x is each of the following expressions defined?

a. $\dfrac{3}{x - 4}$ **b.** $\dfrac{x - 1}{x(x + 2)}$ **c.** $\dfrac{3x}{x^2 + 4x - 5}$ **d.** $\dfrac{1/4}{x^2 + 4x + 4}$

4. Solve the following inequalities for y in terms of the other variables:

a. $3x + 4y \le 12$ **b.** $-x + 3y - z > y - (x - y) + \frac{1}{2}z$

c. $px + qy \le m \quad (q > 0)$

5. Consider Problem 1(c) in Section 1.3. Set up an inequality that determines how many units x the company must produce before the average cost falls below \$$q$. Solve the inequality for x. Put $F = 100{,}000$, $c = 120$, $q = 160$, and solve the problem for this case.

6. Calculate $|2x - 3|$, for $x = 0$, $1/2$, and $7/2$.

7. **a.** Calculate $|5 - 3x|$, for $x = -1, 2$, and 4.
b. Solve the equation $|5 - 3x| = 0$.
c. Rewrite $|5 - 3x|$ by using [1.2].

8. Determine x such that

a. $|3 - 2x| = 5$ **b.** $|x| \le 2$ **c.** $|x - 2| \le 1$

d. $|3 - 8x| \le 5$ **e.** $|x| > \sqrt{2}$ **f.** $|x^2 - 2| \le 1$

9. A 5-meter iron bar is to be produced. It is necessary that the length does not deviate more than 1 mm from its stated size. Write a specification for the rod's length x in meters: (a) by using a double inequality and (b) with the aid of an absolute-value sign.

1.5 A Few Aspects of Logic

An astronomer, a physicist, and a mathematician were travelling on a train in Scotland. Through the window they saw a flock of sheep grazing in a meadow.
The astronomer remarked, "In Scotland all sheep are black."
The physicist protested, "Some Scottish sheep are black."
The mathematician declared, "In Scotland there exists a flock of sheep all of which are black on at least one side."

So far we have emphasized the role of mathematical models in the empirical sciences, especially in economics. The more complicated the phenomena to be described, the more important it is to be exact. Errors in models applied to practical situations can have catastrophic consequences. For example, in the early stages of the U.S. space program, a rocket costing millions of dollars to develop and build had to be destroyed only seconds after launch because a semicolon had been left out of the computer program intended to control the guidance system.

Although the consequences may be less dramatic, errors in mathematical reasoning also occur rather easily. In what follows, we offer a typical example of how a student (or professor) might use faulty logic and thus end up with an incorrect answer to a problem.

Example 1.6

Find a possible solution for the equation $x + 2 = \sqrt{4 - x}$.

"Solution" Squaring each side of the equation gives $(x + 2)^2 = (\sqrt{4-x})^2$, and thus $x^2 + 4x + 4 = 4 - x$. Rearranging this last equation gives $x^2 + 5x = 0$. Canceling x results in $x + 5 = 0$, and therefore $x = -5$.

According to this reasoning, the answer should be $x = -5$. Let us check this. For $x = -5$, we have $x + 2 = -3$. Yet $\sqrt{4-x} = \sqrt{9} = 3$, so this answer is incorrect. In Example 1.9, we explain how the error arose. (Note the wisdom of checking your answer whenever you think you have solved an equation.)

This example highlights the dangers of routine calculation without adequate thought. It may be easier to avoid similar mistakes after studying more closely the structure of logical reasoning.

Propositions

Assertions that are either true or false are called statements, or **propositions.** Most of the propositions in this book are mathematical ones, but others may arise in daily life. "All individuals who breathe are alive" is an example of a true proposition, whereas the assertion "all individuals who breathe are healthy" is an example of a false proposition. It should be noted that if the words used to express such assertions lack a precise meaning, it will often be difficult to distinguish between a true and a false proposition.

Suppose an assertion such as "$x^2 - 1 = 0$" includes one or more variables. By substituting various real numbers for the variable x, we can generate many different propositions, some true and some false. For this reason we say that the assertion is an **open proposition.** In fact, the proposition $x^2 - 1 = 0$ happens to be true if $x = 1$ or -1, but not otherwise. Thus, an open proposition is not simply true or false. It is neither true nor false until we choose a particular value for the variable. In practice we are somewhat careless about this distinction between propositions and open propositions; instead, we simply call both types propositions.

Implications

In order to keep track of each step in a chain of logical reasoning, it often helps to use implication arrows.

Suppose P and Q are two propositions such that whenever P is true, then Q is necessarily true. In this case, we usually write

$$P \implies Q \qquad [*]$$

This is read as "P implies Q," or "if P, then Q," or "Q is a consequence of P." The symbol $\implies$ is an **implication arrow,** and it points in the direction of the logical implication. Here are some examples of correct implications.

Example 1.7

(a) $x > 2 \implies x^2 > 4$.

(b) $xy = 0 \implies x = 0$ or $y = 0$.

(c) x is a square $\implies$ x is a rectangle.

(d) x is a healthy person $\implies$ x is breathing.

Notice that the word "or" in mathematics means the "inclusive or," signifying that "P or Q" means "either P or Q or both."

All the propositions in Example 1.7 are open propositions, just as are most propositions encountered in mathematics. An implication $P \implies Q$ means that for each value of some variable for which P is true, Q is also true.

In certain cases where the implication [*] is valid, it may also be possible to draw a logical conclusion in the other direction:

$$Q \implies P$$

In such cases, we can write both implications together in a single **logical equivalence**:

$$P \iff Q$$

We then say that "P is equivalent to Q," or "P if and only if Q," or just "P iff Q." Note that the statement "P only if Q" expresses the implication $P \Longrightarrow Q$, whereas "P if Q" expresses the implication $Q \Longrightarrow P$.

The symbol $\iff$ is an **equivalence arrow.** In previous Example 1.7, we see that the implication arrow in (b) could be replaced with the equivalence arrow, because it is also true that $x = 0$ or $y = 0$ implies $xy = 0$. Note, however, that no other implication in Example 1.7 can be replaced by the equivalence arrow. For even if x^2 is larger than 4, it is not necessarily true that x is larger than 2 (for instance, x might be -3); also, a rectangle is not necessarily a square; and, finally, just because person x is breathing does not mean that he or she is healthy.

Necessary and Sufficient Conditions

There are other commonly used ways of expressing that proposition P implies proposition Q, or that P is equivalent to Q. Thus, if proposition P implies proposition Q, we state that P is a "sufficient condition" for Q. After all, for Q to be true, it is sufficient that P is true. Accordingly, we know that if P is satisfied, then it is certain that Q is also satisfied. In this case, we say that Q is a "necessary condition" for P. For Q must necessarily be true if P is true. Hence,

> P is a **sufficient condition** for Q means: $P \Longrightarrow Q$
>
> Q is a **necessary condition** for P means: $P \Longrightarrow Q$

For example, if we formulate the implication in Example 1.7(c) in this way, it would read:

A necessary condition for x to be a square is that x be a rectangle.

or

A sufficient condition for x to be a rectangle is that x be a square.

The corresponding verbal expression for $P \iff Q$ is simply: *P is a necessary and sufficient condition for Q, or P if and only if Q, or P iff Q.* It is evident from this that it is very important to distinguish between the propositions "P is a necessary condition for Q" (meaning $Q \Longrightarrow P$) and "P is a sufficient condition for Q" (meaning $P \Longrightarrow Q$). To emphasize the point, consider two propositions:

1. Breathing is a necessary condition for a person to be healthy.
2. Breathing is a sufficient condition for a person to be healthy.

Evidently proposition 1 is true. But proposition 2 is false, because sick (living) people are still breathing. In the following pages, we shall repeatedly refer to necessary and sufficient conditions. Understanding them and the difference between them is a necessary condition for understanding much economic analysis. It is not a sufficient condition, alas!

Solving Equations

We shall now give examples showing how using implication and equivalence arrows can help avoid mistakes in solving equations like that in Example 1.6.

Example 1.8

Find all x such that $(2x-1)^2 - 3x^2 = 2\left(\frac{1}{2} - 4x\right)$.

Solution By expanding $(2x-1)^2$ and also multiplying out the right-hand side, we obtain a new equation that obviously has the same solutions as the original one:

$$(2x-1)^2 - 3x^2 = 2\left(\tfrac{1}{2} - 4x\right) \iff 4x^2 - 4x + 1 - 3x^2 = 1 - 8x$$

Adding $8x - 1$ to each side of the second equality and then gathering terms gives the equivalent expression

$$4x^2 - 4x + 1 - 3x^2 = 1 - 8x \iff x^2 + 4x = 0$$

Now $x^2 + 4x = x(x+4)$, and the latter expression is 0 if and only if $x = 0$ or $x + 4 = 0$. That is,

$$\begin{aligned} x^2 + 4x = 0 \iff x(x+4) = 0 &\iff x = 0 \quad \text{or} \quad x + 4 = 0 \\ &\iff x = 0 \quad \text{or} \quad x = -4 \end{aligned}$$

Putting everything together, we have derived a chain of equivalence arrows showing that the given equation is fulfilled for the two values $x = 0$ and $x = -4$, and for no other values of x. That is,

$$(2x-1)^2 - 3x^2 = 2(\tfrac{1}{2} - 4x) \iff x = 0 \quad \text{or} \quad x = -4$$

Example 1.9

Find all x such that $x + 2 = \sqrt{4 - x}$. (Recall Example 1.6.)

Solution Squaring both sides of the given equation yields

$$(x + 2)^2 = \left(\sqrt{4 - x}\right)^2$$

Consequently, $x^2 + 4x + 4 = 4 - x$, that is, $x^2 + 5x = 0$. From the latter equation it follows that

$$x(x + 5) = 0$$

which implies $x = 0$ or $x = -5$. Thus, a necessary condition for x to solve $x+2 = \sqrt{4 - x}$ is that $x = 0$ or $x = -5$. Inserting these two possible values of x into the original equation shows that only $x = 0$ satisfies the equation. The unique solution to the equation is, therefore, $x = 0$.

In finding the solution to Example 1.9, why was it necessary to test whether the values we found were actually solutions, whereas this step was unnecessary in Example 1.8? To answer this, we must analyze the logical structure of our solution to Example 1.9. With the aid of numbered implication and equivalence arrows, we can express the previous solution as

$$x + 2 = \sqrt{4 - x} \overset{(1)}{\Longrightarrow} (x + 2)^2 = 4 - x \overset{(2)}{\Longrightarrow} x^2 + 4x + 4 = 4 - x$$

$$\overset{(3)}{\Longrightarrow} x^2 + 5x = 0 \overset{(4)}{\Longrightarrow} x(x + 5) = 0 \overset{(5)}{\Longrightarrow} x = 0 \text{ or } x = -5$$

Implication (1) is true (because $a = b \Longrightarrow a^2 = b^2$ and $\left(\sqrt{a}\right)^2 = a$). *It is important to note, however, that the implication cannot be replaced by an equivalence.* If $a^2 = b^2$, then either $a = b$ or $a = -b$; it need not be true that $a = b$. Implications (2), (3), (4), and (5) are also all true; moreover, all could have been written as equivalences, though this is not necessary in order to find the solution. Therefore, a chain of implications has been obtained that leads from the equation $x + 2 = \sqrt{4 - x}$ to the proposition "$x = 0$ or $x = -5$." Because the implication (1) cannot be reversed, there is no corresponding chain of implications going in the opposite direction. We have verified that if the number x satisfies $x+2 = \sqrt{4 - x}$, then x must be either 0 or -5; no other value can satisfy the given equation. However, we have not yet shown that either 0 or -5 really satisfies the equation. Until we try inserting 0 and -5 into the equation, we cannot see that only $x = 0$ is a solution. *Note that in this case, the test we have suggested not only serves to check our calculations, but is also a logical necessity.*

Looking back at Example 1.6, we now realize that two errors were committed. Firstly, the implication $x^2 + 5x = 0 \Rightarrow x + 5 = 0$ is wrong, because $x = 0$ is also a solution of $x^2 + 5x = 0$. Secondly, it is logically necessary to check if 0 or -5 really satisfies the equation.

The method used to solve Example 1.9 is the most common. It involves setting up a chain of implications that starts from the given equation and ends with a set of its possible solutions. By testing each of these trial solutions in turn, we find which of them really do satisfy the equation. Even if the chain of implications is also a chain of equivalences (as it was in Example 1.8), such a test is always a useful check of both logic and calculations.

Problems

1. Implications and equivalences can be expressed in ways that differ from those already mentioned. Use the implication or equivalence arrows to mark in which direction you believe the logical conclusions proceed in the following propositions:
   a. The equation $2x - 4 = 2$ is fulfilled only when $x = 3$.
   b. If $x = 3$, then $2x - 4 = 2$.
   c. The equation $x^2 - 2x + 1 = 0$ is satisfied if $x = 1$.
   d. If $x^2 > 4$, then $x > 2$ or $x < -2$, and conversely.
2. Consider the following six implications and decide in each case: (i) if the implication is true, and (ii) if the converse implication is true. (x, y, and z are real numbers.)
   a. $x = 2$ and $y = 5 \Longrightarrow x + y = 7$
   b. $(x - 1)(x - 2)(x - 3) = 0 \Longrightarrow x = 1$
   c. $x^2 + y^2 = 0 \Longrightarrow x = 0$ or $y = 0$
   d. $x = 0$ and $y = 0 \Longrightarrow x^2 + y^2 = 0$
   e. $xy = xz \Longrightarrow y = z$
   f. $x > y^2 \Longrightarrow x > 0$
3. Consider the proposition $2x + 5 \geq 13$.
   a. Is the condition $x \geq 0$ necessary, sufficient, or both necessary and sufficient for the proposition to be satisfied?
   b. Answer the same question when $x \geq 0$ is replaced by $x \geq 50$.
   c. Answer the same question when $x \geq 0$ is replaced by $x \geq 4$.
4. Solve the equation
$$\frac{(x+1)^2}{x(x-1)} + \frac{(x-1)^2}{x(x+1)} - 2\frac{3x+1}{x^2-1} = 0$$
5. Solve the following equations:
   a. $x + 2 = \sqrt{4x + 13}$ **b.** $|x + 2| = \sqrt{4 - x}$ **c.** $x^2 - 2|x| - 3 = 0$
6. Solve the following equations:
   a. $\sqrt{x - 4} = \sqrt{x + 5} - 9$ **b.** $\sqrt{x - 4} = 9 - \sqrt{x + 5}$
7. Fill in the blank rectangles with "iff" (if and only if) when this results in a true statement, or alternatively with "if" or "only if."
   a. $x = \sqrt{4}$ ☐ $x = 2$

b. $x^2 > 0$ $\square$ $x > 0$

c. $x^2 < 9$ $\square$ $x < 3$

d. $x(x^2 + 1) = 0$ $\square$ $x = 0$

e. $x(x + 3) < 0$ $\square$ $x > -3$

8. Consider the following attempt to solve the equation $x + \sqrt{x+4} = 2$: "From the given equation, it follows that $\sqrt{x+4} = 2 - x$. Squaring both sides gives $x + 4 = 4 - 4x + x^2$. After rearranging the terms, it is seen that this equation implies $x^2 - 5x = 0$. Canceling x, we obtain $x - 5 = 0$ and this equation is satisfied when $x = 5$."
   a. Mark with arrows the implications or equivalences expressed in the text. Which ones are correct?
   b. Give a correct solution to the equation.
9. For each of the following 6 propositions, state the negation as simply as possible.
   a. $x \geq 0$ and $y \geq 0$.
   b. All x satisfy $x \geq a$.
   c. Neither x nor y is less than 5.
   d. For each $\varepsilon > 0$, there exists a $\delta > 0$ such that B is satisfied.
   e. No one can avoid liking cats.
   f. Everyone loves someone at certain times.
10. "Supreme Court refuses to hear challenge to lower court's decision approving a trial judge's refusal to allow a defendant to refuse to speak." Has the defendant the right not to speak?

1.6 Mathematical Proof

In science, what can be proved should not be believed without proof.[4]
—R. Dedekind (1887)

In every branch of mathematics, the most important results are called **theorems.** Constructing logically valid proofs for these results often can be rather complicated. For example, the "four-color theorem" states that any map in the plane needs at most four colors in order that all contiguous regions should have different colors. Proving this involved checking hundreds of thousands of different cases, a task that was impossible without a sophisticated computer program.

In this book, we often omit formal proofs of theorems. Instead, the emphasis is on providing a good intuitive grasp of what the theorems tell us. However,

[4]Here is the German original: "Was beweisbar ist, soll in der Wissenschaft nicht ohne Beweis geglaubt werden."

although proofs do not form a major part of this book, it is still useful to understand something about the different types of proof that are used in mathematics. In fact, a proof that is actually readable is likely to some extent to rely on the reader's intuition. Although many mathematical logicians do take care to present every step and every argument, and this may indeed be a necessary step in enabling computers to check a proof, the overall result is usually unreadable by most people.

Every mathematical theorem can be formulated as an implication

$$P \Longrightarrow Q \qquad [*]$$

where P represents a proposition or a series of propositions called *premises* ("what we know"), and Q represents a proposition or a series of propositions that are called the *conclusions* ("what we want to know"). A statement of the form $P \Longleftrightarrow Q$ can be regarded as two theorems.

Usually, it is most natural to prove a result of the type [*] by starting with the premises P and successively working forward to the conclusion Q; we call this a **direct proof**. Sometimes, however, it is more convenient to prove the implication $P \Longrightarrow Q$ by an **indirect proof**. In this case, we begin by supposing that Q is not true, and on that basis demonstrate that neither can P be true. This is completely legitimate, because we have the following equivalence:

$$P \Longrightarrow Q \quad \text{is equivalent to} \quad \textit{not } Q \Longrightarrow \textit{not } P \qquad [1.6]$$

It is helpful to consider how this rule of logic applies to some concrete examples:

If it is raining, the grass is getting wet

asserts precisely the same thing as

If the grass is not getting wet, then it is not raining.

If T denotes a triangle, then

The base angles of T are equal implies that T is isosceles asserts the same as *If T is not isosceles, then its base angles are not equal.*

There is a third method of proof that is also sometimes useful. It is called **proof by contradiction**. The method is based upon a fundamental logical principle: that it is impossible for a chain of valid inferences to proceed from a true proposition to a false one. Therefore, if we have a proposition R and we can derive a contradiction on the basis of supposing that R is false, then it follows that R must be true.

Example 1.10

Use three different methods to prove that

$$-x^2 + 5x - 4 > 0 \Longrightarrow x > 0$$

Solution

(a) *Direct proof:* Suppose $-x^2 + 5x - 4 > 0$. Adding $x^2 + 4$ to each side of the inequality gives $5x > x^2 + 4$. Because $x^2 + 4 \geq 4$, for all x, we have $5x > 4$, and so $x > 4/5$. In particular, $x > 0$.

(b) *Indirect proof:* Suppose $x \leq 0$. Then $5x \leq 0$ and so $-x^2 + 5x - 4$, as a sum of three nonpositive terms, is ≤ 0.

(c) *Proof by contradiction:* Suppose that the statement is not true. Then there has to exist an x such that $-x^2 + 5x - 4 > 0$ and $x \leq 0$. But if $x \leq 0$, then $-x^2 + 5x - 4 \leq -x^2 - 4 \leq -4$, and we have arrived at a contradiction.

Deductive vs. Inductive Reasoning

The three methods of proof just outlined are all examples of *deductive reasoning*, that is, reasoning based on consistent rules of logic. In contrast, many branches of science use *inductive reasoning*. This process draws general conclusions based only on a few (or even many) observations. For example, the statement that "the price level has increased every year for the last n years; therefore, it will surely increase next year too," demonstrates inductive reasoning. Owners of houses in California know how dangerous such reasoning can be in economics. This inductive approach is nevertheless of fundamental importance in the experimental and empirical sciences, despite the fact that conclusions based upon it never can be absolutely certain.

In mathematics, inductive reasoning is not recognized as a form of proof. Suppose, for instance, that the students taking a course in geometry are asked to show that the sum of the angles of a triangle is always 180 degrees. If they painstakingly measure as accurately as possible 1000 or even 1 million different triangles, demonstrating that in each case the sum of the angles is 180, would this not serve as proof for the assertion? No; although it would represent a very good indication that the proposition is true, it is not a mathematical proof. Similarly, in business economics, the fact that a particular company's profits have risen for each of the past 20 years is no guarantee that they will rise once again this year.

Nevertheless, there is a *mathematical* form of induction that is much used in valid proofs. This is discussed in Section B.5 in Appendix B.

Problems

1. Consider the following (dubious) statement: "If inflation increases, then unemployment decreases." Which of the following statements are equivalent?

a. For unemployment to decrease, inflation must increase.
b. A sufficient condition for unemployment to decrease is that inflation increases.
c. Unemployment can only decrease if inflation increases.
d. If unemployment does not decrease, then inflation does not increase.
e. A necessary condition for inflation to increase is that unemployment decreases.

2. Analyze the following epitaph: (a) using logic and (b) from a poetic viewpoint.

Those who knew him, loved him.
Those who loved him not, knew him not.

3. Fill in the details of the following proof that $\sqrt{2}$ is irrational. Suppose it were true that $\sqrt{2} = p/q$, where p and q are integers with no common factor. Then $p^2 = 2q^2$, which would mean that p^2, and hence p, would have 2 as a factor. Therefore, $p = 2s$ for some integer s, and so $4s^2 = 2q^2$. Thus, $q^2 = 2s^2$. It follows that q would also have 2 as a factor, a contradiction of the hypothesis that p and q have no common factor.

1.7 Set Theory

If you know set theory up to the hilt, and no other mathematics, you would be of no use to anybody. If you knew a lot of mathematics, but no set theory, you might achieve a great deal. But if you knew just some set theory, you would have a far better understanding of the language of mathematics.
—*I. Stewart* (1975)

In daily life, we constantly group together objects of the same kind. For instance, we refer to the university faculty to signify all the members of the academic staff at the university. A garden refers to all the plants that are growing in it. We talk about all firms with more than 1000 employees, all taxpayers in Los Angeles who earned between \$50,000 and \$100,000 in 1992, and so on. In all these cases, we have a collection of objects viewed as a whole. In mathematics, such a collection is called a **set**, and the objects are called the **elements** of, or the **members** of, the set.

How is a set specified? The simplest way is to list its members, in any order, between the two braces { and }. An example is the set

$$S = \{a, b, c\}$$

whose members are the first three letters in the alphabet of most languages of European origin, including English. Or it might be a set consisting of three members represented by the letters a, b, and c. For example, if $a = 0$, $b = 1$, and $c = 2$, then $S = \{0, 1, 2\}$. Also S denotes the set of roots of the cubic equation

$$(x - a)(x - b)(x - c) = 0$$

in the unknown x, where a, b, and c are any three real numbers.

Alternatively, suppose that you are to eat a meal at a restaurant that offers a choice of several main dishes. Four choices might be feasible—fish, pasta, omelette, and chicken. Then the ***feasible set*** F has these four members, and is fully specified as

$$F = \{\text{fish, pasta, omelette, chicken}\}$$

Notice that the order in which the dishes are listed does not matter. The feasible set remains the same even if the order of the items on the menu is changed.

Two sets A and B are considered **equal** if each element of A is an element of B and each element of B is an element of A. In this case, we write $A = B$. This means that the two sets consist of exactly the same elements. Consequently, $\{1, 2, 3\} = \{3, 2, 1\}$, because the order in which the elements are listed has no significance; and $\{1, 1, 2, 3\} = \{1, 2, 3\}$, because a set is not changed if some elements are listed more than once.

Specifying a Property

Not every set can be defined by listing all its members, however. Some sets can be infinite, that is, they contain an infinite number of members.

Actually, such infinite sets are rather common in economics. Take, for instance, the ***budget set*** that arises in consumer theory. Suppose there are two goods with quantities denoted by x and y that can be bought at prices p and q, respectively. A consumption bundle (x, y) is a pair of quantities of the two goods. Its value at prices p and q is $px + qy$. Suppose that a consumer has an amount m to spend on the two goods. Then the ***budget constraint*** is $px + qy \leq m$ (assuming that the consumer is free to underspend). If one also accepts that the quantity consumed of each good must be nonnegative, then the ***budget set***, that will be denoted by B, consists of those consumption bundles (x, y) satisfying the three inequalities $px + qy \leq m$, $x \geq 0$, and $y \geq 0$. (The set B is shown in Fig. 2.41.) Standard notation for such a set is

$$B = \{(x, y) : px + qy \leq m,\ x \geq 0,\ y \geq 0\} \qquad [1.7]$$

The braces { } are still used to denote "the set consisting of." However, instead of listing all the members, which is impossible for the infinite set of points in the triangular budget set B, the set is specified in two parts. To the left of the colon, (x, y) is used to denote the form of the typical member of B, here a consumption bundle that is specified by listing the respective quantities of the two goods. To the right of the colon, the three properties that these typical members must satisfy are all listed, and the set thereby specified. This is an example of the general specification:

$$S = \{\text{typical member : defining properties}\}$$

Note that it is not just infinite sets that can be specified by properties—finite sets can also be specified in this way. Indeed, even some finite sets almost *have* to be specified in this way, such as the set of all human beings currently alive, or even (we hope!), the set of all readers of this book.

Mathematics makes frequent use of infinite sets. For example, in Section 1.4, we studied the set of positive integers, which is often denoted by N, as well as the set of rational numbers, denoted by Q, and the set of real numbers, denoted by R. All these sets are infinite.

Set Membership

As we stated earlier, sets contain members or elements. There is some convenient standard notation that denotes the relation between a set and its members. First,

$$x \in S$$

indicates that x *is an element of* S. Note the special symbol $\in$ (which is a variant of the Greek letter ϵ, or "epsilon"). Occasionally, one sees $S \ni x$ being used to express exactly the same relationship as $x \in S$. The symbol "$\ni$" is generally read as "owns," but is not used very often. To express the fact that x is *not* a member of S, we write $x \notin S$. For example, $d \notin \{a, b, c\}$ says that d is not an element of the set $\{a, b, c\}$.

For additional illustrations of set membership notation, let us return to our earlier examples. Given the budget set B in [1.7], let (x^*, y^*) denote the consumer's actual purchases. Then it must be true that $(x^*, y^*) \in B$. Confronted with the choice from the set of feasible main courses $F = \{\text{fish, pasta, omelette, chicken}\}$, let s denote your actual selection. Then, of course, $s \in F$. This is what we mean by "feasible set"—it is possible only to choose some member of that set but nothing outside it.

In the example of choice from four main courses, it may be argued that if none is to the customer's liking, then she cannot be prevented from ordering nothing at all from the menu. She can eat somewhere else instead, or simply go hungry. If that is what she does, she is not really choosing outside her feasible set. Rather, our previous description of the feasible set should be expanded to include the option of ordering none of the four available dishes. Thus, the customer's true feasible set is

$$F_5 = \{\text{fish, pasta, omelette, chicken, none of the previous four}\}$$

In the end, she can only avoid choosing something from this by choosing more than one item. If this is not allowed, then F_5 is her true feasible set.

Subsets

Let A and B be any two sets. Then A is a **subset** of B if it is true that every member of A is also a member of B. So A is smaller than B in some sense, even though A and B could actually be equal. This relationship is expressed symbolically by $A \subset B$:

$$A \subset B \iff [x \in A \Rightarrow x \in B]$$

A special case of a subset is when A is a *proper subset* of B, meaning that $A \subset B$ and $A \neq B$.[5]

Set Operations

Sets can be combined in many different ways. Especially important are three operations: union, intersection, and the difference of sets, as shown in Table 1.2.

TABLE 1.2

Notation	*Name*	*The set consists of*
$A \cup B$	*A* **union** *B*	The elements that belong to at least one of the sets *A* and *B*.
$A \cap B$	*A* **intersection** *B*	The elements that belong to both *A* and *B*.
$A \setminus B$	*A* **minus** *B*	The elements that belong to *A*, but not to *B*.

Thus,

$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$

$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$

$$A \setminus B = \{x : x \in A \text{ and } x \notin B\}$$

Example 1.11

Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{3, 6\}$. Find $A \cup B$, $A \cap B$, $A \setminus B$, and $B \setminus A$.

Solution $A \cup B = \{1, 2, 3, 4, 5, 6\}$, $A \cap B = \{3\}$, $A \setminus B = \{1, 2, 4, 5\}$, $B \setminus A = \{6\}$.

[5] Sometimes the notation $A \subset B$ is reserved for the case when A is a subset of B satisfying $A \neq B$, just as $a < b$ is reserved for when $a \leq b$ and $a \neq b$. Then $A \subseteq B$ is used to denote that A is a subset of B. However, there is rarely any need to specify that A is a proper subset of B, and when there is, this can easily be done verbally.

An economic example can be obtained by considering particular sets of taxpayers in 1990. Let A be the set of all those taxpayers who had an income of at least \$15,000 and let B be the set of all who had a net worth of at least \$150,000. Then $A \cup B$ would be those taxpayers who earned at least \$15,000 or who had a net worth of at least \$150,000, whereas $A \cap B$ are those taxpayers who earned at least \$15,000 and who also had a net worth of at least \$150,000. Finally, $A \setminus B$ would be those who earned at least \$15,000 but who had less than \$150,000 in net worth.

If two sets A and B have no elements in common, they are said to be **disjoint**. The symbol "$\emptyset$" denotes the set that has no elements. It is called the **empty set**. Thus, sets A and B are disjoint if and only if $A \cap B = \emptyset$.

A collection of sets is often referred to as a family of sets. When considering a certain family of sets, it is usually natural to think of each set in the family as a subset of one particular fixed set Ω, hereafter called the **universal set.** In the previous example, the set of all taxpayers in 1990 would be an obvious choice for a universal set.

If A is a subset of the universal set Ω, then according to the definition of difference, $\Omega \setminus A$ is the set of elements of Ω that are not in A. This set is called the **complement** of A in Ω and is sometimes denoted by $\mathcal{C}A$, so that $\mathcal{C}A = \Omega \setminus A$. Other ways of denoting the complement of A include A^c and $\tilde{A}$.

When using the notation $\mathcal{C}A$, it is important to be clear about which universal set Ω is used to construct the complement.

Example 1.12

Let the universal set Ω be the set of all students at a particular university. Moreover, let F denote the set of female students, M the set of all mathematics students, C the set of students in the university choir, B the set of all biology students, and T the set of all tennis players. Describe the members of the following sets: $\Omega \setminus M$, $M \cup C$, $F \cap T$, $M \setminus (B \cap T)$, and $(M \setminus B) \cup (M \setminus T)$.

Solution $\Omega \setminus M$ consists of those students who are not studying mathematics, $M \cup C$ of those students who study mathematics and/or are in the university choir. The set $F \cap T$ consists of those female students who play tennis. The set $M \setminus (B \cap T)$ has those mathematics students who do not both study biology and play tennis. Finally, the last set $(M \setminus B) \cup (M \setminus T)$ has those students who either are mathematics students not studying biology or mathematics students who do not play tennis. Do you see that the last two sets are equal? (For arbitrary sets M, B, and T, it is true that $(M \setminus B) \cup (M \setminus T) = M \setminus (B \cap T)$. It will be easier to verify this equality after you have read the following discussion of Venn diagrams.)

Venn Diagrams

When considering the relationships between several sets, it is instructive and extremely helpful to represent each set by a region in a plane. The region is drawn so that all the elements belonging to a certain set are contained within some closed region of the plane. Diagrams constructed in this manner are called **Venn diagrams.** The definitions discussed in the previous section can be illustrated as in Fig. 1.6.

By using the definitions directly, or by illustrating sets with Venn diagrams, one can derive formulas that are universally valid regardless of which sets are being considered. For example, the formula $A \cap B = B \cap A$ follows immediately from the definition of the intersection between two sets. It is somewhat more difficult to verify directly from the definitions that the following relationship is valid for all sets A, B, and C:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \qquad [*]$$

With the use of a Venn diagram, however, we easily see that the sets on the right- and left-hand sides of the equality sign both represent the shaded set in Fig. 1.7. The equality in [∗] is therefore valid.

It is important that the three sets A, B, and C in a Venn diagram be drawn in such a way that all possible relations between an element and each of the three sets are represented. In other words, the following eight different sets all should be nonempty: (1): $(A \cap B) \setminus C$; (2): $(B \cap C) \setminus A$; (3): $(C \cap A) \setminus B$; (4): $A \setminus (B \cup C)$; (5): $B \setminus (C \cup A)$; (6): $C \setminus (A \cup B)$; (7): $A \cap B \cap C$; and (8): $C(A \cup B \cup C)$. (See Fig. 1.8.) Notice, however, that this way of representing sets in the plane easily becomes unmanageable if four or more sets are involved, because then there would have to be at least $16\,(= 2^4)$ regions in any such Venn diagram.

From the definition of intersection and union (or by the use of Venn diagrams), it easily follows that $A \cup (B \cup C) = (A \cup B) \cup C$ and that $A \cap (B \cap C) = (A \cap B) \cap C$.

FIGURE 1.6 Venn diagrams.

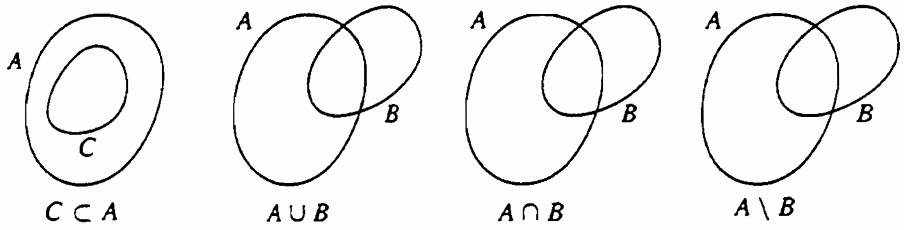


FIGURE 1.7

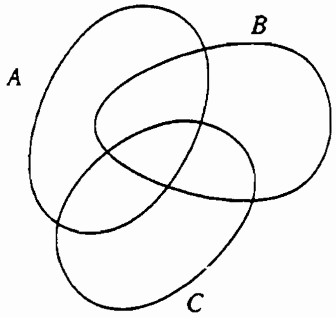
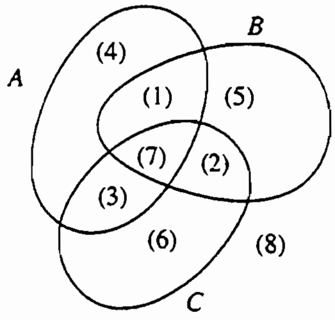


FIGURE 1.8



Consequently, it does not matter where the parentheses are placed. In such cases, the parentheses can be dropped and the expressions written as $A \cup B \cup C$ and $A \cap B \cap C$. Note, however, that the parentheses cannot generally be moved in the expression $A \cap (B \cup C)$, because this set is not always equal to $(A \cap B) \cup C$. Prove this fact by considering the case where $A = \{1, 2, 3\}$, $B = \{2, 3\}$, and $C = \{4, 5\}$, or by using a Venn diagram.

Problems

1. Let $A = \{2, 3, 4\}$, $B = \{2, 5, 6\}$, $C = \{5, 6, 2\}$, and $D = \{6\}$.
   a. Determine if the following statements are true: $4 \in C$; $5 \in C$; $A \subset B$; $D \subset C$; $B = C$; and $A = B$.
   b. Find $A \cap B$; $A \cup B$; $A \setminus B$; $B \setminus A$; $(A \cup B) \setminus (A \cap B)$; $A \cup B \cup C \cup D$; $A \cap B \cap C$; and $A \cap B \cap C \cap D$.
2. **a.** Are the greatest painter among the poets and the greatest poet among the painters one and the same person?
   b. Are the oldest painter among the poets and the oldest poet among the painters one and the same person?
3. With reference to Example 1.12, write the following statements in set terminology:
   a. All biology students are mathematics students.
   b. There are female biology students in the university choir.
   c. Those female students who neither play tennis nor belong to the university choir all study biology.
4. Let F, M, C, B, and T be the sets in Example 1.12. Describe the following sets: $F \cap B \cap C$; $M \cap F$; and $((M \cap B) \setminus C) \setminus T$.
5. Justify the following formulas by either using the definitions or by using Venn diagrams:
   a. $A \cup B = B \cup A$ **b.** $A \cup A = A$
   c. $A \cap A = A$ **d.** $A \cap \emptyset = \emptyset$
   e. $A \cup \emptyset = A$ **f.** $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
6. Determine which of the following formulas are true. If any formula is false, find a counterexample to demonstrate this, using a Venn diagram if you find it helpful.
   a. $A \setminus B = B \setminus A$ **b.** $A \subset B \iff A \cup B = B$
   c. $A \subset B \iff A \cap B = A$ **d.** $A \cap B = A \cap C \implies B = C$
   e. $A \cup B = A \cup C \implies B = C$ **f.** $A \setminus (B \setminus C) = (A \setminus B) \setminus C$
7. Make a complete list of all the different subsets of the set $\{a, b, c\}$. How many are there if the empty set and the set itself are included? Do the same for the set $\{a, b, c, d\}$.

8. A survey revealed that 50 people liked coffee, 40 liked tea, 35 liked both coffee and tea, and 10 did not like either coffee or tea. How many persons in all responded to the survey?

9. If A is a set with a finite number of elements, let $n(A)$ denote the number of elements in A. If A and B are arbitrary finite sets, prove the following:

a. $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

b. $n(A \setminus B) = n(A) - n(A \cap B)$

10. If A and B are two arbitrary sets, define the **symmetric difference** between A and B as

$$A \triangle B = (A \setminus B) \cup (B \setminus A)$$

Obviously, $A \triangle B = B \triangle A$, whereas $A \setminus B \neq B \setminus A$ (in general). Prove by using a Venn diagram, or in some other way, the following:

a. $A \triangle B = (A \cup B) \setminus (A \cap B)$

b. $(A \triangle B) \triangle C$ consists of those elements that occur in just one of the sets A, B, and C, or else in all three.

11. One of the following identities is not generally valid. Which one?

a. $(A \triangle B) \triangle C = A \triangle (B \triangle C)$

b. $(A \cap C) \triangle B = (A \triangle B) \cap (C \triangle B)$

c. $A \triangle A = \emptyset$

12. **a.** A thousand people took part in a survey to reveal which newspaper, A, B, or C, they had read on a certain day. The responses showed that 420 had read A, 316 had read B, and 160 had read C. Of these responses, 116 had read both A and B, 100 had read A and C, 30 had read B and C, and 16 had read all three papers.

(i) How many had read A, but not B?

(ii) How many had read C, but neither A nor B?

(iii) How many had read neither A, B, nor C?

b. Denote the complete set of all 1000 persons in the survey by Ω (the universal set). Applying the notation in Problem 9, we have $n(A) = 420$ and $n(A \cap B \cap C) = 16$, for example. Describe the numbers given in part (a) in a similar manner. Why is the following equation valid?

$$n(\Omega \setminus (A \cup B \cup C)) = n(\Omega) - n(A \cup B \cup C)$$

c. Prove that if A, B, and C are arbitrary finite sets, then

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C)$$
$$-n(B \cap C) + n(A \cap B \cap C)$$

2

Functions of One Variable: Introduction

... mathematics is not so much a subject as a way of studying any subject, not so much a science as a way of life.
—*G. Temple* (1981)

Functions are of fundamental importance in practically every area of pure and applied mathematics, including mathematics applied to economics. The language of mathematical economics is full of terms like supply and demand functions, cost functions, production functions, consumption functions, and so on. Here and in the next chapter, we present a general discussion of functions of one real variable, illustrated by some very important examples.

2.1 Introduction

One variable is a function of another if the first variable *depends* upon the second. For instance, the area of a circle is a function of its radius. If the radius r is given, then the area A is determined. In fact $A = \pi r^2$, where π is the numerical constant 3.14159....

The measurement of temperature provides another example of a function. If C denotes the temperature expressed in degrees Centigrade (or Celsius), this is a function of F, the same temperature measured in degrees Fahrenheit, because $C = \frac{5}{9}(F - 32)$.

In ordinary conversation, we sometimes use the word "function" in a similar way. For example, we might say that the infant mortality rate of a country is a function of the quality of its health care, or that a country's national product is

TABLE 2.1 *Personal consumption expenditure in the United States, 1985–1991*

Year	1985	1986	1987	1988	1989	1990	1991
Personal consumption[1]	2,667.4	2,850.6	3,052.2	3,296.2	3,523.1	3,748.4	3,887.7

[1] In billions of dollars.

a function of the level of investment. In both these cases, it would be a major research task to obtain a formula that represents the function precisely.

One does not need a mathematical formula to convey the idea that one variable is a function of another: A table can also show the relationship. For instance, Table 2.1 shows the growth of annual total personal consumption expenditures, measured in current dollars, in the United States for the period 1985–1991. It is taken from figures in the *Economic Report of the President* dated January 1993. This table defines consumption expenditures as a function of the year. No allowance is made for inflation.

The dependence between two variables can also be illustrated by means of a graph or chart. Consider the following two examples.

In Fig. 2.1, we have drawn a curve that allegedly played an important role some years ago in the discussion of "supply side economics." It shows the presumed relationship between a country's income tax rate and its total income tax revenue. Obviously, if the income tax rate is 0%, then tax revenue is 0. However, if the tax rate is 100%, then tax revenue will also be (about) 0, because virtually no one is willing to work if his or her entire income is going to be confiscated. These ideas are obvious to virtually all competent economists (in cases like Problem 1 of Section 3.2). Nevertheless, a controversy was created by the American economist Arthur Laffer, who claimed to have drawn this curve on a restaurant napkin, and then later popularized its message with the public. Economists have hotly disputed what is the percentage rate a at which the government collects the maximum tax revenue.

FIGURE 2.1 The "Laffer curve," which relates tax revenue to tax rates.

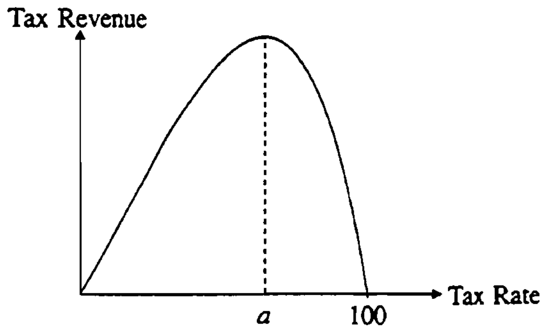


Figure 2.2 reproduces a postage stamp showing how Norway's gross national product grew during the first 100 years of the lifetime of its Central Bureau of Statistics.

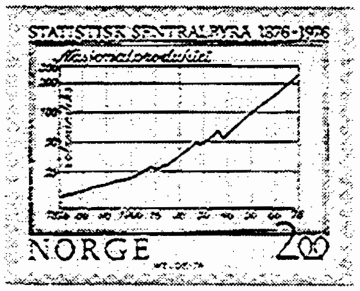


FIGURE 2.2 The national product of Norway (*volume index*) 1876–1976.

All of the relationships just discussed have one characteristic in common: A definite rule relates each value of one variable to a definite value of another variable.

Notice that in all of the examples, it is implicitly assumed that the variables are subject to certain constraints. For instance, in the temperature example, F cannot be less than -459.67, the absolute zero point (which corresponds to -273.15 degrees Centigrade). In Table 2.1, only the years between 1985 and 1991 are relevant.

2.2 Functions of One Real Variable

The examples we studied in the preceding section lead to the following general definition of a real valued function of one real variable:

> A **function** of a real variable x with **domain** D is a rule that assigns a unique real number to each number x in D. [2.1]

The word "rule" is used in a very broad sense. *Every* rule with the properties described in [2.1] is called a function, whether that rule is given by a formula, described in words, defined by a table, illustrated by a curve, or expressed by any other means.

Functions are often given letter names, such as f, g, F, or ϕ. If f is a function and x is a number in its domain D, then $f(x)$ denotes the number that the function f assigns to x. The symbol $f(x)$ is pronounced "f of x." It is important to note the difference between f, which is a symbol for the function (the rule), and $f(x)$, which denotes the value of f at x.

If f is a function, we sometimes let y denote the value of f at x, so

$$y = f(x) \qquad [*]$$

Then we call x the **independent variable**, or the **argument** of f, whereas y is called the **dependent variable**, because the value y (in general) depends on the value of x. In economics, x is often called the *exogenous* variable, whereas y is the *endogenous* variable.

A function is often defined by a particular formula of the type [*], such as $y = 8x^2 + 3x + 2$. The function is then the rule that assigns the number $8x^2 + 3x + 2$ to x.

Functional Notation

To become familiar with the relevant notation, it helps to look at some examples of functions that are defined by formulas.

Example 2.1

A function is defined for all numbers by the following rule:

Assign to any number the third power of that number. [1]

This function will assign $0^3 = 0$ to 0, $3^3 = 27$ to 3, $(-2)^3 = -8$ to -2, and $(1/4)^3 = 1/64$ to $1/4$. In general, it assigns the number x^3 to the number x. If we denote the function by f, then

$$f(x) = x^3 \qquad [2]$$

So $f(0) = 0^3 = 0$, $f(3) = 3^3 = 27$, $f(-2) = (-2)^3 = -8$, $f(1/4) = (1/4)^3 = 1/64$.

Substituting a for x in the formula for f gives $f(a) = a^3$, whereas

$$f(a + 1) = (a + 1)^3 = (a + 1)(a + 1)(a + 1) = a^3 + 3a^2 + 3a + 1 \qquad [3]$$

Note: A common error is to presume that $f(a) = a^3$ implies $f(a + 1) = a^3 + 1$. The error can be illustrated by looking at a simple interpretation of f. If a is the edge of a cube measured in meters, then $f(a) = a^3$ is the volume of the cube measured in cubic meters. Suppose that each edge of the cube has its length increased by 1 m. Then the volume of the new cube is $f(a + 1) = (a + 1)^3$ cubic meters. The number $a^3 + 1$ can be interpreted as the number obtained when the volume of a cube with edge a is increased by 1 m^3. In fact,

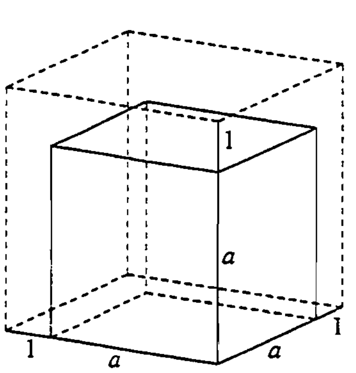


FIGURE 2.3 Volume $f(a+1) = (a+1)^3$.

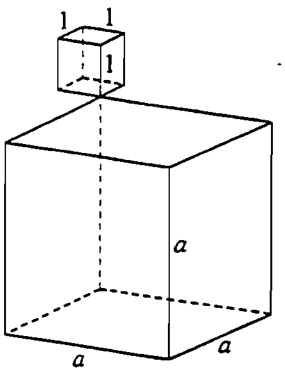


FIGURE 2.4 Volume $a^3 + 1$.

$f(a+1) = (a+1)^3$ is quite different from $a^3 + 1$, as illustrated in Figs. 2.3 and 2.4.

Example 2.2

The total dollar cost of producing x units of a product is given by

$$C(x) = 100x\sqrt{x} + 500$$

Find the cost of producing 16, 100, and a units. Suppose the firm produces a units; find the *increase* in the cost from producing one additional unit.[1]

Solution The cost of producing 16 units is found by substituting 16 for x in the formula for $C(x)$:

$$C(16) = 100 \cdot 16\sqrt{16} + 500 = 100 \cdot 16 \cdot 4 + 500 = 6900$$

Similarly,

$$C(100) = 100 \cdot 100 \cdot \sqrt{100} + 500 = 100{,}500$$
$$C(a) = 100a\sqrt{a} + 500$$

The cost of producing $a+1$ units is $C(a+1)$, so that the increase in cost is

$$C(a+1) - C(a) = 100(a+1)\sqrt{a+1} + 500 - 100a\sqrt{a} - 500$$
$$= 100\left[(a+1)\sqrt{a+1} - a\sqrt{a}\right]$$

[1]This is the concept that economists often call **marginal cost**. However, they should really call it **incremental cost**. In Section 4.3, we will explain the difference between the two.

So far we have used x to denote the independent variable, but we could just as well have used almost any other symbol. For example, all of the following formulas define exactly the same function (and hence we can set $f = g = \phi$):

$$f(x) = \frac{x^2 - 3}{x^4 + 1}, \qquad g(t) = \frac{t^2 - 3}{t^4 + 1}, \qquad \phi(\xi) = \frac{\xi^2 - 3}{\xi^4 + 1} \qquad [*]$$

For that matter, we could also express the function in [*] as follows:

$$f(\cdot) = \frac{(\cdot)^2 - 3}{(\cdot)^4 + 1}$$

Here it is understood that the dot between the parentheses can be replaced by an arbitrary number or an arbitrary letter or even another function (like $1/y$). Thus,

$$f(1) = \frac{(1)^2 - 3}{(1)^4 + 1} = -1, \qquad f(k) = \frac{k^2 - 3}{k^4 + 1}, \qquad \text{and} \qquad f(1/y) = \frac{(1/y)^2 - 3}{(1/y)^4 + 1}$$

In economic theory, we often study functions that depend on a number of parameters in addition to the independent variable. A typical example follows.

Example 2.3

Suppose that the cost of producing x units of a commodity is

$$C(x) = Ax\sqrt{x} + B \qquad (A \text{ and } B \text{ are positive constants}) \qquad [1]$$

Find the cost of producing 0, 10, and $x + h$ units.

Solution The cost of producing 0 units is

$$C(0) = A \cdot 0 \cdot \sqrt{0} + B = 0 + B = B$$

(Parameter B simply represents fixed costs. These are the costs that must be paid whether or not anything is actually produced, such as a taxi driver's annual license fee.) Similarly,

$$C(10) = A \cdot 10\sqrt{10} + B$$

Finally, substituting $x + h$ for x in (1) gives

$$C(x + h) = A(x + h)\sqrt{x + h} + B$$

The Domain and the Range

The definition of a function is incomplete unless its domain has been specified. The domain of the function f defined by $f(x) = x^3$ (see Example 2.1) is the set of all real numbers. In Example 2.2, where $C(x) = 100x\sqrt{x} + 500$ denotes the cost of producing x units of a product, the domain was not specified, but the natural domain is the set of numbers $0, 1, 2, \ldots, x_0$, where x_0 is the maximum number of items the firm can produce. If output x is a continuous variable, the natural domain is the closed interval $[0, x_0]$.

If a function is defined using an algebraic formula, we adopt the convention that the domain consists of all values of the independent variable for which the formula gives a meaningful value (unless another domain is explicitly mentioned).

Example 2.4

Find the domains of

(a) $$f(x) = \frac{1}{x+3}$$

(b) $$g(x) = \sqrt{2x+4}$$

Solution

(a) For $x = -3$, the formula reduces to the meaningless expression "1/0." For all other values of x, the formula makes $f(x)$ a well-defined number. Thus, the domain consists of all numbers $x \neq -3$.

(b) The expression $\sqrt{2x+4}$ is defined for all x such that $2x+4$ is nonnegative. Solving the inequality $2x + 4 \geq 0$ for x gives $x \geq -2$. Hence, the domain of g is the interval $[-2, \infty)$.

Let f be a function with domain D. The set of all values $f(x)$ that the function assumes is called the **range** of f. Often, we denote the domain of f by D_f, and the range by R_f. These concepts are illustrated in Fig. 2.5, using the idea of the graph of a function. (Graphs are discussed in the next section, but you probably have been exposed to them before.)

Alternatively, we can think of any function f as an engine operating so that if the number x in the domain is an input, the output is the number $f(x)$. (See Fig. 2.6.) The range of f is then all the numbers we get as output using all numbers x in the domain as inputs. If we try to use as an input a number not in the domain, the engine does not work, and there is no output.

Example 2.5

Show that the number 4 belongs to the range of the function defined by $g(x) = \sqrt{2x+4}$. Find the entire range of g. (Remember that $\sqrt{u}$ denotes the nonnegative square root of u.)

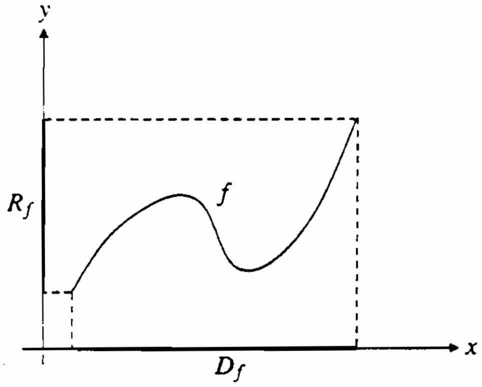


FIGURE 2.5 The domain and the range of *f*.

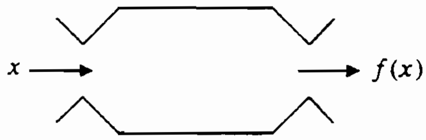


FIGURE 2.6 Function engine.

Solution To show that a number such as 4 is in the range of g, we must find a number x such that $g(x) = 4$. That is, we must solve the equation $\sqrt{2x+4} = 4$ for x. By squaring both sides of the equation, we get $2x + 4 = 4^2 = 16$, that is, $x = 6$. Because $g(6) = 4$, the number 4 does belong to the range R_g.

In order to determine the whole range of g, we must answer the question: As x runs through the whole of the interval $[-2, \infty)$, what are all the possible values of $\sqrt{2x+4}$? For $x = -2$, $\sqrt{2x+4} = 0$, and $\sqrt{2x+4}$ can never be negative. We claim that whatever number $y_0 \geq 0$ is chosen, there exists a number x_0 such that $\sqrt{2x_0+4} = y_0$. Squaring each side of this last equation gives $2x_0 + 4 = y_0^2$. Hence, $2x_0 = y_0^2 - 4$, which implies that $x_0 = \frac{1}{2}(y_0^2 - 4)$. Because $y_0^2 \geq 0$, we have $x_0 = \frac{1}{2}\left(y_0^2 - 4\right) \geq \frac{1}{2}(-4) = -2$. Hence, for every number $y_0 \geq 0$, there is a number $x_0 \geq -2$ such that $g(x_0) = y_0$. The range of g is, therefore, $[0, \infty)$.

Even if we have a function that is completely specified by a formula, including a specific domain, it is not always easy to find the range of the function. For example, without using the methods of differential calculus, it is not at all simple to find R_f when $f(x) = 3x^3 - 2x^2 - 12x - 3$ and $D_f = [-2, 3]$.

Many pocket calculators have some special functions built into them. For example, many have the $\sqrt{\ }$ function that, given a number x, assigns the square root of the number, $\sqrt{x}$. If we enter a nonnegative number such as 25, and press the square-root key, then the number 5 appears. If we enter -3, then the word "Error" is shown, which is the way the calculator tells us that $\sqrt{-3}$ is not defined.

The concept of a function is entirely abstract. In Example 2.2, we studied a function that finds the production cost $C(x)$ in dollars associated with the number of units x of a commodity. Here x and $C(x)$ are concrete, measurable quantities. On the other hand, the letter C, which is the name of the function, does not represent a physical quantity; rather, it represents the dependence of cost upon the number of units produced, a purely abstract concept.

Problems

1. Let $f(x) = x^2 + 1$.
   a. Compute $f(0)$, $f(-1)$, $f(1/2)$, and $f(\sqrt{2})$.
   b. For what x is it true that (i) $f(x) = f(-x)$? (ii) $f(x+1) = f(x)+f(1)$? (iii) $f(2x) = 2f(x)$?
2. Suppose $F(x) = 10$, for all x. Find $F(0)$, $F(-3)$, and $F(a+h) - F(a)$.
3. Let $f(t) = a^2 - (t-a)^2$ (a is a constant).
   a. Compute $f(0)$, $f(a)$, $f(-a)$, and $f(2a)$.
   b. Compute $3f(a) + f(-2a)$.
4. Let f be defined for all x by

$$f(x) = \frac{x}{1+x^2}$$

   a. Compute $f(-1/10)$, $f(0)$, $f(1/\sqrt{2})$, $f(\sqrt{\pi})$, and $f(2)$.
   b. Show that $f(x) = -f(-x)$ for all x, and that $f(1/x) = f(x)$, for $x \neq 0$.
5. The cost of producing x units of a commodity is given by

$$C(x) = 1000 + 300x + x^2$$

   a. Compute $C(0)$, $C(100)$, and $C(101) - C(100)$.
   b. Compute $C(x+1) - C(x)$, and explain in words the meaning of the difference.
6. Let $F(t) = \sqrt{t^2 - 2t + 4}$. Compute $F(0)$, $F(-3)$, and $F(t+1)$.
7. H. Schultz has estimated the demand for cotton in the United States for the period 1915–1919 to be $D(P) = 6.4 - 0.3P$ [with appropriate units for the price P and the quantity $D(P)$].
   a. Find the demand if the price is 8, 10, and 10.22.
   b. If the demand is 3.13, what is the price?
8. The cost of removing $p\%$ of the impurities in a lake is given by

$$b(p) = \frac{10p}{105 - p}$$

   a. Find $b(0)$, $b(50)$, and $b(100)$.
   b. What does $b(50+h) - b(50)$ mean? ($h \geq 0$.)
9. a. If $f(x) = 100x^2$, show that for all t, $f(tx) = t^2 f(x)$.
   b. If $P(x) = x^{1/2}$, show that for all $t \geq 0$, $P(tx) = t^{1/2}P(x)$.
10. Only for special "additive" functions is it true that $f(a+b) = f(a) + f(b)$ for all a and b. Determine whether $f(2+1) = f(2) + f(1)$ for the

following:

a. $f(x) = 2x^2$ b. $f(x) = -3x$ c. $f(x) = \sqrt{x}$

11. a. If $f(x) = Ax$, show that $f(a+b) = f(a) + f(b)$, for all a and b.
b. If $f(x) = 10^x$, show that $f(a+b) = f(a) \cdot f(b)$, for all natural numbers a and b.

12. A student claims that $(x+1)^2 = x^2 + 1$. Can you use a geometric argument to show that this is wrong?

13. Find the domains of the functions defined by the following equations:

a. $y = \sqrt{5-x}$ b. $y = \dfrac{2x-1}{x^2-x}$

c. $y = \sqrt{\dfrac{x-1}{(x-2)(x+3)}}$ d. $y = (x+1)^{1/2} + 1/(x-1)^{1/2}$

14. Consider the function f defined by the formula

$$f(x) = \frac{3x+6}{x-2}$$

a. Find the domain of f.
b. Show that the number 5 is in the range of f by finding a number x such that $(3x+6)/(x-2) = 5$.
c. Show that the number 3 is not in the range of f.

15. Find the domain and the range $g(x) = 1 - \sqrt{x+2}$.

16. Let $f(x) = |x|$. Which of the the following rules are valid for all possible pairs of numbers x and y?

a. $f(x+y) = f(x) + f(y)$ b. $f(x+y) \le f(x) + f(y)$

c. $f(xy) = f(x) \cdot f(y)$ d. $f(2x) = 2f(x)$

e. $f(-2x) = -2f(x)$ f. $f(x) = \sqrt{x^2}$

g. $f(-2x) = 2f(x)$ h. $|f(x) - f(y)| \le |x - y|$

17. Let

$$f(x) = \frac{ax+b}{cx-a}$$

where a, b, and c are constants, and $c \ne 0$. Assuming that $x \ne a/c$, show that

$$f\left(\frac{ax+b}{cx-a}\right) = x$$

2.3 Graphs

Three examples of equations in two variables x and y are

$$y = 2x - 1, \qquad x^2 + y^2 = 16, \qquad x\sqrt{y} = 2 \qquad [*]$$

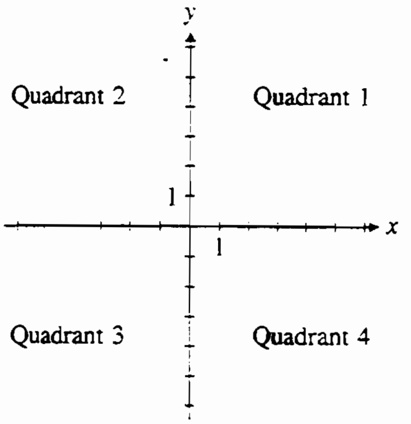
In this section, we shall explain how *any* equation in two variables can be represented by a curve (a graph) in a coordinate system. In particular, any function given by an equation $y = f(x)$ has such a representation, that helps us to visualize the equation or the function. This is because the shape of the graph reflects the properties of the equation or the function.

A Coordinate System in the Plane

In Section 1.4, we claimed that real numbers can be represented by a number line. Analogously, every *pair* of real numbers can be represented by a point in a plane. Draw two perpendicular lines, called respectively the *x-axis* (or the *horizontal axis*) and the *y-axis* (or the *vertical axis*). The intersection point O is called the *origin*. We measure the real numbers along each of these lines, as shown in Fig. 2.7. Often, we measure the numbers on the two axes so that the length on the x-axis that represents the distance between x and $x+1$ is the same length as that along the y-axis that represents the distance between y and $y+1$. But this does not have to be the case.

Figure 2.7 illustrates a **rectangular**, or a **Cartesian, coordinate system**, that we call the **xy-plane**. The coordinate axes separate the plane into four quadrants, which can be numbered as in Fig. 2.7. Any point P in the plane can be represented by a pair (a, b) of real numbers. These can be found by dropping perpendiculars onto the axes. The point represented by (a, b) lies at the intersection of the vertical straight line $x = a$ with the horizontal straight line $y = b$. Conversely, any pair of real numbers represents a unique point in the plane. For example, in Fig. 2.8, the ordered pair (3, 4) corresponds to the point P that lies at the intersection of $x = 3$

FIGURE 2.7 A coordinate system.



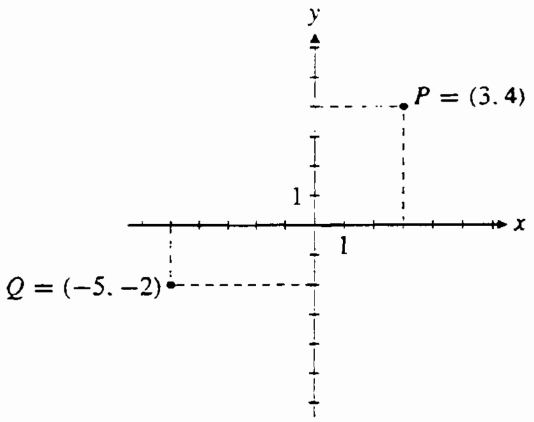


FIGURE 2.8 The points (3, 4) and (−5, −2).

with $y = 4$. Thus, P lies 3 units to the right of the y-axis and 4 units above the x-axis. We call (3, 4) the **coordinates** of P. Similarly, Q lies 5 units to the left of the y-axis and 2 units below the x-axis, so the coordinates of Q are $(-5, -2)$.

Note that we call (a, b) an **ordered pair**, because the order of the two numbers in the pair is important. For instance, (3, 4) and (4, 3) represent two different points.

Example 2.6

Draw coordinate systems and indicate the coordinates (x, y) that satisfy each of the following three conditions:

(a) $x = 3$
(b) $x \geq 0$ and $y \geq 0$
(c) $-2 \leq x \leq 1$ and $-2 \leq y \leq 3$

Solution

(a) See Fig. 2.9, which represents a straight line.
(b) See Fig. 2.10, which represents the first quadrant.
(c) See Fig. 2.11, which represents a rectangle.

FIGURE 2.9 A straight line.

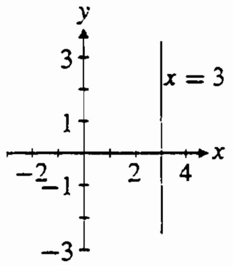


FIGURE 2.10 The first quadrant.

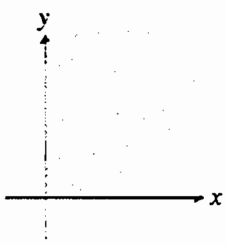
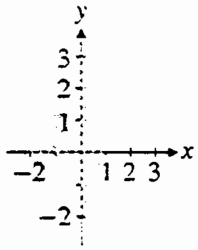


FIGURE 2.11 A rectangle.



Graphs of Equations in Two Variables

A solution of an equation in two variables x and y is a pair (a, b) that satisfies the equation when we substitute a for x and b for y. The **solution set** of the equation is the set of all possible solutions. If we plot all the ordered pairs of the solution set in a coordinate system, we obtain a curve that is called the **graph** of the equation.

Example 2.7

Find some numerical solutions for each of the equations $y = 2x-1$, $x^2+y^2 = 16$, and $x\sqrt{y} = 2$, and try to sketch the graphs.

Solution For $y = 2x - 1$, point $(0, -1)$ is a solution, because if $x = 0$, then $y = 2 \cdot 0 - 1 = -1$. Other solutions are $(1, 1)$, $(3, 5)$, and $(-1, -3)$. In Fig. 2.12, we have plotted the four solutions, and they all appear to lie on a straight line. There exist infinitely many other solutions, so we can never write them all down.

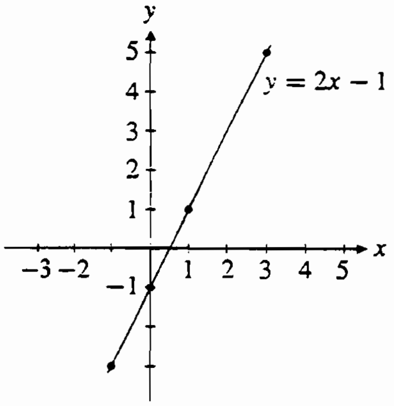


FIGURE 2.12 $y = 2x - 1$.

For $x^2 + y^2 = 16$, point $(4, 0)$ is a solution. Some other solutions are shown in Table 2.2.

TABLE 2.2 *Solutions of* $x^2 + y^2 = 16$

x	-4	-3	-1	0	1	3	4
y	0	$\pm\sqrt{7}$	$\pm\sqrt{15}$	± 4	$\pm\sqrt{15}$	$\pm\sqrt{7}$	0

Figure 2.13 shows the plot of the points in the table, and the graph appears to be a circle.

From $x\sqrt{y} = 2$, we obtain $y = 4/x^2$, and it is easy to fill in Table 2.3. The graph is shown in Fig. 2.14.

Note: When plotting the graph of an equation such as $x^2 + y^2 = 16$, we must try to find a sufficient number of solution pairs (x, y), otherwise we might miss some

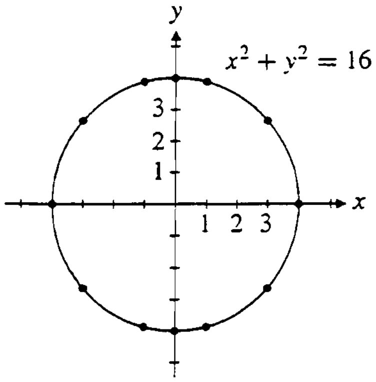


FIGURE 2.13 $x^2 + y^2 = 16$.

TABLE 2.3 *Solutions of* $x\sqrt{y} = 2$

x	1	2	4	6
y	4	1	1/4	1/9

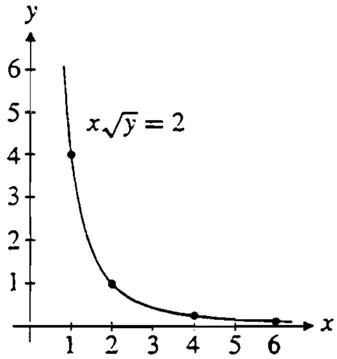


FIGURE 2.14 $x\sqrt{y} = 2$.

important features of the graph. Actually, by merely plotting a finite set of points, we can never be entirely sure that there are no wiggles or bumps we have missed. We shall see in what follows that the graph of the equation $x^2 + y^2 = 16$ really is a circle. For more complicated equations, we have to use differential calculus to decide how many bumps and wiggles there are.

The Distance Between Two Points in the Plane

Let $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ be the two points shown in Fig. 2.15. By Pythagoras' theorem, the distance d between these points satisfies the equation

$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$. Therefore, note that because $(x_1 - x_2)^2 = (x_2 - x_1)^2$ and $(y_1 - y_2)^2 = (y_2 - y_1)^2$, it does not make any difference which point is P_1 and which is P_2.

The Distance Formula

The distance between points (x_1, y_1) and (x_2, y_2) is

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \qquad [2.2]$$

To prove the distance formula, we considered two points in the first quadrant. It turns out that the same formula is valid regardless of where the two points, P_1 and P_2, lie.

Example 2.8

Find the distance d between points $P_1 = (-4, 3)$ and $P_2 = (5, -1)$. (See Fig. 2.16.)

Solution Using [2.2] with $x_1 = -4$, $y_1 = 3$, and $x_2 = 5$, $y_2 = -1$, we have

$$\begin{aligned} d &= \sqrt{(-4-5)^2 + (3-(-1))^2} \\ &= \sqrt{(-9)^2 + 4^2} = \sqrt{81+16} = \sqrt{97} \approx 9.85 \end{aligned}$$

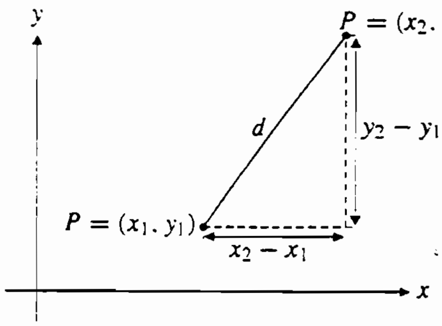


FIGURE 2.15

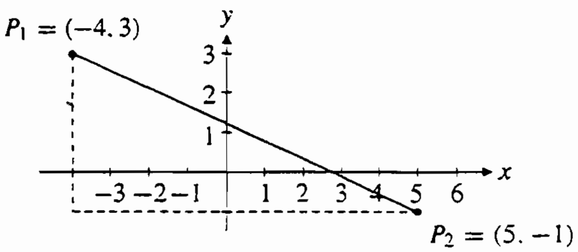


FIGURE 2.16

Circles

Let (a, b) be a point in the plane. *The circle with radius r and center at (a, b) is the set of all points (x, y) whose distance from (a, b) is equal to r.* Considering Fig. 2.17 and using the distance formula gives $\sqrt{(x-a)^2 + (y-b)^2} = r$. Squaring each side yields

The Equation of a Circle

The equation of a circle with center at (a, b) and radius r is

$$(x-a)^2 + (y-b)^2 = r^2 \qquad [2.3]$$

Note that if we let $a = b = 0$ and $r = 4$, then [2.3] reduces to $x^2 + y^2 = 16$. This is the equation of a circle with center at (0, 0) and radius 4, as shown in Fig. 2.13.

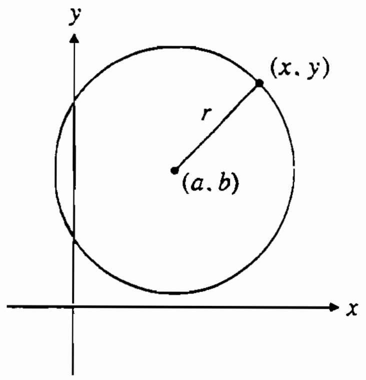


FIGURE 2.17 Circle with center (a, b) and radius r.

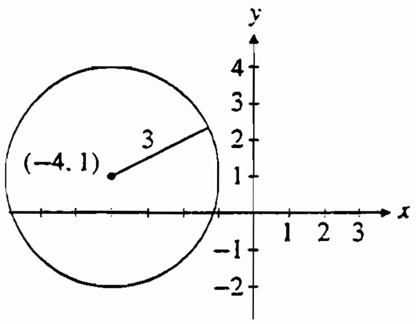
Example 2.9

Find the equation of the circle with center (−4, 1) and radius 3.

Solution Here $a = -4$, $b = 1$, and $r = 3$ (see Fig. 2.18). So the general formula in [2.3] becomes the specific equation

$$(x+4)^2 + (y-1)^2 = 9 \qquad [1]$$

FIGURE 2.18 Circle with center (−4, 1) and radius 3.



Expanding the squares in [1] gives

$$x^2 + 8x + 16 + y^2 - 2y + 1 = 9 \tag{2}$$

This can be written as

$$x^2 + y^2 + 8x - 2y + 8 = 0 \tag{3}$$

Note: The equation of the circle given in [3] has the disadvantage that we cannot immediately read off its center and radius. If we are given equation [3], we can "argue backwards" in order to deduce [1] via [2]. We then say that we have "completed the squares," which is actually one of the oldest tricks in mathematics. (See Section A.8, Appendix A.) The method is illustrated in Problem 9.

Problems

1. Draw a Cartesian coordinate system and plot the points $(2, 3)$, $(-3, 2)$, $(-3/2, 1/4)$, $(4, 0)$, and $(0, 4)$.
2. Sketch the six sets of points (x, y) satisfying the following conditions:
   a. $y = 4$ **b.** $x < 0$ **c.** $x \geq 1$ and $y \geq 2$
   d. $|x| = 2$ **e.** $y = x$ **f.** $y \geq x$
3. Sketch the graphs of each of the following equations:
   a. $y = 4x - 3$ **b.** $xy = 1$ **c.** $y^2 = x$
4. Try to sketch the graphs of each of the following equations:
   a. $x^2 + 2y^2 = 6$ **b.** $y + \sqrt{x - 1} = 0$ **c.** $y^2 - x^2 = 1$
5. Find the distance between each pair of points:
   a. $(1, 3)$ and $(2, 4)$ **b.** $(-1, 2)$ and $(-3, -3)$
   c. $(3/2, -2)$ and $(-5, 1)$ **d.** (x, y) and $(2x, y + 3)$
   e. (a, b) and $(-a, b)$ **f.** $(a, 3)$ and $(2 + a, 5)$
6. The distance between $(2, 4)$ and $(5, y)$ is $\sqrt{13}$. Find y. (Explain geometrically why there must be two values of y. What would happen if the distance were 2?)
7. Find the approximate distance between each pair of points:
   a. $(3.998, 2.114)$ and $(1.130, -2.416)$ **b.** $(\pi, 2\pi)$ and $(-\pi, 1)$
8. Find the equations of the following circles:
   a. Center at $(2, 3)$ and radius 4.
   b. Center at $(2, 5)$ and passes through $(-1, 3)$.
9. We can show that the graph of $x^2 + y^2 + 8x - 2y + 8 = 0$ is a circle by arguing like this: First, rearrange the equation to read $(x^2 + 8x \ldots) +$

$(y^2 - 2y \ldots) = -8$. Completing the two squares gives $(x^2 + 8x + 4^2) + (y^2 - 2y + (-1)^2) = -8 + 4^2 + (-1)^2 = 9$. Thus, the equation becomes $(x+4)^2 + (y-1)^2 = 9$, whose graph is a circle with center $(-4, 1)$ and radius $\sqrt{9} = 3$. Use this method to find the center and the radius of the two circles with equations:

a. $x^2 + y^2 + 10x - 6y + 30 = 0$ **b.** $3x^2 + 3y^2 + 18x - 24y = -39$

10. Point P moves in the plane so that it is always equidistant from each of the points $A = (3, 2)$ and $B = (5, -4)$. Find a simple equation that the coordinates (x, y) of P must satisfy. Illustrate the problem and its solution geometrically. (*Hint:* Compute the square of the distance from P to A and to B, respectively.)

11. Prove that if the distance from a point (x, y) to the point $(-2, 0)$ is twice the distance from (x, y) to $(4, 0)$, then (x, y) must lie on the circle with center $(6, 0)$ and radius 4.

Harder Problems

12. Try to sketch the graph of the equation $\sqrt{x} + \sqrt{y} = 1$.

13. A firm has two plants A and B located 60 kilometers apart at the two points $(0, 0)$ and $(60, 0)$. See Fig. 2.19. It supplies one identical product priced at $\$p$ per unit. Shipping costs per kilometer per unit are \$10 from A and \$5 from B. An arbitrary purchaser is located at point (x, y).

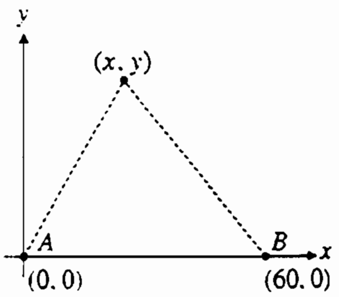
a. Give economic interpretations for the expressions:

$$p + 10\sqrt{x^2 + y^2} \qquad \text{and} \qquad p + 5\sqrt{(x-60)^2 + y^2}$$

b. Find the equation for the curve that separates the markets of the two firms, assuming that customers buy from the firm for which total costs are lower.

14. Generalize Problem 13 to the case where $A = (0, 0)$ and $B = (a, 0)$, and assume that shipping costs per kilometer are r and s dollars, respectively. Show that the curve separating the markets is a circle, and find its center and radius.

FIGURE 2.19



15. Show that the graph of

$$x^2 + y^2 + Ax + By + C = 0 \qquad (A,\ B,\ \text{and } C \text{ are constants})$$

is a circle if $A^2 + B^2 > 4C$. Find its center and radius. (See Problem 9.) What happens if $A^2 + B^2 \leq 4C$?

2.4 Graphs of Functions

The **graph** of a function f is the set of all points $(x, f(x))$, where x belongs to the domain of f. This is simply the graph of the equation $y = f(x)$. Typical examples of graphs of functions are given in Figs. 2.20 and 2.21.

In Fig. 2.20, we show the graph of $f(x) = x^2 - 3x$. It is found by computing points $(x, f(x))$ on the graph and then drawing a smooth curve through the points.

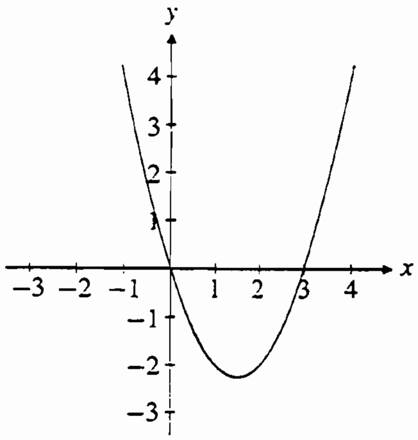
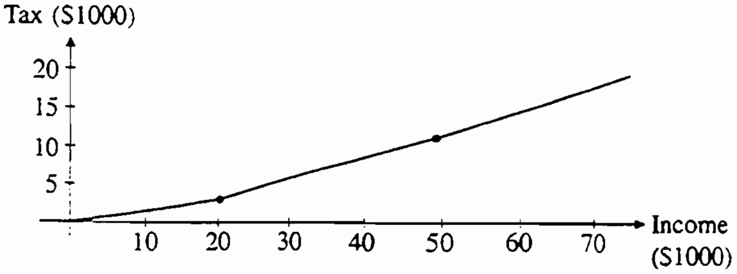


FIGURE 2.20 The graph of $f(x) = x^2 - 3x$.

The function whose graph is shown in Fig. 2.21 is of a type often encountered in economics. It is defined by different formulas on different intervals.

Example 2.10 (U.S. Federal Income Tax (1991) for Single Persons)
In Fig. 2.21, we show the graph of this income tax function. Income up to

FIGURE 2.21 U.S. federal income tax.



\$20,250 was taxed at 15%, income between \$20,251 and \$49,300 was taxed at 28%, and income above \$49,300 was taxed at 31%.

Graphs of different functions can have innumerable different shapes. However, not all curves in the plane are graphs of functions. A function assigns to each point x in the domain only one y-value. *The graph of a function therefore has the property that a vertical line through any point on the x-axis has at most one point of intersection with the graph.* This simple *vertical line test* is illustrated in Figs. 2.22 and 2.23.

The graph of the circle $x^2 + y^2 = 16$, as shown in Fig. 2.13, is a typical example of a graph that does *not* represent a function, because it does not pass the vertical line test. The vertical line $x = a$ for any a with $-4 < a < 4$ intersects the circle at *two* points. When we solve the equation $x^2 + y^2 = 16$ for y, we obtain $y = \pm\sqrt{16 - x^2}$. Note that the upper semicircle alone is the graph of the function $y = \sqrt{16 - x^2}$ and the lower semicircle is the graph of the function $y = -\sqrt{16 - x^2}$. Both these functions are defined on the interval $[-4, 4]$.

Choice of Units

A function of one variable is a rule assigning numbers in its range to numbers in its domain. When we describe an empirical relationship by means of a function, we must first choose the units of measurement. For instance, we might measure

FIGURE 2.22 This graph represents a function.

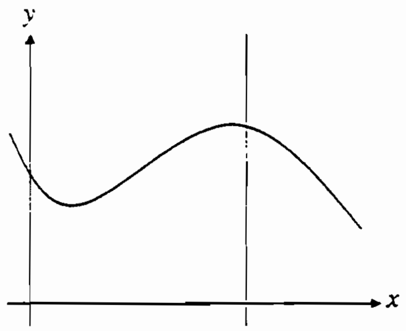
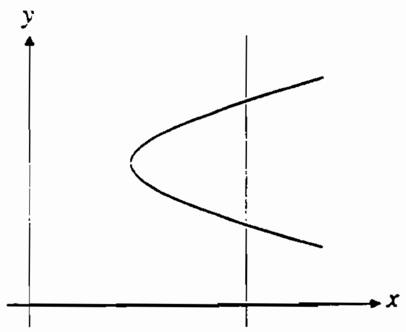


FIGURE 2.23 This graph does not represent a function.



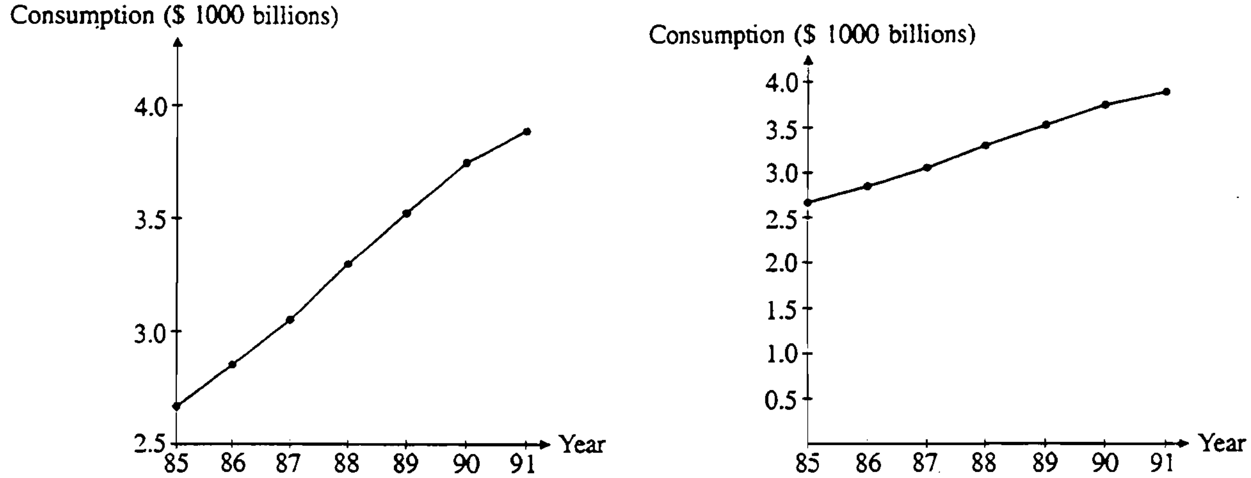


FIGURE 2.24 Graphical representations of the function defined in Table 2.1 with different units of measurement.

time in years, days, or weeks. We might measure money in dollars, yen, or francs. The choice we make may influence the visual impression conveyed by the graph of the function.

Figure 2.24 illustrates a standard trick that is often used to influence people's impressions of empirical relationships. In both diagrams, time is measured in years and consumption in billions of dollars. They both graph the same function. (Which graph would you think the Republicans in the United States might prefer for their advertising, and which is more to the liking of the Democrats?)

Shifting Graphs

Given the graph of a function f, it is sometimes useful to know how to find the graphs of the related functions:

$$f(x)+c, \qquad f(x+c), \qquad -f(x), \qquad \text{and} \qquad f(-x) \qquad [2.4]$$

Problem 3 of this section asks you to study these graphs in general. Here we consider a simple economic example.

Example 2.11

Suppose a person earning y (dollars) in a given year pays $f(y)$ (dollars) in income tax. It is decided to reduce taxes. One proposal is to allow all individuals to deduct d dollars from their taxable income before tax is calculated. An alternative proposal involves calculating income tax on the full amount of taxable income and then allowing each person a "tax credit" that deducts d dollars from the total tax due. Illustrate graphically the two proposals for a "normal" tax function f, and mark off the the income y^*, where the two proposals give the same tax.

Solution Figure 2.25 illustrates the solution. First, draw the graph of the original tax function, $T = f(y)$. If taxable income is y and the deduction is

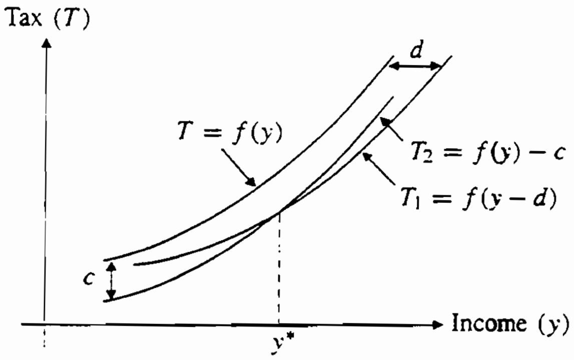


FIGURE 2.25 The graphs of $T_1 = f(y - c)$ and $T_2 = f(y) - d$.

d, then $y-d$ is the reduced taxable income, and so the tax liability is $f(y-d)$. By shifting the graph of the original tax function d units to the right, we obtain the graph of $T_1 = f(y-d)$.[2] The graph of $T_2 = f(y)-c$ is obtained by lowering the graph of $T = f(y)$ by c units. The income y^* that gives the same tax under the two different schemes is the value of y satisfying the equation

$$f(y-d) = f(y) - c$$

This value of y is marked y^* in the figure.

Problems

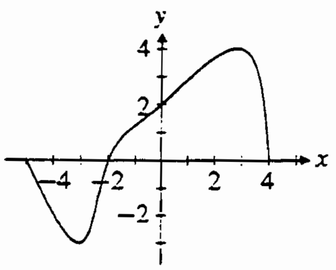
1. Determine the domain on which each of the following equations defines y as a function of x:

   a. $y = x + 2$ **b.** $y = \pm\sqrt{x}$ **c.** $y = x^4$ **d.** $y^4 = x$

   e. $x^2 - y^2 = 1$ **f.** $y = \dfrac{x}{x-3}$ **g.** $y^3 = x$ **h.** $x^3 + y^3 = 1$

2. The graph of the function f is given in Fig. 2.26.

FIGURE 2.26



[2]As an example: $y = x^2$ is a parabola, whereas $y = (x-1)^2$ is a parabola obtained by shifting the first parabola 1 unit to the right.

a. Find $f(-5)$, $f(-3)$, $f(-2)$, $f(0)$, $f(3)$, and $f(4)$ by examining the graph.
b. Find the domain and the range of f.

3. Explain how to get the graphs of the four functions defined by [2.4] based on the graph of $y = f(x)$.

4. Use the rules obtained in Problem 3 to sketch the graphs of the following:
   a. $y = x^2 + 1$ **b.** $y = (x+3)^2$
   c. $y = 3 - (x+1)^2$ **d.** $y = 2 - (x+2)^{-2}$

2.5 Linear Functions

A *linear relationship* between the variables x and y takes the form

$$y = ax + b \qquad (a \text{ and } b \text{ are constants})$$

The graph of the equation is a straight line. If we let f denote the function that assigns y to x, then $f(x) = ax + b$, and f is called a **linear** function.[3] The number a is called the **slope** of the function and of the line. Take an arbitrary value of x. Then $f(x+1) - f(x) = a(x+1) + b - ax - b = a$. This shows that the slope a measures the change in the value of the function when x increases by 1 unit.

If the slope a is positive, the line slants upward to the right, and the larger the value of a, the steeper is the line. On the other hand, if a is negative, then the line slants downward to the right, and the absolute value of a measures the steepness of the line. For example, when $a = -3$, the steepness is 3. In the special case when $a = 0$, then $y = ax + b = b$ for all x, and the line is parallel to the x-axis. The three cases are illustrated in Figs. 2.27 to 2.29. If $x = 0$, then $y = ax + b = b$, and b is called the y-**intercept** (or often just the intercept).

FIGURE 2.27

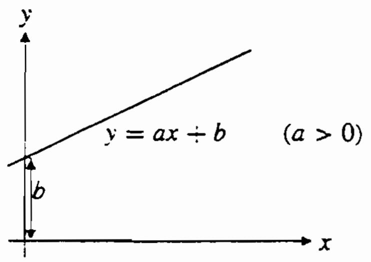
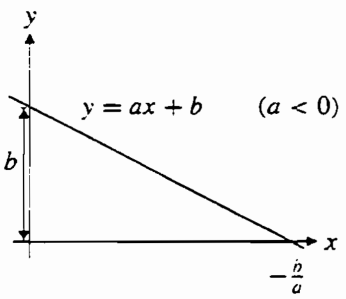


FIGURE 2.28



[3]Actually, mathematicians usually reserve the term "linear" for the functions defined by $y = ax$ (with the y-intercept $b = 0$). They call $y = ax + b$ with $b \neq 0$ an "affine" function. Most economists call $f(x) = ax + b$ a linear function.

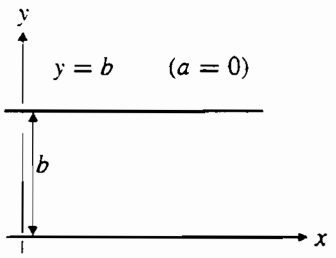


FIGURE 2.29

Example 2.12

Find and interpret the slopes of the following straight lines:

(a)

$$C = 55.73x + 182{,}100{,}000$$

which is the estimated cost function for the U.S. Steel Corp. over the period 1917–1938 (C is the total cost in dollars per year, and x is the production of steel in tons per year).

(b)

$$q = -0.15p + 0.14$$

which is the estimated annual demand function for rice in India for the period 1949–1964 (p is the price, and q is consumption per person).

Solution

(a) The slope is 55.73, which means that if production increases by 1 ton, then the cost *increases* by \$55.73.

(b) The slope is -0.15, which tells us that if the price increases by 1 unit, then the quantity demanded *decreases* by 0.15 unit.

Finding the Slope

Consider an arbitrary, nonvertical (straight) line in the plane. Pick two distinct points on the line, $P = (x_1, y_1)$ and $Q = (x_2, y_2)$, as shown in Fig. 2.30. Because the line is not vertical and because P and Q are distinct, $x_1 \neq x_2$. The slope of the line is the ratio $(y_2 - y_1)/(x_2 - x_1)$. If we denote the slope by a, then the following holds.

The **slope of a straight line** l is

$$a = \frac{y_2 - y_1}{x_2 - x_1}, \qquad x_1 \neq x_2 \qquad [2.5]$$

where (x_1, y_1) and (x_2, y_2) are any two distinct points on l.

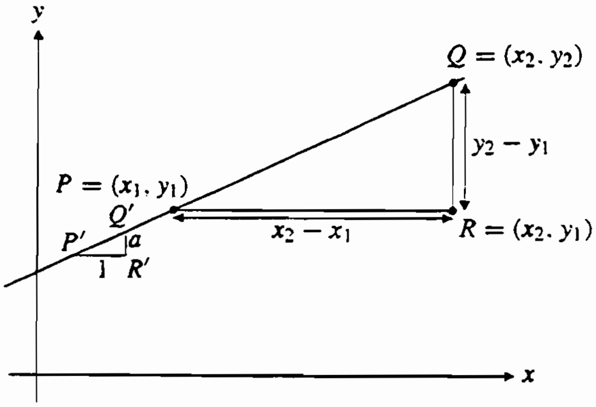


FIGURE 2.30 **Slope $a = (y_2 - y_1)/(x_2 - x_1)$.**

Multiplying both the numerator and the denominator of the fraction in [2.5] by -1, we obtain the fraction $(y_1 - y_2)/(x_1 - x_2)$. This shows that it does not make any difference which point is P and which is Q. Moreover, using the properties of similar triangles, we see by studying the two triangles PQR and $P'Q'R'$ in Fig. 2.30 that the number a in [2.5] is equal to the change in the value of y when x increases by 1 unit.

Example 2.13

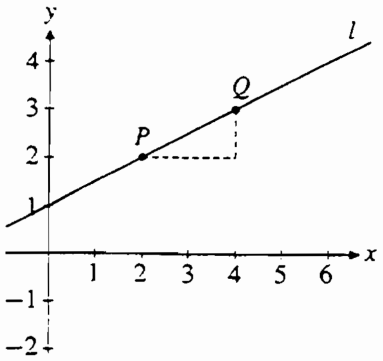
Determine the slopes of the three straight lines l, m, and n in Figs. 2.31–2.33 using [2.5].

Solution The lines l, m, and n all pass through $P = (2, 2)$. In Fig. 2.31, Q is $(4, 3)$. In Fig. 2.32, Q is $(1, -2)$. And in Fig. 2.33, Q is $(5, -1)$. Therefore, the respective slopes of the lines l, m, and n are

$$a_l = \frac{3-2}{4-2} = \frac{1}{2}, \qquad a_m = \frac{-2-2}{1-2} = 4, \qquad a_n = \frac{-1-2}{5-2} = -1$$

The following example illustrates a problem that is important in differential calculus, as will be seen in Chapter 4.

FIGURE 2.31 **The line l.**



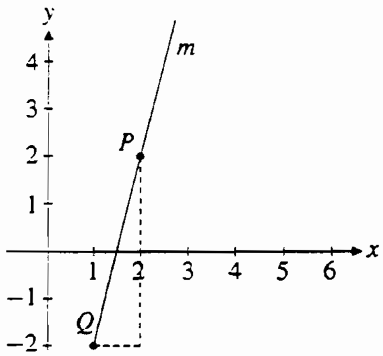


FIGURE 2.32 **The line *m*.**

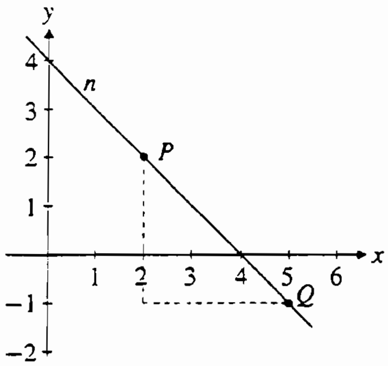


FIGURE 2.33 **The line *n*.**

Example 2.14

Find an expression for the slope of the line through the two points (x_0, x_0^2) and $\left(x_0 + h, (x_0 + h)^2\right)$, where $h \neq 0$.

Solution Apply formula [2.5] with $(x_1, y_1) = (x_0, x_0^2)$ and $(x_2, y_2) = \left(x_0 + h, (x_0 + h)^2\right)$ to obtain

$$a = \frac{(x_0 + h)^2 - x_0^2}{x_0 + h - x_0} = \frac{x_0^2 + 2x_0h + h^2 - x_0^2}{h} = \frac{2x_0h + h^2}{h} = 2x_0 + h$$

The Point–Slope and Point–Point Formulas

Let us find the equation of a straight line l passing through point $P = (x_1, y_1)$ with slope a. If (x, y) is any other point on the line, the slope a is given by formula [2.5]:

$$\frac{y - y_1}{x - x_1} = a$$

Multiplying each side by $x - x_1$, we obtain $y - y_1 = a(x - x_1)$. Hence:

> **Point–Slope Formula of a Straight Line**
>
> The equation of the straight line passing through (x_1, y_1) with slope a is
>
> $$y - y_1 = a(x - x_1) \qquad [2.6]$$

Note that when using equation [2.6], x_1 and y_1 are fixed numbers giving the coordinates of the fixed point. On the other hand, x and y are variables denoting an arbitrary point on the line.

Example 2.15

Find the equation of the line through $(-2, 3)$ with slope -4. Then find the point at which this line intersects the x-axis.

Solution The point–slope formula with $(x_1, y_1) = (-2, 3)$ and $a = -4$ gives

$$y - 3 = (-4)[x - (-2)] \quad \text{or} \quad y - 3 = -4(x + 2) \qquad [1]$$

The line intersects the x-axis at the point where $y = 0$, that is, where $0 - 3 = -4(x + 2)$ or $-3 = -4x - 8$. Solving for x, we get $x = -5/4$, so the point of intersection with the x-axis is $(-5/4, 0)$.

Often we need to find the equation of a straight line that passes through two given points. Combining [2.5] with [2.6], we obtain the following formula:

Point–Point Formula of a Straight Line

The equation of the straight line passing through (x_1, y_1) and (x_2, y_2), where $x_1 \neq x_2$, is obtained as follows:

1. Compute the slope of the line:

$$a = \frac{y_2 - y_1}{x_2 - x_1}$$

2. Substitute the expression for a into the point–slope formula $y - y_1 = a(x - x_1)$. The result is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) \qquad [2.7]$$

Example 2.16

Find the equation of the line passing through $(-1, 3)$ and $(5, -2)$.

Solution Let $(x_1, y_1) = (-1, 3)$ and $(x_2, y_2) = (5, -2)$. Then the point–point formula gives

$$y - 3 = \frac{-2 - 3}{5 - (-1)}[x - (-1)] \quad \text{or} \quad y - 3 = -\frac{5}{6}(x + 1)$$

$$\text{or} \quad 5x + 6y = 13$$

Linear Models

Linear relations occur frequently in applied models. The relationship between the Celsius and Fahrenheit temperature scales is an example of an exact linear relation between two variables. Most of the linear models in economics are approximations to more complicated models. Two typical relations are those shown in Example 2.12. Statistical methods have been devised to construct linear functions that approximate the actual data as closely as possible. Let us consider a very naïve attempt to construct a linear model based on some data.

Example 2.17

In a United Nations report, the European population in 1960 was estimated as 641 million, and in 1970 the estimate was 705 million. Use these estimates to construct a linear function of t that approximates the population in Europe (in millions), where t is the number of years from 1960 ($t = 0$ is 1960, $t = 1$ is 1961, and so on). Make use of the equation to estimate the population in 1975 and in 2000. How do you estimate the population in 1930 on the basis of this linear relationship?

Solution If P denotes the population in millions, we construct an equation of the form $P = at + b$. We know that the graph must pass through the points $(t_1, P_1) = (0, 641)$ and $(t_2, P_2) = (10, 705)$. Using the formula in [2.7], replacing x and y with t and P, respectively, we obtain

$$P - 641 = \frac{705 - 641}{10 - 0}(t - 0) = \frac{64}{10}t$$

or

$$P = 6.4t + 641 \qquad [1]$$

In Table 2.4, we have compared our estimates with UN forecasts. Note that because $t = 0$ corresponds to 1960, $t = -30$ will correspond to 1930.

Note that the slope of line [1] is 6.4. This means that if the European population had developed according to [1], then the annual increase in the population would have been constant and equal to 6.4 million.

Actually, Europe's population grew unusually fast during the 1960s. Of course, it grew unusually slowly when millions died during the war years 1939–1945. We see that formula [1] does not give very good results compared to the UN estimates. (For a better way to model population growth, see Example 3.12 in Section 3.5.)

Example 2.18 (The Consumption Function)

In Keynesian macroeconomic theory, total consumption expenditure on goods and services, C, is assumed to be a function of national income

TABLE 2.4 *Population estimates for Europe*

Year	*1930*	*1975*	*2000*
t	−30	15	40
UN estimates	573	728	854
Formula [1] gives	449	737	897

Y, with

$$C = f(Y) \tag{2.8}$$

In many models, the consumption function is assumed to be linear, so that

$$C = a + bY$$

The slope b is called the **marginal propensity to consume**. If C and Y are measured in millions of dollars, the number b tells us by how many millions of dollars consumption increases if the national income increases by 1 million dollars. The number b will usually lie between 0 and 1.

In a study of the U.S. economy for the period 1929–1941, T. Haavelmo found the following consumption function:

$$C = 95.05 + 0.712\,Y$$

Here, the marginal propensity to consume is equal to 0.712.

Example 2.19

Some other economic examples of linear functions are the following demand and supply schedules:

$$D = a - bP$$

$$S = \alpha + \beta P$$

Here a and b (both positive) are parameters of the demand function D, while α and β (both positive) are parameters of the supply function. Such functions play an important role in quantitative economics. It is often the case that the market for a particular commodity, such as a specific brand of $3\frac{1}{2}$-inch computer diskettes, can be represented approximately by linear demand and supply functions. The equilibrium price P^e must equate demand and supply, so that $D = S$ at $P = P^e$. Thus,

$$a - bP^e = \alpha + \beta P^e$$

Adding $bP^e - \alpha$ to each side gives

$$a - bP^e + bP^e - \alpha = \alpha + \beta P^e + bP^e - \alpha$$

Thus, $a - \alpha = (\beta + b)P^e$. The corresponding equilibrium quantity is $Q^e = a - bP^e$. Hence, equilibrium occurs when

$$P^e = \frac{a - \alpha}{\beta + b}, \qquad Q^e = a - b\frac{a - \alpha}{\beta + b} = \frac{a\beta + \alpha b}{\beta + b}$$

If the four parameters, a, b, α, and β, were all known, then the model would be complete and the equilibrium price and quantity could be predicted. Suppose that there is a later shift in the supply or demand function—for instance, suppose supply increases so that S becomes $\tilde{\alpha} + \beta P$, where $\tilde{\alpha} > \alpha$. Then we could predict that the new equilibrium price and quantity would be

$$\tilde{P}^e = \frac{a - \tilde{\alpha}}{\beta + b}, \qquad \tilde{Q}^e = \frac{a\beta + \tilde{\alpha} b}{\beta + b}$$

Here $\tilde{P}^e$ is less than P^e, whereas $\tilde{Q}^e$ is greater than Q^e. In fact,

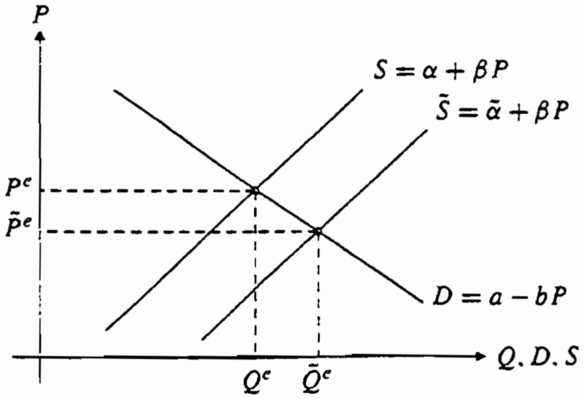
$$\tilde{P}^e - P^e = \frac{\alpha - \tilde{\alpha}}{\beta + b} \quad \text{and} \quad \tilde{Q}^e - Q^e = \frac{(\tilde{\alpha} - \alpha)b}{\beta + b} = -b(\tilde{P}^e - P^e)$$

This is in accord with Fig. 2.34. The rightward shift in the supply curve from S to $\tilde{S}$ moves the equilibrium down and to the right along the unchanged demand curve.

A peculiarity of Fig. 2.34 is that, although quantity is a function of price, here we measure price on the vertical axis and quantity on the horizontal axis. This has been standard practice in elementary price theory since the work of Alfred Marshall late in the nineteenth century.

The trouble with this method of analysis comes when the parameters are not known, so the supply and demand curves cannot be drawn with any certainty. Indeed, if all an economist observes is a decrease in price and an increase in quantity from (P^e, Q^e) to $(\tilde{P}^e, \tilde{Q}^e)$, there is no way of knowing (without more information) whether this results from just a rightward shift in the supply curve, as illustrated in Fig. 2.34, or from some combination of a shift to the right (or left) in demand *and* a shift to

FIGURE 2.34



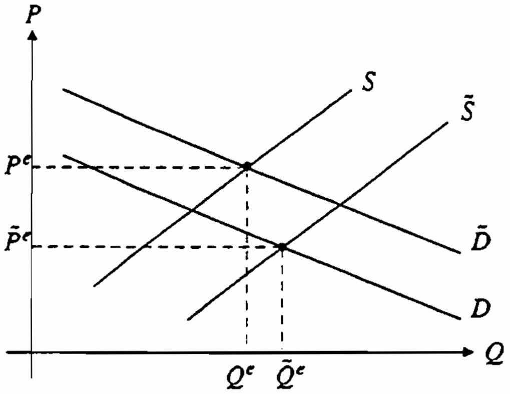


FIGURE 2.35

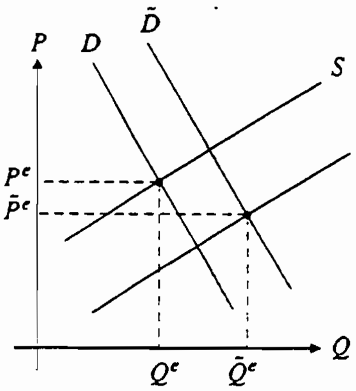


FIGURE 2.36

the right in supply, as illustrated in Figs. 2.35 and 2.36. All that can be said is that, because the equilibrium price falls and the quantity increases, there must have been some rightward shift in supply—but demand could have fallen, risen, or stayed the same. Moreover, there is also the possibility that the demand and supply curves could have changed their slopes—that is, the parameters b and β could also have changed.

The General Equation for a Straight Line

Any nonvertical line in the plane has the equation $y = ax + b$. A vertical line, that is parallel to the y-axis, will intersect the x-axis at some point $(c, 0)$. Every point on the line has the same x-coordinate c, so the line must be

$$x = c$$

This is the equation for a straight line through $(c, 0)$ *parallel to the* y*-axis.*

The equations $y = ax + b$ and $x = c$ can both be written as

$$Ax + By + C = 0 \qquad [2.9]$$

for suitable values of the constants A, B, and C. Specifically, $y = ax + b$ corresponds to $A = a$, $B = -1$, and $C = b$, whereas $x = c$ corresponds to $A = 1$, $B = 0$, and $C = -c$. Conversely, every equation of the form [2.9] represents a straight line in the plane, disregarding the uninteresting case when $A = B = 0$. If $B = 0$, it follows from [2.9] that $Ax = -C$, or $x = -C/A$. This is the equation for a straight line parallel to the y-axis. On the other hand, if $B \neq 0$, solving [2.9] for y yields

$$y = -\frac{A}{B}x - \frac{C}{B}$$

This is the equation for a straight line with slope $-A/B$. Equation [2.9] thus deserves to be called the *general equation for a straight line in the plane.*

Graphical Solutions of Linear Equations

Section A.9 of Appendix A deals with algebraic methods for solving a system of linear equations in two unknowns. The equations are linear, so their graphs are straight lines. The coordinates of any point on a line satisfy the equation of that line. Thus, the coordinates of any point of intersection of these lines will satisfy both equations. This means that a point of intersection solves the system.

Example 2.20

Solve each of the following three pairs of equations graphically:

(a) $x + y = 5$ and $x - y = -1$

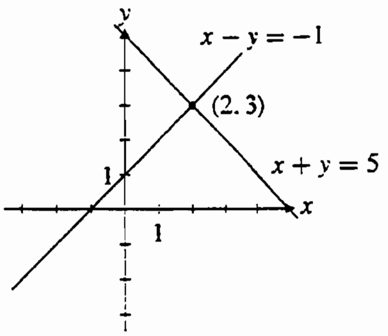
(b) $3x + y = -7$ and $x - 4y = 2$

(c) $3x + 4y = 2$ and $6x + 8y = 24$

Solution

(a) Figure 2.37 shows the graphs of the straight lines $x + y = 5$ and $x - y = -1$. There is only one point of intersection $(2, 3)$. The solution of the system is, therefore, $x = 2$, $y = 3$.

FIGURE 2.37



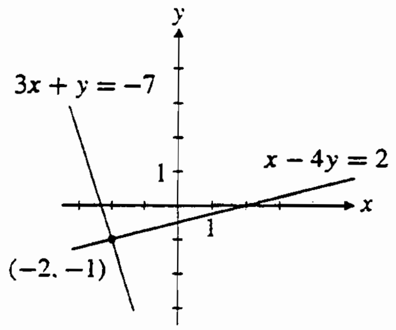


FIGURE 2.38

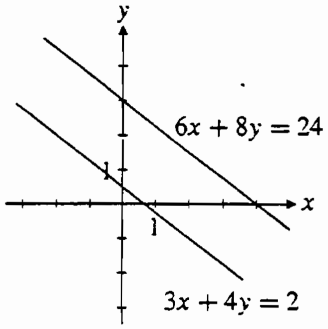


FIGURE 2.39

(b) Figure 2.38 shows the graphs of the straight lines $3x + y = -7$ and $x - 4y = 2$. There is only one point of intersection $(-2, -1)$. The solution of the system is, therefore, $x = -2$, $y = -1$.

(c) Figure 2.39 shows the graphs of the straight lines $3x + 4y = 2$ and $6x + 8y = 24$. These lines are parallel and have no point of intersection. The system has no solutions.

Linear Inequalities

This chapter concludes by discussing how to represent linear inequalities geometrically. Consider two examples.

Example 2.21

Sketch in the xy-plane the set of all pairs of numbers (x, y) that satisfy the inequality $2x + y \leq 4$. (Using set notation, this is $\{(x, y) : 2x + y \leq 4\}$.)

Solution The inequality can be written as $y \leq -2x + 4$. The set of points (x, y) that satisfy the equation $y = -2x + 4$ is a straight line. Therefore, the set of points (x, y) that satisfy the inequality $y \leq -2x + 4$ must have y-values below those of points on the line $y = -2x + 4$. So it must consist of all points that lie on or below this straight line. See Fig. 2.40.

Example 2.22

A person has $\$m$ to spend on the purchase of two commodities. The prices of the two commodities are $\$p$ and $\$q$ per unit. Suppose x units of the first

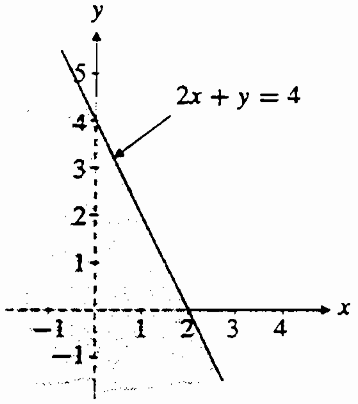


FIGURE 2.40 $\{(x, y) : 2x + y \le 4\}$.

commodity and y units of the second commodity are bought. Assuming one cannot purchase negative units of x and y, the *budget set* is

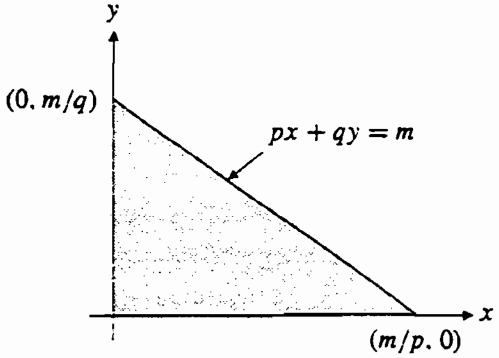
$$B = \{(x, y) : \; px + qy \le m, \; x \ge 0, \; y \ge 0\}$$

as in (1.7) in Section 1.7. Sketch the budget set B in the xy-plane. Find the slope of the budget line $px + qy = m$, and its points of intersection with the two coordinate axes.

Solution The set of points (x, y) that satisfy $x \ge 0$ and $y \ge 0$ was sketched in Fig. 2.10. It is the first (nonnegative) quadrant. If we impose the additional requirement that $px + qy \le m$, we obtain the triangular domain B shown in Fig. 2.41.

If we solve $px + qy = m$ for y, we get $y = (-p/q)x + m/q$, so the slope is $-p/q$. The budget line intersects the x-axis when $y = 0$. Then $px = m$, so $x = m/p$. The budget line intersects the y-axis when $x = 0$. Then $qy = m$, so $y = m/q$. So the two points of intersection are $(m/p, 0)$ and $(0, m/q)$, as shown in Fig. 2.41.

FIGURE 2.41 **Budget set:** $px + qy \le m$, $x \ge 0$, **and** $y \ge 0$.



Problems

1. Find the slopes of the lines passing through the following points by using the formula in [2.5].

   a. (2, 3) and (5, 8) **b.** (−1, −3) and (2, −5) **c.** $\left(\frac{1}{2}, \frac{3}{2}\right)$ and $\left(\frac{1}{3}, -\frac{1}{5}\right)$

2. The consumption function $C = 4141 + 0.78\,Y$ for the UK was estimated for the period 1949–1975. What is the marginal propensity to consume?

3. Find the slopes of the five lines L_1 to L_5 shown in Fig. 2.42, and give equations describing them. (L_3 is horizontal.)

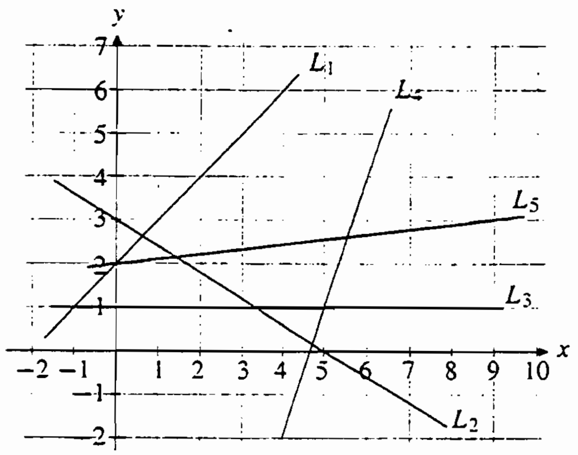


FIGURE 2.42

4. Draw graphs for the following equations:

   a. $3x + 4y = 12$ **b.** $\dfrac{x}{10} - \dfrac{y}{5} = 1$ **c.** $x = 3$

5. Decide which of the following relationships are linear:

   a. $5y + 2x = 2$ **b.** $P = 10(1 - 0.3t)$ **c.** $C = (0.5x + 2)(x - 3)$

   d. $p_1x_1 + p_2x_2 = R$ (p_1, p_2, and R constants)

6. **a.** Determine the relationship between Centigrade and Fahrenheit temperature scales when you know that (i) the relation is linear; (ii) water freezes at 0°C and 32°F; and (iii) water boils at 100°C and 212°F.

   b. Which temperature is measured by the same number in both Centigrade and Fahrenheit scales?

7. Determine the equations and draw graphs for the following straight lines:

   a. L_1 passes through (1, 3) and has a slope of 2.

   b. L_2 passes through (−2, 2) and (3, 3).

   c. L_3 passes through the origin and has a slope of −1/2.

   d. L_4 passes through $(a, 0)$ and $(0, b)$ (suppose $a \neq 0$).

8. A line L passes through the point (1, 1) and has a slope of 3. A second line M passes through (−1, 2) and (3, −1). Find the equations for L and M

and their point of intersection, P. Also determine the equation for the line N that passes through $(-1, -1)$ and is parallel to M. Draw the figure.

9. The total cost y of producing x units of some commodity is a linear function. Records show that on one occasion, 100 units were made at a total cost of \$200, and on another occasion, 150 units were made at a total cost of \$275. Express the linear equation for total cost in terms of the number of units x produced.

10. Find the equilibrium price in the model in Example 2.19 for the following.

a. $D = 75 - 3P, \quad S = 20 + 2P$ **b.** $D = 100 - 0.5P, \quad S = 10 + 0.5P$

11. According to 20th report of the International Commission on Whaling, the number N of fin whales in the Antarctic for the period 1958–1963 was given by

$$N = -17{,}400t + 151{,}000, \qquad 0 \leq t \leq 5$$

where $t = 0$ corresponds to January 1958, $t = 1$ corresponds to January 1959, and so on.

a. According to this equation, how many fin whales would there be left in April 1960?

b. If the decrease continued at the same rate, when would there be no fin whales left? (Actually, the 1993 estimate was approximately 21,000.)

12. The expenditure of a household on consumer goods, C, is related to the household's income, y, in the following way: When the household's income is \$1000, the expenditure on consumer goods is \$900, and whenever income is increased by \$100, the expenditure on consumer goods is increased by \$80. Express the expenditure on consumer goods as a function of income, assuming a linear relationship.

13. Solve the following three systems of equations graphically:

a. $x - y = 5$ and $x + y = 1$

b. $x + y = 2$, $x - 2y = 2$ and $x - y = 2$

c. $3x + 4y = 1$ and $6x + 8y = 6$

14. Show that $-1/[x_0(x_0 + h)]$ is the slope of the line passing through P and Q in Fig. 2.43.

15. The following table shows the total consumption and net national income in some country for the period from 1955–1960, measured in millions of dollars. Plot the points from the table in the YC-plane. Draw the straight line through the "extreme points" $(21.3, 17.4)$ and $(24.7, 20.4)$. Find the equation for this line. What is the interpretation of its slope?

Year	*1955*	*1956*	*1957*	*1958*	*1959*	*1960*
Total consumption (C)	17.4	18.0	18.4	18.6	19.3	20.4
Net national product (Y)	21.3	22.4	23.0	22.6	23.4	24.7

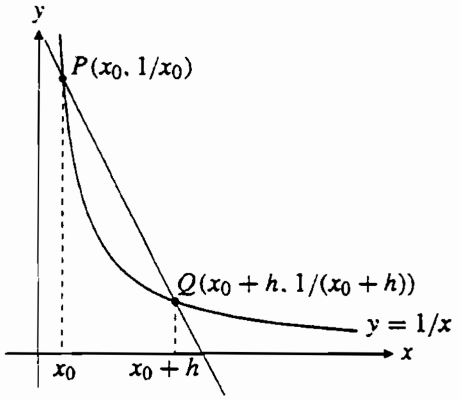


FIGURE 2.43

16. Sketch in the xy-plane the set of all pairs of numbers (x, y) that satisfy the following inequalities:

a. $2x + 4y \geq 5$ **b.** $x - 3y + 2 \leq 0$ **c.** $100x + 200y \leq 300$

17. Sketch in the xy-plane the set of all pairs of numbers (x, y) that satisfy all the following three inequalities: $3x + 4y \leq 12$; $x - y \leq 1$; and $3x + y \geq 3$.

3

Polynomials, Powers, and Exponentials

The paradox is now fully established that the utmost abstractions are the true weapons with which to control our thought of concrete facts.
—A. N. Whitehead

The linear functions and associated linear models that were studied in some detail in the previous chapter are particularly simple. Not surprisingly, most economic applications require much more accuracy than is possible with only linear functions, and so economists most often use more complicated functions.

3.1 Quadratic Functions

Many economic models involve functions that either decrease down to some minimum value and then increase, or else increase up to some maximum value and then decrease. Simple functions with this property are the general **quadratic** functions

$$f(x) = ax^2 + bx + c \qquad (a,\ b,\ \text{and are } c \text{ constants},\ a \neq 0) \qquad [3.1]$$

(If $a = 0$, the function is linear, hence, the restriction $a \neq 0$.) Figure 2.20 of Section 2.4 shows the graph of $f(x) = x^2 - 3x$, which is obtained from [3.1] by choosing $a = 1$, $b = -3$, and $c = 0$. In general, the graph of $f(x) = ax^2 + bx + c$ is called a **parabola**. The shape of this parabola roughly

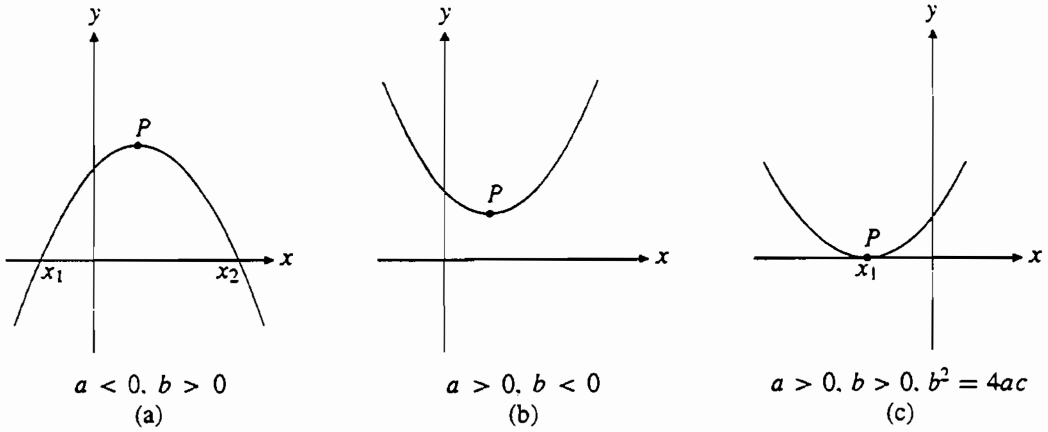


FIGURE 3.1 Graphs of the parabola $y = ax^2 + bx + c$.

resembles ∩ when $a < 0$ and ∪ when $a > 0$. Three typical cases are illustrated in Fig. 3.1.

In order to understand the function $f(x) = ax^2 + bx + c$ in more detail, we are interested in the answers to the following questions:

1. For what values of x (if any) is $ax^2 + bx + c = 0$?
2. What are the coordinates of the maximum/minimum point P?

In the case of question 1, we have to find solutions to the equation $f(x) = 0$. Geometrically, this involves determining points of intersection of the parabola with the x-axis. These points are called the **zeros** of the quadratic function. In Fig. 3.1(a), the zeros are given by x_1 and x_2, in Fig. 3.1(b) there are no zeros, whereas the graph in Fig. 3.1(c) has x_1 as its only point of intersection with the x-axis. In Section A.8 of Appendix A it is proved that, in the case when $b^2 \geq 4ac$ and $a \neq 0$, then

$$ax^2 + bx + c = 0 \iff x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad [3.2]$$

To derive this formula, we used the method known as "completing the square." This technique will also help us answer question 2. In fact, when $a \neq 0$, the function defined by [3.1] can be expressed as

$$f(x) = a\left[x^2 + 2\left(\frac{b}{2a}\right)x + \left(\frac{b}{2a}\right)^2\right] - a\left(\frac{b}{2a}\right)^2 + c = a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a} \qquad [3.3]$$

Consider the expression after the second equality sign of [3.3]. When x varies, only the value of $a(x + b/2a)^2$ changes. This term is equal to 0 when $x = -b/2a$, and if $a > 0$, it is never less than 0. This means that when $a > 0$, then the function $f(x)$ attains its minimum when $x = -b/2a$, and the value of $f(x)$ is then equal to $f(-b/2a) = -(b^2 - 4ac)/4a = c - b^2/4a$. If $a < 0$ on the other hand, then $a(x + b/2a)^2 \leq 0$ for all x, and the squared term is equal to 0 when $x = -b/2a$. Hence, $f(x)$ attains its maximum when $x = -b/2a$ in this second case. To summarize, we have shown the following:

If $a > 0$, then $f(x) = ax^2 + bx + c$ has a **minimum** at

$$\left(-\frac{b}{2a}, c - \frac{b^2}{4a}\right)$$

If $a < 0$, then $f(x) = ax^2 + bx + c$ has a **maximum** at

$$\left(-\frac{b}{2a}, c - \frac{b^2}{4a}\right)$$

[3.4]

If you find it difficult to follow the argument leading up to [3.4], you should study the following special examples very carefully.

Example 3.1

Complete the square as in [3.3] for the following functions and then find the maximum/minimum point of each:

(a) $f(x) = x^2 - 4x + 3$
(b) $f(x) = -2x^2 + 40x - 600$
(c) $f(x) = \frac{1}{3}x^2 + \frac{2}{3}x - \frac{8}{3}$

Solution

(a) $x^2 - 4x + 3 = (x^2 - 4x) + 3 = (x^2 - 4x + 4) - 4 + 3 = (x - 2)^2 - 1$

The expression $(x - 2)^2 - 1$ attains its smallest value, which is -1, at $x = 2$.

(b)
$$\begin{aligned} -2x^2 + 40x - 600 &= -2(x^2 - 20x) - 600 \\ &= -2(x^2 - 20x + 100) + 200 - 600 \\ &= -2(x - 10)^2 - 400 \end{aligned}$$

The expression $-2(x - 10)^2 - 400$ attains its largest value, which is -400, at $x = 10$.

(c) $\frac{1}{3}x^2 + \frac{2}{3}x - \frac{8}{3} = \frac{1}{3}(x^2 + 2x) - \frac{8}{3}$

$$= \frac{1}{3}(x^2 + 2x + 1) - \frac{1}{3} - \frac{8}{3}$$

$$= \frac{1}{3}(x + 1)^2 - 3$$

The expression $\frac{1}{3}(x + 1)^2 - 3$ attains its smallest value, which is -3, at $x = -1$.

A useful exercise is to solve the three cases in Example 3.1 by using the expressions set out in [3.4] directly, substituting appropriate values for the three parameters a, b, and c. You should then check that the same results are obtained.

Problems

1. a. Let $f(x) = x^2 - 4x$. Complete the following table:

x	−1	0	1	2	3	4	5
$f(x)$							

b. Using the table in part (a), sketch the graph of f.
c. Using [3.3], determine the minimum point.
d. Solve the equation $f(x) = 0$.

2. a. Let $f(x) = -\frac{1}{2}x^2 - x + \frac{3}{2}$. Complete the following table:

x	−4	−3	−2	−1	0	1	2
$f(x)$							

b. Use the information in part (a) to sketch the graph of f.
c. Using [3.3], determine the maximum point.
d. Solve the equation $-\frac{1}{2}x^2 - x + \frac{3}{2} = 0$ for x.
e. Show that $f(x) = -\frac{1}{2}(x - 1)(x + 3)$, and use this to study how the sign of f varies with x. Compare the result with the graph.

3. Complete the squares as in [3.3] for the following quadratic functions, and then determine the maximum/minimum points:

a. $x^2 + 4x$　　b. $x^2 + 6x + 18$　　c. $-3x^2 + 30x - 30$
d. $9x^2 - 6x - 44$　　e. $-x^2 - 200x + 30{,}000$　　f. $x^2 + 100x - 20{,}000$

4. Find the zeros of each quadratic function in Problem 3, and write each function in the form $a(x - x_1)(x - x_2)$ (if possible).

5. Use the formula in [3.2] to find solutions to the following equations, where p and q are positive parameters.
   a. $x^2 - 3px + 2p^2 = 0$ **b.** $x^2 - (p+q)x + pq = 0$
   c. $x^2 + px + q = 0$
6. A person is given a rope of length L with which to enclose a rectangular area.
   a. If one of the sides is x, show that the area of the enclosure is $A(x) = Lx/2 - x^2$, where $0 \leq x \leq L/2$. Find x such that the area of the rectangle is maximized.
   b. Will a circle of circumference L enclose an area that is larger than the one we found in part (a)? (It is reported that certain surveyors in antiquity wrote contracts with farmers to sell them rectangular pieces of land in which only the circumference was specified. As a result, the lots were long narrow rectangles.)
7. Consider the function given by the formula $A = 500x - x^2$ in Example 1.1 of Section 1.3. What choice of x gives the largest value for the area A?
8. **a.** Solve $x^4 - 5x^2 + 4 = 0$. (*Hint:* Put $x^2 = u$ and form a quadratic equation in u.)
   b. Solve the equations (i) $x^4 - 8x^2 - 9 = 0$ and (ii) $x^6 - 9x^3 + 8 = 0$.
9. A model occurring in the theory of efficient loan markets involves the function

$$U(x) = 72 - (4+x)^2 - (4-rx)^2$$

   where r is a constant. Find the value of x for which $U(x)$ attains its largest value.
10. Find the equation for the parabola $y = ax^2 + bx + c$ that passes through the three points $(1, -3)$, $(0, -6)$, and $(3, 15)$. (*Hint:* Determine a, b, and c.)

Harder Problems

11. The graph of a function f is said to be *symmetric* about the line $x = p$ if

$$f(p-t) = f(p+t) \qquad \text{(for all } t\text{)}$$

   Show that the parabola $f(x) = ax^2 + bx + c$ is symmetric about the line $x = -b/2a$. (*Hint:* Use [3.3].)
12. Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be arbitrary real numbers. We claim that the following inequality (called the **Cauchy–Schwarz inequality**) is always valid:

$$(a_1b_1 + a_2b_2 + \cdots + a_nb_n)^2 \leq (a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2) \qquad [3.5]$$

a. Check the inequality for (i) $a_1 = 1$, $a_2 = 3$, $b_1 = 2$, and $b_2 = 5$; and for (ii) $a_1 = -3$, $a_2 = 2$, $b_1 = 5$, and $b_2 = -2$. (In both cases, $n = 2$.)

b. Prove [3.5] by means of the following trick: first, define f for all x by

$$f(x) = (a_1 x + b_1)^2 + \cdots + (a_n x + b_n)^2$$

We see that $f(x) \geq 0$ for all x. Write $f(x)$ as $Ax^2 + Bx + C$, where the expressions for A, B, and C are related to the terms in [3.5]. Because $Ax^2 + Bx + C \geq 0$ for all x, we must have $B^2 - 4AC \leq 0$. Why? The conclusion follows.

3.2 Examples of Quadratic Optimization Problems

Much of mathematical economics is concerned with optimization problems. Economics, after all, is the science of choice, and optimization problems are the form in which choice is usually expressed mathematically.

A general discussion of such problems must be postponed until we have developed the necessary tools from calculus. Here we show how the simple results from the previous section on maximizing quadratic functions can be used to illustrate some basic economic ideas.

Example 3.2 (A Monopoly Problem)

Consider a firm that is the only seller of the commodity it produces, possibly a patented medicine, and so enjoys a monopoly. The total costs of the monopolist are assumed to be given by the quadratic function

$$C = \alpha Q + \beta Q^2, \qquad Q \geq 0 \tag{1}$$

of its output level Q, where α and β are positive constants. For each Q, the price P at which it can sell its output is assumed to be determined from the linear "inverse" demand function

$$P = a - bQ, \qquad Q \geq 0 \tag{2}$$

where a and b are constants with $a > 0$ and $b \geq 0$. So for any nonnegative Q, the total revenue R is given by the quadratic function

$$R = PQ = (a - bQ)Q$$

and profit by the quadratic function[1]

$$\pi(Q) = R - C = (a - bQ)Q - \alpha Q - \beta Q^2$$
$$= (a - \alpha)Q - (b + \beta)Q^2 \qquad [3]$$

The monopolist's objective is to maximize $\pi = \pi(Q)$. By using [3.4], we see that there is a maximum of π (for the monopolist M) at

$$Q^M = \frac{a - \alpha}{2(b + \beta)} \quad \text{with} \quad \pi^M = \frac{(a - \alpha)^2}{4(b + \beta)} \qquad [4]$$

This is valid if $a > \alpha$; if $a \leq \alpha$, the firm will not produce, but will have $Q^M = 0$ and $\pi^M = 0$. The two cases are illustrated in Figs. 3.2 and 3.3. The associated price and cost can be found by routine algebra.

If we put $b = 0$ in [2], then $P = a$ for all Q. In this case, the firm's choice of quantity does not influence the price at all and so the firm is said to be *perfectly competitive*. By replacing a by P in [3] and putting $b = 0$, we see that profit is maximized for a perfectly competitive firm at

$$Q^* = \frac{P - \alpha}{2\beta} \quad \text{with} \quad \pi^* = \frac{(P - \alpha)^2}{4\beta} \qquad [5]$$

provided that $P > \alpha$. If $P \leq \alpha$, then $Q^* = 0$ and $\pi^* = 0$.

FIGURE 3.2 The profit function, $a > \alpha$.

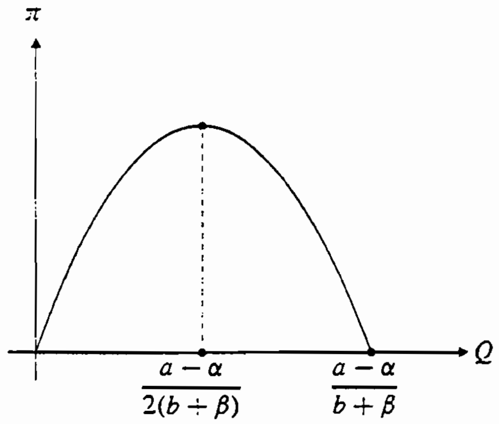
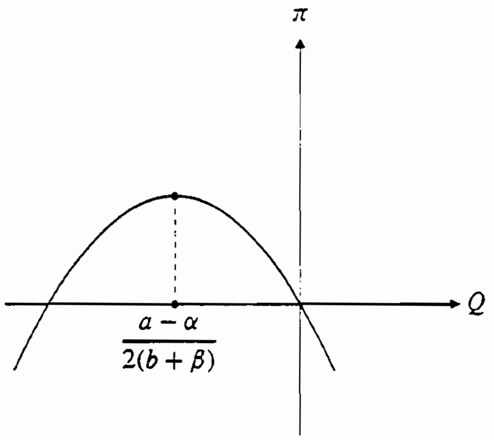


FIGURE 3.3 The profit function, $a \leq \alpha$.



[1]Previously, π has been used to denote the constant ratio 3.14159... between the circumference of a circle and its diameter. In economics, this constant is not used very often, so π has come to denote profit, or probability.

Solving the first equation in [5] for P yields $P = \alpha + 2\beta Q^*$. Thus,

$$P = \alpha + 2\beta Q \qquad [6]$$

represents the supply curve of this perfectly competitive firm for $P > \alpha$ when $Q^* > 0$, whereas for $P \leq \alpha$, the profit maximizing output Q^* is 0. The supply curve relating the price on the market to the firm's choice of output quantity is shown in Fig. 3.4; it includes points between the origin and $(0, x)$.

Let us return to the monopoly firm (which has no supply curve). If it could somehow be made to act like a competitive firm, taking price as given, it would be on the supply curve [6]. Given the demand curve $P = a - bQ$, equilibrium between supply and demand occurs when [6] is also satisfied, and so $P = a - bQ = \alpha + 2\beta Q$. Solving the second equation for Q, and then substituting for P and π in turn, we see that the equilibrium level of output, the corresponding price, and the profit would be

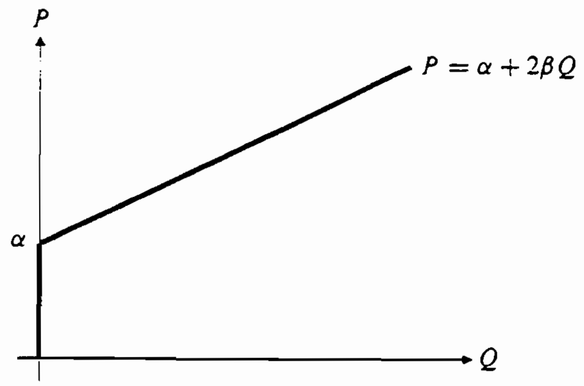
$$Q^e = \frac{a-\alpha}{b+2\beta}, \qquad P^e = \frac{2a\beta + \alpha b}{b+2\beta}, \qquad \pi^e = \frac{\beta(a-\alpha)^2}{(b+2\beta)^2} \qquad [7]$$

In order to have the monopolist mimic a competitive firm by choosing to be at (Q^e, P^e), it may be desirable to tax (or subsidize) the output of the monopolist. Suppose that the monopolist is required to pay a specific tax of t per unit of output. Because the tax payment tQ is added to the firm's costs, the new total cost function is

$$\begin{aligned} C &= \alpha Q + \beta Q^2 + tQ \\ &= (\alpha + t)Q + \beta Q^2 \qquad [8] \end{aligned}$$

Carrying out the same calculations as before, but with α replaced by $\alpha + t$,

FIGURE 3.4 The supply curve of a perfectly competitive firm.



gives the monopolist's choice of output as

$$Q_t^M = \begin{cases} \dfrac{a-\alpha-t}{2(b+\beta)}, & \text{if } a \geq \alpha + t \\ 0, & \text{otherwise} \end{cases} \qquad [9]$$

So $Q_t^M = Q^e$ when $(a-\alpha-t)/2(b+\beta) = (a-\alpha)/(b+2\beta)$. Solving this equation for t yields $t = -(a-\alpha)b/(b+2\beta)$. Note that t is actually negative, indicating the desirability of *subsidizing* the output of the monopolist in order to encourage additional production. (Of course, subsidizing monopolists is usually felt to be unjust, and many additional complications need to be considered carefully before formulating a desirable policy for dealing with monopolists. Still the previous analysis suggests that if justice requires lowering a monopolist's price or profit, this is much better done directly than by taxing output.)

Problem

1. If a cocoa shipping firm sells Q tons of cocoa in England, the price received is given by $P = \alpha_1 - \frac{1}{3}Q$. On the other hand, if it buys Q tons from its only source in Ghana, the price it has to pay is given by $P = \alpha_2 + \frac{1}{6}Q$. In addition, it costs γ per ton to ship cocoa from its supplier in Ghana to its customers in England (its only market). The numbers α_1, α_2, and γ are all positive.
   a. Express the cocoa shipper's profit as a function of Q, the number of tons shipped.
   b. Assuming that $\alpha_1 - \alpha_2 - \gamma > 0$, find the profit maximizing shipment of cocoa. What happens if $\alpha_1 - \alpha_2 - \gamma \leq 0$?
   c. Suppose the government of Ghana imposes an export tax on cocoa of t per ton. Find the new expression for the shipper's profits and the new quantity shipped.
   d. Calculate the government's export tax revenue as a function of t, and advise it on how to obtain as much tax revenue as possible.

3.3 Polynomials

After considering linear and quadratic functions, the logical next step is to examine **cubic functions** of the form

$$f(x) = ax^3 + bx^2 + cx + d \qquad (a, b, c, \text{ and } d \text{ are constants; } a \neq 0) \qquad [3.6]$$

It is relatively easy to understand the behavior of linear and quadratic functions from their graphs. Cubic functions are considerably more complicated, because

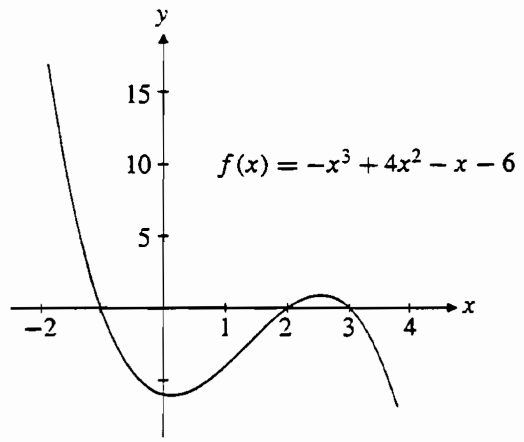


FIGURE 3.5 A cubic function.

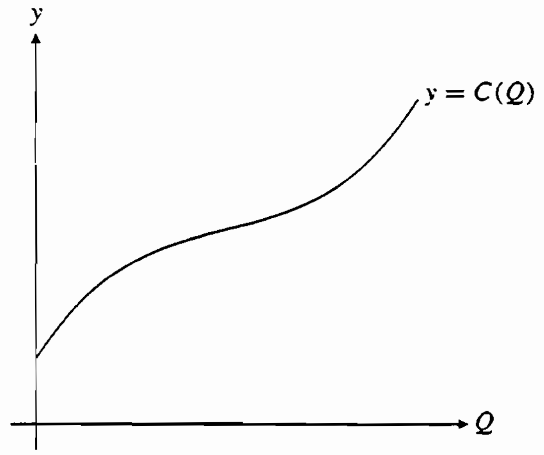


FIGURE 3.6 A cubic cost function.

the shape of their graphs changes drastically as the coefficients a, b, c, and d vary. Two examples are given in Figs. 3.5 and 3.6.

Cubic functions do occasionally appear in economic models. Let us look at a typical example.

Example 3.3

Consider a firm producing a single commodity. The total cost of producing Q units of the commodity is $C(Q)$. Cost functions often have the following properties: First, $C(0)$ is positive, because an initial fixed expenditure is involved. When production increases, costs also increase. In the beginning, costs increase rapidly, but the rate of increase slows down as production equipment is used for a higher proportion of each working week. However, at high levels of production, costs again increase at a fast rate, because of technical bottlenecks and overtime payments to workers, for example. The cubic cost function $C(Q) = aQ^3 + bQ^2 + cQ + d$ exhibits this type of behavior provided that $a > 0$, $b < 0$, $c > 0$, and $d > 0$ with $3ac > b^2$. Such a function is sketched in Fig. 3.6.

General Polynomials

Linear, quadratic, and cubic functions are all examples of **polynomials**. The function P defined for all x by

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \qquad (a\text{'s are constants; } a_n \neq 0) \quad [3.7]$$

is called the **general polynomial of degree n**. When $n = 4$, we obtain $P(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$, which is the general polynomial of degree 4.

Numerous problems in mathematics and its applications involve polynomials. Often, one is particularly interested in finding the number and location of the zeros of $P(x)$—that is, the values of x such that $P(x) = 0$. The equation

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0 \qquad [3.8]$$

is called the **general nth-order equation**. It will soon be shown that this equation has *at most n* (real) solutions, also called **roots**, but it need not have any.

According to the **fundamental theorem of algebra**, every polynomial of the form [3.7] can be written as a product of polynomials of first or second degree. Here is a somewhat complicated case:

$$x^5 - x^4 + x - 1 = (x-1)(x^4+1) = (x-1)\left(x^2 - \sqrt{2}x + 1\right)\left(x^2 + \sqrt{2}x + 1\right)$$

Integer Roots

Suppose that x_0 is an integer that satifies the cubic equation $-x^3 + 4x^2 - x - 6 = 0$, or, equivalently, $-x^3 + 4x^2 - x = 6$. Then x_0 must also satisfy the equation

$$x_0(-x_0^2 + 4x_0 - 1) = 6 \qquad [*]$$

Because x_0 is an integer, it follows that x_0^2, $4x_0$, and $-x_0^2 + 4x_0 - 1$ must also be integers. But because x_0 multiplied by the integer $-x_0^2 + 4x_0 - 1$ is equal to 6, the number x_0 must be a factor of 6—that is, 6 must be divisible by x_0. Now, the only integers by which 6 is divisible are ± 1, ± 2, ± 3, and ± 6. Direct substitution into the left-hand side (LHS) of equation [*] reveals that of these eight possibilities, -1, 2, and 3 are roots of the equation. A third degree equation has at most three roots, so we have found all of them. In general, we can state the following result:

> Suppose that $a_n, a_{n-1}, \ldots, a_1, a_0$ are all integers. Then all possible integer roots of the equation
>
> $$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$$
>
> must be factors of the constant term a_0.

[3.9]

Proof If x_0 is an integer root, then x_0 must satisfy the equation

$$x_0(a_n x_0^{n-1} + a_{n-1}x_0^{n-2} + \cdots + a_1) = -a_0$$

Both factors on the left are integers, so $-a_0$ must be divisible by each of them, and in particular by x_0. So must a_0.

Example 3.4
Find all possible integer roots to the equation $\frac{1}{2}x^3 - x^2 + \frac{1}{2}x - 1 = 0$.

Solution We multiply both sides of the equation by 2 to obtain an equation whose coefficients are all integers:

$$x^3 - 2x^2 + x - 2 = 0$$

According to [3.9], all integer solutions of the equation must be factors of -2. So only ± 1 and ± 2 can be integer solutions. A check shows that $x = 2$ is the only integer solution. In fact, because $x^3 - 2x^2 + x - 2 = (x-2)(x^2+1)$, there is only one real root.

The Remainder Theorem

Let $P(x)$ and $Q(x)$ be two polynomials for which the degree of $P(x)$ is greater than or equal to the degree of $Q(x)$. Then there always exist unique polynomials $q(x)$ and $r(x)$ such that

$$P(x) = q(x)Q(x) + r(x) \qquad [3.10]$$

where the degree of $r(x)$ is less than the degree of $Q(x)$. This fact is called the **remainder theorem**. When x is such that $Q(x) \neq 0$, then [3.10] can be written in the form

$$\frac{P(x)}{Q(x)} = q(x) + \frac{r(x)}{Q(x)} \qquad [3.11]$$

If $r(x) = 0$ in [3.10] and [3.11], we say that *$Q(x)$ is a factor of $P(x)$*, or that *$P(x)$ is divisible by $Q(x)$*. Then $P(x) = q(x)Q(x)$ or $P(x)/Q(x) = q(x)$, which is the *quotient*. When $r(x) \neq 0$, it is the *remainder*.

An important special case is when $Q(x) = x - a$. Then $Q(x)$ is of degree 1, so the remainder $r(x)$ must have degree 0, and is therefore a constant. In this special case, for all x,

$$P(x) = q(x)(x - a) + r$$

For $x = a$ in particular, we get $P(a) = r$. *Hence, $x - a$ divides $P(x)$ if and only if $P(a) = 0$.* This is an important observation that can be formulated as follows:

$$\text{Polynomial } P(x) \text{ has the factor } x - a \iff P(a) = 0 \qquad [3.12]$$

It follows from [3.12] that an nth-degree polynomial $P(x)$ can have *at most* n different zeros. To see this, note that each zero $x = a_1$, $x = a_2$, ..., $x = a_k$ gives rise to a different factor of the form $x - a$. From this it follows that $P(x)$ can be expressed as $P(x) = A(x)(x - a_1)\dots(x - a_k)$ for some polynomial $A(x)$. Thus, $P(x)$ has degree $\geq k$, and so k cannot exceed n.

Example 3.5

Prove that the polynomial $f(x) = -2x^3 + 2x^2 + 10x + 6$ has a zero at $x = 3$, and factorize the polynomial.

Solution Inserting $x = 3$ into the polynomial yields

$$f(3) = -2 \cdot 3^3 + 2 \cdot 3^2 + 10 \cdot 3 + 6 = -54 + 18 + 30 + 6 = 0$$

So $x - 3$ is a factor. It follows that the cubic function $f(x)$ can be expressed as the product of $(x - 3)$ with a second degree polynomial. In fact,

$$f(x) = -2x^3 + 2x^2 + 10x + 6 = -2(x - 3)(x^2 + ax + b)$$

We must determine a and b. Expanding the last expression yields

$$f(x) = -2x^3 + (6 - 2a)x^2 + (6a - 2b)x + 6b$$

If this polynomial $f(x)$ is to equal $-2x^3 + 2x^2 + 10x + 6$ for all x, then the coefficients of like powers of x must be equal; thus, $6 - 2a = 2$, $6a - 2b = 10$, and $6b = 6$. Hence, $b = 1$ and $a = 2$. Because $x^2 + 2x + 1 = (x + 1)^2$, we conclude that

$$f(x) = -2x^3 + 2x^2 + 10x + 6 = -2(x - 3)(x^2 + 2x + 1)$$
$$= -2(x - 3)(x + 1)^2$$

The factorization procedure used in this example is called the *method of undetermined coefficients*. (Here a and b were the undetermined coefficients.) The alternative "long-division" method for factorizing polynomials will be considered next.

Polynomial Division

One can divide polynomials in much the same way as one divides numbers. Consider first a simple numerical example:

$$
\begin{array}{rl}
2735 \div 5 & = 500 + 40 + 7 \\
\underline{2500} & \\
235 & \\
\underline{200} & \\
35 & \\
\underline{35} & \\
0 & \quad \text{remainder}
\end{array}
$$

Hence, $2735 \div 5 = 547$. Note that the horizontal lines instruct you to subtract the numbers above the lines. (You might be more accustomed to a different way of arranging the numbers, but the idea is the same.)

Consider next

$$(-x^3 + 4x^2 - x - 6) \div (x - 2)$$

We write the following:

$$
\begin{array}{llll}
(-x^3 + 4x^2 - x - 6) \div (x - 2) & & = -x^2 & + 2x + 3 \\
\underline{-x^3 + 2x^2} & \longleftarrow & \boxed{-x^2(x-2)} & \\
\quad 2x^2 - x - 6 & & & \\
\quad \underline{2x^2 - 4x} & \longleftarrow & \boxed{2x(x-2)} & \longleftarrow \\
\qquad 3x - 6 & & & \\
\qquad \underline{3x - 6} & \longleftarrow & \boxed{3(x-2)} & \longleftarrow \\
\qquad\quad 0 & \text{remainder} & &
\end{array}
$$

(You can omit the boxes, but they should help you to see what is going on.) We conclude that $(-x^3 + 4x^2 - x - 6) \div (x - 2) = -x^2 + 2x + 3$. Because it is easy to see that $-x^2 + 2x + 3 = -(x + 1)(x - 3)$, we have

$$-x^3 + 4x^2 - x - 6 = -(x + 1)(x - 3)(x - 2)$$

Polynomial Division with a Remainder

The division $2734 \div 5$ gives 546 and leaves the remainder 4. So $2734/5 = 546 + 4/5$. We consider a similar form of division for polynomials.

Example 3.6

$(x^4 + 3x^2 - 4) \div (x^2 + 2x)$

Solution

$$
\begin{array}{rl}
(x^4 \qquad + 3x^2 \qquad\quad - 4) \div (x^2 + 2x) = x^2 - 2x + 7 & \\
\underline{x^4 + 2x^3 \qquad\qquad\qquad\quad} & \\
\underline{-2x^3 + 3x^2 \qquad - 4} & \\
\underline{-2x^3 - 4x^2 \qquad\qquad} & \\
7x^2 \qquad - 4 & \\
\underline{7x^2 + 14x \quad} & \\
-14x - 4 & \text{remainder}
\end{array}
$$

(The polynomial $x^4 + 3x^2 - 4$ has no terms in x^3 and x, so we inserted some extra space between the powers of x to make room for the terms in x^3 and x that arise in the course of the calculations.) We conclude that

$$x^4 + 3x^2 - 4 = (x^2 - 2x + 7)(x^2 + 2x) + (-14x - 4)$$

Hence,

$$\frac{x^4 + 3x^2 - 4}{x^2 + 2x} = x^2 - 2x + 7 - \frac{14x + 4}{x^2 + 2x} \qquad [*]$$

Rational Functions

A **rational function** is a function $R(x) = P(x)/Q(x)$ that can be expressed as the ratio of two polynomials $P(x)$ and $Q(x)$. This function is defined for all x where $Q(x) \neq 0$. The rational function $R(x)$ is called **proper** if the degree of $P(x)$ is less than the degree of $Q(x)$. When the degree of $P(x)$ is greater than or equal to that of $Q(x)$, then $R(x)$ is called an **improper** rational function. By using polynomial division, any improper rational function can be written as a polynomial plus a proper rational function, as in [3.11] and Example 3.6.

Problems

1. By making use of [3.9], find all integer roots of the following equations:

   a. $x^2 + x - 2 = 0$ **b.** $x^3 - x^2 - 25x + 25 = 0$ **c.** $x^5 - 4x^3 - 3 = 0$

2. Find all integer roots of the following equations:

   a. $x^4 - x^3 - 7x^2 + x + 6 = 0$ **b.** $2x^3 + 11x^2 - 7x - 6 = 0$

   c. $x^4 + x^3 + 2x^2 + x + 1 = 0$ **d.** $\frac{1}{4}x^3 - \frac{1}{4}x^2 - x + 1 = 0$

3. Perform the following divisions:

   a. $(x^2 - x - 20) \div (x - 5)$ **b.** $(x^3 - 1) \div (x - 1)$

   c. $(-3x^3 + 48x) \div (x - 4)$

4. Perform the following divisions:

   a. $(2x^3 + 2x - 1) \div (x - 1)$ b. $(x^4 + x^3 + x^2 + x) \div (x^2 + x)$

   c. $(3x^8 + x^2 + 1) \div (x^3 - 2x + 1)$ d. $(x^5 - 3x^4 + 1) \div (x^2 + x + 1)$

5. Which of the following divisions leave no remainder? (a and b are constants; n is a natural number.)

   a. $(x^3 - x - 1)/(x - 1)$ b. $(2x^3 - x - 1)/(x - 1)$

   c. $(x^3 - ax^2 + bx - ab)/(x - a)$ d. $(x^{2n} - 1)/(x + 1)$

6. Write the following polynomials as products of linear factors:

   a. $p(x) = x^3 + x^2 - 12x$ b. $q(x) = 2x^3 + 3x^2 - 18x + 8$

7. Find possible formulas for each of the three polynomials with graphs in Fig. 3.7.

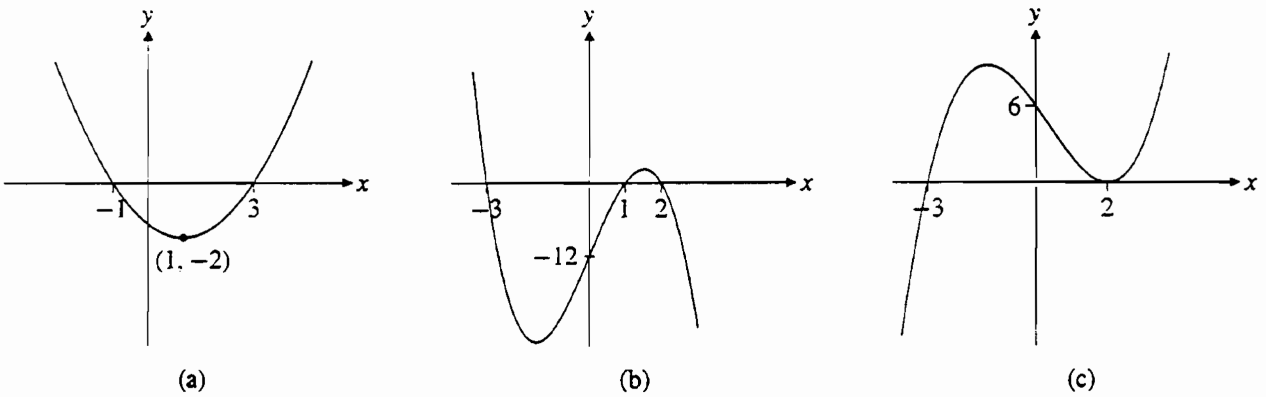


FIGURE 3.7

3.4 Power Functions

Consider the power function f defined by the formula

$$f(x) = x^r \qquad [3.13]$$

We know the meaning of x^r if r is any integer—that is, $r = 0, \pm 1, \pm 2, \ldots$. In fact, if r is a natural number, x^r is the product of r x's. Also if $r = 0$, then $x^r = x^0 = 1$ for all $x \neq 0$, and if $r = -n$, then $x^r = 1/x^n$ for $x \neq 0$. In addition, for $r = 1/2$, $x^r = x^{1/2} = \sqrt{x}$, defined for all $x \geq 0$. (See Section A.2 of Appendix A.) This section extends the definition of x^r so that it has meaning for any rational number r.

Here are some examples of why powers with rational exponents are needed:

1. The flow of blood (in liters per second) through the heart of an individual is approximately proportional to $x^{0.7}$, where x is the body weight.
2. The formula $S \approx 4.84V^{2/3}$ gives the approximate surface S of a ball as a function of its volume V. (See Example 3.10, which follows.)

3. The formula $Y = 2.262K^{0.203}L^{0.763}(1.02)^t$ appears in a study of the growth of national output, and shows how powers with fractional exponents can arise in economics. (Here Y is the net national product, K is capital stock, L is labor, and t is time.)

These examples illustrate the need to define x^r for $r = 0.7 = 7/10$, $r = 2/3$, $r = 0.203$, and $r = 0.763 = 763/1000$. In general, we want to define x^r for $x > 0$ when r is an arbitrary rational number.

The following basic power rules (discussed in Section A.1, Appendix A) are valid for all integers r and s:

$$\text{(i) } a^r a^s = a^{r+s} \qquad \text{(ii) } (a^r)^s = a^{rs} \tag{3.14}$$

When extending the definition of x^r so that it also applies to rational exponents r, it is natural to require that these rules retain their validity.

Let us first examine the meaning of $a^{1/n}$, where n is a natural number, and a is positive. For example, what does $5^{1/3}$ mean? If rule [3.14](ii) is still to apply in this case, we must have $(5^{1/3})^3 = 5$. This implies that $5^{1/3}$ must be a solution of the equation $x^3 = 5$. This equation can be shown to have a unique positive solution, denoted by $\sqrt[3]{5}$, the *cube root of 5*. (See Example 7.2 in Section 7.1.) Therefore, we must define $5^{1/3}$ as $\sqrt[3]{5}$. In general, $(a^{1/n})^n = a$. Thus, $a^{1/n}$ is a solution of the equation $x^n = a$. This equation can be shown to have a unique positive solution denoted by $\sqrt[n]{a}$, the *nth root of a*:

$$a^{1/n} = \sqrt[n]{a} \tag{3.15}$$

In words: *if a is positive and n is a natural number, then $a^{1/n}$ is the unique positive number that, raised to the nth power, gives a—that is, $(a^{1/n})^n = a^1 = a$.* For example,

$$27^{1/3} = \sqrt[3]{27} = 3 \qquad \text{because} \qquad 3^3 = (27^{1/3})^3 = 27$$

$$\left(\tfrac{1}{625}\right)^{1/4} = \sqrt[4]{\tfrac{1}{625}} = \tfrac{1}{5} \qquad \text{because} \qquad \left(\tfrac{1}{5}\right)^4 = \left[\left(\tfrac{1}{625}\right)^{1/4}\right]^4 = \tfrac{1}{625}$$

Usually, we write $a^{1/2}$ as $\sqrt{a}$ rather than $\sqrt[2]{a}$ (see Section A.2 of Appendix A).

We proceed to define $a^{p/q}$ whenever p is an integer, q is a natural number, and $a > 0$. Consider $5^{2/3}$, for example. We have already defined $5^{1/3}$. For rule [3.14](ii) to apply, we must have $5^{2/3} = (5^{1/3})^2$. So we must define $5^{2/3}$ as $(\sqrt[3]{5})^2$. In general, for $a > 0$, we define

$$a^{p/q} = \left(a^{1/q}\right)^p = \left(\sqrt[q]{a}\right)^p, \qquad p \text{ an integer, } q \text{ a natural number} \tag{3.16}$$

Note: If q is an odd number and p is an integer, $a^{p/q}$ can be defined even when $a < 0$. For example, $(-8)^{1/3} = \sqrt[3]{-8} = -2$, because $(-2)^3 = -8$. However, in defining $a^{p/q}$ when $a < 0$, the fraction p/q must be reduced to lowest terms. If not, we would get contradictions such as "$-2 = (-8)^{1/3} = (-8)^{2/6} = \sqrt[6]{(-8)^2} = \sqrt[6]{64} = 2$."

Example 3.7

Compute $625^{0.75}$ and $32^{-3/5}$.

Solution

$$625^{0.75} = 625^{3/4} = (625^{1/4})^3 = 5^3 = 125$$

$$32^{-3/5} = \left(32^{1/5}\right)^{-3} = 2^{-3} = 1/8$$

Many scientific calculators have a power key, often denoted by $\boxed{y^x}$. For instance, suppose we let $y = 625$ and $x = 0.75$, then instruct the calculator to compute y^x (the way this is done varies from calculator to calculator). The display may show the number 125.000—or possibly, if 7 decimals are shown, 125.0000001. This shows that the key $\boxed{y^x}$ does not always give an exact answer, even in simple cases. Try it with 2^3, and check the value for $32^{-3/5}$. Simple pocket calculators are usually exact enough for practical purposes, however.

With this definition of $a^{p/q}$, we can show that rules [3.14] are still valid when r and s are rational numbers. In particular,

$$a^{p/q} = \left(a^{1/q}\right)^p = \left(a^p\right)^{1/q} = \sqrt[q]{a^p}$$

Thus, to compute $a^{p/q}$, we could either first take the qth root of a and raise the result to p, or first raise a to the power p and then take the qth root of the result. We obtain the same answer either way. For example,

$$625^{0.75} = 625^{3/4} = \left(625^3\right)^{1/4} = (244140625)^{1/4} = \sqrt[4]{244140625} = 125$$

Note that this procedure involves more difficult computations than the one used in Example 3.7.

Example 3.8

If z denotes demand for coffee in tons per year and p denotes its price per ton, the approximate relationship between them over a specific time period is

$$z = 694{,}500p^{-0.3}$$

(a) Write the formula using roots.

(b) Use a calculator to compute demand when $p = 35{,}000$ and when $p = 55{,}000$.

Solution

(a) $p^{-0.3} = \dfrac{1}{p^{0.3}} = \dfrac{1}{p^{3/10}} = \dfrac{1}{\sqrt[10]{p^3}}$.

so we obtain

$$z = \frac{694,500}{\sqrt[10]{p^3}}$$

(b) $p = 35,000$ gives $z = 694,500 \cdot (35,000)^{-0.3} \approx 30,092$ (tons)

$p = 55,000$ gives $z = 694,500 \cdot (55,000)^{-0.3} \approx 26,276$ (tons)

Note that when price increases, demand decreases.

Using the Power Rules

Powers with rational exponents often occur in economic applications, so you must learn to use them correctly. Before we consider some more examples, note that the power rules can easily be extended to more factors. For instance, we have

$$(abcd)^p = (ab)^p(cd)^p = a^p b^p c^p d^p$$

Example 3.9

Simplify the following expression so that the answer contains only a single exponent for each variable x and y:

$$\left(\frac{5x^{-2}y^{2/3}}{625x^4y^{-4/3}}\right)^{-1/3}$$

Solution One method begins by simplifying the expression inside the parentheses:

$$\left(\frac{5x^{-2}y^{2/3}}{625x^4y^{-4/3}}\right)^{-1/3} = \left(\frac{1}{125}\cdot\frac{x^{-2}}{x^4}\cdot\frac{y^{2/3}}{y^{-4/3}}\right)^{-1/3} = \left(\frac{1}{125}\cdot x^{-6}\cdot y^2\right)^{-1/3}$$

$$= \left(\frac{1}{125}\right)^{-1/3}\left(x^{-6}\right)^{-1/3}\left(y^2\right)^{-1/3} = (125)^{1/3}x^2y^{-2/3} = \frac{5x^2}{y^{2/3}}$$

Alternatively, we can also raise all the factors to the power $-1/3$ and use the relation $625 = 5^4$ to obtain

$$\left(\frac{5x^{-2}y^{2/3}}{625x^4y^{-4/3}}\right)^{-1/3} = \frac{5^{-1/3}x^{2/3}y^{-2/9}}{(5^4)^{-1/3}x^{-4/3}y^{4/9}} = 5^{-1/3-(-4/3)}\cdot x^{2/3-(-4/3)}\cdot y^{-2/9-4/9}$$

$$= 5^1x^2y^{-2/3} = \frac{5x^2}{y^{2/3}}$$

Example 3.10

The formulas for the surface S and the volume V of a ball with radius r are $S = 4\pi r^2$ and $V = (4/3)\pi r^3$. Express S in terms of V.

Solution We must eliminate r. From $V = (4/3)\pi r^3$ we obtain $r^3 = 3V/4\pi$. By raising each side of this equation to the power 1/3 and using $(r^3)^{1/3} = r$, we obtain $r = (3V/4\pi)^{1/3}$. Hence,

$$S = 4\pi r^2 = 4\pi\left[\left(\frac{3V}{4\pi}\right)^{1/3}\right]^2 = 4\pi\frac{(3V)^{2/3}}{(4\pi)^{2/3}}$$

$$= (4\pi)^{1-(2/3)}3^{2/3}V^{2/3} = (4\pi)^{1/3}(3^2)^{1/3}V^{2/3} = \sqrt[3]{36\pi}\, V^{2/3}$$

We have thus shown that

$$S = \sqrt[3]{36\pi}\, V^{2/3} \approx 4.84\, V^{2/3} \qquad [1]$$

Note: Perhaps the most commonly committed error in elementary algebra is to replace $(x+y)^2$ by x^2+y^2 and hence lose the term $2xy$. If we replace $(x+y)^3$ by x^3+y^3, then we lose the terms $3x^2y+3xy^2$. What error do we commit if we replace $(x-y)^3$ by x^3-y^3? Tests also reveal that students who are able to handle these simple power expressions often make mistakes when dealing with more complicated powers. A surprisingly common error is replacing $\left(25-\frac{1}{2}x\right)^{1/2}$ by $25^{1/2}-\left(\frac{1}{2}x\right)^{1/2}$, for example. In general:

> $(x+y)^\alpha$ is usually NOT equal to $x^\alpha + y^\alpha$
> $(x-y-z)^{1/\alpha}$ is usually NOT equal to $x^{1/\alpha} - y^{1/\alpha} - z^{1/\alpha}$

The *only* exception, for general values of x, y, and z, occurs when $\alpha = 1$.

Graphs of Power Functions

We return to the power function $f(x) = x^r$ in [3.13], which is now defined for all rational numbers r provided that $x > 0$. We always have $f(1) = 1^r = 1$, so the graph of the function passes through the point $(1, 1)$ in the xy-plane. The behavior of the graph depends crucially on whether r is positive or negative.

Example 3.11

Sketch the graphs $y = x^{0.3}$ and $y = x^{-1.3}$.

Solution Using a pocket calculator allows us to complete the following table:

x	0	1/3	2/3	1	2	3	4
$y = x^{0.3}$	0	0.72	0.89	1	1.23	1.39	1.52
$y = x^{-1.3}$	*	4.17	1.69	1	0.41	0.24	0.16

*Not defined.

The graphs are shown in Figs. 3.8 and 3.9.

Figure 3.10 illustrates how the graph of $y = x^r$ changes with changing values of the exponent. Try to draw the graphs of $y = x^{-3}$, $y = x^{-1}$, $y = x^{-1/2}$, and $y = x^{-1/3}$.

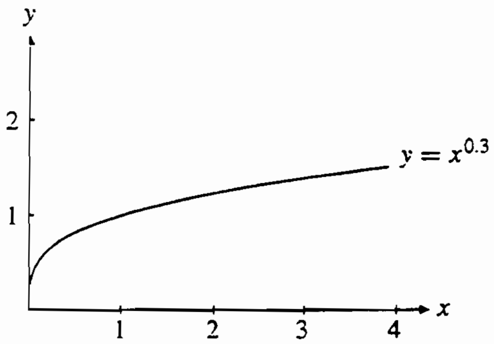


FIGURE 3.8

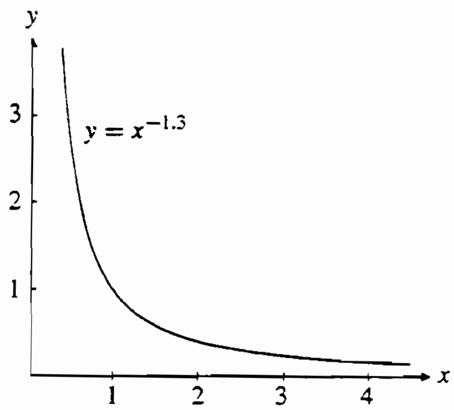


FIGURE 3.9

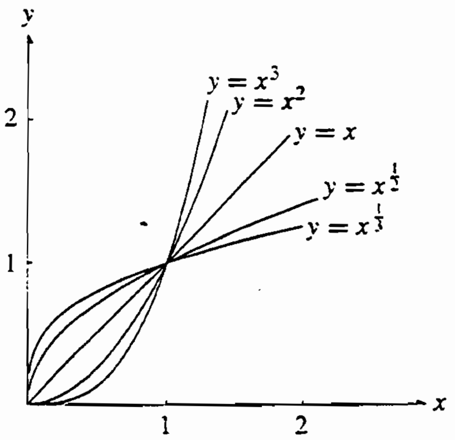


FIGURE 3.10

Problems

1. Compute the following:

   a. $16^{1/4}$ b. $243^{-1/5}$ c. $5^{1/7} \cdot 5^{6/7}$ d. $(4^8)^{-3/16}$

2. Using a pocket calculator or computer, find approximate values for the following:

   a. $100^{1/5}$ b. $16^{-3.33}$ c. $5.23^{1.02} \cdot 2.11^{-3.11}$

3. Compute the following:

   a. $\dfrac{4 \cdot 3^{-1/3}}{\sqrt[6]{81}}$ b. $(0.064)^{-1/3}$ c. $\left(3^2 + 4^2\right)^{-1/2}$

4. How can the number $50^{0.16}$ be expressed as a root?

5. Simplify the following expressions so that each contains only a single exponent of a.

   a. $\{[(a^{1/2})^{2/3}]^{3/4}\}^{4/5}$ b. $a^{1/2}a^{2/3}a^{3/4}a^{4/5}$

   c. $\{[(3a)^{-1}]^{-2}(2a^{-2})^{-1}\}/a^{-3}$ d. $\dfrac{\sqrt[3]{a}\, a^{1/12} \sqrt[4]{a^3}}{a^{5/12}\sqrt{a}}$

6. Solve the following equations for x:

   a. $2^{2x} = 8$ b. $3^{3x+1} = 1/81$ c. $10^{x^2-2x+2} = 100$

7. Which of the following equations are valid for all x and y?

   a. $(2^x)^2 = 2^{x^2}$ b. $3^{x-3y} = \dfrac{3^x}{3^{3y}}$

   c. $3^{-1/x} = \dfrac{1}{3^{1/x}}$ $(x \neq 0)$ d. $5^{1/x} = \dfrac{1}{5^x}$ $(x \neq 0)$

   e. $a^{x+y} = a^x + a^y$ f. $2^{\sqrt{x}} \cdot 2^{\sqrt{y}} = 2^{\sqrt{xy}}$ (x and y positive)

8. Solve the following equations for the variables indicated:
   a. $3K^{-1/2}L^{1/3} = 1/5$ for K
   b. $p - abx_0^{b-1} = 0$ for x_0
   c. $ax(ax+b)^{-2/3} + (ax+b)^{1/3} = 0$ for x
   d. $\left[(1-\lambda)a^{-\rho} + \lambda b^{-\rho}\right]^{-1/\rho} = c$ for b

9. A sphere of capacity 100 m^3 is to have its outside surface painted. One liter of paint covers 5 m^2. How many liters of paint are needed? (*Hint:* Use formula [1] in Example 3.10.)

10. Show by using a pocket calculator (or a computer) that the equation

$$Y = 2.262K^{0.203}L^{0.763}(1.02)^t$$

has an approximate solution for K given by $K \approx 0.018Y^{4.926}L^{-3.759}(0.907)^t$. Then determine K numerically when $Y = 100$, $L = 6$, and $t = 10$.

11. Simplify the following expressions:

   a. $\left(a^{1/3} - b^{1/3}\right)\left(a^{2/3} + a^{1/3}b^{1/3} + b^{2/3}\right)$

   b. $\dfrac{bx^{1/2} - (x-a)b\frac{1}{2}x^{-1/2}}{\left(bx^{1/2}\right)^2}$ $(x > 0)$

3.5 Exponential Functions

A quantity that increases (or decreases) by a fixed factor per unit of time is said to *increase* (or decrease) *exponentially*. If this fixed factor is a, this leads to the study of the exponential function f defined by

$$f(t) = Aa^t \tag{3.17}$$

where a and A are positive constants. Note that if $f(t) = Aa^t$, then $f(t+1) = Aa^{t+1} = Aa^t \cdot a^1 = af(t)$, so the value of f at time $t+1$ is a times the value of f at time t. If $a > 1$, then f is increasing; if $0 < a < 1$, then f is decreasing. Because $f(0) = Aa^0 = A$, we can write $f(t) = f(0)a^t$.

Exponential functions appear in many important economic, social, and physical models. For instance, economic growth, population growth, continuously accumulated interest, radioactive decay, and decreasing illiteracy have all been described by exponential functions. In addition, the exponential function is one of the most important in statistics.

Example 3.12 (Population Growth)

Consider a growing population like that of Europe. In Example 2.13, we constructed a linear function

$$P = 6.4t + 641$$

where P denotes the population in millions, $t = 0$ corresponds to the year 1960 when the population was 641 million, and $t = 10$ corresponds to the year 1970 when the population estimate was 705 million. According to this formula, the annual increase in population would be constant and equal to 6.4 million. This is a very unreasonable assumption. After all, populations tend to grow faster as they get bigger because there are more people to have babies, and the death rate usually decreases or stays the same. In fact, according to UN estimates, the European population was expected to grow by approximately 0.72% annually during the period 1960 to 2000. With a population of 641 million in 1960, the population in 1961 would then be

$$641 + \frac{641 \cdot 0.72}{100} = 641 \cdot \left(1 + \frac{0.72}{100}\right) = 641 \cdot 1.0072$$

which is approximately 645 million. Next year, in 1962, it would have grown to

$$\begin{aligned} 641 \cdot 1.0072 + \frac{641 \cdot 1.0072 \cdot 0.72}{100} &= 641 \cdot 1.0072 \cdot (1 + 0.0072) \\ &= 641 \cdot 1.0072^2 \end{aligned}$$

which is approximately 650 million. Note how the population figure grows by the factor 1.0072 each year. If the growth rate were to continue at 0.72% annually, then t years after 1960 the population would be given by

$$P(t) = 641 \cdot 1.0072^t \qquad [1]$$

Thus, $P(t)$ is an exponential function of the form [3.17]. For the year 2000, corresponding to $t = 40$, the formula yields the estimate $P(40) \approx 854$ million.

Many countries, particularly in Africa and Latin America, have recently had far faster population growth than Europe. For instance, during the 1970s and 1980s, the growth rate of Zimbabwe's population was close to 3.5% annually. If we let $t = 0$ correspond to the census year 1969 when the population was 5.1 million, the population t years after 1969 is given by

$$P(t) = 5.1 \cdot 1.035^t$$

If we calculate $P(20)$, $P(40)$, and $P(60)$ using this formula, we get roughly 10, 20, and 40. Thus, the population of Zimbabwe roughly doubles after 20 years; during the next 20 years, it doubles again, and so on. We say that the *doubling time* of the population is approximately 20 years. Of course, extrapolating so far into the future is quite dubious, because exponential growth of population cannot go on forever. (If the growth rate continued at 3.5% annually, and the Zimbabwean territory did not expand, in the year 2697, each Zimbabwean would on average have only 1 square meter of land. See Problem 7.)

If $a > 1$ and $A > 0$, the exponential function $f(t) = Aa^t$ is increasing. Its **doubling time** is the time it takes for it to double. Its value at $t = 0$ is A, so the doubling time t^* is given by the equation $f(t^*) = Aa^{t^*} = 2A$, or after cancelling A, by $a^{t^*} = 2$. Thus the doubling time of the exponential function $f(t) = Aa^t$ is the power to which a must be raised in order to get 2.[2] (In Problem 8 you will be asked to show that the doubling time is independent of which year you take as the base.)

Example 3.13

Use your calculator to find the doubling time of

(a) a population (like that of Zimbabwe) increasing at 3.5% annually (thus confirming the earlier calculations)

(b) the population of Kenya in the 1980s (which had the world's highest annual growth rate of 4.2%).

Solution

(a) The doubling time t^* is given by the equation $1.035^{t^*} = 2$. Using a calculator shows that $1.035^{15} \approx 1.68$, whereas $1.035^{25} \approx 2.36$. Thus,

[2]By using natural logarithms as explained in Section 8.2, we find that $t^* = \ln 2/\ln a$.

t^* must lie between 15 and 25. Because $1.035^{20} \approx 1.99$, t^* is close to 20. In fact, $t^* \approx 20.15$.

(b) The doubling time t^* is given by the equation $1.042^{t^*} = 2$. Using a calculator, we find that $t^* \approx 16.85$. Thus, with a growth rate of 4.2%, Kenya's population would double in less than 17 years.

Example 3.14 (Compound Interest)

A savings account of $\$K$ that increases by $p\%$ interest each year will have increased after t years to

$$K\,(1 + p/100)^t \qquad [1]$$

(see Section A.1 of Appendix A). According to this formula, \$1 ($K = 1$) earning interest at 8% per annum ($p = 8$) will have increased after t years to

$$(1 + 8/100)^t = 1.08^t \qquad [2]$$

Table 3.1 indicates how this dollar grows over time:

TABLE 3.1 *How \$1 of savings increases with time*

t	1	2	5	10	20	30	50	100	200
$(1.08)^t$	1.08	1.17	1.47	2.16	4.66	10.06	46.90	2,199.76	4,838,949.60

After 30 years, \$1 of savings has increased to more than \$10, and after 200 years, it has grown to more than \$4.8 million! This growth is illustrated in Fig. 3.11. Observe that the expression 1.08^t defines an exponential function of the type [3.17] with $a = 1.08$. Even if a is only slightly larger than 1, $f(t)$ will increase very quickly when t is large.

Example 3.15 (Radioactive Decay)

Measurements indicate that radioactive materials decay by a fixed percentage per unit of time. Plutonium 239, which is a waste product of certain nuclear power plants and is used in the production of nuclear weapons, decays by 50% every 24,400 years. We say, therefore, that the *half-life* of plutonium 239 is 24,400 years. If there are I_0 units of plutonium 239 at time $t = 0$, then after t years, there will be

$$I(t) = I_0 \cdot \left(\tfrac{1}{2}\right)^{t/24{,}400} = I_0 \cdot 0.9999716^t$$

units remaining. (Observe that this is consistent with $I(24{,}400) = \frac{1}{2} I_0$.)

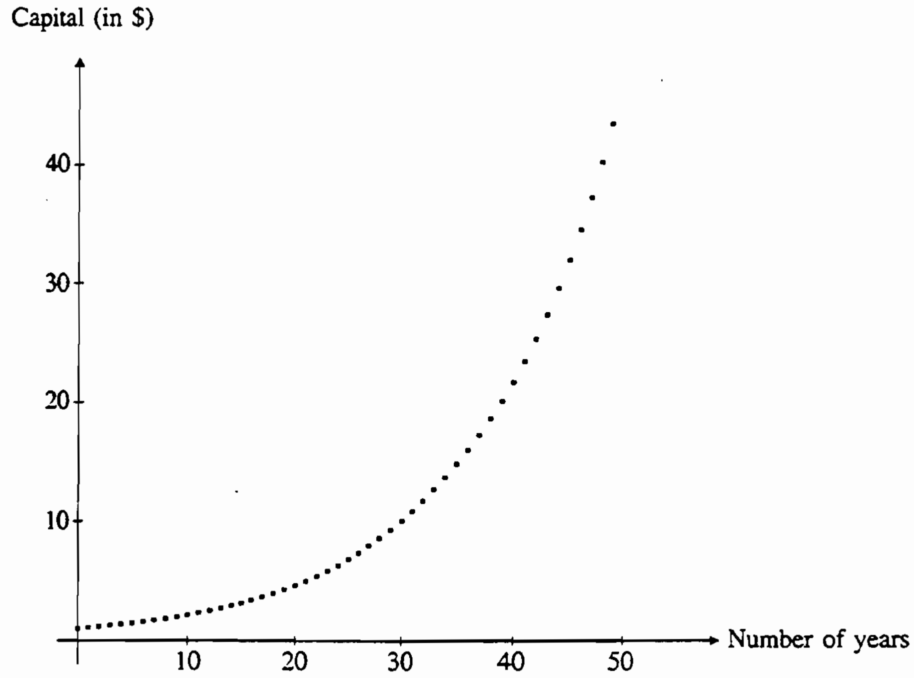


FIGURE 3.11 The growth of $1 of savings after t years when the interest rate is 8% per year.

Chapter 8 discusses the exponential function in much greater detail. Observe the fundamental difference between the two functions

$$f(x) = a^x \qquad \text{and} \qquad g(x) = x^a$$

The second of these two is the **power function** discussed in Section 3.4. For the exponential function a^x, it is the exponent that varies, while the base is constant. For the power function x^a, on the other hand, the exponent is constant, while the base varies.

The most important properties of the exponential function are summed up by the following:

The **general exponential function** with base $a > 0$ is

$$f(x) = Aa^x$$

where $f(0) = A$, and a is the factor by which $f(x)$ changes when x increases by 1.

If $a = 1 + p/100$, where $p > 0$ and $A > 0$, then $f(x)$ will increase by $p\%$ for each unit increase in x.

If $a = 1 - p/100$, where $p > 0$ and $A > 0$, then $f(x)$ will decrease by $p\%$ for each unit increase in x.

Problems

1. If the population of Europe grew at the rate of 0.72% annually, what would be the doubling time?
2. The population of Botswana was estimated to be 1.22 million in 1989, and to be growing at the rate of 3.4% annually.
   a. If $t = 0$ denotes 1989, find a formula for the population at date t.
   b. What is the doubling time?
3. A savings account with an initial deposit of \$100 earns 12% interest per year.
   a. What is the amount of savings after t years?
   b. Make a table similar to Table 3.1. (Stop at 50 years.)
4. Suppose that you are promised \$2 on the first day, \$4 on the second day, \$8 on the third day, \$16 on the fourth day, and so on (so that every day you get twice as much as the day before).
   a. How much will you receive on the tenth day?
   b. Find a function $f(t)$ that indicates how much you will obtain on the tth day.
   c. Explain why $f(20)$ is more than \$1 million. (*Hint:* 2^{10} is a little larger than 10^3.)
5. Fill in the following table and then make a rough sketch of the graphs of $y = 2^x$ and $y = 2^{-x}$.

x	-3	-2	-1	0	1	2	3
2^x							
2^{-x}							

6. Fill in the following table and then sketch the graph of $y = 2^{x^2}$.

x	-2	-1	0	1	2
2^{x^2}					

7. The area of Zimbabwe is approximately $3.91 \cdot 10^{11}$ square meters. Referring to the text following Example 3.12 and using a calculator, solve the equation $5.1 \cdot 1.035^t = 3.91 \cdot 10^{11}$ for t, and interpret your answer. (Recall that $t = 0$ corresponds to 1969.)
8. With $f(t) = Aa^t$, if $f(t + t^*) = 2f(t)$, prove that $a^{t^*} = 2$. Explain why this shows that the doubling time of the general exponential function is independent of the initial time.

9. In 1964 a five-year plan was introduced in Tanzania. One objective was to double the real per capita income over the next 15 years. What is the average annual rate of growth of real income per capita required to achieve this objective?

10. Consider the function f defined for all x by $f(x) = 1 - 2^{-x}$.
    a. Make a table of function values for $x = 0, \pm 1, \pm 2$, and ± 3. Then sketch the graph of f.
    b. What happens to $f(x)$ as x becomes very large and very small?

11. Which of the following equations do *not* define exponential functions of x?
    a. $y = 3^x$ b. $y = x^{\sqrt{2}}$ c. $y = (\sqrt{2})^x$
    d. $y = x^x$ e. $y = (2.7)^x$ f. $y = 1/2^x$

12. Fill in the following table and then sketch the graph of $y = x^2 2^x$.

x	−10	−5	−4	−3	−2	−1	0	1	2
$x^2 2^x$									

13. Find possible exponential functions for the graphs of Fig. 3.12.

14. The radioactive isotope iodine 131, which has a half-life of 8 days, is often used to diagnose disease in the thyroid gland. If there are I_0 units of the material at time $t = 0$, how much remains after t days?

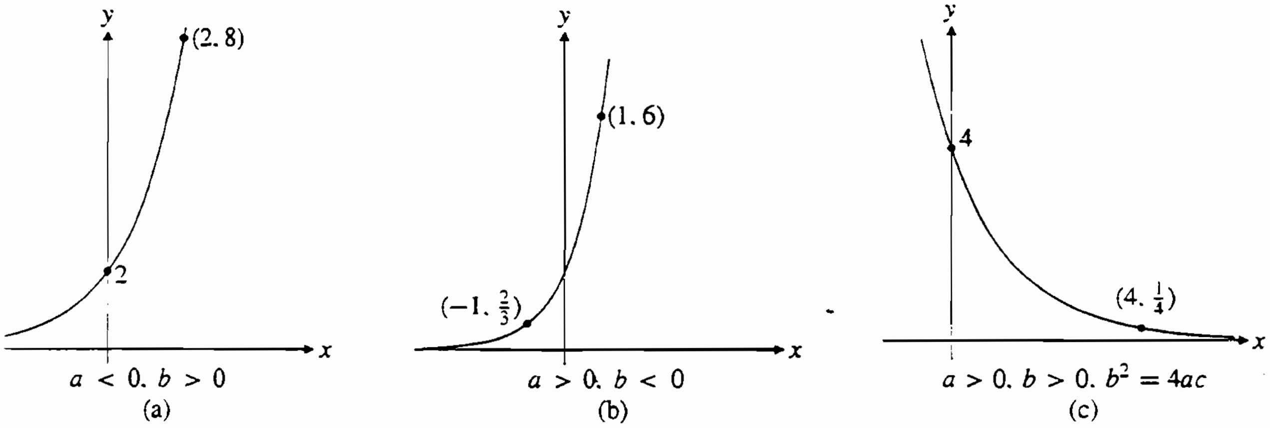


FIGURE 3.12

3.6 The General Concept of a Function

So far we have studied functions of one variable. These are functions whose domain is a set of real numbers, and whose range is also a set of real numbers. Yet a realistic description of many economic phenomena requires considering a large number of variables simultaneously. For example, the demand for a good like butter is a function of several variables such as the price of the good, the prices of complements and substitutes, consumers' incomes, and so on.

Actually, we have already seen many special functions of several variables. For instance, the formula $V = \pi r^2 h$ for the volume V of a cylinder with base radius r and height h involves a function of two variables. (Of course, in this case $\pi \approx 3.14159$ is a mathematical constant.) A change in one of these variables will not affect the value of the other variable. For each pair of positive numbers (r, h), there is a definite value for the volume V. To emphasize that V depends on the values of both r and h, we write

$$V(r, h) = \pi r^2 h$$

For $r = 2$ and $h = 3$, we obtain $V(2, 3) = 12\pi$, whereas $r = 3$ and $h = 2$ give $V(3, 2) = 18\pi$. Also, $r = 1$ and $h = 1/\pi$ give $V(1, 1/\pi) = 1$. Note in particular that $V(2, 3) \neq V(3, 2)$.

In some abstract economic models, it may be enough to know that there is some functional relationship between variables, without specifying the dependence more closely. For instance, suppose a market sells three commodities whose prices per unit are respectively p, q, and r. Then economists generally assume that the demand for one of the commodities by an individual with income m is given by a function $f(p, q, r, m)$ of four variables, without specifying the precise form of that function.

An extensive discussion of functions of several variables begins in Chapter 15. This section introduces an even more general type of function. In fact, general functions of the kind presented here are of fundamental importance in practically every area of pure and applied mathematics, including mathematics applied to economics.

Example 3.16

The following examples indicate how very wide is the concept of a function.

- **(a)** The function that assigns to each triangle in a plane the area of that triangle (measured, say, in cm^2).
- **(b)** The function that determines the social security number, or other identification number, of each taxpayer.
- **(c)** The function that for each point P in a plane determines the point lying 3 units above P.
- **(d)** Let A be the set of possible actions that a person can choose in a certain situation. Suppose that every action $a \in A$ produces a certain result (say, a certain profit $\varphi(a)$). In this way, we have defined a function φ with domain A.

Here is a general definition:

> A **function** from A to B is a rule that assigns to each element of the set A one and only one element of the set B. [3.18]

If we denote the function by f, the set A is called the **domain** of f, and B is called the **target**. The two sets A and B need not consist of numbers, but can be sets of quite arbitrary elements.

The definition of a function requires three objects to be specified:

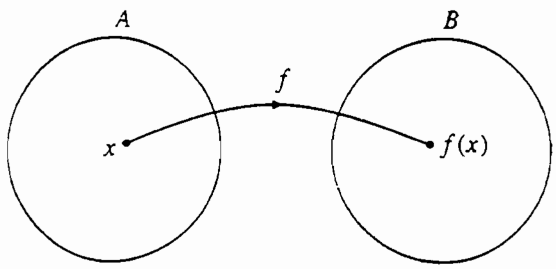
1. A domain A
2. A target B
3. A rule that assigns a *unique* element in B to *each* element in A.

Nevertheless, in many cases, we refrain from specifying the sets A and/or B explicitly when it obvious from the context what these sets are.

An important requirement in the definition of a function is that to each element in domain A, there corresponds a *unique* element in target B. While it is meaningful to talk about the function that assigns the natural mother to every child, the rule that assigns the aunt to any child does not, in general, define a function, because many children have several aunts. Explain why the following rule, as opposed to the one in Example 3.16(c), does not define a function: "to a point P in a horizontal plane, assign a point that lies 3 units from P."

If f is a function with domain A and target B, we often say that f is a **function from A to B**, and write $f : A \to B$. The functional relationship is often represented as in Fig. 3.13. Other words that are sometimes used instead of "function" include **transformation** and **map** or **mapping**. The particular value $f(x)$ is often called the **image** of the element x by the function f. The set of elements in B that are images of at least one element in A is called

FIGURE 3.13 A function from A to B.



the **range** of the function. Thus, the range is a subset of the target. If we denote the range of f by R_f, then $R_f = \{f(x) : x \in A\}$. This is also written as $f(A)$. The range of the function in Example 3.16(a) is the set of all positive numbers. Explain why the range of the function in (c) must be the entire plane.

The definition of a function requires that only *one* element in B be assigned to each element in A. However, different elements in A might be mapped to the same element in B. In Example 3.16(a), for instance, many different triangles have the same area. If each element of B is the image of at most one element in A, function f is called **one-to-one**. Otherwise, if one or more elements of B are the images of more than one element in A, the function is many-to-one.

The social security function in Example 3.16(b) is intended to be one-to-one, because two different taxpayers should always have different numbers. (In very rare instances, errors cause this function to be many-to-one. These always create a great deal of confusion when they are noticed!) Can you explain why the function defined in Example 3.16(c) is also one-to-one, whereas the function that assigns to each child his or her mother is not?

Suppose f is a one-to-one function from a set A to a set B, and assume that the range of f is all of B. Thus:

1. f maps each element of A into an element of B (so f is a function).
2. Two different elements of A are always mapped into different elements of B (so f is one-to-one).
3. For each element v in B, there is an element u in A such that $f(u) = v$ (so the range of f is the whole of B).

We can then define a function g from B to A by the following obvious rule: Assign to each element v of B the element $u = g(v)$ of A that f maps to v—that is, the u satisfying $v = f(u)$. Because of rule 2, there can be only one u in A such that $v = f(u)$, so g is a function. Its domain is B and its target and range are both equal to A. The function g is called the **inverse function** of f. For instance, the inverse of the social security function mentioned earlier is the function that, to each social security number, assigns the person carrying that number. Section 7.6 provides more detail about inverse functions and their properties.

Problems

1. Decide which of the following rules defines a function:
   a. The rule that assigns to each person in a classroom his or her height.
   b. The rule that assigns to a mother her youngest child.
   c. The rule that assigns the circumference of a rectangle to its area.
   d. The rule that assigns the surface area of a spherical ball to its volume.

e. The rule that assigns the pair of numbers $(x+3, y)$ to the pair of numbers (x, y).

2. Decide which of the functions defined in Problem 1 is one-to-one, and which then have an inverse. Determine the inverse when it exists.

3. Each person has red blood cells that belong to one and only one of four blood groups denoted A, B, AB, and O. Consider the function that assigns each person in a team to his or her blood group. Can this function be one-to-one if the team consists of at least five persons?

4

Single-Variable Differentiation

To think of it [differential calculus] merely as a more advanced technique is to miss its real content. In it, mathematics becomes a dynamic mode of thought, and that is a major mental step in the ascent of man.
—*J. Bronowski* (1973)

An important topic in scientific disciplines including economics is the study of how quickly quantities change over time. In order to compute the future position of a planet, to predict the growth in population of a biological species, or to estimate the future demand for a commodity, we need information about rates of change.

The mathematical concept used to describe the rate of change of a function is the derivative, which is *the* central concept in mathematical analysis. This chapter defines the derivative of a function and presents some of the simpler rules for calculating it. The next chapter develops some further rules allowing derivatives of quite complicated functions to be computed.

Isaac Newton (1642–1727) and Gottfried Leibniz (1646–1716) discovered most of these general rules independently of each other. This initiated the development of differential and integral calculus.

4.1 Slopes of Curves

Even though in economics we are usually interested in the derivative as a rate of change, we begin this chapter with a geometrical motivation for the concept.

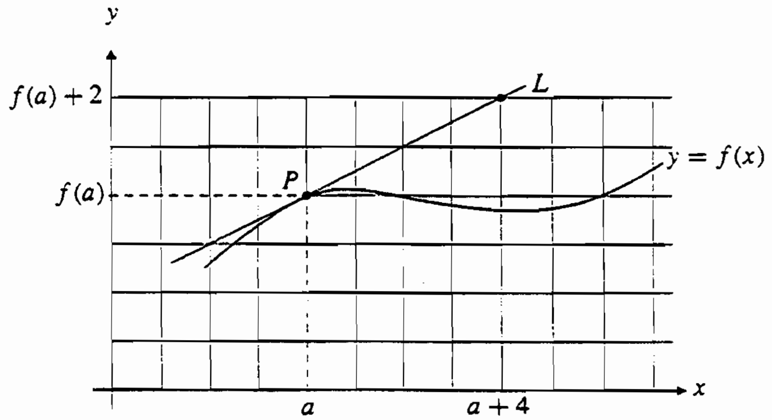


FIGURE 4.1 $f'(a) \approx 1/2$.

When we study the graph of a function, we would like to have a precise measure of the steepness of a graph at a point. We know that for the line $y = ax+b$, the number a denotes its slope. If a is large and positive, then the line rises steeply from left to right; if a is large and negative, the line falls steeply. But for an arbitrary function f, what is the steepness of its graph? A natural answer is to define the steepness of a curve at a particular point as the slope of the tangent to the curve at that point—that is, as the slope of the straight line that just touches the curve at that point. For the curve in Fig. 4.1 the steepness at point P is therefore $1/2$, because the tangent passes through the pair of points $(a, f(a))$ and $(a+4, f(a)+2)$, for instance. In Fig. 4.1, point P has coordinates $\big(a, f(a)\big)$. The slope of the tangent to the graph at P is called the **derivative** of f at point a and we denote this number by $f'(a)$ (read as "*f prime a*"). In general, we have

$$f'(a) = \text{the slope of the tangent to the curve } y = f(x) \text{ at the point } \big(a, f(a)\big) \qquad [4.1]$$

Thus, in Fig. 4.1, we have $f'(a) = [f(a) + 2 - f(a)]/(a + 4 - a) = 2/4 = 1/2$.

Example 4.1

Use definition [4.1] to determine $f'(1)$, $f'(4)$, and $f'(7)$ for the function whose graph is shown in Fig. 4.2.

Solution At $P = (1, 2)$, the tangent goes through $(0, 1)$, and so has slope 1. At $Q = (4, 3)$ the tangent is horizontal, and so has slope 0. At $R = (7, 2\frac{1}{2})$, the tangent goes through $(8, 2)$, and so has slope $-\frac{1}{2}$. Therefore, we obtain: $f'(1) = 1$, $f'(4) = 0$, and $f'(7) = -1/2$.

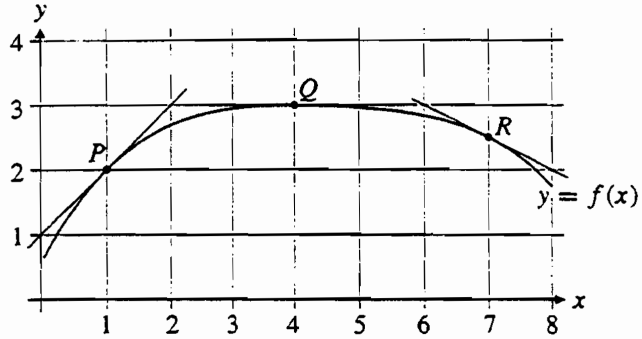


FIGURE 4.2

4.2 The Slope of the Tangent and the Derivative

The previous section gave a rather vague definition of the tangent to a curve at a point, because we said that it is a straight line that just touches the curve at that point. We now give a more formal definition of the same concept.

The geometrical idea behind the definition is easy to understand. Consider a point P on a curve in the xy-plane (see Fig. 4.3). Take another point Q on the curve. The entire straight line through P and Q is called a *secant* (from a Latin word meaning "cutting"). If we keep P fixed, but let Q move along the curve toward P, then the secant will rotate around P, as indicated in Fig. 4.4. The limiting straight line PT toward which the secant tends is called the **tangent (line)** to the curve at P. Suppose that P is a point on the graph of the function f. We shall see how the preceding considerations enable us to find the slope of the tangent at P. This is shown in Fig. 4.5.

Point P has the coordinates $(a, f(a))$. Point Q lies close to P and is also on the graph of f. Suppose that the x-coordinate of Q is $a + h$, where h is a small number $\neq 0$. Then the x-coordinate of Q is not a (because $Q \neq P$), but a

FIGURE 4.3

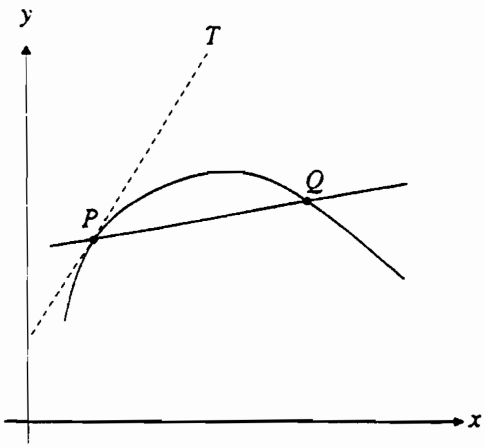
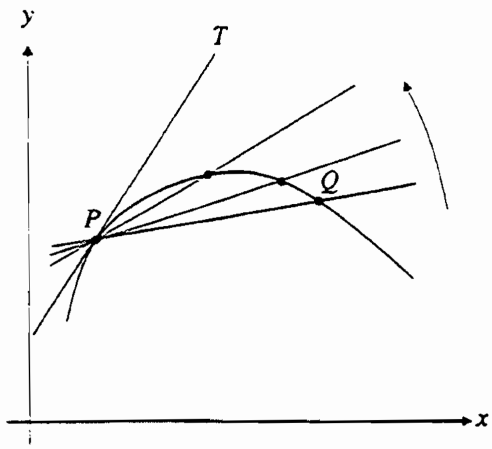


FIGURE 4.4



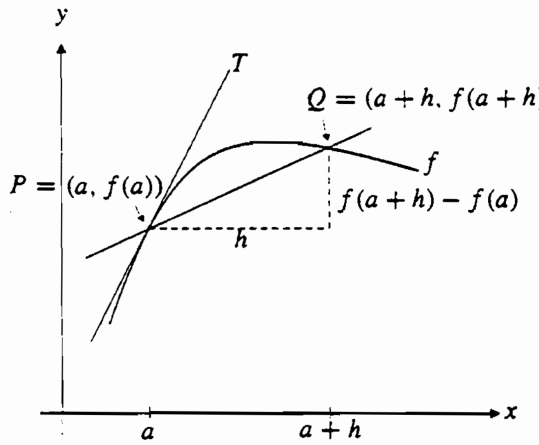


FIGURE 4.5

number close to a. Because Q lies on the graph of f, the y-coordinate of Q is equal to $f(a+h)$. Hence, the coordinates of the points are $P = \big(a, f(a)\big)$ and $Q = \big(a+h, f(a+h)\big)$. The slope m_{PQ} of the secant PQ is

$$m_{PQ} = \frac{f(a+h) - f(a)}{h} \qquad [4.2]$$

In mathematics, this fraction is often called a **Newton** (or **differential**) **quotient of f**. Note that when $h = 0$, the fraction in [4.2] becomes 0/0 and so is undefined. But choosing $h = 0$ corresponds to letting $Q = P$. When Q moves toward P (Q tends to P) along the graph of f, the x-coordinate of Q, which is $a+h$, must tend to a, and so h tends to 0. Simultaneously, the secant PQ tends to the tangent to the graph at P. This suggests that we ought to *define* the slope of the tangent at P as the number that m_{PQ} in [4.2] approaches as h tends to 0. In [4.1], we called this value the slope $f'(a)$. So we propose the following definition of $f'(a)$:

$$f'(a) = \begin{Bmatrix}\text{the limit as } h \\ \text{tends to 0 of}\end{Bmatrix} \frac{f(a+h) - f(a)}{h}$$

In mathematics, it is common to use the abbreviated notation $\lim_{h \to 0}$ for "the limit as h tends to zero" of an expression involving h. We therefore have the following definition:

The **derivative** $f'(a)$ of the function f at point a of its domain is given by the formula

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \qquad [4.3]$$

As in [4.1], the number $f'(a)$ gives the slope of the tangent to the curve $y = f(x)$ at the point $(a, f(a))$. The equation for a straight line passing through (x_1, y_1) and having a slope b is given by $y - y_1 = b(x - x_1)$. Hence, we obtain:

The equation for the **tangent** to the graph of $y = f(x)$ at the point $(a, f(a))$ is

$$y - f(a) = f'(a)(x - a) \tag{4.4}$$

So far the concept of a limit in the definition of $f'(a)$ is not quite clear. Section 6.7 gives a precise definition. Because it is relatively complicated, we rely on intuition for the time being. Consider a simple example.

Example 4.2

Use [4.3] to compute $f'(a)$ when $f(x) = x^2$. Find in particular $f'(1/2)$, $f'(0)$, and $f'(-1)$. Give geometric interpretations, and find the equation for the tangent at the point $(1/2, 1/4)$.

Solution For $f(x) = x^2$, we have $f(a+h) = (a+h)^2 = a^2+2ah+h^2$, and so $f(a+h) - f(a) = (a^2+2ah+h^2) - a^2 = 2ah + h^2$. Hence, for all $h \neq 0$, we obtain

$$\frac{f(a+h) - f(a)}{h} = \frac{2ah + h^2}{h} = \frac{h(2a+h)}{h} = 2a + h \qquad [1]$$

because we can cancel h whenever $h \neq 0$. But as h tends to 0, so $2a + h$ obviously tends to $2a$. Thus, we can write

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0}(2a + h) = 2a \qquad [2]$$

This shows that when $f(x) = x^2$, then $f'(a) = 2a$. For $a = 1/2$, we obtain $f'(1/2) = 2 \cdot 1/2 = 1$. Similarly, $f'(0) = 2 \cdot 0 = 0$ and $f'(-1) = 2 \cdot (-1) = -2$.

In Fig. 4.6, we provide the geometric interpretation of [1]. In Fig. 4.7, we have drawn the tangents to the curve $y = x^2$ corresponding to $a = 1/2$ and $a = -1$. At $a = 1/2$, we have $f(a) = (1/2)^2 = 1/4$ and $f'(1/2) = 1$. According to [4.4], the equation of the tangent is $y - 1/4 = 1 \cdot (x - 1/2)$ or $y = x - 1/4$. (Show that the other tangent drawn in Fig. 4.7 has the equation $y = -2x - 1$.) Note that the formula $f'(a) = 2a$ shows that $f'(a) < 0$ when $a < 0$, and $f'(a) > 0$ when $a > 0$. Does this agree with the graph?

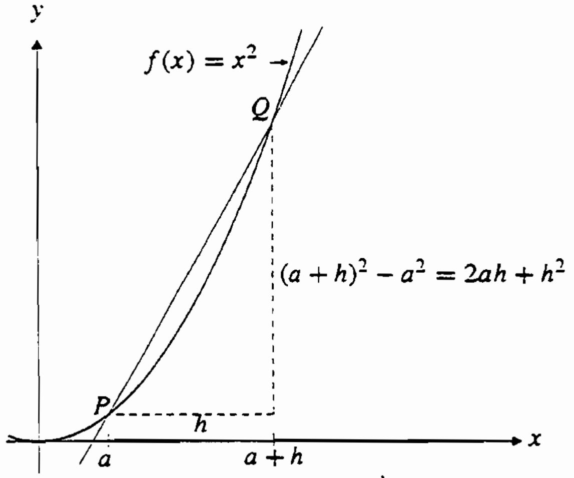


FIGURE 4.6 $f(x) = x^2$.

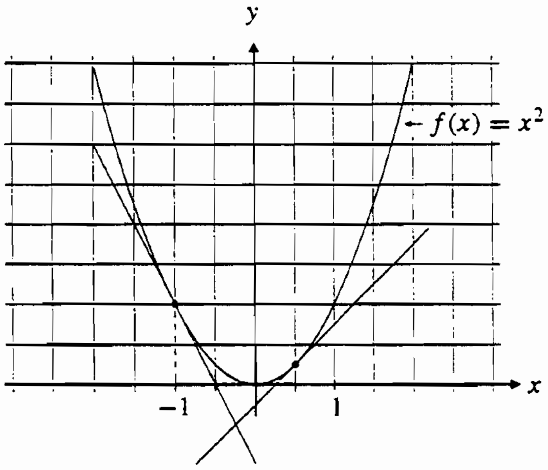


FIGURE 4.7 $f(x) = x^2$.

If f is a relatively simple function, we can find $f'(a)$ as follows:

Recipe for computing $f'(a)$:

1. Add h to a ($h \neq 0$) and compute $f(a+h)$.
2. Compute the corresponding change in the function value: $f(a+h) - f(a)$.
3. For $h \neq 0$, form the Newton quotient

$$\frac{f(a+h) - f(a)}{h} \tag{4.5}$$

4. Simplify the fraction in step 3 as much as possible. Wherever possible, cancel h from both numerator and denominator.
5. Then $f'(a)$ is the number that

$$\frac{f(a+h) - f(a)}{h}$$

approaches as h tends to 0.

Let us apply this recipe to another example.

Example 4.3

Compute $f'(a)$ when $f(x) = x^3$.

Solution We follow the recipe in [4.5].

1. $f(a+h) = (a+h)^3 = a^3 + 3a^2h + 3ah^2 + h^3$

2. $f(a+h) - f(a) = (a^3 + 3a^2h + 3ah^2 + h^3) - a^3$
$= 3a^2h + 3ah^2 + h^3$

3–4. $\dfrac{f(a+h) - f(a)}{h} = \dfrac{3a^2h + 3ah^2 + h^3}{h} = 3a^2 + 3ah + h^2$

5. As h tends to 0, so $3ah + h^2$ will also tend to 0; hence, the entire expression $3a^2 + 3ah + h^2$ tends to $3a^2$. Therefore, $f'(a) = 3a^2$.

We have thus shown that the graph of the function $f(x) = x^3$ at the point $x = a$ has a tangent with slope $3a^2$. Note that $f'(a) = 3a^2 > 0$ when $a \neq 0$, and $f'(0) = 0$. The tangent points upwards to the right for all $a \neq 0$, and is horizontal at the origin. You should draw the graph of $f(x) = x^3$ to confirm this behavior.

It is easy to use the recipe in [4.5] on simple functions. However, the recipe becomes difficult or even impossible if we try to use it on slightly more complicated functions such as $f(x) = \sqrt{3x^2 + x + 1}$. The next chapter develops rules for computing the derivative of quite complicated functions, without the need to use [4.5]. Before presenting such rules, however, we must examine the concept of a limit a little more carefully. This is done in Section 4.4.

On Notation

We showed in Example 4.2 that, if $f(x) = x^2$, then for every a, we have $f'(a) = 2a$. We frequently use x as the symbol for a quantity that can take any value, so we write $f'(x) = 2x$. If we use this new notation for the function in Example 4.3, we can briefly formulate our main results from the two last examples as follows:

$$f(x) = x^2 \implies f'(x) = 2x \qquad [4.6]$$

$$f(x) = x^3 \implies f'(x) = 3x^2 \qquad [4.7]$$

Equation [4.6] is a special case of the following rule, which you are asked to show in Problem 6.

$$f(x) = ax^2 + bx + c \implies f'(x) = 2ax + b \qquad (a, b, \text{ and } c \text{ are constants}) \qquad [4.8]$$

For $a = 1$, $b = c = 0$, this reduces to [4.6]. Here are some special cases of [4.8]:

$$f(x) = 3x^2 + 2x + 5 \implies f'(x) = 3 \cdot 2x + 2 = 6x + 2$$

$$f(x) = -16 + \tfrac{1}{2}x - \tfrac{1}{16}x^2 \implies f'(x) = -\tfrac{1}{8}x + \tfrac{1}{2}$$

$$f(x) = (x - p)^2 = x^2 - 2px + p^2 \implies f'(x) = 2x - 2p \qquad (p \text{ constant})$$

If we use y to denote the value of the function $y = f(x)$, we often denote the derivative by y'. We can then write $y = x^3 \Rightarrow y' = 3x^2$.

Several other forms of notation for the derivative are often used in mathematics and its applications. One of them, originally due to Leibniz, is called the **differential notation**. If $y = f(x)$, we write

$$\frac{dy}{dx} = dy/dx \text{ or } \frac{df(x)}{dx} = df(x)/dx \text{ or } \frac{d}{dx}f(x) \text{ in place of } f'(x)$$

For instance, if $y = x^2$, then

$$\frac{dy}{dx} = 2x = \frac{d}{dx}(x^2) = 2x$$

At this point, we will only think of the symbol dy/dx as meaning $f'(x)$ and will not consider it as dy divided by dx. Later chapters discuss this notation in greater detail. In fact, d/dx really denotes an instruction to differentiate what follows with respect to x. Differentiation with respect to a variable occurs so often in mathematics that it has become standard to use **w.r.t.** as an abbreviation for "with respect to."

When we use letters other than f, x, and y, the notation for the derivative changes accordingly. For example: $P(t) = t^2 \Rightarrow P'(t) = 2t$; $Y = K^3 \Rightarrow Y' = 3K^2$; and $A = r^2 \Rightarrow dA/dr = 2r$.

Problems

1. Let $f(x) = 4x^2$. Show that $f(5+h) - f(5) = 40h + 4h^2$. Hence,

$$\frac{f(5+h) - f(5)}{h} = 40 + 4h$$

Using this result, find $f'(5)$. Compare the answer with [4.8].

2. Let $f(x) = 3x^2 + 2x - 1$. Show that for $h \neq 0$,

$$\frac{f(x+h) - f(x)}{h} = 6x + 2 + 3h$$

Use this result to find $f'(x)$. Find in particular $f'(0)$, $f'(-2)$, and $f'(3)$, and the equation for the tangent to the graph at the point $(0, -1)$.

3. Figure 4.8 shows the graph of a function f. Determine whether the following derivatives are > 0, $= 0$, or < 0: $f'(a)$, $f'(b)$, $f'(c)$, and $f'(d)$.

4. Show that

$$f(x) = \frac{1}{x} \Longrightarrow f'(x) = -\frac{1}{x^2}$$

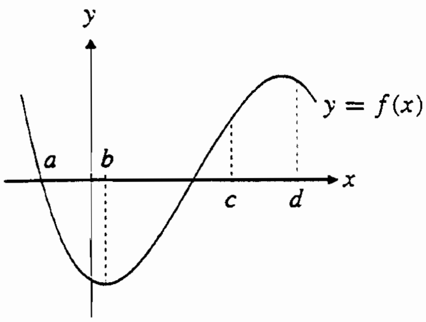


FIGURE 4.8

Hint: Show that $[f(x+h) - f(x)]/h = -1/x(x+h)$. (See Problem 14, Section 2.5.)

5. Find the slope for the tangent to the graph of the following functions at the specified points:

   a. $f(x) = 3x + 2$ at $(0, 2)$ **b.** $f(x) = x^2 - 1$ at $(1, 0)$

   c. $f(x) = \dfrac{3}{x} + 2$ at $(3, 3)$ **d.** $f(x) = x^3 - 2x$ at $(0, 0)$

   e. $f(x) = x + \dfrac{1}{x}$ at $(-1, -2)$ **f.** $f(x) = x^4$ at $(1, 1)$

6. **a.** If $f(x) = ax^2 + bx + c$, show that $[f(x+h) - f(x)]/h = 2ax + b + ah$. Use this to show that $f'(x) = 2ax + b$.
   b. For what value of x is $f'(x) = 0$? Explain this result in the light of [3.4] in Section 3.1.

7. **a.** The demand function for a commodity with price P is given by the formula $D(P) = a - bP$. Find $dD(P)/dP$.
   b. The cost of producing x units of a commodity is given by the formula $C(x) = p + qx^2$. Find $C'(x)$.

8. **a.** Show that $(\sqrt{x+h} - \sqrt{x})(\sqrt{x+h} + \sqrt{x}) = h$.
   b. If $f(x) = \sqrt{x}$, show that $[f(x+h) - f(x)]/h = 1/(\sqrt{x+h} + \sqrt{x})$.
   c. Use the result in part (b) to show that for $x > 0$,

$$f(x) = \sqrt{x} \Longrightarrow f'(x) = \frac{1}{2\sqrt{x}}$$

   d. Show that the result could also be written as

$$\frac{d}{dx}x^{1/2} = \frac{1}{2}x^{-1/2}$$

9. **a.** If $f(x) = ax^3 + bx^2 + cx + d$, show that

$$[f(x+h) - f(x)]/h = 3ax^2 + 2bx + c + 3axh + ah^2 + bh$$

   and hence that $f'(x) = 3ax^2 + 2bx + c$.

b. Show that the result in part (a) generalizes the results in Example 4.3 and in Problem 6.

Harder Problems

10. **a.** If $f(x) = x^{1/3}$, show that

$$\frac{f(x+h) - f(x)}{h} = \frac{1}{(x+h)^{2/3} + (x+h)^{1/3} \cdot x^{1/3} + x^{2/3}}$$

by using the result in Problem 11(a), Section 3.14, with $a = x + h$ and $b = x$.

b. Use the result in part (a) to show that

$$\frac{d}{dx}(x^{1/3}) = \frac{1}{3}x^{-2/3}$$

4.3 Rates of Change and Their Economic Significance

We have so far interpreted the derivative of a function as the slope of the tangent to its graph at a particular point. In economics, other interpretations are more important. Let us first see how the derivative in general can be interpreted as a rate of change.

Suppose that a quantity y is related to a quantity x by $y = f(x)$. If x has a given value a, then the value of the function is given by $f(a)$. Suppose that a is changed to $a+h$. The new value of y is $f(a+h)$, and the change in the value of the function when x is changed from a to $a+h$ is $f(a+h) - f(a)$. The change in y per unit change in x has a particular name, the *average rate of change of f over the interval from a to $a+h$*. It is equal to

$$\frac{f(a+h) - f(a)}{h} \tag{4.9}$$

Note that this fraction is precisely the Newton quotient of f. Taking the limit as h tends to 0 gives the derivative of f at a. Therefore:

> The **instantaneous rate of change** of f at a is $f'(a)$ [4.10]

This very important concept appears whenever we study quantities that change. When time is the independent variable, we often use the "dot notation" for differentiation with respect to time. For example, if $x(t) = t^2$, we write $\dot{x}(t) = 2t$.

Sometimes we are interested in studying the proportion $f'(a)/f(a)$. We introduce a name for this:

The **proportional rate of change** of f at a is $f'(a)/f(a)$ [4.11]

In economics, proportional rates of change are seen very often. Sometimes they are called **relative rates of change**. They are usually quoted in percentages—or when time is the independent variable, as percentages per year, or *per annum*. Often we will describe a variable as increasing at, say, 3% a year if there is a proportional rate of change of 3/100 each year.

Example 4.4

Let $N(t)$ be the number of individuals in a population (of humans, animals, or plants) at time t. If t increases to $t + h$, then the change in population is equal to $N(t + h) - N(t)$ individuals. Hence, $[N(t + h) - N(t)]/h$ is *the average rate of change*. Taking the limit as h tends to 0 gives $\dot{N}(t) = dN/dt$ for *the rate of change of population at time* t. (At the end of this section, we will discuss the problem that arises when $N(t)$ takes only integer values.)

Example 6 of Section 2.5 was based on the case when P denotes the number (in millions) of inhabitants of Europe, which was given by the formula

$$P = 6.4t + 641 \qquad [1]$$

Here t is the number of years, as computed from 1960. In this case, the rate of change is the same for all t:

$$\frac{dP}{dt} = 6.4 \text{ million per year}$$

Economic Interpretations

Example 4.5

Consider a firm producing some commodity in a given period. Let

$C(x) =$ cost of producing x units

$R(x) =$ revenue from selling x units

$\pi(x) = R(x) - C(x) =$ profit from producing (and selling) x units

We call $C'(x)$ the **marginal cost** (at x), $R'(x)$ the **marginal revenue**, and $\pi'(x)$ the **marginal profit**. Economists often use the word **marginal** in this way in order to signify a derivative.

Other examples of the derivative in economics include the following. The **marginal propensity to consume** is the derivative of the consumption function with respect to income; similarly, the **marginal product** (or **productivity) of labor** is the derivative of the production function with respect to labor input.

According to the definition, marginal cost is equal to

$$C'(x) = \lim_{h\to 0} \frac{C(x+h) - C(x)}{h} \qquad \textbf{(marginal cost)} \qquad [4.12]$$

Usually, the firm will produce many units of x. Then $h = 1$ can be considered a number close to 0, and we obtain the approximation

$$C'(x) \approx \frac{C(x+1) - C(x)}{1} = C(x+1) - C(x)$$

Marginal cost is then approximately equal to the **incremental cost** $C(x+1) - C(x)$, that is, the *additional cost of producing one more unit of* x. In elementary economics courses, marginal cost is often defined as the difference $C(x+1) - C(x)$ because more appropriate concepts from differential calculus cannot be used.

Example 4.6

Let $K(t)$ be the capital stock in an economy at time t. The rate of change $\dot{K}(t)$ of $K(t)$ is called the **rate of investment** at time t. It is usually denoted by $I(t)$, so

$$\dot{K}(t) = I(t) \qquad [4.13]$$

Differentiability and Empirical Functions

The very definition of derivative assumes that arbitrary small increments in the independent variable are possible. In practical problems, however, it is usually impossible to implement, or even measure, arbitrary small changes in the variable. For example, economic quantities that vary with time, such as the price of a commodity or the national income of a country, are usually measured at intervals of days, weeks, or years. Further, the cost functions of the type we discussed in Example 4.5 are often properly defined only for integer values of x. In all these cases, the variables only take discrete values. The graphs of such functions, therefore, will only consist of discrete points. For functions of this type in which time and numbers both change discretely, the concept of the derivative is not defined. To remedy this, the actual function is usually replaced by a differentiable function that is a "good approximation" to the original function. As an illustration, Fig. 4.9 graphs observations of the number of registered unemployed in Norway for each month of the years 1928–1929. In Fig. 4.10

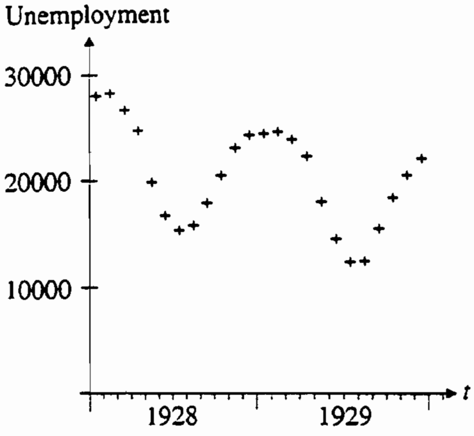


FIGURE 4.9 Unemployment in Norway (1928–1929).

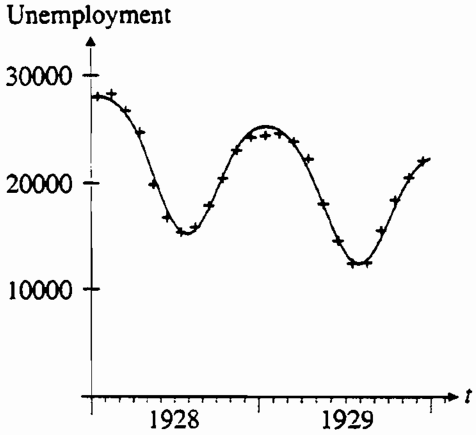


FIGURE 4.10 A smooth curve approximating the points in Fig. 4.9.

we show the graph of a differentiable function that is an approximation to the points plotted in Fig. 4.9. (The graph in Fig. 4.10 is drawn using a computer program.)

Problems

1. Let $C(x) = x^2 + 3x + 100$ be the cost function of a firm. Show that the average rate of change when x is changed from 100 to $100 + h$ is

$$\frac{C(100+h) - C(100)}{h} = 203 + h \qquad (h \neq 0)$$

What is the marginal cost $C'(100)$? Use [4.8] to find $C'(x)$ and in particular $C'(100)$.

2. If the cost function of a firm is $C(x) = kx + I$, give economic interpretations of the parameters k and I.

3. If the total savings of a country is a function $S(Y)$ of the national product Y, then $S'(Y)$ is called the *marginal propensity to save* (MPS). Find the MPS

for the following functions:

a. $S(Y) = a + bY$ b. $S(Y) = 100 + 10Y + 2Y^2$

4. If the tax a family pays is a function of its income y given by $T(y)$, then $dT(y)/dy$ is called the *marginal tax rate*. Characterize the following tax function by determining its marginal rate:

$$T(y) = ty \qquad (t \text{ is a constant number } \in (0, 1))$$

5. Refer to the definitions given in Example 4.5. Compute the marginal revenue, marginal cost, and marginal profit in the following two cases (where p, a, b, a_1, b_1, and c_1 are all positive constants), and in each case find an expression for the value of x at which the marginal profit is equal to 0:

a. $R(x)$, $C(x) = a_1x^2 + b_1x + c_1$

b. $R(x) = ax^2 - bx^2$, $C(x) = a_1x + b_1$

4.4 A Dash of Limits

The previous section defined the derivative of a function based on the concept of a limit. The same concept is important for other reasons as well, so now we should take a closer look. Here we give a preliminary definition and formulate some important rules for limits. In Chapter 6, we discuss the limit concept more closely, as well as the related concept of continuity.

As an example, consider the formula

$$F(x) = \frac{x^2 - 16}{4\sqrt{x} - 8}$$

Note that if $x = 4$, then the fraction collapses to the absurd expression "0/0." Thus, the function F is not defined for $x = 4$, but one can still ask what happens to $F(x)$ when x is close to 4. Using a calculator (except when $x = 4$), we find the values shown in Table 4.1.

TABLE 4.1 *Values of $F(x) = (x^2 - 16)/(4\sqrt{x} - 8)$ when x is close to 4*

x	3.9	3.99	3.999	3.9999	4.0	4.0001	4.001	4.01	4.1
$F(x)$	7.850	7.985	7.998	8.000	*	8.000	8.002	8.015	8.150

*Not defined.

It seems obvious from the table that as x gets closer and closer to 4, so the fraction $F(x)$ gets closer and closer to 8. It therefore seems reasonable to say that

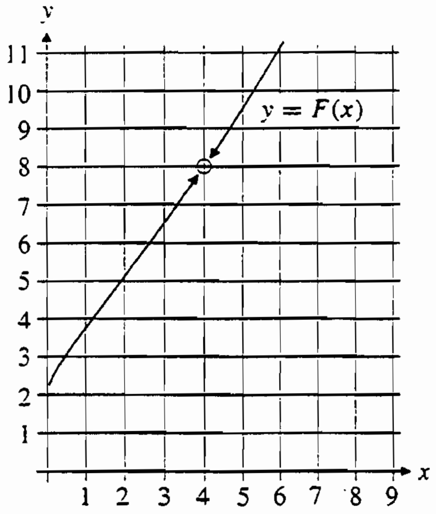


FIGURE 4.11 $y = F(x) = \dfrac{x^2 - 16}{4\sqrt{x} - 8}$

$F(x)$ tends to 8 in the limit as x tends to 4. We write

$$\lim_{x \to 4} \frac{x^2 - 16}{4\sqrt{x} - 8} = 8 \quad \text{or} \quad \frac{x^2 - 16}{4\sqrt{x} - 8} \to 8 \quad \text{as} \quad x \to 4$$

In Fig. 4.11 we have drawn a portion of the graph of F. The function F is defined for all $x \geq 0$, except at $x = 4$. Also $\lim_{x \to 4} F(x) = 8$. (A small circle is used to indicate that the corresponding point $(4, 8)$ is not in the graph of F.)

A Preliminary Definition of the Limit Concept

Suppose, in general, that a function f is defined for all x near a, but not necessarily at $x = a$. *Then we say that $f(x)$ has the number A as its limit as x tends to a, if $f(x)$ tends to A as x tends to (but is not equal to) a.* We write

$$\lim_{x \to a} f(x) = A \quad \text{or} \quad f(x) \to A \quad \text{as} \quad x \to a$$

It is possible, however, that the value of $f(x)$ does not tend to any fixed number as x tends to a. Then we say that $\lim_{x \to a} f(x)$ *does not exist*, or that $f(x)$ *does not have a limit as x tends to a.*

Example 4.7

Use a calculator to examine the following limits:

(a) $\lim_{x \to 3}(3x - 2)$

(b) $\lim_{h \to 0} \dfrac{\sqrt{h+1} - 1}{h}$

(c) $\lim_{x \to -2} \dfrac{1}{(x+2)^2}$

Solution

(a) We obtain Table 4.2 when x is a number close to 3. As $x \to 3$, it seems that $3x - 2$ tends to 7, so that $\lim_{x\to 3}(3x - 2) = 7$. (If x is precisely equal to 3, then $3x - 2$ is equal to 7. But the definition of $\lim_{x\to 3}(3x - 2)$ ignores the value of $3x - 2$ at $x = 3$.)

TABLE 4.2 *Values of* $3x - 2$ *when* x *is close to 3*

x	2.9	2.95	2.99	2.999	3.001	3.01	3.05	3.1
$3x - 2$	6.7	6.85	6.97	6.997	7.003	7.03	7.15	7.3

(b) Some values of h close to 0 give the values in Table 4.3. This suggests that

$$\lim_{h\to 0} \frac{\sqrt{h+1} - 1}{h} = 0.5$$

TABLE 4.3 *Values of* $(\sqrt{h+1} - 1)/h$ *when* h *is close to 0*

h	−0.5	−0.2	−0.1	−0.01	0.0	0.01	0.1	0.2	0.5
$\frac{\sqrt{h+1}-1}{h}$	0.586	0.528	0.513	0.501	*	0.499	0.488	0.477	0.449

*Not defined.

(c) We choose x values close to -2 and obtain the values in Table 4.4. As x gets closer and closer to -2, we see that the value of the fraction becomes larger and larger. By extending the values in the table, it is clear, for example, that for $x = -2.0001$ and $x = -1.9999$, the value of the fraction is 100 million. Hence, we conclude that $1/(x+2)^2$ does not tend to any limit as x tends to -2. Because the fraction becomes larger and larger as x tends to -2, we say that it tends to infinity, and write $\lim_{x\to -2} 1/(x+2)^2 = \infty$.

TABLE 4.4 *Values of* $1/(x+2)^2$ *when* x *is close to* -2

x	−1.8	−1.9	−1.99	−1.999	−2.0	−2.001	−2.01	−2.1	−2.2
$\frac{1}{(x+2)^2}$	25	100	10,000	1,000,000	*	1,000,000	10,000	100	25

*Not defined.

The limits found previously were all based on shaky numerical foundations. For instance, in Example 4.7(b), can we really be certain that our guess is correct? Could it be that if we chose h values even closer to 0, the fraction would tend to a limit other than 0.5, or maybe not have any limit at all? Further numerical computations will support our belief that the initial guess is correct, but we can never make a table that has *all* the values of h close to 0, so numerical computation alone can never establish with certainty what the limit is. This illustrates the need to have a rigorous procedure for finding limits. First of all, however, a precise mathematical definition of the limit concept is required. One such definition is given in Section 6.7. Meanwhile, here is a preliminary definition.

> Writing $\lim_{x\to a} f(x) = A$ means that we can make $f(x)$ as close to A as we want for all x sufficiently close to (but not equal to) a. [4.14]

We emphasize:

1. The number $\lim_{x\to a} f(x)$ depends on the value of $f(x)$ for x values close to a, but not on how f behaves at the precise value of $x = a$. When finding $\lim_{x\to a} f(x)$, we are simply not interested in the value $f(a)$, or even in whether f is actually defined at a.
2. When we compute $\lim_{x\to a} f(x)$, we must take into consideration x values on both sides of a.

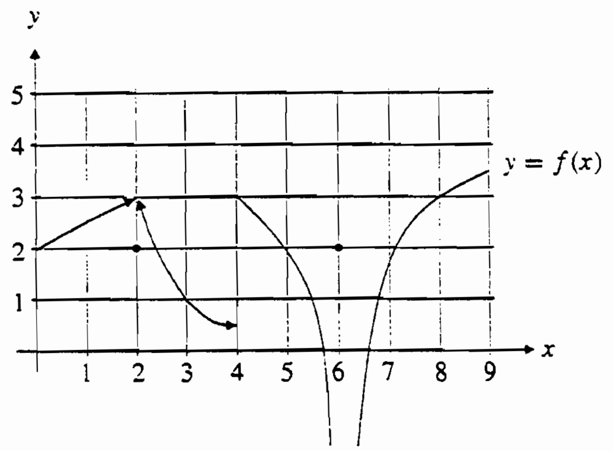
The next example illustrates the limit concept geometrically.

Example 4.8

Figure 4.12 shows the graph of a particular function f, defined in the closed interval $[0, 9]$. Determine $\lim_{x\to a} f(x)$ for $a = 2, 3, 4$, and 6. (The point at the end of each arrow is not part of the graph, but is the limit of points on the graph.)

Solution We see that $\lim_{x\to 2} f(x) = 3$. Note that $f(2) = 2$. Also $\lim_{x\to 3} f(x) = 1$. Here $f(3) = 1$. The limit $\lim_{x\to 4} f(x)$ does not exist.

FIGURE 4.12



For x close to 4 and $x < 4$, $f(x)$ tends to 1/2, and for x close to 4 and $x > 4$, $f(x)$ tends to 3. Thus, $f(x)$ does not tend to *one* specific number as x tends to 4. Finally, $\lim_{x\to 6} f(x)$ does not exist. As x tends to 6, $f(x)$ will decrease without limit. We write $\lim_{x\to 6} f(x) = -\infty$.

Rules for Limits

Of course, one cannot really determine limits by means of numerical computations. Instead, we use some simple rules for finding limits whose validity can be shown once we have a precise definition of the limit concept. These rules are very straightforward and we have even used a few of them already in the previous section. Let us briefly discuss some of them.

Suppose that f and g are defined as functions in the neighborhood of a (but not necessarily at a). Then we have the following rules:

Rules for Limits

If $\lim_{x\to a} f(x) = A$ and $\lim_{x\to a} g(x) = B$, then

i. $\lim_{x\to a} [f(x) + g(x)] = A + B$ [4.15]
ii. $\lim_{x\to a} [f(x) - g(x)] = A - B$
iii. $\lim_{x\to a} [f(x)g(x)] = A \cdot B$
iv. $\lim_{x\to a}[f(x)/g(x)] = A/B$ (provided $B \neq 0$)
v. $\lim_{x\to a} [f(x)]^{p/q} = A^{p/q}$ (if $A^{p/q}$ is defined)

It is easy to give intuitive explanations for these rules. If $\lim_{x\to a} f(x) = A$ and $\lim_{x\to a} g(x) = B$, then we know that, when x is close to a, then $f(x)$ is close to A and $g(x)$ is close to B. So presumably the sum $f(x) + g(x)$ is close to $A + B$, the product $f(x)g(x)$ is close to $A \cdot B$, and so on.

The rules in [4.15] can be used repeatedly to obtain new extended rules such as

$$\lim_{x\to a} [f_1(x) + f_2(x) + \cdots + f_n(x)] = \lim_{x\to a} f_1(x) + \lim_{x\to a} f_2(x) + \cdots + \lim_{x\to a} f_n(x) \qquad [4.16]$$

$$\lim_{x\to a} [f_1(x) \cdot f_2(x) \cdots f_n(x)] = \lim_{x\to a} f_1(x) \cdot \lim_{x\to a} f_2(x) \cdots \lim_{x\to a} f_n(x) \qquad [4.17]$$

In words, we can say that *the limit of a sum is the sum of the limits, and the limit of a product is equal to the product of the limits.*

Suppose the function $f(x)$ is equal to the same constant value c for every x. Then

$$\lim_{x \to a} c = c \qquad \text{(at every point } a\text{)} \tag{4.18}$$

It is also evident that if $f(x) = x$, then

$$\lim_{x \to a} f(x) = \lim_{x \to a} x = a \qquad \text{(at every point } a\text{)} \tag{4.19}$$

Combining these two simple limits with the general rules [4.15]–[4.17] allows easy computation of the limits for certain combinations of functions.

Example 4.9

Compute the following limits:

(a) $\lim_{x \to -2} (x^2 + 5x)$

(b) $\lim_{x \to 4} \dfrac{2x^{3/2} - \sqrt{x}}{x^2 - 15}$

(c) $\lim_{x \to a} Ax^n$

Solution Using the rules in [4.15]–[4.17], we get

(a)
$$\begin{aligned}\lim_{x \to -2} (x^2 + 5x) &= \lim_{x \to -2} (x \cdot x) + \lim_{x \to -2} (5 \cdot x) \\ &= \big(\lim_{x \to -2} x\big)\big(\lim_{x \to -2} x\big) + \big(\lim_{x \to -2} 5\big)\big(\lim_{x \to -2} x\big) \\ &= (-2)(-2) + 5 \cdot (-2) = -6\end{aligned}$$

(b)
$$\lim_{x \to 4} \frac{2x^{3/2} - \sqrt{x}}{x^2 - 15} = \frac{2 \lim_{x \to 4} x^{3/2} - \lim_{x \to 4} \sqrt{x}}{\lim_{x \to 4} x^2 - 15} = \frac{2 \cdot 4^{3/2} - \sqrt{4}}{4^2 - 15} = 14$$

(c)
$$\lim_{x \to a} Ax^n = \big(\lim_{x \to a} A\big)\big(\lim_{x \to a} x^n\big) = A \cdot \big(\lim_{x \to a} x\big)^n = A \cdot a^n$$

It was easy to find the limits in this example by using rules [4.15]–[4.19]. The example that started this section and Example 4.7(b) both present more difficulties. They involve a fraction whose numerator and denominator both tend to 0. Rule [4.15](iv) cannot be applied directly in such cases. However, a simple observation can still help us find the limit (provided that it exists). Because $\lim_{x \to a} f(x)$ can only depend on the values of f when x is close to, but not equal to a, we have the following:

> If the functions f and g are equal for all x close to a (but not necessarily at $x = a$), then $\lim_{x \to a} f(x) = \lim_{x \to a} g(x)$ whenever either limit exists. [4.20]

Here are some examples of how this rule works.

Example 4.10

Compute the following limits:

(a) $\lim_{x\to 2} \dfrac{3x^2 + 3x - 18}{x - 2}$

(b) $\lim_{h\to 0} \dfrac{\sqrt{h+1} - 1}{h}$

(c) $\lim_{x\to 4} \dfrac{x^2 - 16}{4\sqrt{x} - 8}$

Solution

(a) We see that both numerator and denominator tend to 0 when x tends to 2. Because the numerator $3x^2 + 3x - 18$ is equal to 0 for $x = 2$, it has $x - 2$ as a factor. In fact, $3x^2 + 3x - 18 = 3(x-2)(x+3)$. Hence,

$$f(x) = \frac{3x^2 + 3x - 18}{x - 2} = \frac{3(x-2)(x+3)}{x-2}$$

For $x \neq 2$, we can cancel $x - 2$ from both numerator and denominator to obtain $3(x + 3)$. So the functions $f(x)$ and $g(x) = 3(x + 3)$ are equal for all $x \neq 2$. According to [4.20], this implies that

$$\lim_{x\to 2} \frac{3x^2 + 3x - 18}{x - 2} = \lim_{x\to 2} 3(x+3) = 3(2+3) = 15$$

(b) Again both numerator and denominator tend to 0 as h tends to 0. Here we must use a little trick. We multiply both numerator and denominator by $\sqrt{h+1} + 1$ to get

$$\frac{\sqrt{h+1} - 1}{h} = \frac{\left(\sqrt{h+1} - 1\right)\left(\sqrt{h+1} + 1\right)}{h\left(\sqrt{h+1} + 1\right)}$$
$$= \frac{h + 1 - 1}{h\left(\sqrt{h+1} + 1\right)} = \frac{1}{\sqrt{h+1} + 1}$$

where the common factor h has been canceled. For all $h \neq 0$ (and $h \geq -1$), the given function is equal to $1/(\sqrt{h+1} + 1)$, which tends to $1/2$ as h tends to 0. We conclude that the limit of our function is equal to $1/2$, which confirms the result in Example 4.7(b).

(c) We must try to simplify the fraction because $x = 4$ gives $0/0$. Again we can use a trick to factorize the fraction as follows:

$$\frac{x^2 - 16}{4\sqrt{x} - 8} = \frac{(x+4)(x-4)}{4\left(\sqrt{x} - 2\right)} = \frac{(x+4)\left(\sqrt{x} + 2\right)\left(\sqrt{x} - 2\right)}{4\left(\sqrt{x} - 2\right)} \qquad [*]$$

Here we have used the factorization $x-4 = \left(\sqrt{x}+2\right)\left(\sqrt{x}-2\right)$, which is correct for $x \geq 0$. In the last fraction of [*], we can cancel $\sqrt{x}-2$ when $\sqrt{x}-2 \neq 0$—that is, when $x \neq 4$. Using [4.20] again gives

$$\lim_{x\to 4} \frac{x^2-16}{4\sqrt{x}-8} = \lim_{x\to 4} \frac{1}{4}(x+4)(\sqrt{x}+2) = \frac{1}{4}(4+4)(\sqrt{4}+2) = 8$$

This confirms the claim we made in the introduction to this section. Section 7.5 treats more systematically limits of fractions of the type studied in Example 4.10.

Problems

1. Determine the following by using the rules for limits:

a. $\lim_{x\to 0} (3+2x^2)$ **b.** $\lim_{x\to -1} \frac{3+2x}{x-1}$ **c.** $\lim_{x\to 2} (2x^2+5)^3$

d. $\lim_{t\to 8} \left(5t+t^2-\frac{1}{8}t^3\right)$ **e.** $\lim_{y\to 0} \frac{(y+1)^5-y^5}{y+1}$ **f.** $\lim_{z\to -2} \frac{1/z+2}{z}$

2. Consider the following limit:

$$\lim_{x\to 1} \frac{x^2+7x-8}{x-1}$$

a. Examine the limit numerically by making a table of values of the fraction when x is close to 1.

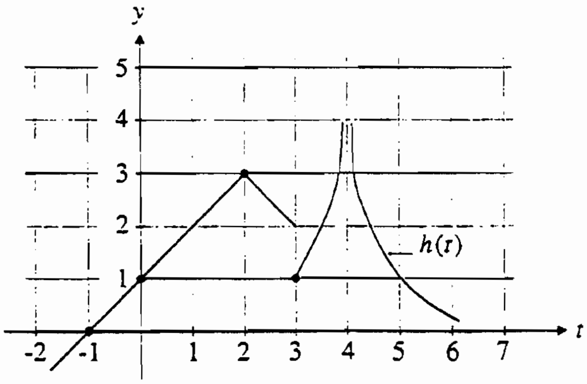
b. Find the limit by using [4.20].

3. For the function h whose graph is given in Fig. 4.13, examine $\lim_{t\to a} h(t)$ for $a = -1, 0, 2, 3$, and 4.

4. Compute the following limits:

a. $\lim_{x\to 2}(x^2+3x-5)$ **b.** $\lim_{y\to -3} \frac{1}{y+8}$

FIGURE 4.13



c. $\lim_{x \to 0} \dfrac{x^3 - 2x - 1}{x^5 - x^2 - 1}$ d. $\lim_{x \to 0} \dfrac{x^3 + 3x^2 - 2x}{x}$

e. $\lim_{h \to 0} \dfrac{(x+h)^3 - x^3}{h}$ f. $\lim_{x \to 0} \dfrac{(x+h)^3 - x^3}{h}$ $(h \neq 0)$

5. Compute the following limits:

a. $\lim_{h \to 2} \dfrac{1/3 - 2/3h}{h - 2}$ b. $\lim_{x \to 0} \dfrac{x^2 - 1}{x^2}$ c. $\lim_{t \to 3} \sqrt[3]{\dfrac{32t - 96}{t^2 - 2t - 3}}$

d. $\lim_{h \to 0} \dfrac{\sqrt{h+3} - \sqrt{3}}{h}$ e. $\lim_{t \to -2} \dfrac{t^2 - 4}{t^2 + 10t + 16}$ f. $\lim_{x \to 4} \dfrac{2 - \sqrt{x}}{4 - x}$

6. If $f(x) = x^2 + 2x$, compute the following limits:

a. $\lim_{x \to 1} \dfrac{f(x) - f(1)}{x - 1}$ b. $\lim_{x \to 2} \dfrac{f(x) - f(1)}{x - 1}$

c. $\lim_{h \to 0} \dfrac{f(2+h) - f(2)}{h}$ d. $\lim_{x \to a} \dfrac{f(x) - f(a)}{x - a}$

e. $\lim_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$ f. $\lim_{h \to 0} \dfrac{f(a+h) - f(a-h)}{h}$

7. Compute the following limits numerically by using a calculator:

a. $\lim_{h \to 0} \dfrac{2^h - 1}{h}$ b. $\lim_{h \to 0} \dfrac{3^h - 1}{h}$ c. $\lim_{h \to 0} (1 + h)^{1/h}$

Harder Problems

8. Compute the following limits. (*Hint:* For part (b), substitute $\sqrt[3]{27 + h} = u$.)

a. $\lim_{x \to 2} \dfrac{x^2 - 2x}{x^3 - 8}$ b. $\lim_{h \to 0} \dfrac{\sqrt[3]{27 + h} - 3}{h}$

c. $\lim_{x \to 1} \dfrac{x^n - 1}{x - 1}$ (n is a natural number)

4.5 Simple Rules for Differentiation

In Section 4.2, we defined the derivative of a function f by the formula

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \qquad [*]$$

If this limit exists, we say that f is **differentiable** at x. The process of finding the derivative of a function is called **differentiation.** It is useful to think of this as an operation that transforms one function f into a new function f'. The function f' is then defined for the values of x where the limit in [*] exists. If $y = f(x)$, we can use the symbols y' and dy/dx as alternatives to $f'(x)$.

In the examples and problems in Section 4.2, we used formula [*] and the recipe in [4.5] in order to find the derivatives of some simple functions. However, it is often difficult to apply the definition directly. The next chapter uses the recipe in [4.5] systematically to devise rules that can be used to find the derivatives of quite complicated functions. Here we only consider some very simple rules.

If f is the constant function $f(x) = A$, then the derivative $f'(x)$ is equal to 0:

$$f(x) = A \Longrightarrow f'(x) = 0$$ [4.21]

The result is easy to see geometrically. The graph of $f(x) = A$ is a straight line parallel to the x-axis. The tangent to the graph has slope 0 at each point (see Fig. 4.14). You should now use the definition of $f'(x)$ to get the same answer.

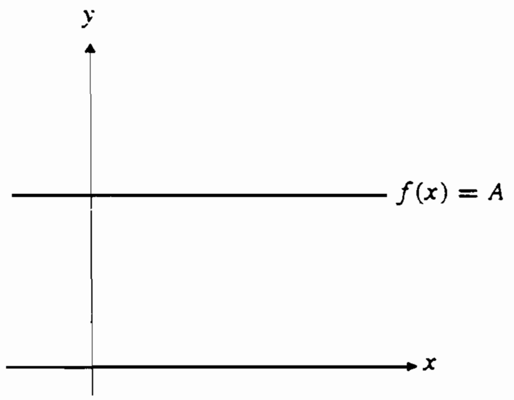
Additive constants disappear with differentiation:

$$y = A + f(x) \Longrightarrow y' = f'(x)$$ [4.22]

Multiplicative constants are preserved by differentiation:

$$y = Af(x) \Longrightarrow y' = Af'(x)$$ [4.23]

FIGURE 4.14 The derivative of a constant is 0.



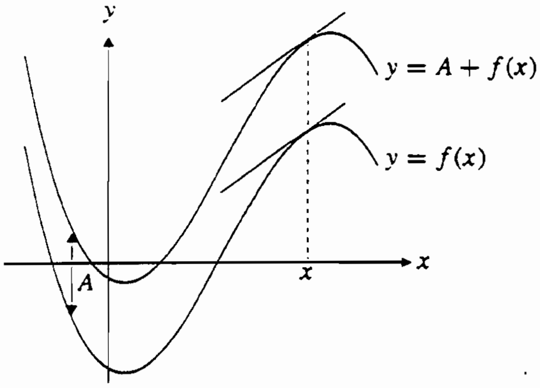


FIGURE 4.15 The graphs of the functions are parallel, and the functions have the same derivative at each point.

Rule [4.22] is illustrated in Fig. 4.15. The graph of $A + f(x)$ is that of $f(x)$ shifted upwards by A units in the direction of the y-axis. The graphs of $f(x)$ and $f(x) + A$ are thus parallel, and the tangents to the two curves at each x value must have the same slope. Again you should use the definition of $f'(x)$ to give a formal proof of this assertion.

We prove the rule in [4.23] by using the definition of a derivative. If $g(x) = Af(x)$, then $g(x+h) - g(x) = Af(x+h) - Af(x) = A[f(x+h) - f(x)]$, and so

$$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} = A \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = Af'(x)$$

In Leibniz's notation, the three results are as follows:

$$\frac{d}{dx} A = 0, \qquad \frac{d}{dx}[A + f(x)] = \frac{d}{dx} f(x), \qquad \frac{d}{dx}[Af(x)] = A \frac{d}{dx} f(x)$$

Example 4.11

Suppose we know $f'(x)$. Use rules [4.22] and [4.23] to find the derivatives of

(a) $5 + f(x)$

(b) $f(x) - 1/2$

(c) $3f(x)$

(d) $-\dfrac{f(x)}{5}$

(e) $\dfrac{Af(x) + B}{C}$

Solution With a little mixed notation, we obtain the following:

(a) $\dfrac{d}{dx}[5 + f(x)] = f'(x)$

(b) $\frac{d}{dx}[f(x) - 1/2] = \frac{d}{dx}[(-1/2) + f(x)] = f'(x)$

(c) $\frac{d}{dx}[3f(x)] = 3f'(x)$

(d) $\frac{d}{dx}\left[-\frac{f(x)}{5}\right] = \frac{d}{dx}\left[-\frac{1}{5}f(x)\right] = -\frac{1}{5}f'(x)$

(e) $\frac{d}{dx}\left[\frac{Af(x)+B}{C}\right] = \frac{d}{dx}\left[\frac{A}{C}f(x) + \frac{B}{C}\right] = \frac{A}{C}f'(x)$

Power Rule

Few rules of differentiation are more useful than the following:

Power Rule

$$f(x) = x^a \Longrightarrow f'(x) = ax^{a-1} \quad (a \text{ is an arbitrary constant}) \qquad [4.24]$$

For the examples of $a = 2$ and $a = 3$, this rule was confirmed in Section 4.2. The method used for these two examples can be generalized, as the following proof shows.

Proof of [4.24] when a is a natural number n: We put $f(x) = x^n$ and form the Newton quotient

$$\frac{f(x+h) - f(x)}{h} = \frac{(x+h)^n - x^n}{h} \qquad [*]$$

Let us multiply out $(x + h)^n = (x + h)(x + h)\cdots(x + h)$. The resulting expression must contain the term x^n that results from choosing x from each of the n terms in parentheses. The expression will also contain terms of the type $x^{n-1}h$. There are n such terms obtained by choosing $n-1$ of the x-only terms together with *one* h. All remaining terms must contain at least two h's, so

$$(x+h)^n = x^n + nx^{n-1}h + (\text{terms that contain } h^2 \text{ as a factor})$$

Hence,

$$(x+h)^n - x^n = nx^{n-1}h + (\text{terms that contain } h^2 \text{ as a factor})$$

So, whenever $h \neq 0$, we have

$$\frac{(x+h)^n - x^n}{h} = nx^{n-1} + (\text{terms that contain } h \text{ as a factor})$$

Now let h tend to 0. Then each term that contains h as a factor also tends to 0, and the sum of all these terms will tend to 0. Thus, the right-hand side tends to the expression $nx^{n-1} + 0 = nx^{n-1}$. Because the fraction in [*] tends to nx^{n-1} as h tends to 0, [4.24] *is* true when a is a natural number, according to the definition of $f'(x)$.

Example 4.12

Compute the derivative of the following:

(a) $y = x^5$

(b) $y = 3x^8$

(c) $y = \dfrac{x^{100}}{100}$

Solution .

(a) $y = x^5 \Longrightarrow y' = 5x^{5-1} = 5x^4$

(b) $y = 3x^8 \Longrightarrow y' = 3 \cdot 8x^{8-1} = 24x^7$

(c) $y = \dfrac{x^{100}}{100} = \dfrac{1}{100}x^{100} \Longrightarrow y' = \dfrac{1}{100}100x^{100-1} = x^{99}$

The previous proof covers only the case when a is a natural number. But the result in [4.24] is also valid if a is a negative integer, or even if a is a positive or negative rational number. Actually [4.24] is also correct even if a is an irrational number. All these cases will be considered later.

Example 4.13

Compute the following:

(a) $\dfrac{d}{dx}\left(x^{-0.33}\right)$

(b) $\dfrac{d}{dr}(-5r^{-3})$

(c) $\dfrac{d}{dp}(Ap^{\alpha} + B)$

(d) $\dfrac{d}{dx}\left(\dfrac{A}{\sqrt{x}}\right)$

Solution

(a) $\dfrac{d}{dx}\left(x^{-0.33}\right) = -0.33x^{-0.33-1} = -0.33x^{-1.33}$

(b) $\dfrac{d}{dr}(-5r^{-3}) = (-5)(-3)r^{-3-1} = 15r^{-4}$

(c) $\dfrac{d}{dp}(Ap^{\alpha} + B) = A\alpha p^{\alpha-1}$

(d) $\dfrac{d}{dx}\left(\dfrac{A}{\sqrt{x}}\right) = \dfrac{d}{dx}(Ax^{-1/2}) = A\left(-\dfrac{1}{2}\right)x^{-1/2-1} = -\dfrac{A}{2}x^{-3/2} = \dfrac{-A}{2x\sqrt{x}}$

Problems

1. Compute the derivative of the following:

   a. $y = 5$ b. $y = x^4$ c. $y = 9x^{10}$ d. $y = \pi^7$

2. Suppose we know $g'(x)$. Find an expression for the derivative of the following:

   a. $2g(x) + 3$ b. $-\frac{1}{6}g(x) + 8$ c. $\frac{g(x) - 5}{3}$

3. Find the derivative of the following:

   a. x^6 b. $3x^{11}$ c. x^{50} d. $-4x^{-7}$

   e. $\frac{x^{12}}{12}$ f. $-\frac{2}{x^2}$ g. $\frac{3}{\sqrt{x}}$ h. $-\frac{2}{x\sqrt{x}}$

4. Compute the following:

   a. $\frac{d}{dr}(4\pi r^2)$ b. $\frac{d}{dy}\left(Ay^{b+1}\right)$ c. $\frac{d}{dA}\left(\frac{1}{A^2\sqrt{A}}\right)$

5. Explain why

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

   Use this equation to find $f'(a)$ when $f(x) = x^2$.

6. For each of the following functions, find a function $F(x)$ that has $f(x)$ as its derivative. (Note that you are not asked to find $f'(x)$.)

   a. $f(x) = x^2$ b. $f(x) = 2x + 3$ c. $f(x) = x^a$ $(a \neq -1)$

Harder Problems

7. The following limits are all written in the form $\lim_{h \to 0}[f(a + h) - f(a)]/h$. Use your knowledge of derivatives to find the limits.

   a. $\lim_{h \to 0} \frac{(5 + h)^2 - 5^2}{h}$ b. $\lim_{s \to 0} \frac{(s + 1)^5 - 1}{s}$

   c. $\lim_{h \to 0} \frac{5(x + h)^2 + 10 - 5x^2 - 10}{h}$

4.6 Differentiation of Sums, Products, and Quotients

If we know $f'(x)$ and $g'(x)$, then what are the derivatives of $f(x) + g(x)$, $f(x) - g(x)$, $f(x) \cdot g(x)$, and $f(x)/g(x)$? You will probably guess the first two correctly, but are less likely to be right about the last two (unless you have already learned the answers).

Sums and Differences

Suppose that f and g are functions both defined on a set A of real numbers. The function F defined by the formula $F(x) = f(x) + g(x)$ is called the *sum* of f and g, written as $F = f + g$. The function G defined by $G(x) = f(x) - g(x)$ is called the *difference* between f and g, written as $G = f - g$. The following rules of differentiation are important.

Differentiation of Sums and Differences

If f and g are both differentiable at point x, then the sum $F = f + g$ and the difference $G = f - g$ are also differentiable at x, with

$$F(x) = f(x) + g(x) \Longrightarrow F'(x) = f'(x) + g'(x) \tag{4.25}$$

$$G(x) = f(x) - g(x) \Longrightarrow G'(x) = f'(x) - g'(x) \tag{4.26}$$

In Leibniz's notation:

$$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

$$\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$$

Proof In [4.25], the Newton quotient of F is

$$\frac{F(x+h) - F(x)}{h} = \frac{[f(x+h) + g(x+h)] - [f(x) + g(x)]}{h} = \frac{f(x+h) - f(x)}{h} + \frac{g(x+h) - g(x)}{h}$$

When $h \to 0$, the last two fractions tend to $f'(x)$ and $g'(x)$, respectively, and thus the sum of the fractions tends to $f'(x) + g'(x)$. Hence,

$$F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = f'(x) + g'(x)$$

The proof of [4.26] is similar—only some of the signs change in the obvious way.

Example 4.14

Compute

$$\frac{d}{dx}\left(3x^8 + \frac{x^{100}}{100}\right) \quad \text{and} \quad \frac{d}{dx}\left(3x^8 - \frac{x^{100}}{100}\right)$$

Solution

$$\frac{d}{dx}\left(3x^8 + \frac{x^{100}}{100}\right) = \frac{d}{dx}(3x^8) + \frac{d}{dx}\left(\frac{x^{100}}{100}\right) = 24x^7 + x^{99}$$

where we used [4.25] and results from Example 4.12. Similarly,

$$\frac{d}{dx}\left(3x^8 - \frac{x^{100}}{100}\right) = 24x^7 - x^{99}$$

Example 4.15

In Example 4.5, we defined $\pi(x) = R(x) - C(x)$, and so [4.26] implies that $\pi'(x) = R'(x) - C'(x)$. In particular, $\pi'(x) = 0$ when $R'(x) = C'(x)$. In words: *Marginal profit is 0 when marginal revenue is equal to marginal cost.*

Rule [4.25] can be extended to the sum of an arbitrary number of terms:

The derivative of a sum is the sum of the derivatives:

$$\frac{d}{dx}[f_1(x) + \cdots + f_n(x)] = \frac{d}{dx}f_1(x) + \cdots + \frac{d}{dx}f_n(x)$$

The rules previously developed can now be used to differentiate any polynomial.

Example 4.16

Find the derivative of a general nth-degree polynomial.

Solution

$$\frac{d}{dx}(a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0)$$
$$= na_nx^{n-1} + (n-1)a_{n-1}x^{n-2} + \cdots + 2a_2x + a_1$$

There is usually no reason to use so general a formula, because it is quite easy to apply the earlier rules to each specific case.

Products

If f and g are defined in a set A, then the function F defined by the formula $F(x) = f(x) \cdot g(x)$ is called the *product* of f and g, and we write $F = f \cdot g$ (or $F = fg$). For example, if $f(x) = x$ and $g(x) = x^2$, then $(f \cdot g)(x) = x^3$. Here $f'(x) = 1$, $g'(x) = 2x$ and $(f \cdot g)'(x) = 3x^2$. Hence, we see that the derivative of $(f \cdot g)(x)$ is *not* equal to $f'(x) \cdot g'(x) = 2x$. The correct rule for differentiating a product is a little more complicated.

The Derivative of a Product

If f and g are both differentiable at point x, then the function $F = f \cdot g$ is also differentiable at x, and [4.27]

$$F(x) = f(x) \cdot g(x) \Longrightarrow F'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$

Briefly formulated in words: *The derivative of the product of two functions is equal to the derivative of the first times the second, plus the first times the derivative of the second.* The formula, however, is much easier to understand than these words.

In Leibniz's notation, the product rule is expressed as:

$$\frac{d}{dx}[f(x) \cdot g(x)] = \frac{d}{dx}f(x) \cdot g(x) + f(x) \cdot \frac{d}{dx}g(x)$$

Before demonstrating why [4.27] is valid, here are two examples:

Example 4.17

Find $h'(x)$ when $h(x) = (x^3 - x) \cdot (5x^4 + x^2)$.

Solution We see that $h(x) = f(x) \cdot g(x)$ with $f(x) = x^3 - x$ and $g(x) = 5x^4 + x^2$. Here $f'(x) = 3x^2 - 1$ and $g'(x) = 20x^3 + 2x$. Thus,

$$\begin{aligned} h'(x) &= f'(x) \cdot g(x) + f(x) \cdot g'(x) \\ &= (3x^2 - 1) \cdot (5x^4 + x^2) + (x^3 - x) \cdot (20x^3 + 2x) \end{aligned}$$

Usually (but not always), we can simplify the answer by multiplying out to obtain just one polynomial. Simple computation gives

$$h'(x) = 35x^6 - 20x^4 - 3x^2$$

Alternatively, we can begin by multiplying out the expression for $h(x)$ to get

$$h(x) = (x^3 - x)(5x^4 + x^2) = 5x^7 - 4x^5 - x^3$$

Differentiating this polynomial gives the same expression for $h'(x)$ as before.

Example 4.18

We will illustrate the product rule for differentiation by considering the extraction of oil from a well. Suppose that both the amount of oil extracted per unit of time and the price per unit change with time t. We define

$$x(t) = \text{rate of extraction in barrels per day at time } t \qquad [1]$$

$$p(t) = \text{price in dollars per barrel at time } t \qquad [2]$$

Then we obtain

$$R(t) = p(t)x(t) \text{ as the revenue in dollars per day} \tag{3}$$

and according to the product rule (recalling that we often use "dot notation" for differentiation with respect to time),

$$\dot{R}(t) = \dot{p}(t)x(t) + p(t)\dot{x}(t) \tag{4}$$

The right-hand side of [4] can be interpreted as follows. Suppose that $p(t)$ and $x(t)$ both increase over time, because of inflation and because the oil company operating the well steadily expands the capacity of its extraction equipment. Then $R(t)$ increases for two reasons. First, $R(t)$ increases because of the price increase. This increase is proportional to the amount of extraction $x(t)$ and is equal to $\dot{p}(t)x(t)$. But $R(t)$ also rises because extraction increases. Its contribution to the rate of change of $R(t)$ must be proportional to the price, and is equal to $p(t)\dot{x}(t)$. Equation (4) merely expresses the simple fact that $\dot{R}(t)$, the total rate of change of $R(t)$, is the sum of these two parts.

Note too that the proportional rate of growth of revenue can be found by dividing [4] by [3] to obtain

$$\frac{\dot{R}}{R} = \frac{\dot{p}x + p\dot{x}}{px} = \frac{\dot{p}}{p} + \frac{\dot{x}}{x}$$

In words, the proportional rate of growth of revenue is the sum of the proportional rates at which the price and quantity are changing.

We have now seen how to differentiate products of two functions. What about products of more than two functions? For example, suppose that

$$y = f(x)g(x)h(x)$$

What is y'? We extend the same technique shown earlier and put $y = [f(x)g(x)]$ $h(x)$. Then the product rule gives

$$\begin{aligned} y' &= [f(x)g(x)]'\, h(x) + [f(x)g(x)]\, h'(x) \\ &= [f'(x)g(x) + f(x)g'(x)]\, h(x) + f(x)g(x)h'(x) \\ &= f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x) \end{aligned}$$

If none of the three functions is equal to 0, we can write the result in the following way:

$$\frac{(fgh)'}{fgh} = \frac{f'}{f} + \frac{g'}{g} + \frac{h'}{h}$$

By analogy, it is easy to write down the corresponding result for a product of n functions.

Proof of [4.27] Suppose f and g are differentiable at x, so that the two Newton quotients

$$\frac{f(x+h)-f(x)}{h} \quad \text{and} \quad \frac{g(x+h)-g(x)}{h} \qquad [1]$$

tend to the limits $f'(x)$ and $g'(x)$, respectively, as h tends to 0. We must show that the Newton quotient of F also tends to a limit, which is given by $f'(x)g(x)+f(x)g'(x)$. But the Newton quotient of F is

$$\frac{F(x+h)-F(x)}{h} = \frac{f(x+h)g(x+h)-f(x)g(x)}{h} \qquad [2]$$

To proceed further we must somehow transform the right-hand side (RHS) to involve the Newton quotients of f and g. We use a trick: The numerator of the RHS of (2) is unchanged if we both subtract and add the number $f(x)g(x+h)$. Hence, with suitable regrouping of the terms, we have

$$\begin{aligned} &\frac{F(x+h)-F(x)}{h} \\ &= \frac{f(x+h)g(x+h)-f(x)g(x+h)+f(x)g(x+h)-f(x)g(x)}{h} \qquad [3] \\ &= \left[\frac{f(x+h)-f(x)}{h}\right] g(x+h) + f(x)\left[\frac{g(x+h)-g(x)}{h}\right] \end{aligned}$$

As h tends to 0, the two Newton quotients in the square brackets tend to $f'(x)$ and $g'(x)$, respectively. Now we can write $g(x+h)$ for $h \neq 0$ as

$$g(x+h) = \left[\frac{g(x+h)-g(x)}{h}\right] h + g(x)$$

which tends to $g'(x)\cdot 0 + g(x) = g(x)$ as h tends to 0. It follows that the Newton quotient of F in (3) tends to $f'(x)g(x)+f(x)g'(x)$ as h tends to 0.

Quotients

Let f and g be functions which are differentiable at x, and define $F(x) = f(x)/g(x)$. We naturally assume that $g(x) \neq 0$, so that F is defined at x. We call F the *quotient* of f and g and write $F = f/g$. We would like to find a formula for $F'(x)$. Bearing in mind the complications in the formula for the derivative of a product, one should be somewhat reluctant to make a quick guess as to how the correct formula for $F'(x)$ will turn out.

In fact, it is quite easy to find the formula for $F'(x)$ if we *suppose* that $F(x)$ *is* differentiable, for $F(x) = f(x)/g(x)$ implies that $f(x) = F(x)g(x)$. Thus, the product rule gives

$$f'(x) = F'(x) \cdot g(x) + F(x) \cdot g'(x)$$

Solving for $F'(x)$ in terms of the other functions yields

$$F'(x) = \frac{f'(x) - F(x)g'(x)}{g(x)} = \frac{f'(x) - [f(x)/g(x)]g'(x)}{g(x)}$$

Multiplying both numerator and denominator of the last fraction by $g(x)$ gives

$$F'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

Formally, the theorem can be stated as follows.

The Derivative of a Quotient

If f and g are differentiable at x and $g(x) \neq 0$, then $F = f/g$ is differentiable at x, and

$$F(x) = \frac{f(x)}{g(x)} \Longrightarrow F'(x) = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{[g(x)]^2} \qquad [4.28]$$

In words: *The derivative of a quotient is equal to the derivative of the numerator times the denominator minus the numerator times the derivative of the denominator, this difference then being divided by the square of the denominator.* (To prove that F *is* differentiable in x, under the earlier assumptions, we must study the Newton quotient of F as we did for the product rule. See Problem 12.) In simpler notation, we have

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

Note: In the product rule formula, the two functions appear symmetrically, so that it is easy to remember. In the formula for the derivative of a quotient, the expressions in the numerator must be in the right order. The following suggestion checks whether you have the order right. Write down the formula you believe is correct. Consider the case when $g \equiv 1$. Then $g' \equiv 0$, and your formula ought to reduce to f'. If you get $-f'$, then your signs are wrong.

Example 4.19

Compute $F'(x)$ and $F'(4)$ when $F(x) = (3x - 5)/(x - 2)$.

Solution We apply [4.28] with $f(x) = 3x - 5$, $g(x) = x - 2$. Then $f'(x) = 3$ and $g'(x) = 1$. So we obtain, for $x \neq 2$:

$$F'(x) = \frac{3 \cdot (x-2) - (3x-5) \cdot 1}{(x-2)^2}$$

$$= \frac{3x - 6 - 3x + 5}{(x-2)^2} = -\frac{1}{(x-2)^2}$$

To find $F'(4)$, we put $x = 4$ in the formula for $F'(x)$ to get $F'(4) = -1/(4-2)^2 = -1/4$.

Example 4.20

Let $C(Q)$ be the total cost of producing Q units of a commodity. (See Example 3.3.) The quantity $C(Q)/Q$ is called the *average cost* of producing Q units. Find an expression for

$$\frac{d}{dQ}[C(Q)/Q]$$

Solution

$$\frac{d}{dQ}\left[\frac{C(Q)}{Q}\right] = \frac{QC'(Q) - C(Q)}{Q^2} = \frac{1}{Q}\left[C'(Q) - \frac{C(Q)}{Q}\right]$$

Note that for positive output levels Q, the marginal cost $C'(Q)$ exceeds the average cost $C(Q)/Q$ if and only if the rate of change of the average cost as output increases is positive. (In a similar way, if a basketball team recruits a new player, the average height of the team increases if and only if the new player's height exceeds the old average height.)

The formula for the derivative of a quotient becomes easier to understand if we consider proportional rates of change. (See [4.11].) By using [4.28], simple computation shows that

$$F(x) = \frac{f(x)}{g(x)} \Longrightarrow \frac{F'(x)}{F(x)} = \frac{f'(x)}{f(x)} - \frac{g'(x)}{g(x)} \qquad [4.29]$$

The proportional rate of change of a quotient is equal to the proportional rate of change of the numerator minus the proportional rate of change of the denominator.

An economic application of rule [4.29] is as follows. Let $W(t)$ be the nominal wage rate and $P(t)$ the price index at time t. Then $w(t) = W(t)/P(t)$ is called the **real wage rate**. According to [4.29],

$$\frac{\dot{w}(t)}{w(t)} = \frac{\dot{W}(t)}{W(t)} - \frac{\dot{P}(t)}{P(t)}$$

The proportional rate of change of the real wage is equal to the difference between the proportional rates of change of the nominal wage and the price index. Thus, if nominal wages increase at the rate of 5% per year but prices rise by 6%, then real wages fall by 1%. (Recall from Section 4.3 that these are proportional rates of change.)

Problems

In Problems 1–4, differentiate the functions defined by the various formulas.

1. **a.** $x+1$ **b.** $x+x^2$ **c.** $3x^5+2x^4+5$
   d. $8x^4+2\sqrt{x}$ **e.** $\frac{1}{2}x-\frac{3}{2}x^2+5x^3$ **f.** $1-3x^7$

2. **a.** $\frac{3}{5}x^2-2x^7+\frac{1}{8}-\sqrt{x}$ **b.** $(2x^2-1)(x^4-1)$ **c.** $\left(x^5+\dfrac{1}{x}\right)(x^5+1)$

3. **a.** $\dfrac{1}{x^6}$ **b.** $x^{-1}(x^2+1)\sqrt{x}$ **c.** $\dfrac{1}{\sqrt{x^3}}$ **d.** $\dfrac{x+1}{x-1}$
   e. $\dfrac{x+1}{x^5}$ **f.** $\dfrac{3x-5}{2x+8}$ **g.** $3x^{-11}$ **h.** $\dfrac{3x-1}{x^2+x+1}$

4. **a.** $\dfrac{\sqrt{x}-2}{\sqrt{x}+1}$ **b.** $\dfrac{(x+1)(x-1)}{(x^2+2)(x+3)}$ **c.** $(3x+1)\left(\dfrac{1}{x^2}+\dfrac{1}{x}\right)$
   d. $\dfrac{x^2-1}{x^2+1}$ **e.** $\dfrac{x^2+x+1}{x^2-x+1}$ **f.** $\dfrac{1}{2}+\dfrac{1}{3}\left(\dfrac{x-1}{x+1}\right)(1+x^{-2})$

5. If $D(P)$ denotes the demand for a product when the price per unit is P, then the revenue function $R(P)$ is given by $R(P)=PD(P)$. Find an expression for $R'(P)$.

6. For each of the following functions, determine the value(s) of x at which $f'(x)=0$.
   a. $f(x)=3x^2-12x+13$ **b.** $f(x)=\frac{1}{4}(x^4-6x^2)$
   c. $f(x)=\dfrac{2x}{x^2+2}$ **d.** $f(x)=\dfrac{x^2-x^3}{2(x+1)}$

7. Find the equations for the tangents to the graphs of the following functions at the specified points:
   a. $y=3-x-x^2$ at $x=1$ **b.** $y=\dfrac{x^2-1}{x^2+1}$ at $x=1$
   c. $y=\left(\dfrac{1}{x^2}+1\right)(x^2-1)$ at $x=2$ **d.** $y=\dfrac{x^4+1}{(x^2+1)(x+3)}$ at $x=0$

8. Differentiate the following functions of t:
   a. $\dfrac{at+b}{ct+d}$ **b.** $t^n\left(a\sqrt{t}+b\right)$ **c.** $\dfrac{1}{at^2+bt+c}$

9. Compute the following:

a. $\dfrac{d}{dp}\left(\dfrac{Ap^2+B}{Cp^2+D}\right)$ **b.** $\dfrac{d}{dy}\left(\dfrac{y^2+2}{y^8}\right)$ **c.** $\dfrac{d}{dx}\left(\dfrac{1-f(x)}{1+f(x)}\right)$

10. If $f(x) = \sqrt{x}$, then $f(x) \cdot f(x) = x$. Use the product rule to find a formula for $f'(x)$. Compare this with the result in Problem 8 of Section 4.2.

11. Prove the power rule [4.24] for $a = -n$, where n is a natural number, by using the relation $f(x) = x^{-n} = 1/x^n$ and the quotient rule [4.28].

Harder Problems

12. Let $F(x) = f(x)/g(x)$. Write out the Newton quotient of F, and show that it tends to the expression for $F'(x)$ in [4.28]. *Hint:* The Newton quotient of F is equal to

$$\frac{1}{g(x)g(x+h)}\left[g(x)\frac{f(x+h)-f(x)}{h} - f(x)\frac{g(x+h)-g(x)}{h}\right]$$

Then use the same idea as in the proof of [4.27].

4.7 Second- and Higher-Order Derivatives

The derivative of a function f is often called the **first derivative** of f. If f' is also differentiable, then we can differentiate f' in turn. In fact, we call $(f')'$ the **second derivative** of f. We write f'' instead of $(f')'$, and let $f''(x)$ denote the second derivative of f evaluated at the particular point x.

Example 4.21

Find $f'(x)$ and $f''(x)$ when $f(x) = 2x^5 - 3x^3 + 2x$.

Solution Using the rules for differentiating polynomials, we first differentiate $2x^5 - 3x^3 + 2x$ to get

$$f'(x) = 10x^4 - 9x^2 + 2$$

Then we differentiate $10x^4 - 9x^2 + 2$ to get

$$f''(x) = 40x^3 - 18x$$

The different forms of notation for the second derivative are analogous to those for the first derivative. For example, we write $y'' = f''(x)$ in order to denote the second derivative of $y = f(x)$. The Leibniz notation for the second derivative is also used. In the notation dy/dx or $df(x)/dx$ for the first derivative, we interpreted the symbol d/dx as an operator indicating that what follows is to be differentiated with respect to x. The second derivative is obtained by using the operator d/dx twice: $f''(x) = (d/dx)(d/dx)f(x)$. We usually think of this as $f''(x) = (d/dx)^2 f(x)$, and so write it as follows:

$$f''(x) = \frac{d^2 f(x)}{dx^2} = d^2 f(x)/dx^2 \qquad \text{or} \qquad y'' = \frac{d^2 y}{dx^2} = d^2 y/dx^2$$

Pay special attention to where the superscripts 2 are placed.

Of course, the notation for the second derivative must change if the variables have other names.

Example 4.22

(a) Find Y'' when $Y = AK^a$ is a function of K ($K > 0$), with A and a as constants.

(b) Find d^2L/dt^2 when $L = t/t + 1$, and $t \geq 0$.

Solution

(a) Differentiating $Y = AK^a$ with respect to K gives

$$Y' = AaK^{a-1}$$

A second differentiation with respect to K yields

$$Y'' = Aa(a-1)K^{a-2}$$

(b) First, we use the quotient rule to find that

$$\frac{dL}{dt} = \frac{d}{dt}\left(\frac{t}{t+1}\right) = \frac{1 \cdot (t+1) - t \cdot 1}{(t+1)^2} = \frac{1}{t^2 + 2t + 1}$$

The quotient rule can be used again to yield

$$\frac{d^2L}{dt^2} = \frac{0 \cdot (t^2 + 2t + 1) - 1 \cdot (2t + 2)}{(t^2 + 2t + 1)^2} = \frac{-2(t+1)}{(t+1)^4} = -2\frac{1}{(t+1)^3}$$

Later, both first and second derivatives will be given important geometric and economic interpretations. Corresponding simple interpretations are not available for derivatives of higher order, but they are used from time to time.

Higher-Order Derivatives

The derivative of $y'' = f''(x)$ is called the **third derivative**, and we use the notation $y''' = f'''(x)$ for this. It is notationally cumbersome to continue using primes to indicate differentiation, so the **fourth derivative** is usually denoted by $y^{(4)} = f^{(4)}(x)$. (We must put the number 4 in parentheses so that it will not get confused with y^4, the fourth power of y.) The same derivative can be expressed as d^4y/dx^4. In general, let

$$y^{(n)} = f^{(n)}(x) \quad \text{or} \quad d^ny/dx^n \quad \text{denote the } \textbf{\textit{n}th derivative} \text{ of } f \text{ at } x$$

The number n is called the **order** of the derivative. For example, $f^{(6)}(x_0)$ denotes the sixth derivative of f calculated at x_0, found by differentiating six times.

Example 4.23

Compute all the derivatives up to and including order 4 of

$$f(x) = 3x^{-1} + 6x^3 - x^2 \qquad (x \neq 0)$$

Solution Repeated differentiation gives

$$f'(x) = -3x^{-2} + 18x^2 - 2x$$

$$f''(x) = 6x^{-3} + 36x - 2$$

$$f'''(x) = -18x^{-4} + 36$$

$$f^{(4)}(x) = 72x^{-5}$$

In the same way that a function need not be differentiable at x_0, a higher-order derivative need not exist at x_0. If $f'(x_0)$, $f''(x_0)$, ..., $f^{(n)}(x_0)$ all exist, then we say that f is *n times differentiable* at x_0. If $f^{(n)}(x_0)$ is continuous, then f is said to be *n times continuously differentiable* at x_0—or more concisely, a C^n **function** at x_0.

Example 4.24

Differentiate $f(x) = 3x^{11/3}$ four times.

Solution

$$f'(x) = 11x^{8/3}$$

$$f''(x) = (88/3)x^{5/3}$$

$$f'''(x) = (440/9)x^{2/3}$$

$$f^{(4)}(x) = (880/27)x^{-1/3}$$

Note that $f'(0) = f''(0) = f'''(0) = 0$, but $f^{(4)}(0)$ does not exist. Hence, f is three times differentiable everywhere, but it is not four times differentiable at 0.

Problems

1. Find the second derivative of the following:

   a. $y = x^5 - 3x^4 + 2$ **b.** $y = \sqrt{x}$ **c.** $y = \dfrac{x}{x+1}$

2. Find d^2y/dx^2 when $y = x^a + x^{-a}$.
3. Compute the following:
   a. y'' for $y = 3x^3 + 2x - 1$
   b. Y''' for $Y = 1 - 2x^2 + 6x^3$
   c. d^3z/dt^3 for $z = 120t - (1/3)t^3$
   d. $f^{(4)}(1)$ for $f(z) = 100z^{-4}$
4. Find $g''(2)$ when $g(t) = t^2/(t - 1)$.
5. Find formulas for y'' and y''' when $y = f(x)g(x)$.
6. If n is a natural number, let $n!$ (read as "n factorial") be defined as

$$n! = 1 \cdot 2 \cdot 3 \cdots (n - 1) \cdot n$$

For example, $5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$. Show (by using mathematical induction) that

$$y = x^n \Longrightarrow y^{(n)} = n!$$

Harder Problems

7. Find a function that is five times differentiable, but not six times differentiable at $x = 0$. (*Hint:* See Example 4.24.)

5

More on Differentiation

Although this may seem a paradox, all science
is dominated by the idea of approximation.
—Bertrand Russell

This chapter presents some extensively used techniques of differentiation. It begins with the generalized power rule and then proceeds to a discussion of the highly useful chain rule. In many economic models, functions are defined implicitly by one or more equations. In some simple but economically relevant cases, we show how to compute derivatives of such functions. Next we consider differentials and linear, quadratic, or higher-order approximations, all of which occur in many applications of mathematics to economics. A discussion of the important economic concept of elasticity ends the chapter.

5.1 The Generalized Power Rule

It is often necessary to differentiate expressions of the form

$$y = [g(x)]^a$$

where g is a differentiable function, and a is a constant. For $a = 1$, the derivative is just $g'(x)$. For $a = 2$, we can use the product rule as follows:

$$y = [g(x)]^2 = g(x) \cdot g(x) \Longrightarrow y' = g'(x) \cdot g(x) + g(x) \cdot g'(x) = 2g(x) \cdot g'(x)$$

For $a = 3$, we can combine the previous result with the product rule as follows:

$$y = [g(x)]^3 = [g(x)]^2 \cdot g(x) \Longrightarrow y' = [2g(x) \cdot g'(x)] \cdot g(x) + [g(x)]^2 \cdot g'(x)$$
$$= 3[g(x)]^2 \cdot g'(x)$$

See if you can discern a pattern here. In general, we have the following rule (where a is an arbitrary real number):

The Generalized Power Rule

$$y = [g(x)]^a \Longrightarrow y' = a[g(x)]^{a-1} \cdot g'(x) \qquad [5.1]$$

Note this important formula. If we put $g(x) = x$, then $g'(x) = 1$ and [5.1] reduces to $y = x^a \Rightarrow y' = ax^{a-1}$, which is the power of Section 4.5. A generalization of [5.1] is proved in Section 5.2. In the meantime, ambitious students may want to try proving [5.1] by mathematical induction for the case when a is a natural number. (See Problem 10.)

Example 5.1

Differentiate the functions:

(a) $y = (x^3 + x^2)^{50}$

(b) $y = \left(\dfrac{x-1}{x+3}\right)^{1/3}$

(c) $y = \sqrt{x^2 + 1}$

Solution The key to applying the generalized power rule is to determine how the given function can be expressed as a power. In the first problem, it is rather obvious:

(a) $y = (x^3 + x^2)^{50} = [g(x)]^{50}$ where $g(x) = x^3 + x^2$. Differentiating this directly gives $g'(x) = 3x^2 + 2x$, and so formula [5.1] yields

$$y' = 50[g(x)]^{50-1} \cdot g'(x) = 50(x^3 + x^2)^{49}(3x^2 + 2x)$$

(b) Again it is obvious how to apply [5.1]:

$$y = \left(\frac{x-1}{x+3}\right)^{1/3} = [g(x)]^{1/3}$$

where $g(x) = (x - 1)/(x + 3)$. In this case, the quotient rule implies that

$$g'(x) = \frac{1 \cdot (x+3) - (x-1) \cdot 1}{(x+3)^2} = \frac{4}{(x+3)^2}$$

Hence, [5.1] gives

$$y' = \frac{1}{3}\left[g(x)\right]^{(1/3)-1} \cdot g'(x) = \frac{1}{3}\left(\frac{x-1}{x+3}\right)^{-2/3} \cdot \frac{4}{(x+3)^2}$$
$$= \frac{4}{3}(x+3)^{-4/3}(x-1)^{-2/3}$$

(c) Here we first notice that $y = \sqrt{x^2+1} = (x^2+1)^{1/2}$. So $y = \left[g(x)\right]^{1/2}$, where $g(x) = x^2 + 1$. Hence,

$$y' = \frac{1}{2}\left[g(x)\right]^{(1/2)-1} \cdot g'(x) = \frac{1}{2}(x^2+1)^{-1/2} \cdot 2x = \frac{x}{\sqrt{x^2+1}}$$

The generalized power rule can also be formulated in Leibniz's notation.

The Generalized Power Rule (Leibniz's Notation)

When $u = g(x)$ is a function of x, then

$$y = u^a \Longrightarrow \frac{dy}{dx} = au^{a-1}\frac{du}{dx} \tag{5.2}$$

Often we need to combine the generalized power rule with the other rules of differentiation shown earlier. Here is an example from economics.

Example 5.2

Suppose that the relationship between gross income Y and total income tax T is for taxpayers with incomes between 80,000 and 120,000. The following values for the constants in [*] were estimated, given by the equation

$$T = a(bY + c)^p + kY \qquad [*]$$

where a, b, c, p, and k are positive constants.

(a) Find an expression for the *marginal tax rate*, dT/dY.

(b) In an empirical study

$a = 0.000338$, $b = 0.81$, $c = 6467$, $p = 1.61$, $k = 0.053$

Use these numbers to find the values of T and dT/dY when $Y = 100{,}000$.

Solution

(a) Let $z = (bY + c)^p = u^p$ with $u = bY + c$. Then [5.2] gives

$$\frac{dz}{dY} = pu^{p-1}\frac{du}{dY} = p(bY + c)^{p-1}b$$

Because $T = az + kY$, differentiation of [*] gives

$$\frac{dT}{dY} = a\frac{dz}{dY} + k = apb(bY + c)^{p-1} + k$$

(b) We have

$$T = 0.000338(0.81 \cdot 100{,}000 + 6467)^{1.61} + 0.053 \cdot 100{,}000 \approx 35{,}869.33$$

and

$$\frac{dT}{dY} = 0.000338 \cdot 0.81 \cdot 1.61 \cdot (0.81 \cdot 100{,}000 + 6467)^{0.61} + 0.053 \approx 0.51$$

Thus, the marginal tax rate on an income of 100,000 is approximately 51%.

Problems

1. Compute $f'(x)$ when $f(x) = (3x^2 + 1)^2$ by (a) expanding the square and then differentiating; (b) using [5.1]. Compare the answers.
2. Find the derivatives of the functions defined by the following:

   a. $(2x + 1)^3$ **b.** $(1 - x)^5$ **c.** $(x^2 - 2x + 2)^2$

   d. $\dfrac{(x + 1)^5}{x}$ **e.** $(3x - 4)^{-7}$ **f.** $(2x^2 + 3x - 4)^{-2}$
3. Find the derivatives of the functions defined by the following:

   a. $(1 + x)^{1/2}$ **b.** $\sqrt{x^3 + 1}$ **c.** $\left(\dfrac{2x + 1}{x - 1}\right)^{1/2}$

   d. $(1 - x^2)^{33}$ **e.** $x^3\sqrt{1 - x}$ **f.** $\sqrt[3]{1 + x} \cdot \sqrt[5]{1 - x}$
4. Find the derivatives of the following functions of t (where a, b, and n are constants):

   a. $(at^2 + 1)^{-3}$ **b.** $(at + b)^n$ **c.** $\left(\dfrac{at + b}{nt}\right)^{a+1}$

5. If f is differentiable at x, find expressions for the derivatives of the following functions:

a. $x + f(x)$ **b.** $[f(x)]^2 - x$ **c.** $[f(x)]^4$

d. $x^2 f(x) + [f(x)]^3$ **e.** $xf(x)$ **f.** $\sqrt{f(x)}$

g. $\dfrac{x^2}{f(x)}$ **h.** $\dfrac{[f(x)]^2}{x^3}$ **i.** $\left\{f(x) + [f(x)]^3 + x\right\}^{1/3}$

6. Let $x = (Ap + B)^r$ and $p = at^2 + bt + c$. Find an expression for dx/dt.

7. Compute dy/dv when $y = A(av^p + b)^q$.

Harder Problems

8. Suppose that [5.1] has already been proved when a is a natural number. Prove that [5.1] is then also valid when $a = -n$, where n is a natural number. (*Hint:* Put $y = [g(x)]^{-n} = 1/[g(x)]^n$, and then use the quotient rule.)

9. Let a, b, m, and n be fixed numbers, where $a < b$, and m and n are positive. Define the function f for all x by $f(x) = (x - a)^m \cdot (x - b)^n$. For the equation $f'(x) = 0$, find a solution x_0 that lies between a and b.

10. Prove by induction that [5.1] holds when a is a natural number.

11. Prove that

$$\frac{d}{dx}[f(x)]^m [g(x)]^n = [mf'(x)g(x) + nf(x)g'(x)]\,[f(x)]^{m-1}[g(x)]^{n-1}$$

What do you get if $m = n = 1$, and if $m = -n = 1$?

5.2 Composite Functions and the Chain Rule

If y is a function of u, and u is a function of x, then y is a function of x. In this case, we call y a **composite function** of x. (In the previous section, we considered the special case where y was given by u^a.) Suppose that x changes. This will lead to a change in u and hence a change in y. A change in x, therefore, causes a "chain reaction." If we know the rates of change du/dx and dy/du, then what is the rate of change dy/dx? It turns out that the relationship between these rates of change is simply:

The Chain Rule

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \qquad [5.3]$$

A slightly more detailed formulation of the rule says that *if y is a differentiable* *function of u, and u is a differentiable function of x, then y is a differentiable function of x*, and [5.3] holds.

The chain rule is a further generalization of the generalized power rule from the previous section. In the special case where $y = u^a$, we have $dy/du = au^{a-1}$, and substituting this expression into [5.3] gives the formula in [5.2].

It is easy to remember the chain rule when using Leibniz's notation. The left-hand side of [5.3] is exactly what results if we "cancel" the du on the right side. Of course, because dy/du and du/dx are not fractions (but merely symbols for derivatives) and du is not a number, canceling is not defined.

When we interpret the derivatives involved in [5.3] as rates of change, the chain rule becomes rather intuitive, as the next example from economics will indicate.

Example 5.3

The demand x for a commodity depends on price p. Suppose that price p is not constant, but depends on time t. Then x is a composite function of t, and according to the chain rule,

$$\frac{dx}{dt} = \frac{dx}{dp} \cdot \frac{dp}{dt} \tag{[*]}$$

Suppose, for instance, that the demand for butter decreases by 5000 pounds if the price goes up by \$1 per pound. So $dx/dp \approx -5000$. Suppose further that the price per pound increases by \$0.05 per week, so $dp/dt \approx 0.05$. What is the decrease in demand in pounds per week?

Solution: Because the price per pound increases by \$0.05 per week, and the demand decreases by 5000 pounds for every dollar increase in the price, the demand *decreases* by $5000 \cdot 0.05 \approx 250$ pounds per week. This means that $dx/dt \approx -250$ (measured in pounds per week). Note how this argument roughly confirms [*].

The chain rule is very powerful. Facility in applying it comes only from a lot of practice.

Example 5.4

(a) Find dy/dx when $y = u^5$ and $u = 1 - x^3$.

(b) Find dy/dx when

$$y = \frac{10}{(x^2 + 4x + 5)^7}$$

Solution

(a) We can use [5.3] directly. Because $dy/du = 5u^4$ and $du/dx = -3x^2$, we have

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 5u^4(-3x^2) = -15x^2u^4 = -15x^2(1 - x^3)^4$$

(b) In this case, it is not immediately obvious how to apply the chain rule. However, if we rewrite y as $y = 10(x^2 + 4x + 5)^{-7}$, then

$$y = 10u^{-7}$$

where $u = x^2 + 4x + 5$. Thus,

$$\frac{dy}{du} = 10(-7)u^{-7-1} = -70u^{-8} \quad \text{and} \quad \frac{du}{dx} = 2x + 4$$

So using [5.3] yields

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = -70u^{-8} \cdot (2x + 4) = -140(x + 2)/(x^2 + 4x + 5)^8$$

Note 1: After a little training, the intermediate steps become unnecessary. For example, to differentiate

$$y = (\underbrace{1 - x^3}_{u})^5$$

we can *think* of y as $y = u^5$, where $u = 1 - x^3$. We can then differentiate both u^5 and $1 - x^3$ in our heads, and immediately write down $y' = 5(1 - x^3)^4(-3x^2)$.

Note 2: Of course, one could differentiate $y = x^5/5$ using the quotient rule, rather than writing y as $y = (1/5)x^5$ to get $y' = (1/5)5x^4 = x^4$. But the latter method is much easier. In the same way, it is unnecessarily cumbersome to apply the quotient rule to the function given in Example 5.4(b). The chain rule is much more effective.

The next example uses the chain rule several times.

Example 5.5

Find $x'(t)$ when $x(t) = 5\left(1 + \sqrt{t^3 + 1}\right)^{25}$.

Solution The initial step is easy. Let $x(t) = 5u^{25}$, where $u = 1 + \sqrt{t^3 + 1}$, to obtain

$$x'(t) = 5 \cdot 25u^{24}\frac{du}{dt} = 125u^{24}\frac{du}{dt} \qquad [1]$$

The new feature in this example is that we cannot write down du/dt at once. Finding du/dt requires using the chain rule a second time. Let $u = 1 + \sqrt{v} = 1 + v^{1/2}$, where $v = t^3 + 1$. Then

$$\frac{du}{dt} = \frac{1}{2}v^{1/2-1} \cdot \frac{dv}{dt} = \frac{1}{2}v^{-1/2} \cdot 3t^2 = \frac{1}{2}(t^3 + 1)^{-1/2} \cdot 3t^2 \qquad [2]$$

From [1] and [2], we get

$$x'(t) = 125\left(1+\sqrt{t^3+1}\right)^{24}\cdot\tfrac{1}{2}(t^3+1)^{-1/2}\cdot 3t^2$$

Suppose, as in the last example, that x is a function of u, u is a function of v, and v is in turn a function of t. Then x is a composite function of t, and the chain rule can be used twice to obtain

$$\frac{dx}{dt}=\frac{dx}{du}\cdot\frac{du}{dv}\cdot\frac{dv}{dt}$$

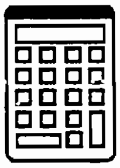
This is precisely the formula used in the last example. Again the notation is suggestive because the left-hand side is exactly what results if we "cancel" both du and dv on the right-hand side.

An Alternative Formulation of the Chain Rule

Although Leibniz's notation makes it very easy to remember the chain rule, it suffers from the defect of not specifying where each derivative is evaluated. We remedy this by introducing names for the functions involved. So let $y = f(u)$ and $u = g(x)$. Then y can be written in the form

$$y = f\big(g(x)\big)$$

Note that when we compute $f\big(g(x)\big)$, we *first* compute $g(x)$, and then *second*, we apply f to the result. We say that we have a **composite function**, with $g(x)$ as the the **kernel**, and f as the **exterior function**.



Most scientific calculators have several built-in functions. When we punch a number x_0 and strike the key for the function f, we obtain $f(x_0)$. When we compute a composite function given f and g, and try to obtain the value of $f\big(g(x)\big)$, we proceed in a similar manner: punch the number x_0, then strike the g key to get $g(x_0)$, and again strike the f key to get $f\big(g(x_0)\big)$. Suppose the machine has the functions $\boxed{1/x}$ and $\boxed{\sqrt{x}}$. If we press the number 9, then strike the button $\boxed{1/x}$ followed by $\boxed{\sqrt{x}}$, we get $1/3 = 0.33\ldots$. The computation we have performed can be illustrated as follows:

$$9 \xrightarrow{\boxed{1/x}} 1/9 \xrightarrow{\boxed{\sqrt{x}}} 1/3$$

Using function notation, $f(x) = \sqrt{x}$ and $g(x) = 1/x$, so $f\big(g(x)\big) = f(1/x) = \sqrt{1/x} = 1/\sqrt{x}$. In particular, $f\big(g(9)\big) = 1/\sqrt{9} = 1/3$.

The Chain Rule

If g is differentiable at x_0 and f is differentiable at $u_0 = g(x_0)$, then $F(x) = f(g(x))$ is differentiable at x_0 and

$$F'(x_0) = f'(g(x_0))g'(x_0) \tag{5.4}$$

In words: *to differentiate a composite function, first differentiate the exterior function and substitute in the value of the kernel, then multiply by the derivative of the kernel.* It is important to notice that the derivatives f' and g' appearing in formula [5.4] are evaluated at *different* points; the derivative g' is evaluated at x_0, whereas f' is evaluated at $g(x_0)$.

Example 5.6

Find the derivative of $F(x) = f(g(x))$ at $x_0 = -3$ if $f(u) = u^3$ and $g(x) = 2 - x^2$.

Solution In this case, $f'(u) = 3u^2$ and $g'(x) = -2x$. So [5.4] gives

$$F'(-3) = f'(g(-3))\,g'(-3)$$

Now $g(-3) = 2 - (-3)^2 = 2 - 9 = -7$; $g'(-3) = 6$; and $f'(g(-3)) = f'(-7) = 3(-7)^2 = 3 \cdot 49 = 147$. So $F'(-3) = f'(g(-3))\,g'(-3) = 147 \cdot 6 = 882$.

Note: The function that maps x to $f(g(x))$ is often denoted by $f \circ g$, and is read as "f of g" or "f compounded with g." Correspondingly, $g \circ f$ denotes the function that maps x to $g(f(x))$. Thus, we have

$$(f \circ g)(x) = f(g(x)) \qquad \text{and} \qquad (g \circ f)(x) = g(f(x))$$

Usually, $f \circ g$ and $g \circ f$ are quite different functions. For instance, the functions used in Example 5.6 give $(f \circ g)(x) = (2-x^2)^3$, whereas $(g \circ f)(x) = 2-(x^3)^2 = 2-x^6$; the two resulting polynomials are not the same.

It is easy to confuse $f \circ g$ with $f \cdot g$, especially typographically. But these two functions are defined in entirely different ways. When we evaluate $f \circ g$ at x, we first compute $g(x)$ and then evaluate f at $g(x)$. On the other hand, the product $f \cdot g$ of f and g is the function whose value at a particular number x is simply the product of $f(x)$ and $g(x)$, so $(f \cdot g)(x) = f(x) \cdot g(x)$.

Proof of the chain rule To find the derivative of $F(x) = f(g(x))$ at $x = x_0$, we must examine the limit of the following Newton quotient as h

tends to 0:

$$N = \frac{F(x_0 + h) - F(x_0)}{h} = \frac{f\big(g(x_0 + h)\big) - f\big(g(x_0)\big)}{h}$$

The change in x from x_0 to x_0+h causes the value of g to change by the amount $k = g(x_0 + h) - g(x_0)$. As h tends to 0, so $k = \big\{\big[g(x_0 + h) - g(x_0)\big]/h\big\} \cdot h$ tends to $g'(x_0) \cdot 0 = 0$. *Suppose that $k \neq 0$ whenever $h \neq 0$ is small enough.* Because $g(x_0 + h) = g(x_0) + k$, we can write the Newton quotient as

$$\begin{aligned} N &= \frac{f\big(g(x_0) + k\big) - f\big(g(x_0)\big)}{k} \cdot \frac{k}{h} \\ &= \frac{f\big(g(x_0) + k\big) - f\big(g(x_0)\big)}{k} \cdot \frac{g(x_0 + h) - g(x_0)}{h} \end{aligned}$$

As $h \to 0$, so $k \to 0$, and the last two fractions tend to $f'\big(g(x_0)\big)$ and $g'(x_0)$, respectively. This yields the desired formula.

We cannot divide by 0, so the argument fails if $g(x_0 + h) = g(x_0)$ for arbitrary small values of h, because then $k = 0$. A more complicated proof takes care of this case as well.

Problems

1. Use the chain rule [5.3] to find dy/dx for the following:

a. $y = 5u^4$ and $u = 1 + x^2$

b. $y = u - u^6$ and $u = 1/x + 1$

2. Compute the following:

a. dY/dt when $Y = -3(V + 1)^5$ and $V = \frac{1}{3}t^3$.

b. dK/dt when $K = AL^a$ and $L = bt + c$ (A, a, b, and c are positive constants).

3. Find the derivatives of the following functions, where a, p, q, and b are constants:

a. $y = \dfrac{1}{(x^2 + x + 1)^5}$ **b.** $y = \sqrt{x + \sqrt{x + \sqrt{x}}}$ **c.** $y = x^a(px + q)^b$

4. If Y is a function of K, and K is a function of t, find the formula for the derivative of Y with respect to t at $t = t_0$.

5. If $Y = F(K)$ and $K = h(t)$, find the formula for dY/dt.

6. Compute dx/dp for the demand function

$$x = b - \sqrt{ap - c} \qquad (a,\ b, \text{ and } c \text{ are positive constants})$$

where x is the number of units demanded, and p is the price per unit, with $p \geq c/a$.

7. If $h(x) = f(x^2)$, find a formula for $h'(x)$.

8. Let $s(t)$ be the distance in kilometers traveled by a car in t hours. Let $B(s)$ be the number of liters of fuel the car uses to go s kilometers.
   a. Provide an interpretation of the function $b(t) = B(s(t))$.
   b. Find and interpret the formula for $b'(t)$.

9. If $a(t)$ and $b(t)$ are positive-valued differentiable functions of t, and if A, α, and β are constants, find expressions for $\dot{x}/x$ where:

   a. $x = [a(t)]^2 b(t)$ **b.** $x = \dfrac{[(a(t)]^5}{b(t)}$

   c. $x = A\left\{[a(t)]^\alpha + [b(t)]^\beta\right\}^{\alpha+\beta}$ **d.** $x = A\,[a(t)]^\alpha\,[b(t)]^\beta$

10. Suppose that $f(x) = 3x+7$. Compute $f(f(x))$. Find x such that $f(f(x)) = 100$.

11. Express (in at least one way) the following functions as composites of simpler functions, and find $h'(x)$ in each case:
    a. $h(x) = (1 + x + x^2)^{1/2}$ **b.** $h(x) = 1/(x^{100} + 28)$

12. Suppose that $C = 20q - 4q\left(25 - \frac{1}{2}x\right)^{1/2}$, where q is a constant and $x < 50$. Find dC/dx.

13. Differentiate each of the following in two different ways:
    a. $y = (x^4)^5 = x^{20}$ **b.** $y = (1-x)^3 = 1 - 3x + 3x^2 - x^3$

14. If $p(x) = (x-a)^2 q(x)$ and q is differentiable at $x = a$, show that $p'(a) = 0$.

15. If $R = S^\alpha$, $S = 1 + \beta K^\gamma$, and $K = At^p + B$, find an expression for dR/dt.

16. If $F(x) = f(x^n g(x))$, find a formula for $F'(x)$.

5.3 Implicit Differentiation

We know how to differentiate functions given explicitly by certain formulas. Now we consider how to differentiate functions defined implicitly by an equation.

An Introductory Example

The following equation was studied in Example 2.7 of Section 2.3:

$$x\sqrt{y} = 2 \qquad (x > 0,\ \ y > 0) \qquad [*]$$

Note that $y = 4$ when $x = 1$, that $y = 1$ when $x = 2$, that $y = 1/4$ when $x = 4$, and $y = 1/9$ when $x = 6$. In general, for each positive number x, there is a unique number y such that the pair (x, y) satisfies the equation. We say that equation [*] *defines y implicitly as a function of x*. The graph of equation [*] shown in Fig. 5.1 is reproduced from Fig. 2.14.

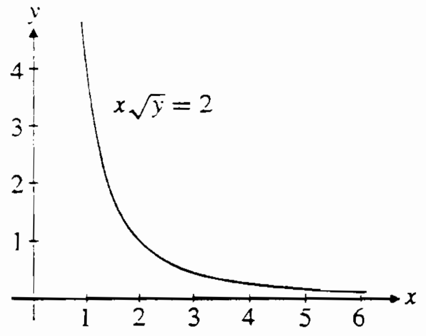


FIGURE 5.1 $x\sqrt{y} = 2$.

It is natural to ask what is the slope of the tangent at an arbitrary point on the graph. In other words, what is the derivative of y as function of x? The answer can be found by implicit differentiation of equation [$*$]. Let f denote the function defined by equation [$*$]. Substituting $f(x)$ for y gives

$$x\sqrt{f(x)} = 2 \qquad \text{(for all } x > 0)$$

The derivative of the left-hand side of this identity must be equal to the derivative of the right-hand side, for all $x > 0$. Now use the product rule to differentiate $x\sqrt{f(x)} = 2$ w.r.t. x. The implication is that

$$1 \cdot \sqrt{f(x)} + x\frac{d}{dx}\sqrt{f(x)} = 0 \qquad [**]$$

But the chain rule yields

$$\frac{d}{dx}\sqrt{f(x)} = \frac{1}{2\sqrt{f(x)}} \cdot f'(x)$$

Inserting this into [$**$] and rearranging gives

$$\frac{x}{2\sqrt{f(x)}}f'(x) = -\sqrt{f(x)}$$

When $x > 0$, solving for $f'(x)$ leads to

$$f'(x) = \frac{-2f(x)}{x}$$

For $x = 2$, we get $f(2) = 1$, and hence $f'(2) = -1$, which agrees with Fig. 5.1.

Usually, we do not introduce a name for y as a function of x. Instead, we differentiate directly, using the following reasoning. Differentiating [$*$] with respect to x, while recalling that y is a differentiable function of x, gives

$$1 \cdot \sqrt{y} + x \cdot \frac{1}{2\sqrt{y}}y' = 0$$

Solving for y' gives

$$y' = -\frac{2y}{x} \qquad [***]$$

For this particular example, there is another way to find the answer. Squaring each side of equation [*] yields $x^2y = 4$, and so $y = 4/x^2 = 4x^{-2}$ for $x > 0$. Differentiating w.r.t. x yields $y' = 4(-2)x^{-3} = -8/x^3$. Note that substituting $4/x^2$ for y in [***] yields $y' = -8/x^3$ again.

The method used to derive [***] can be summarized as follows:

The Method of Implicit Differentiation

If two variables x and y are related by an equation, to find dy/dx:

1. Differentiate each side of the equation w.r.t. x, considering y as a function of x. (Usually, you will need the chain rule.)
2. Solve the resulting equation for dy/dx.

Further Examples

It is important for economists to master the technique of implicit differentiation, so here are some further examples.

Example 5.7

Suppose that y is a differentiable function of x given by

$$x + y^3 = y^5 - x^2 + 2y \qquad [1]$$

for all x in a given interval I. Find an expression for y'. The graph of Equation [1] passes through $(x, y) = (1, 1)$. Find y' at this point.

Solution In this case, it is impossible to solve the equation explicitly for y. It is still possible, however, to find an explicit expression for y'. We suppose that an (unspecified) function of x is substituted for y. Then $x + y^3$ and $y^5 - x^2 + 2y$ are both functions of x, and these expressions must be equal for all x in I. This implies that their derivatives must be equal. According to the chain rule, the derivative of y^3 with respect to x is $3y^2y'$ and the derivative of y^5 is equal to $5y^4y'$. Thus,

$$1 + 3y^2y' = 5y^4y' - 2x + 2y'$$

To find y', collect all terms containing y' on the right-hand side and all others on the left. The result is

$$1 + 2x = (5y^4 + 2 - 3y^2)y'$$

Solving for y' gives

$$y' = \frac{2x + 1}{5y^4 - 3y^2 + 2}$$

Because we have no explicit expression for y as a function of x, we cannot express y' explicitly as a function of x. At $(x, y) = (1, 1)$, however, we get $y' = 3/4$.

Example 5.8

Consider the following standard macroeconomic model for determining national income in a closed economy:

$$[1]\ Y = C + I$$

$$[2]\ C = f(Y)$$

Here [2] is the consumption function discussed in Example 2.18 of Section 2.5, whereas [1] states that the national income Y goes either to consumption C or to investment I. We suppose that $f'(Y)$, *the marginal propensity to consume*, is between 0 and 1.

(a) Suppose first that $C = f(Y) = 95.05 + 0.712\,Y$ (see Example 2.18), and use equations [1] and [2] to find Y in terms of I. Find the change ΔY in Y if I is changed by ΔI units.

(b) Equations [1] and [2] define Y as a differentiable function of I. Find an expression for dY/dI.

Solution

(a) In this case, we find that $Y = 95.05 + 0.712Y + I$. Solving for Y yields

$$Y = (95.05 + I)/(1 - 0.712) \approx 3.47\,I + 330.03 \qquad [3]$$

Suppose now that I changes by ΔI. The corresponding change ΔY in Y satisfies

$$Y + \Delta Y \approx 3.47(I + \Delta I) + 330.03 \qquad [4]$$

Subtracting [3] from [4] gives

$$\Delta Y \approx 3.47\,\Delta I \qquad [5]$$

In particular, if I is changed by one unit (for example, \$1 billion) so that $\Delta I = 1$, then the corresponding change in the national product is $\Delta Y \approx 3.47$ (billion).

(b) Inserting the expression for C from [2] into [1] gives

$$Y = f(Y) + I \tag{6}$$

Suppose that this equation defines Y as a differentiable function of I. Differentiating [6] with respect to I, and using the chain rule, we have

$$\frac{dY}{dI} = f'(Y)\frac{dY}{dI} + 1 \qquad \text{or} \qquad \frac{dY}{dI}\left[1 - f'(Y)\right] = 1$$

Solving for dY/dI yields

$$\frac{dY}{dI} = \frac{1}{1 - f'(Y)} \tag{7}$$

For example, if $f'(Y) = 1/2$, then $dY/dI = 2$. Also $f'(Y) = 0.712$ gives $dY/dI \approx 3.47$. In general, we see that because $f'(Y)$ lies between 0 and 1, so $1 - f'(Y)$ also lies between 0 and 1. Hence, $1/\left[1 - f'(Y)\right]$ is always larger than 1. In this model, therefore, a \$1 billion increase in investment will always lead to a more than \$1 billion increase in the national product. The larger is $f'(Y)$, the marginal propensity to consume, the larger is dY/dI.

Example 5.9

In the linear supply and demand model of Example 2.19, Section 2.5, suppose that a tax of t per unit is imposed on consumers. Then

$$D = a - b(P + t), \qquad S = \alpha + \beta P \tag{1}$$

Here a, b, α, and β are positive constants. The equilibrium price is determined by equating supply and demand, so that

$$a - b(P + t) = \alpha + \beta P \tag{2}$$

(a) Equation [2] implicitly defines the price P as a function of the unit tax t. Compute dP/dt by implicit differentiation. What is its sign? Check the result by first solving Equation [2] for P and then finding dP/dt explicitly.

(b) Compute tax revenue T as a function of t. For what value of t does the quadratic function T reach its maximum?

(c) Generalize the foregoing model by assuming that

$$D = f(P + t) \qquad \text{and} \qquad S = g(P)$$

where f and g are differentiable functions with $f' < 0$ and $g' > 0$. The equilibrium condition

$$f(P + t) = g(P) \tag{3}$$

defines P implicitly as a differentiable function of t. Find an expression for dP/dt by implicit differentiation.

Solution

(a) Differentiating [2] w.r.t. t yields $-b(dP/dt + 1) = \beta\, dP/dt$. Solving for dP/dt gives

$$\frac{dP}{dt} = \frac{-b}{b+\beta}$$

We see that dP/dt is negative. Because P is the price received by the producer, this price will go down if the tax rate t increases. But $P + t$ is the price paid by the consumer. Because

$$\frac{d}{dt}(P+t) = \frac{dP}{dt} + 1 = \frac{-b}{b+\beta} + 1 = \frac{-b+b+\beta}{b+\beta} = \frac{\beta}{b+\beta} > 0$$

it follows that $0 < d(P+t)/dt < 1$. Thus, the consumer price increases, but by less than the increase in the tax.

If we solve [2] for P, we obtain

$$P = \frac{a - \alpha - bt}{b+\beta} = \frac{a-\alpha}{b+\beta} - \frac{b}{b+\beta}t$$

This equation shows that the equilibrium price is a linear function of the tax per unit with slope $-b/(b+\beta)$.

(b) The total tax revenue is $T = St = (\alpha + \beta P)t$, where P is the equilibrium price. Thus,

$$T = \left[\alpha + \beta\left(\frac{-bt}{b+\beta} + \frac{a-\alpha}{b+\beta}\right)\right]t = \frac{-b\beta t^2}{b+\beta} + \frac{(a\beta + \alpha b)t}{b+\beta}$$

This quadratic function has its maximum at $t = (\alpha b + \beta a)/2b\beta$.

(c) Differentiating [3] w.r.t. t yields $f'(P+t)\,(dP/dt + 1) = g'(P)\,dP/dt$. Solving for dP/dt gives

$$\frac{dP}{dt} = \frac{f'(P+t)}{g'(P) - f'(P+t)}$$

Because $f' < 0$ and $g' > 0$, we see that dP/dt is negative in this case as well. Moreover,

$$\frac{d}{dt}(P+t) = \frac{dP}{dt} + 1 = \frac{f'(P+t)}{g'(P) - f'(P+t)} + 1$$
$$= \frac{g'(P)}{g'(P) - f'(P+t)}$$

which implies that $0 < d(P+t)/dt < 1$ in this case also.

The Second Derivative of Functions Defined Implicitly

The following examples suggest how to compute the second derivative of a function that is defined implicitly by an equation.

Example 5.10

Compute y'' when y is given implicitly as function of x by

$$x\sqrt{y} = 2 \tag{1}$$

Solution In the introductory example to this section, we found that $y' = -2y/x$ by implicit differentiation. Using the quotient rule to differentiate this equation implicitly w.r.t. x, while taking into account how y is a function of x, we obtain

$$y'' = -\frac{2y'x - 2y \cdot 1}{x^2}$$

Inserting the expression $-2y/x$ we already have for y' gives

$$y'' = -\frac{2(-2y/x)x - 2y}{x^2} = \frac{6y}{x^2} \tag{2}$$

In this case, we can check the answer directly. From [1], we have $y = 4/x^2$, which when inserted into [2] gives $y'' = 24/x^4$. On the other hand, because $y = 4/x^2 = 4x^{-2}$, direct differentiation gives $y' = -8x^{-3}$ and $y'' = 24x^{-4} = 24/x^4$.

Example 5.11

Find d^2Y/dI^2 when $Y = f(Y) + I$.

Solution We found in Example 5.8 that $dY/dI = 1/[1 - f'(Y)] = [1 - f'(Y)]^{-1}$. Differentiation with respect to I using the chain rule yields

$$\frac{d^2Y}{dI^2} = (-1)[1 - f'(Y)]^{-2} \cdot [-f''(Y)]\frac{dY}{dI} = f''(Y)[1 - f'(Y)]^{-2}\frac{dY}{dI}$$

(We had to differentiate $1 - f'(Y)$ with respect to I. The result is $0 - f''(Y)(dY/dI)$.) Using the expression for dY/dI gives

$$\frac{d^2Y}{dI^2} = f''(Y)\left[1 - f'(Y)\right]^{-3}$$

$$= \frac{f''(Y)}{\left[1 - f'(Y)\right]^3}$$

Problems

1. For the following equations, find dy/dx by implicit differentiation:

   a. $xy = 1$ **b.** $x - y + 3xy = 2$ **c.** $y^6 = x^5$

   Check by solving each equation w.r.t. y and then differentiating.

2. Suppose that y is a differentiable function of x that satisfies the equation

$$2x^2 + 6xy + y^2 = 18$$

   Find an expression for y' by implicit differentiation. The point $(x, y) = (1, 2)$ lies on the graph of the equation. Find y' at this particular point.

3. A curve in the uv-plane is given by

$$u^2 + uv - v^3 = 0$$

   Compute dv/du by implicit differentiation. Find the point (u, v) on the curve where $dv/du = 0$ and $u \neq 0$.

4. For each of the following equations, answer the question: If $y = f(x)$ is a differentiable function that satifies the equation, what is y'? (a is a positive constant.)

   a. $x^2 + y^2 = a^2$ **b.** $\sqrt{x} + \sqrt{y} = \sqrt{a}$ **c.** $x^4 - y^4 = x^2y^3$

5. According to Wold, the demand Q for butter in Stockholm during the period 1925–1937 was related to the price P by the equation

$$Q \cdot P^{1/2} = 38$$

   Find dQ/dP by implicit differentiation. Check the answer by using a different method to compute the derivative.

6. Suppose that f and g are two functions defined in an open interval I.

   a. If $f(x_0) = g(x_0)$ for some $x_0 \in I$, what can you conclude about $f'(x_0)$ and $g'(x_0)$?

   b. If $f(x) = g(x)$ for all $x \in I$, and if $x_0 \in I$, what can you conclude about $f'(x_0)$ and $g'(x_0)$?

7. A standard model for income determination in an open economy is

$$Y = C + I + \bar{X} - M \tag{[1]}$$

$$C = f(Y) \tag{[2]}$$

$$M = g(Y) \tag{[3]}$$

where $0 < f'(Y) < 1$. Here $\bar{X}$ is an exogeneous constant that denotes exports, whereas M denotes the volume of imports. The function g in [3] is called an *import function*. By inserting [2] and [3] into [1], we obtain an equation that defines Y as a function of exogeneous investment I. Find a expression for dY/dI by implicit differentiation. What is the likely sign of $g'(Y)$? Discuss the sign of dY/dI.

8. If $a = m/n$, where m and n are integers, the power rule (4.24) gives

$$y = x^{m/n} \Longrightarrow y' = (m/n)x^{(m/n)-1}$$

Verify this result (assuming that y *is* differentiable) by differentiating the equation $y^n = x^m$ implicitly with respect to x.

9. If f and g are differentiable and $g(f(x)) = x$ for all x, find an expression for $f'(x)$ in terms of the derivative of g.

5.4 Linear Approximations and Differentials

When a complicated function is difficult to work with, we sometimes try to find a simpler function that in some sense approximates the original one. Linear functions are very easy to manipulate. It is therefore natural first to try to find a "linear approximation" to a given function.

Consider a function $f(x)$ that is differentiable at $x = a$. The tangent to the graph at $(a, f(a))$ has the equation $y = f(a) + f'(a)(x - a)$ (see [4.4] of Section 4.2). If we approximate the graph of f by its tangent line at $x = a$, as shown in Fig. 5.2, the resulting approximation has a special name.

The **linear approximation** to f about a is

$$f(x) \approx f(a) + f'(a)(x - a) \qquad (x \text{ close to } a) \tag{[5.5]}$$

Note that if $p(x)$ is the linear function $f(a) + f'(a)(x - a)$ of x, then f and p have both the same value and the same derivative at $x = a$.

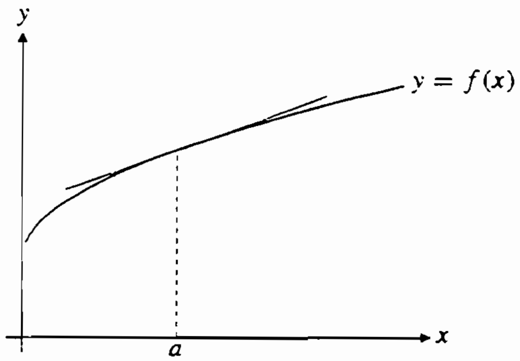


FIGURE 5.2 The approximation of a function by its tangent.

Example 5.12

Find the linear approximation to $f(x) = \sqrt[3]{x}$ about $a = 1$.

Solution We have $f(x) = \sqrt[3]{x} = x^{1/3}$, so $f'(x) = \frac{1}{3}x^{-2/3}$ and $f'(1) = 1/3$. Because $f(1) = 1$, using [5.5] yields

$$\sqrt[3]{x} \approx 1 + \tfrac{1}{3}(x-1) \qquad (x \text{ close to } 1)$$

For example, $\sqrt[3]{1.03} \approx 1 + \frac{1}{3}(1.03 - 1) = 1 + \frac{1}{3}(0.03) = 1.01$. The correct value to 4 decimals is 1.0099.

Example 5.13

In a paper by economists Samuelson and Swamy, the authors were concerned with the behavior of the following function about $\varepsilon = 0$:

$$f(\varepsilon) = \left(1 + \tfrac{3}{2}\varepsilon + \tfrac{1}{2}\varepsilon^2\right)^{1/2}$$

Find the linear approximation to $f(\varepsilon)$ about $\varepsilon = 0$.

Solution Here $f'(\varepsilon) = \frac{1}{2}\left(1 + \frac{3}{2}\varepsilon + \frac{1}{2}\varepsilon^2\right)^{-1/2} \cdot (\frac{3}{2} + \varepsilon)$ and so $f'(0) = \frac{3}{4}$. Because $f(0) = 1$, using [5.5] yields

$$\left(1 + \tfrac{3}{2}\varepsilon + \tfrac{1}{2}\varepsilon^2\right)^{1/2} \approx 1 + \tfrac{3}{4}\varepsilon \qquad (\varepsilon \text{ close to } 0)$$

The Differential of a Function

Consider a differentiable function $f(x)$, and let dx denote an arbitrary change in the variable x. In this notation, "dx" is not a product of d and x. Rather, dx is a single symbol representing the change in the value of x. The expression $f'(x)\,dx$ is called the **differential** of $y = f(x)$, and it is denoted by dy (or df), so that

$$dy = f'(x)\,dx \qquad [5.6]$$

Note that dy is proportional to dx, with $f'(x)$ as the factor of proportionality.

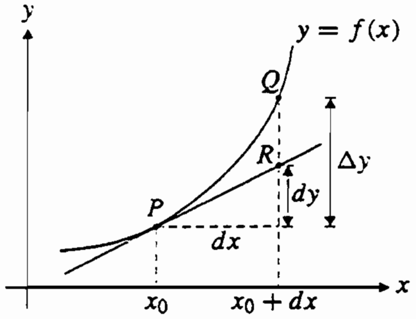


FIGURE 5.3 A geometric representation of the differential.

Now if x changes by dx, then the corresponding change in $y = f(x)$ is

$$\Delta y = f(x + dx) - f(x) \qquad [5.7]$$

By using the definitions of dy and Δy, replacing x by $x + dx$ and a by x, the approximation in [5.5] takes the form

$$\Delta y \approx dy = f'(x)\,dx$$

The differential dy is not the actual increment in y as x is changed to $x+dx$, but rather the change in y that would occur if y continued to change at the fixed rate $f'(x)$ as x changes to $x + dx$. Figure 5.3 illustrates the difference between Δy and dy. Consider, first, the movement from P to Q along the curve $y = f(x)$: as x changes by dx, the actual change in the vertical height of the point is Δy. Suppose instead that we are only allowed to move along the tangent to the graph at P. Thus, as we move from P to R along the tangent, the change in height that corresponds to dx is dy. Note that, as in Fig. 5.3, the approximation $\Delta y \approx dy$ is usually better if dx is smaller in absolute value, because the length of line segment RQ representing the difference between Δy and dy tends to 0 as dx tends to 0.

Rules for Differentials

The notation $(d/dx)(\quad)$ calls for the expression in parentheses to be differentiated with respect to x. In the same way, we let $d(\quad)$ denote the differential of whatever is inside the parentheses.

Example 5.14

Compute the following:

(a) $d\big(Ax^a + B\big)$ $\quad$ (A, B, and a are constants)

(b) $d\big(f(K)\big)$ $\quad$ (f a differentiable function of K)

Solution

(a) Putting $f(x) = Ax^a + B$, we get $f'(x) = Aax^{a-1}$, so $d(Ax^a + B) = Aax^{a-1}\,dx$.

(b) $d(f(K)) = f'(K)\,dK$.

All the usual rules for differentiation can be expressed in terms of differentials. If f and g are two differentiable functions of x, then the following holds.

Rules for Differentials

$$
\begin{aligned}
d(af + bg) &= a\,df + b\,dg \qquad (a \text{ and } b \text{ are constants}) \\
d(fg) &= g\,df + f\,dg \\
d\left(\frac{f}{g}\right) &= \frac{g\,df - f\,dg}{g^2} (g \neq 0)
\end{aligned}
\qquad [5.8]
$$

Here is a proof of the second of these formulas:

$$d(fg) = (fg)'\,dx = (f'g + fg')\,dx = gf'\,dx + fg'\,dx = g\,df + f\,dg$$

You should now prove the other rules in the same way.

Suppose that $y = f(x)$ and $x = g(t)$ is a function of t. Then $y = h(t) = f(g(t))$ is a function of t. The differential of $y = h(t)$ is $dy = h'(t)\,dt$. According to the chain rule, $h'(t) = f'(g(t))\,g'(t)$, so that $dy = f'(g(t))g'(t)\,dt$. Because $x = g(t)$, the differential of x is equal to $dx = g'(t)\,dt$, hence

$$dy = f'(x)\,dx$$

This shows that if $y = f(x)$, then the differential of y is equal to $dy = f'(x)\,dx$, whether x depends on another variables or not.

Economists often use differentials in their models. A typical example follows.

Example 5.15

Consider again the model in Example 5.8 of Sec. 5.3:

$$[1]\ Y = C + I \qquad [2]\ C = f(Y)$$

Find the differential dY expressed in terms of dI. If in addition to [1] and [2] it is assumed that employment $N = g(Y)$ is a function of Y, find also the differential dN expressed in terms of dI.

Solution Differentiating [1] and [2], we obtain

$$[3]\ dY = dC + dI \qquad [4]\ dC = f'(Y)\,dY$$

Substituting dC from [4] into [3] and solving for dY yields

$$dY = \frac{1}{1 - f'(Y)}\, dI \tag{5}$$

which corresponds exactly to [7] in Example 5.8. From $N = g(Y)$, we get $dN = g'(Y)\,dY$, so

$$dN = \frac{g'(Y)}{1 - f'(Y)}\, dI \tag{6}$$

Provided that $g'(Y) > 0$ and $f'(Y)$, the marginal propensity to consume, is between 0 and 1, we see from [6] that if investment increases, then employment increases.

Problems

1. Prove that $\sqrt{1+x} \approx 1 + \frac{1}{2}x$, for x close to 0, and illustrate this approximation by drawing the graphs of $y = 1 + \frac{1}{2}x$ and $y = \sqrt{1+x}$ in the same coordinate system.
2. Use [5.5] to find the linear approximation to $f(x) = (5x+3)^{-2}$ about $x_0 = 0$.
3. Find the linear approximation to the following functions about $x_0 = 0$:
   a. $f(x) = (1+x)^{-1}$ **b.** $f(x) = (1+x)^5$ **c.** $f(x) = (1-x)^{1/4}$
4. Find the linear approximation to $F(K) = AK^{\alpha}$ about $K_0 = 1$.
5. Prove that $(1+x)^m \approx 1 + mx$, for x close to 0, and use this approximation to find approximations to the following numbers:
   a. $\sqrt[3]{1.1} = \left(1 + \frac{1}{10}\right)^{1/3}$ **b.** $\sqrt[5]{33} = 2\left(1 + \frac{1}{32}\right)^{1/5}$
   c. $\sqrt[3]{9} = \sqrt[3]{8+1}$ **d.** $(1.02)^{25}$
   e. $\sqrt{37} = \sqrt{36+1}$ **f.** $(26.95)^{1/3} = \left(27 - \frac{5}{100}\right)^{1/3}$
6. Compute $\Delta y = f(x+dx) - f(x)$ and the differential $dy = f'(x)\,dx$ for the following:
   a. $f(x) = x^2 + 2x - 3$ when (i) $x = 2$, $dx = 1/10$, and (ii) $x = 2$, $dx = 1/100$.
   b. $f(x) = 1/x$ when (i) $x = 3$, $dx = -1/10$, and (ii) $x = 3$, $dx = -1/100$.
   c. $f(x) = \sqrt{x}$ when (i) $x = 4$, $dx = 1/20$, and (ii) $x = 4$, $dx = 1/100$.
7. The radius of a ball increases from 2 to 2.03. Estimate the increase in volume of the ball by using a linear approximation. Compare with the actual increase in volume. (Hint: See Appendix D.)

Harder Problems

8. Find the linear approximation to the function

$$g(\mu) = A(1+\mu)^{a/(1+b)} - 1 \qquad (A,\ a,\ \text{and } b \text{ are positive constants})$$

about the the point $\mu = 0$.

5.5 Polynomial Approximations

The previous section discussed approximations of functions of one variable by linear functions. In particular, Example 5.12 established the approximation

$$\sqrt[3]{x} \approx 1 + \tfrac{1}{3}(x-1) \qquad (x \text{ close to } 1) \tag{1}$$

In this case, at $x = 1$, the functions $y = \sqrt[3]{x}$ and $y = 1 + \frac{1}{3}(x-1)$ have the same value, 1, and the same derivative, 1/3.

If approximation by linear functions is insufficiently accurate, it is natural to try quadratic approximations, or approximations by polynomials of higher order.

Quadratic Approximations

We begin by showing how a twice differentiable function $y = f(x)$ can be approximated near $x = a$ by a quadratic polynomial

$$f(x) \approx p(x) = A + B(x-a) + C(x-a)^2$$

There are three coefficients, A, B, and C, to be determined. So we are free to impose three conditions on the polynomial. We will assume that $f(x)$ and $p(x) = A + B(x-a) + C(x-a)^2$ have the same value, the same derivative, and the same second derivative at $x = a$. In symbols, we require that $f(a) = p(a)$, $f'(a) = p'(a)$, and $f''(a) = p''(a)$. Now

$$p'(x) = B + 2C(x-a), \qquad p''(x) = 2C$$

So, after inserting $x = a$ into our expressions for $p(x)$, $p'(x)$, and $p''(x)$, it follows that $A = p(a)$, $B = p'(a)$, and $C = \frac{1}{2}p''(a)$. Hence:

The **quadratic approximation** to $f(x)$ about $x = a$ is

$$f(x) \approx f(a) + f'(a)(x-a) + \tfrac{1}{2}f''(a)(x-a)^2 \qquad (x \text{ close to } a) \tag{5.9}$$

For $a = 0$, in particular, we obtain the following:

$$f(x) \approx f(0) + f'(0)x + \tfrac{1}{2}f''(0)x^2 \qquad (x \text{ close to } 0) \qquad [5.10]$$

Example 5.16

Find the quadratic approximation to $f(x) = \sqrt[3]{x}$ about $a = 1$.

Solution Here $f'(x) = \frac{1}{3}x^{-2/3}$ and $f''(x) = \frac{1}{3}(-\frac{2}{3})x^{-5/3}$. It follows that $f'(1) = \frac{1}{3}$ and $f''(1) = -\frac{2}{9}$. Because $f(1) = 1$, using [5.9] yields

$$\sqrt[3]{x} \approx 1 + \tfrac{1}{3}(x-1) - \tfrac{1}{9}(x-1)^2 \qquad (x \text{ close to } 1)$$

For example, $\sqrt[3]{1.03} \approx 1 + \frac{1}{3}\cdot 0.03 - \frac{1}{9}(0.03)^2 = 1 + 0.01 - 0.0001 = 1.0099$, which is correct to 4 decimals.

Example 5.17

Find the quadratic approximation to $f(x) = (5x+3)^{-2}$ about $x = 0$.

Solution Here $f'(x) = -10(5x+3)^{-3}$ and $f''(x) = 150(5x+3)^{-4}$, so that $f(0) = 1/9$, $f'(0) = -10/27$, and $f''(0) = 50/27$. Hence, [5.10] gives

$$\frac{1}{(5x+3)^2} \approx \frac{1}{9} - \frac{10}{27}x + \frac{25}{27}x^2 \qquad [*]$$

Example 5.18

Find the quadratic approximation to $y = y(x)$ about $x = 0$ when y is defined implicitly as a function of x near $(x, y) = (0, 1)$ by

$$xy^3 + 1 = y \qquad [1]$$

Solution Implicit differentiation of [1] with respect to x yields

$$y^3 + 3xy^2y' = y' \qquad [2]$$

Substituting $x = 0$ and $y = 1$ into [2] gives $y' = 1$. Differentiating [2] with respect to x now yields

$$3y^2y' + (3y^2 + 6xyy')y' + 3xy^2y'' = y''$$

Substituting $x = 0$, $y = 1$, and $y' = 1$, we obtain $y'' = 6$. Hence, according to [5.10],

$$y(x) \approx y(0) + y'(0)x + \tfrac{1}{2}y''(0)x^2 = 1 + x + 3x^2$$

Higher-Order Approximations

So far, we have considered linear and quadratic approximations. We can find better approximations near one point by using polynomials of a higher degree. Suppose we want to approximate a function $f(x)$ over an interval centered at $x = a$ with an nth-degree polynomial of the form

$$p(x) = A_0 + A_1(x-a) + A_2(x-a)^2 + A_3(x-a)^3 + \cdots + A_n(x-a)^n \quad [1]$$

Because $p(x)$ has $n+1$ coefficients, we can impose the following $n+1$ conditions on this polynomial:

$$f(a) = p(a), \qquad f'(a) = p'(a), \qquad \ldots, \qquad f^{(n)}(a) = p^{(n)}(a) \quad [2]$$

These conditions require that $p(x)$ and its first n derivatives agree with the value of $f(x)$ and its first n derivatives at $x = a$. Let us see what these conditions become when $n = 3$. In this case,

$$p(x) = A_0 + A_1(x-a) + A_2(x-a)^2 + A_3(x-a)^3$$

and we find that

$$\begin{aligned} p'(x) &= A_1 + 2A_2(x-a) + 3A_3(x-a)^2 \\ p''(x) &= 2A_2 + 2\cdot 3A_3(x-a) \\ p'''(x) &= 2\cdot 3A_3 \end{aligned}$$

Thus, when $x = a$, we have[1]

$$p(a) = A_0, \qquad p'(a) = 1!\,A_1, \qquad p''(a) = 2!\,A_2, \qquad p'''(a) = 3!\,A_3$$

This implies the following approximation:

$$f(x) \approx f(a) + \frac{1}{1!}f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2 + \frac{1}{3!}f'''(a)(x-a)^3$$

The general case follows the same pattern. When $p(x)$ is given by [1], yields

[1]For the definition of $n!$, see Problem 6 of Section 4.7.

successive differentiation of $p(x)$

$$\begin{aligned} p'(x) &= A_1 + 2A_2(x-a) + 3A_3(x-a)^2 + \cdots + nA_n(x-a)^{n-1} \\ p''(x) &= 2A_2 + 3\cdot 2A_3(x-a) + \cdots + n(n-1)A_n(x-a)^{n-2} \\ p'''(x) &= 3\cdot 2A_3 + \cdots + n(n-1)(n-2)A_n(x-a)^{n-3} \\ p^{(4)}(x) &= 4\cdot 3\cdot 2A_4 + \cdots + n(n-1)(n-2)(n-3)A_n(x-a)^{n-4} \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ p^{(n)}(x) &= n(n-1)(n-2)\cdots 2\cdot 1A_n \end{aligned} \tag{3}$$

Substituting $x = a$ into [3] gives

$$p'(a) = 1!\,A_1, \quad p''(a) = 2!\,A_2, \quad \ldots, \quad p^{(n)}(a) = n!\,A_n \tag{4}$$

This leads to the following approximation to $f(x)$ by an nth degree polynomial.

Approximation to $f(x)$ about $x = a$:

$$f(x) \approx f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n \tag{5.11}$$

The polynomial on the right-hand side of [5.11] is called the **nth-order Taylor polynomial** for f about $x = a$.

The function f and its nth-order Taylor polynomial have such a high degree of contact at $x = a$ that it is reasonable to expect the approximation in [5.11] to be good over some (possibly small) interval centered about $x = a$. Section 7.4 analyses the error that results from using such polynomial approximations. In the case when f is itself a polynomial whose degree does not exceed n, the formula becomes exact, without any approximation error at any point.

Example 5.19

Find the third-order Taylor approximation of $f(x) = \sqrt{1+x}$ about $a = 0$.

Solution We write $f(x) = \sqrt{1+x} = (1+x)^{1/2}$. Then we have $f'(x) = (1/2)(1+x)^{-1/2}$, $f''(x) = (1/2)(-1/2)(1+x)^{-3/2}$, and $f'''(x) = (1/2)(-1/2)(-3/2)(1+x)^{-5/2}$. Putting $x = 0$ gives $f(0) = 1$, $f'(0) = 1/2$, $f''(0) = (1/2)(-1/2) = -1/4$, and finally $f'''(0) = (1/2)(-1/2)(-3/2) = 3/8$. Hence, by [5.11] for the case $n = 3$, we have

$$f(x) \approx 1 + \frac{1}{1!}\frac{1}{2}x + \frac{1}{2!}\left(-\frac{1}{4}\right)x^2 + \frac{1}{3!}\frac{3}{8}x^3 = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3$$

Problems

1. Find quadratic approximations to the following functions about the given points:

   a. $f(x) = (1+x)^5,\ a = 0$ b. $F(K) = AK^{\alpha},\ K_0 = 1$

   c. $f(\varepsilon) = \left(1 + \frac{3}{2}\varepsilon + \frac{1}{2}\varepsilon^2\right)^{1/2},\ \varepsilon_0 = 0$ d. $H(x) = (1-x)^{-1},\ a = 0$

2. In connection with a study of attitudes to risk, the following approximation to a consumer's utility function is encountered. Explain how to derive this approximation.

$$U(y + M - s) \approx U(y) + U'(y)(M - s) + \tfrac{1}{2}U''(y)(M - s)^2$$

3. Find the quadratic approximation about $x = 0$, $y = 1$ for y when y is defined implicitly as a function of x by the equation $1 + x^3y + x = y^{1/2}$.

4. Let the function $x(t)$ be given by the conditions $x(0) = 1$ and

$$\dot{x}(t) = tx(t) + 2[x(t)]^2$$

   Determine the second order Taylor polynomial for $x(t)$ about $t = 0$.

5. Establish the approximation

$$\left(1 + \frac{p}{100}\right)^n \approx 1 + n\frac{p}{100} + \frac{n(n-1)}{2}\left(\frac{p}{100}\right)^2$$

6. The function h is defined for all $x > 0$ by

$$h(x) = \frac{x^p - x^q}{x^p + x^q} \qquad (p > q > 0)$$

   Find the first-order Taylor polynomial about $x = 1$ for $h(x)$.

5.6 Elasticities

Why do economists so often use elasticities instead of derivatives? Suppose we study how demand for a certain commodity reacts to price changes. We can ask by how many units the quantity demanded will change when the price increases by \$1. In this way, we obtain a concrete number, a certain number of units. There are, however, several unsatisfactory aspects of this way of measuring the sensitivity of demand to price changes. For instance, a \$1 price increase per pound of coffee may be considerable, whereas a \$1 increase in the price of a car is insignificant.

This problem arises because the sensitivity of demand to price changes is being measured in the same arbitrary units as those used to measure both quantity demanded and price. The difficulties are eliminated if we use relative changes instead. We ask by what percentage the quantity demanded changes when the price increases by 1%. The number we obtain in this way will be independent of the units in which both quantities and prices are measured. It is called the **price elasticity of demand**, measured at a given price.

In 1960, the price elasticity of butter in a certain country was calculated to be approximately -1. This means that an increase of 1% in the price would lead to a decrease of 1% in the quantity of butter demanded, if all the other factors that influence the demand for butter remained constant. In the same year, the demand elasticity for potatoes was calculated to be -0.2. What does this mean? Why do you think the absolute value of this elasticity is so much less than that for butter?

Assume now that the demand for a commodity can be described by the function

$$x = D(p) \tag{1}$$

When the price changes from p to $p+\Delta p$, the quantity demanded, x, also changes. The absolute change in x is $\Delta x = D(p + \Delta p) - D(p)$, and the *relative* (or proportional) change is

$$\frac{\Delta x}{x} = \frac{D(p+\Delta p) - D(p)}{D(p)}$$

The ratio between the relative change in the quantity demanded and the relative change in the price is

$$\frac{\Delta x}{x} \Big/ \frac{\Delta p}{p} = \frac{p}{x}\frac{\Delta x}{\Delta p} = \frac{p}{D(p)}\frac{D(p+\Delta p) - D(p)}{\Delta p} \tag{2}$$

When $\Delta p = p/100$ so that p increases by 1%, then [2] becomes $(\Delta x/x)\cdot 100$, which is the percentage change in the quantity demanded. We call the proportion in [2] *the average elasticity of x in the interval* $[p, p+\Delta p]$. Observe that the number defined in [2] depends both on the price change Δp and on the price p, but is unit-free. Thus, it makes no difference whether quantities are measured in tons, kilograms, or pounds, or whether the prices are measured in dollars, pounds, or crowns.

We would like to define the elasticity of D at p so that it does not depend on the size of the increase in p. We can do this if D is a differentiable function of p. For then it is natural to define the elasticity of D in p as the limit of the ratio in [2] as Δp tends to 0. Because the Newton quotient $\left[D(p + \Delta p) - D(p)\right]/\Delta p$ tends to $D'(p)$ as Δp tends to 0, we obtain

$$\textit{the elasticity of } D(p) \textit{ with respect to } p \textit{ is } \frac{p}{D(p)}\frac{dD(p)}{dp}$$

Usually, we get a good approximation to the elasticity by letting $\Delta p/p = 1/100 = 1\%$ and computing $p\,\Delta x/x\,\Delta p$.

Example 5.20

Assume that the quantity demanded for a particular commodity is given by the formula

$$D(p) = 8000p^{-1.5}$$

Compute the elasticity of $D(p)$ and find the percentage change in quantity demanded when the price increases by 1% from $p = 4$.

Solution We find that

$$\frac{dD(p)}{dp} = 8000 \cdot (-1.5)p^{-1.5-1} = -12{,}000p^{-2.5}$$

so that the elasticity of $D(p)$ with respect to p is

$$\frac{p}{D(p)} \cdot \frac{dD(p)}{dp} = \frac{p}{8000 \cdot p^{-1.5}} \cdot (-12{,}000)p^{-2.5}$$

$$= -\frac{12{,}000}{8000} \frac{p \cdot p^{-2.5}}{p^{-1.5}} = -1.5$$

The elasticity is a constant equal to -1.5, so that an increase in the price of 1% causes quantity demanded to decrease by about 1.5%.

In this case we, can compute the decrease in demand exactly. When the price is 4, the quantity demanded is $D(4) = 8000 \cdot 4^{-1.5} = 1000$. If the price $p = 4$ is increased by 1%, the new price will be $4 + 4/100 = 4.04$, so that the *change* in demand is

$$D(4.04) - D(4) = 8000 \cdot 4.04^{-1.5} - 1000 \approx -14.81$$

The percentage change in demand from $D(4) = 1000$ is approximately $-(14.81/1000) \cdot 100 = -1.481$.

The General Definition of Elasticity

Suppose function f is differentiable at x. If $f(x) \neq 0$, we define the following:

The **elasticity** of f with respect to x is

$$\mathrm{El}_x f(x) = \frac{x}{f(x)} f'(x) \qquad [5.12]$$

Other notation used instead of $\mathrm{El}_x f(x)$ for the elasticity when $y = f(x)$ includes $\mathrm{El}_x y$ and ε_{yx}.

Example 5.21
Find the elasticity of $f(x) = ax^b$ (a and b are constants, and $a \neq 0$).

Solution In this case, $f'(x) = abx^{b-1}$. Hence,

$$\mathrm{El}_x\left(ax^b\right) = \frac{x}{ax^b}abx^{b-1} = b$$

Example 5.22
Let $D(p)$ denote the demand function for a product. By selling $D(p)$ units at price p, the producer earns revenue $R(p)$ given by

$$R(p) = pD(p)$$

By the product rule,

$$R'(p) = D(p) + pD'(p) = D(p)\left[1 + \frac{p}{D(p)}D'(p)\right]$$

so that

$$R'(p) = D(p)\left[1 + \mathrm{El}_p D(p)\right]$$

and

$$\mathrm{El}_p R(p) = \frac{pR'(p)}{R(p)} = \frac{R'(p)}{D(p)} = 1 + \mathrm{El}_p D(p)$$

Observe that if $\mathrm{El}_p D(p) = -1$, then $R'(p) = 0$. When the price elasticity of the demand at a point is equal to -1, a small price change will have (almost) no influence on the revenue. More generally, the marginal revenue generated by a price change is positive if the price elasticity of demand is greater than -1, and negative if the elasticity is less than -1. And the elasticity of revenue w.r.t. price is exactly one greater than the price elasticity of demand.

There are some rules for elasticities of sums, products, quotients, and composite functions that are occasionally useful. You are encouraged to derive these rules in Problem 7.

Problems

1. Find the elasticities of the functions given by the following formulas:

   a. $3x^{-3}$ b. $-100x^{100}$ c. $\sqrt{x}$ d. $\dfrac{A}{x\sqrt{x}}$ (A constant)

2. A study of transport economics uses the relation $T = 0.4K^{1.06}$, where K is expenditure on building roads, and T is a measure of traffic volume. Find the elasticity of T w.r.t. K. An increase in expenditure of 1% corresponds

in this model to an increase in the volume of traffic of approximately how many percent?

3. A study of Norway's State Railways reveals that, for rides up to 60 km, the price elasticity of the volume of traffic is approximately -0.4.
   a. According to this study, what is the consequence of a 10% increase in fares?
   b. The corresponding elasticity for journeys over 300 km is calculated to be approximately -0.9. Can you think of a reason why this elasticity is larger in absolute value than the previous one ?
4. Use the definition [5.12] to find $\mathrm{El}_x f(x)$ for the following:
   a. $f(x) = A$ (A constant) b. $f(x) = x + 1$ c. $f(x) = (1 - x^2)^{10}$
5. Prove that $\mathrm{El}_x f(x)^p = p\, \mathrm{El}_x f(x)$ (p constant).
6. Compute $\mathrm{El}_x Af(x)$ and $\mathrm{El}_x \left[A + f(x)\right]$ (A constant).

Harder Problems

7. Prove that if f and g are differentiable functions of x and A is a constant, then the following rules hold (where we write, for instance, $\mathrm{El}_x f$ instead of $\mathrm{El}_x f(x)$).
   a. $\mathrm{El}_x A = 0$
   b. $\mathrm{El}_x(fg) = \mathrm{El}_x f + \mathrm{El}_x g$
   c. $\mathrm{El}_x (f/g) = \mathrm{El}_x f - \mathrm{El}_x g$
   d. $\mathrm{El}_x(f + g) = \dfrac{f\, \mathrm{El}_x f + g\, \mathrm{El}_x g}{f + g}$
   e. $\mathrm{El}_x(f - g) = \dfrac{f\, \mathrm{El}_x f - g\, \mathrm{El}_x g}{f - g}$
   f. $\mathrm{El}_x f\big(g(x)\big) = \mathrm{El}_u f(u)\, \mathrm{El}_x u$ (where $u = g(x)$)
8. Use the rules in Problem 7 to calculate the following:
   a. $\mathrm{El}_x 3x^{-3}$ b. $\mathrm{El}_x(x + x^2)$ c. $\mathrm{El}_x(x^3 + 1)^{10}$
   d. $\mathrm{El}_x \mathrm{El}_x 5x^2$ e. $\mathrm{El}_x(1 + x^2)$ f. $\mathrm{El}_x \left(\dfrac{x - 1}{x^5 + 1}\right)$
9. Assume that f is a differentiable function with $f(x) \neq 0$. Find expressions for the elasticity of the following:
   a. $x^5 f(x)$ b. $\big(f(x)\big)^{3/2}$ c. $x + \sqrt{f(x)}$ d. $1/f(x)$
10. Find the elasticity of y with respect to x for the following:
    a. $y^6 = x^5$ b. $\dfrac{y}{x} = (x + 1)^a (y - 1)^b$ (a and b are constants)

6

Limits, Continuity, and Series

We could, of course, dismiss the rigorous proof as being superfluous: if a theorem is geometrically obvious why prove it? This was exactly the attitude taken in the eighteenth century. The result, in the nineteenth century, was chaos and confusion: for intuition, unsupported by logic, habitually assumes that everything is much nicer behaved than it really is.
—*I. Stewart* (1975)

This chapter is concerned with limits, continuity, and series—key ideas in mathematics, and also very important in the application of mathematics to economic problems. The preliminary discussion of limits in Section 4.4 was necessarily very sketchy. In this chapter, we take a closer look at this concept and extend it in several directions.

Without limits, the real number system would be seriously incomplete. We would essentially be confined to those numbers that can be calculated precisely in a finite number of steps—for example, integers and rational numbers. In order to assert that the equation $x^2 - 2 = 0$ has a positive solution $x = \sqrt{2}$, and (perhaps more important) to be able to give arbitrarily accurate approximations to $\sqrt{2}$, we really need to be able to define $\sqrt{2}$ as a limit. This is implicitly what we do when we write $\sqrt{2} \approx 1.41421\ldots$. We have in mind an infinite sequence of decimal expansions, starting with 1, 1.4, 1.41, 1.414, ..., which get closer and closer to the *limit* $\sqrt{2}$. In this way, $\sqrt{2}$ is effectively regarded as a limit of a sequence of rational numbers; the same is true for all irrational numbers. Thus, limits arise in the study of infinite series, another topic in this chapter.

Recall too that the derivative of a function, measuring its rate of change, was defined using limits. In fact, in Section 4.2 we defined the derivative of f at a as $f'(a) = \lim_{h\to 0}[f(a+h) - f(a)]/h$. There is in addition a close connection between the limit concept and the idea of *continuity*, which will also be discussed. An optional section giving a precise definition of limits ends this chapter.

6.1 Limits

Section 4.4 gave a preliminary discussion of limits. We now supplement this with some additional concepts and results, still keeping the discussion at an intuitive level. The reason for this gradual approach is that it is important and quite easy to acquire a working knowledge of limits. Experience suggests, however, that the precise definition is rather difficult to understand, as are proofs based on this definition.

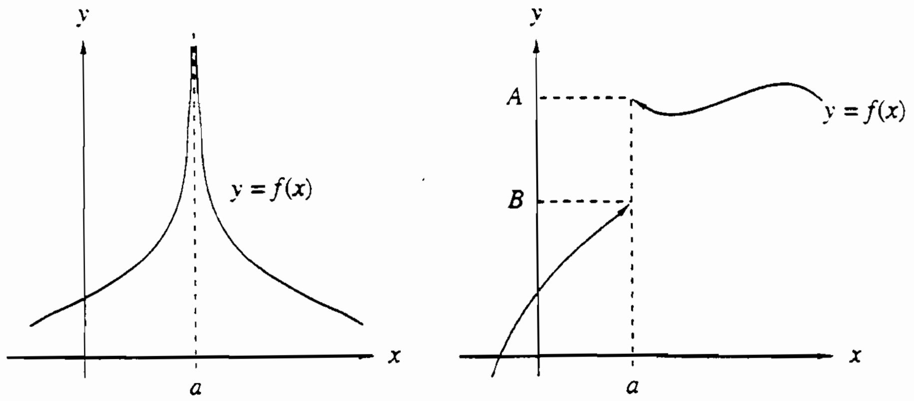
Limits That Do Not Exist: One-Sided Limits

Suppose f is defined for all x close to a, but not necessarily at a. According to [4.14] of Section 4.4, the function $f(x)$ has the number A as its limit as x tends to a, provided that the number $f(x)$ can be made as close to A as one pleases for all x sufficiently close to (but not equal to) a. Then we write

$$\lim_{x\to a} f(x) = A \qquad \text{or} \qquad f(x) \to A \quad \text{as} \quad x \to a$$

In this case, we say that the limit exists. The graphs of Figs. 6.1 and 6.2 show two cases where $f(x)$ *does not* tend to any limit as x tends to a.

FIGURE 6.1 $\lim_{x\to a} f(x) = \infty$.



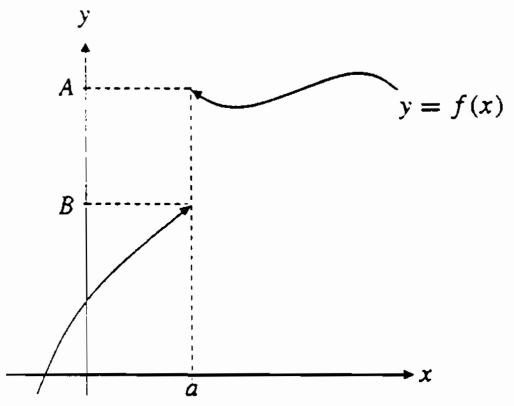


FIGURE 6.2 $\lim_{x \to a} f(x)$ **does not exist.**

Figure 6.1 shows the graph of a function such as $f(x) = 1/|x - a|$ or $f(x) = 1/(x - a)^2$, which increases without bound as x tends to a (either from the right or from the left). We write $f(x) \to \infty$ as $x \to a$, or $\lim_{x \to a} f(x) = \infty$. Because $f(x)$ does not tend to a definite (finite) number as x tends to a, we say that the limit does not exist. (In a sense, there is an infinite limit, but we follow standard mathematical practice in insisting that limits be *finite* numbers.) The straight line $x = a$ is called a **vertical asymptote** for the graph of f.

The function whose graph is shown in Fig. 6.2 also fails to have a limit as x tends to a. However, it seems from the figure that if x tends to a from below, then $f(x)$ tends to the number B. We say, therefore, that the *limit of $f(x)$ as x tends to a from below is B*, and we write

$$\lim_{x \to a^-} f(x) = B \qquad \text{or} \qquad f(x) \to B \quad \text{as} \quad x \to a^-$$

Analogously, also referring to Fig. 6.2, we say that the *limit of $f(x)$ as x tends to a from above is A*, and we write

$$\lim_{x \to a^+} f(x) = A \qquad \text{or} \qquad f(x) \to A \quad \text{as} \quad x \to a^+$$

We call these *one-sided limits*, the first from below and the second from above. They can also be called *left limits* and *right limits*, respectively.

Necessary and sufficient conditions for the (ordinary) limit to exist are that the two one-sided limits of f at a exist and are equal:

$$\lim_{x \to a} f(x) = A \iff \lim_{x \to a^-} f(x) = A \quad \text{and} \quad \lim_{x \to a^+} f(x) = A \qquad [6.1]$$

It should now also be clear what is meant by

$$\lim_{x \to a^-} f(x) = \infty \;\; (\text{or } -\infty) \qquad \text{and} \qquad \lim_{x \to a^+} f(x) = \infty \;\; (\text{or } -\infty)$$

In these cases, despite the notation, we say that the limits do not exist.

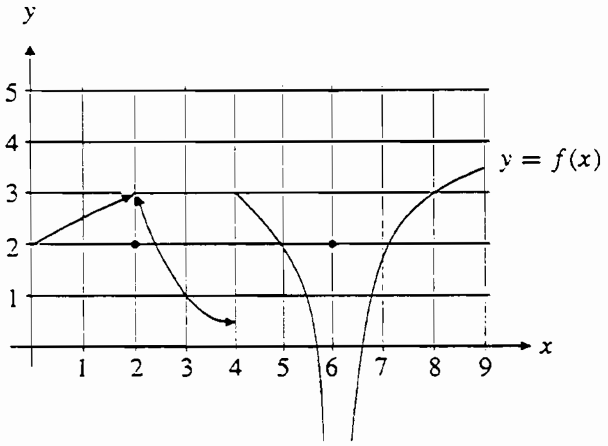


FIGURE 6.3

Example 6.1

Figure 6.3 reproduces Fig. 4.12 of Section 4.4, and shows the graph of a function f defined on $[0, 9]$. Using the figure, verify (roughly) that the following limits are correct:

$$\lim_{x \to 4^-} f(x) = 1/2, \qquad \lim_{x \to 4^+} f(x) = 3, \qquad \lim_{x \to 9^-} f(x) = 3.5$$

Example 6.2

Explain the following limits:

$$\lim_{x \to 0^-} \frac{1}{x} = -\infty, \qquad \lim_{x \to 0^+} \frac{1}{x} = \infty,$$

$$\lim_{x \to 2^-} \frac{1}{\sqrt{2 - x}} = \infty, \qquad \lim_{x \to 2^+} \frac{-1}{\sqrt{x - 2}} = -\infty$$

Solution If x is negative and close to 0, then $1/x$ is a large negative number. For example, $1/(-0.001) = -1000$. In fact, $1/x$ decreases without bound as x tends to zero from below, and it is reasonable to say that $1/x$ tends to minus infinity as x tends to 0 from below.

The second limit is very similar, except that $1/x$ is large and positive when x is positive and close to 0.

If x is slightly smaller than 2, then $2 - x$ is positive, so $\sqrt{2-x}$ is close to 0, and $1/\sqrt{2-x}$ is a large positive number. For example, $1/\sqrt{2 - 1.9999} = 1/\sqrt{0.0001} = 100$. As x tends to 2^-, so $1/\sqrt{2-x}$ tends to ∞.

The fourth limit is similar, because when x is slightly larger than 2, then $\sqrt{x-2}$ is positive and close to 0, so $-1/\sqrt{x-2}$ is a large negative number.

Limits at Infinity

We can also use the language of limits to describe the behavior of a function as its argument becomes infinitely large through positive or negative values. Let f be defined for arbitrarily large positive numbers x. We say that $f(x)$ *has the limit A as x tends to infinity* if $f(x)$ can be made arbitrary close to A for all x sufficiently large. We write

$$\lim_{x\to\infty} f(x) = A \quad \text{or} \quad f(x) \to A \quad \text{as} \quad x \to \infty$$

In the same way,

$$\lim_{x\to-\infty} f(x) = B \quad \text{or} \quad f(x) \to B \quad \text{as} \quad x \to -\infty$$

indicates that $f(x)$ can be made arbitrary close to B for all x sufficiently large and negative. The two limits are illustrated in Fig. 6.4. The horizontal line $y = A$ is a (horizontal) **asymptote** for the graph of f as x tends to ∞, whereas $y = B$ is a (horizontal) asymptote for the graph as x tends to $-\infty$.

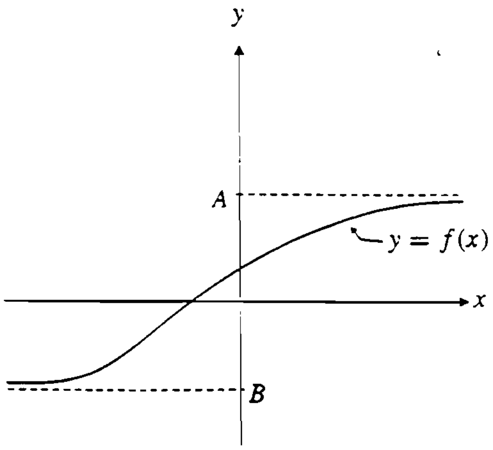
Example 6.3

Examine the behavior of the following functions, both as $x \to \infty$ and as $x \to -\infty$:

(a) $f(x) = \dfrac{3x^2 + x - 1}{x^2 + 1}$

(b) $g(x) = \dfrac{1 - x^5}{x^4 + x + 1}$

FIGURE 6.4 ***y* = *A* and *y* = *B* are horizontal asymptotes.**



Solution

(a) A rough argument is as follows: If x is a large negative or a large positive number, then the term $3x^2$ "dominates" in the numerator, whereas x^2 dominates in the denominator. Thus, if $|x|$ is a large number, $f(x)$ behaves like the fraction $3x^2/x^2 = 3$. We conclude that $f(x)$ tends to 3 as $|x|$ tends to ∞.

More formally we argue as follows. First, divide each term in the numerator and the denominator by the highest power of x, which is x^2, to obtain

$$f(x) = \frac{3x^2 + x - 1}{x^2 + 1} = \frac{3 + (1/x) - (1/x^2)}{1 + (1/x^2)}$$

If x is large in absolute value, then both $1/x$ and $1/x^2$ will be close to 0. Thus, $f(x)$ is arbitrarily close to 3 if $|x|$ is sufficiently large, and

$$\lim_{x \to \infty} f(x) = \lim_{x \to -\infty} f(x) = 3$$

(b) A first rough argument is that if $|x|$ is a large number, then $g(x)$ behaves like the fraction $-x^5/x^4 = -x$. Therefore, $g(x) \to -\infty$ as $x \to \infty$, whereas $g(x) \to \infty$ as $x \to -\infty$. Alternatively,

$$g(x) = \frac{1 - x^5}{x^4 + x + 1} = \frac{(1/x^4) - x}{1 + (1/x^3) + 1/x^4}$$

You should now finish the argument yourself along the lines given in part (a).

Warnings

We have extended the original definition of a limit in several different directions. For these extended limit concepts, the previous limit rules set out in Section 4.4 still apply. For example, all the results in [4.15] on sums, products, and ratios of limits as $x \to a$ are valid if we consider only left limits with $x \to a^-$, or only right limits with $x \to a^+$. Also, if we replace $x \to a$ by $x \to \infty$ or $x \to -\infty$ in [4.15], then again the corresponding limit properties are valid.

When $f(x)$ and $g(x)$ both tend to ∞ as x tends to a (possibly with $x \to a$ replaced by $x \to a^-$ or $x \to a^+$), we must be much more careful. Because $f(x)$ and $g(x)$ each can be made arbitrarily large if x is sufficiently close to a, both $f(x) + g(x)$ and $f(x) \cdot g(x)$ also can be made arbitrarily large. But, in general, we cannot say what are the limits of $f(x) - g(x)$ and $f(x)/g(x)$. The limits of these expressions will depend on how "fast" $f(x)$ and $g(x)$, respectively, tend to

∞ as x tends to a. Briefly formulated:

$$\lim_{x\to a} f(x) = \infty \quad \text{and} \quad \lim_{x\to a} g(x) = \infty \implies \begin{cases} \lim_{x\to a} [f(x) + g(x)]) = \infty \\ \lim_{x\to a} [f(x) \cdot g(x)] = \infty \\ \lim_{x\to a} [f(x) - g(x)] = ? \\ \lim_{x\to a} [f(x)/g(x)] = ? \end{cases}$$

The two question marks mean that we cannot determine the limits of $f(x) - g(x)$ and $f(x)/g(x)$ without having more information about f and g. We do not even know if these limits exist or not. The following example illustrates some of the possibilities.

Example 6.4

Let $f(x) = 1/x^2$ and $g(x) = 1/x^4$. As $x \to 0$, then $f(x) \to \infty$ and $g(x) \to \infty$. Determine the limits as $x \to 0$ of the following:

(a) $f(x) - g(x)$
(b) $g(x) - f(x)$
(c) $f(x)/g(x)$
(d) $g(x)/f(x)$

Solution

(a) $$f(x) - g(x) = \frac{x^2 - 1}{x^4} \to -\infty \quad \text{as} \quad x \to 0$$

(b) $$g(x) - f(x) = \frac{1 - x^2}{x^4} \to \infty \quad \text{as} \quad x \to 0$$

(c) $$f(x)/g(x) = x^2 \to 0 \quad \text{as} \quad x \to 0$$

(d) $$g(x)/f(x) = 1/x^2 \to \infty \quad \text{as} \quad x \to 0$$

These examples serve to illustrate that infinite limits require extreme care. Let us consider some other tricky examples.

Suppose we study the product $f(x) \cdot g(x)$ of two functions, where $g(x)$ tends to 0 as x tends to a. Will $f(x) \cdot g(x)$ also tend to 0 as x tends to a? Not necessarily. If $f(x)$ tends to a limit A, then by rule [4.15](iii) of Sec. 4.4, $f(x) \cdot g(x)$ tends to $A \cdot 0 = 0$. On the other hand, if $f(x)$ tends to $\pm\infty$, then it is easy to construct examples in which the product $f(x) \cdot g(x)$ does not tend to 0 at all. (You should try to construct some examples of your own before turning to Problem 4.)

The rules for limits in [4.15] are fundamental. However, one must be careful not to read more into them than what they actually say. If $f(x)$ tends to the number A and $g(x)$ tends to the number B as x tends to a, then by [4.15](a) we see that $f(x) + g(x)$ tends to $A + B$ as x tends to a. But the sum $f(x) + g(x)$ might very well tend to a limit even though $f(x)$ and $g(x)$ do not tend to a limit. The same goes for the fraction $f(x)/g(x)$.

Example 6.5

Let $f(x) = 3 + 1/x$ and $g(x) = 5 - 1/x$. Examine the limits as $x \to 0$ of **(a)** $f(x) + g(x)$ and **(b)** $f(x)/g(x)$.

Solution We find that

$$\text{(a)} \qquad f(x) + g(x) = 8 \to 8 \quad \text{as} \quad x \to 0$$

$$\text{(b)} \qquad \frac{f(x)}{g(x)} = \frac{3x+1}{5x-1} \to -1 \quad \text{as} \quad x \to 0$$

But in this case neither $f(x)$ nor $g(x)$ tends to a limit as x tends to 0. In fact, $f(x) \to \infty$ and $g(x) \to -\infty$ as $x \to 0^+$, whereas $f(x) \to -\infty$ and $g(x) \to \infty$ as $x \to 0^-$.

Problems

1. Evaluate the following limits:

   a. $\lim_{x\to 0^+} (x^2 + 3x - 4)$ b. $\lim_{x\to 0^-} \dfrac{x + |x|}{x}$ c. $\lim_{x\to 0^-} \dfrac{x + |x|}{x}$

   d. $\lim_{x\to 0^+} \dfrac{-1}{\sqrt{x}}$ e. $\lim_{x\to 3^+} \dfrac{x}{x-3}$ f. $\lim_{x\to 3^-} \dfrac{x}{x-3}$

2. Evaluate the following limits:

   a. $\lim_{x\to\infty} \dfrac{x-3}{x^2+1}$ b. $\lim_{x\to -\infty} \sqrt{\dfrac{2+3x}{x-1}}$ c. $\lim_{x\to\infty} \dfrac{(ax-b)^2}{(a-x)(b-x)}$

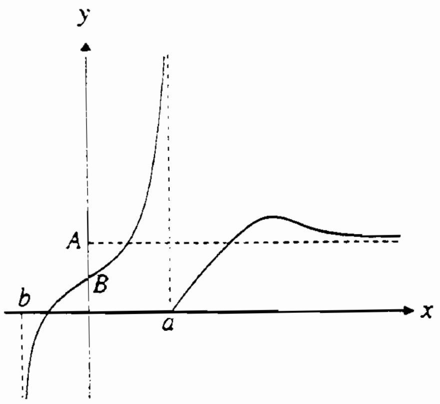
3. A function f defined for $x > b$ has a graph indicated by Fig. 6.5.

   a. Determine likely values of the following limits: (i) $\lim_{x\to b^-} f(x)$, (ii) $\lim_{x\to a^-} f(x)$, (iii) $\lim_{x\to a^+} f(x)$, (iv) $\lim_{x\to\infty} f(x)$.

   b. Only one of the following limits is defined. Which one?

$$\lim_{x\to -\infty} f(x), \quad \lim_{x\to 0} f(x), \quad \lim_{x\to b^-} f(x)$$

FIGURE 6.5



4. Let $f_1(x) = x$, $f_2(x) = x$, $f_3(x) = x^2$, and $f_4(x) = 1/x$. For $i = 1, 2, 3, 4$, determine $\lim_{x\to\infty} f_i(x)$. Then examine the limits of the following functions as $x \to \infty$:

   a. $f_1(x) + f_2(x)$ b. $f_1(x) - f_2(x)$ c. $f_1(x) - f_3(x)$
   d. $f_1(x)/f_2(x)$ e. $f_1(x)/f_3(x)$ f. $f_1(x) \cdot f_2(x)$
   g. $f_1(x) \cdot f_4(x)$ h. $f_3(x) \cdot f_4(x)$

5. The nonvertical line $y = ax + b$ is said to be an **asymptote** as $x \to \infty$ (or $x \to -\infty$) to the curve $y = f(x)$ if

$$f(x) - (ax + b) \to 0 \quad \text{as} \quad x \to \infty \ (\text{or } x \to -\infty)$$

   This condition means that the vertical distance between point $(x, f(x))$ on the curve and point $(x, ax + b)$ on the line tends to 0 as $x \to \pm\infty$. (See Figure 6.6.)

   If $f(x) = P(x)/Q(x)$ is a rational function where the degree of the polynomial $P(x)$ is *one greater* than that of the polynomial $Q(x)$, then $f(x)$ will have an asymptote that can be found by performing the long division $P(x) \div Q(x)$ and ignoring the remainder. Use this method to find asymptotes for the graph of each of the functions defined by the following formulas:

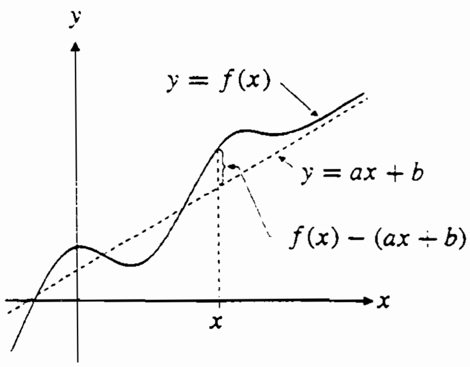
   a. $\dfrac{x^2}{x+1}$ b. $\dfrac{2x^3 - 3x^2 + 3x - 6}{x^2 + 1}$ c. $\dfrac{3x^2 + 2x}{x - 1}$ d. $\dfrac{5x^4 - 3x^2 + 1}{x^3 - 1}$

6. Consider the following cost function defined for $x \geq 0$ by

$$C(x) = A\frac{x(x + b)}{x + c} + d$$

   Here A, b, c, and d are positive constants. Find the asymptotes.

FIGURE 6.6



6.2 Continuity

The word *continuous* is rather common in everyday language. We use it, in particular, to characterize changes that are gradual rather than sudden. This usage is closely related to the idea of a continuous function. Roughly speaking, a function is continuous if small changes in the independent variable produce small changes in the function values. Geometrically, *a function is continuous if its graph is connected—that is, it has no breaks*. An example is indicated in Fig. 6.7.

It is often said that a function is continuous if its graph can be drawn without lifting one's pencil off the paper. On the other hand, if the graph makes one or more jumps, we say that f is *discontinuous*. Thus, the function whose graph is shown in Fig. 6.8 is discontinuous at $x = a$, but continuous at all other points of the interval that constitutes its domain.

Why are we interested in distinguishing between continuous and discontinuous functions? One important reason is that we must usually work with numerical approximations. For instance, if a function f is given by some formula and we wish to compute $f(\sqrt{2})$, we usually take it for granted that we can compute $f(1.4142)$

FIGURE 6.7 A continuous function.

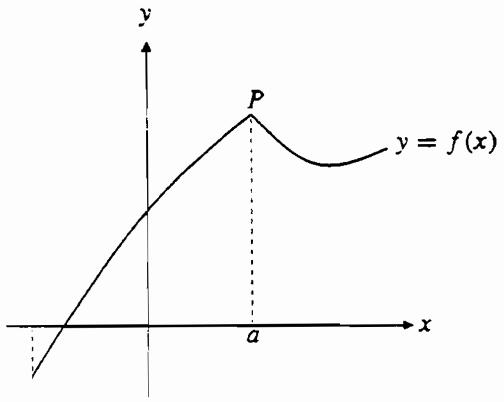
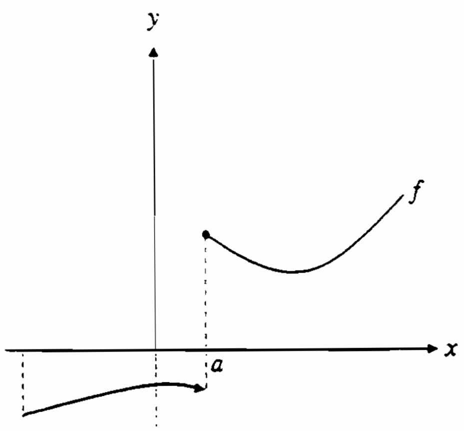


FIGURE 6.8 A discontinuous function.



and obtain a good approximation to $f(\sqrt{2})$. In fact, this implicitly assumes that f is continuous. Then, because 1.4142 is close to $\sqrt{2}$, the value $f(1.4142)$ must be close to $f(\sqrt{2})$.

In applications of mathematics to natural sciences and economics, a function will often represent the change in some phenomenon over time. The continuity of the function will then reflect the continuity of the phenomenon, in the sense of a gradual development without sudden changes. We might, for example, think of a person's body temperature as a function of time. Here we may assume that it changes continuously and that it does not jump from one value to another without passing through the intermediate values. On the other hand, if we consider the price of a barrel of oil in a certain market, this function of time will be discontinuous. One reason is that the price (measured in dollars or some other currency) must always be a rational number. A second, more interesting, reason for occasional large jumps in the price is the sudden arrival of news or a rumor that significantly affects either the demand or supply function.

The concept of continuity just discussed must obviously be made more precise before we can operate with it as a mathematical concept. We must search for a definition of continuity not solely based on intuitive geometric ideas.

Continuous Functions

We suggested earlier that a function is continuous if its graph is a "connected" curve. In particular, we say that f is continuous at a point a if the graph of f has no break at a. How do we define this precisely? It is evident that we must consider the value of f at points x close to a. If the graph of f has no break at a, then $f(x)$ cannot differ much from $f(a)$ when x is close to a. Stated differently, if x is close to a, then $f(x)$ must be close to $f(a)$. This motivates the following definition:

Suppose f is defined on a domain that includes an open interval around a. Then f is **continuous** at $x = a$ provided that $f(x)$ tends to $f(a)$ in the limit as x tends to a:

$$f \text{ is } \textbf{continuous} \text{ at } x = a \text{ if } \lim_{x \to a} f(x) = f(a) \qquad [6.2]$$

Hence, we see that in order for f to be continuous at $x = a$, the following three conditions must all be fulfilled:

1. the function f must be defined at $x = a$
2. the limit of $f(x)$ as x tends to a must exist
3. this limit must be exactly equal to $f(a)$

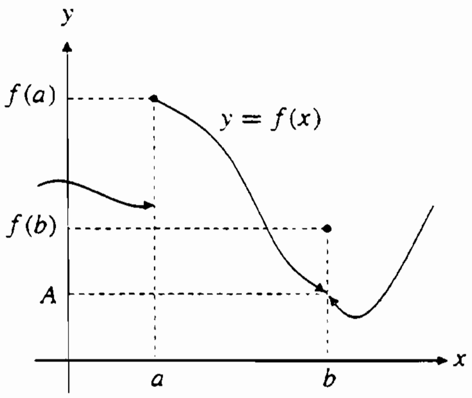


FIGURE 6.9 ***f* has two points of discontinuity. *x* = *a* is an irremovable discontinuity and *x* = *b* is a removable discontinuity.**

Unless all three of these conditions are satisfied, we say that f is **discontinuous** at a. Figure 6.9 indicates two important types of discontinuity that can occur. At $x = a$, the function is discontinuous because $f(x)$ clearly has no limit as x tends to a. Hence, condition 2 is not satisfied. This is an "irremovable" discontinuity. On the other hand, the limit of $f(x)$ as x tends to b exists and is equal to A. But because $A \neq f(b)$, condition 3 is not satisfied, so f is discontinuous at b. This is a "removable" discontinuity that would disappear if $f(b)$ were redefined as A.

Example 6.6

Let $f(x) = 3x - 2$. In Example 4.7(a) of Sec. 4.4, we argued that $f(x)$ tends to 7 as x tends to 3. Because $f(3) = 7$, this means that f is continuous at $a = 3$. Actually, this function is continuous at all points a, because $f(x) = 3x - 2$ always tends to $3a - 2 = f(a)$ as x tends to a.

Properties of Continuous Functions

Many of the central results of mathematical analysis are true only for continuous functions. It is therefore important to be able to decide whether or not a given function is continuous. The rules for limits given in Section 4.4 make it is easy to prove continuity of many types of functions. Note that because of [4.18] and [4.19],

$$f(x) = c \quad \text{and} \quad f(x) = x \text{ are continuous everywhere} \qquad [6.3]$$

This is as it should be, because the graphs of these functions are straight lines. Now, using definition [6.2] and the limit rules in [4.15], we have the following:

Results on Continuous Functions

If f and g are continuous at a, then

(a) $f+g$ and $f-g$ are continuous at a [6.4]
(b) $f \cdot g$ is continuous at a
(c) f/g is continuous at a if $g(a) \neq 0$
(d) $[f(x)]^{p/q}$ is continuous at a if $[f(a)]^{p/q}$ is defined

The proofs of these properties are straightforward if we use the limit laws from Section 4.4. For instance, to prove (b), if both f and g are continuous at a, then $\lim_{x\to a} f(x) = f(a)$ and $\lim_{x\to a} g(x) = g(a)$. According to [4.15](iii), therefore, $\lim_{x\to a} f(x)g(x) = f(a)g(a)$, which means that $f \cdot g$ is continuous at a.

By combining [6.3] and [6.4], it follows that, say, $h(x) = x + 8$ and $k(x) = 3x^3 + x + 8$ are continuous. In general, because a polynomial $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ is a sum of continuous functions, it is continuous everywhere. Moreover, a rational function

$$R(x) = \frac{P(x)}{Q(x)} \qquad (P(x) \text{ and } Q(x) \text{ are polynomials})$$

is continuous at all x where $Q(x) \neq 0$.

Consider a composite function $f(g(x))$ where f and g are assumed to be continuous. If x is close to a, then by the continuity of g at a, $g(x)$ is close to $g(a)$. In turn, $f(g(x))$ becomes close to $f(g(a))$ because f is continuous at $g(a)$, and thus $f(g(x))$ is continuous at a. A more formal proof of this result requires the $\varepsilon\delta$-definition of limits (see Section 6.7). For future reference:

Composites of continuous functions are continuous:

If g is continuous at $x = a$, and f is continuous at $g(a)$, then $f\big(g(x)\big)$ is continuous at $x = a$. [6.5]

By using the results just discussed, a mere glance at the formula defining a function will usually suffice to determine the points at which it is continuous.

In general:

> Any function that can be constructed from continuous functions by combining one or more operations of addition, subtraction, multiplication, division (except by zero, of course), and composition is continuous at all points where it is defined. [6.6]

Example 6.7

Determine at which values of x the functions given by the following formulas are continuous:

(a) $$f(x) = \frac{x^4 + 3x^2 - 1}{(x-1)(x+2)}$$

(b) $$g(x) = (x^2 + 2)(x^3 + 1/x)^4 + 1/\sqrt{x+1}$$

Solution

(a) This is a rational function that is continuous at all x, except where the denominator $(x-1)(x+2) = 0$. Hence, f is continuous at all x different from 1 and -2.

(b) This function is defined when $x \neq 0$ and $x + 1 > 0$, or when $x \neq 0$ and $x > -1$. Hence, g is continuous in the domain $(-1, 0) \cup (0, \infty)$.

Knowing where a function is continuous simplifies the computation of many limits. For instance, using the rules for limits, Example 4.9(a) showed the result that $\lim_{x\to -2}(x^2 + 5x) = -6$. Because $f(x) = x^2 + 5x$ is a continuous function of x, we know that $\lim_{x\to -2}(x^2+5x)$ is simply $f(-2) = (-2)^2+5(-2) = 4-10 = -6$. Thus, we find the limit by just evaluating $f(x) = x^2 + 5x$ at $x = -2$.

Functions that are defined "piecewise" by different formulas applying to different intervals are frequently discontinuous at the junction points. For example, the amount of postage you pay for a letter is a discontinuous function of the weight. (As long as we use preprinted stamps, it would be extremely inconvenient to have the "postage function" be even approximately continuous.) On the other hand, the tax you pay as a function of your net income is (essentially) a continuous function (although many people seem to believe that it is not). An actual tax function for the U.S. is shown in Fig. 6.13 at the end of Sec. 6.3.

Example 6.8

For what values of a is the following function continuous everywhere?

$$f(x) = \begin{cases} ax^2 + 4x - 1, & \text{if } x \leq 1 \\ -x + 3, & \text{if } x > 1 \end{cases}$$

Solution The function is obviously continuous at all $x \neq 1$. For $x = 1$, the function is given by the upper formula, so $f(1) = a + 3$. If x is slightly larger than 1, then $f(x) = -x + 3$ is close to 2, and $f(x) \to 2$ as $x \to 1^+$. In order to have f continuous at $x = 1$, we must have $f(1) = a + 3 = 2$, which requires $a = -1$. Thus, for $a = -1$ the function is continuous at all x, including at $x = 1$. If $a \neq -1$, the function is discontinuous at $x = 1$, but continuous at all other points. (Draw the graph of f for $a = 1$ and for $a = -1$.)

One-Sided Continuity

Section 6.1 introduced one-sided limits. These allow us to define one-sided continuity. Suppose f is defined on a domain including the half-open interval $(c, a]$. If $f(x)$ tends to $f(a)$ as x tends to a^-, we say that f is **left-continuous** at a. Similarly, if f is defined on a domain including $[a, d)$, we say that f is **right-continuous** at a if $f(x)$ tends to $f(a)$ as x tends to a^+. For example, the function f indicated earlier in Fig. 6.8 is right-continuous at a. Although f tends to a limit as x tends to a from the left, f is not left-continuous at a, because the limit is different from $f(a)$.

Making use of [6.1] in Section 6.1, we readily see that a function f is continuous at a if and only if f is both left- and right-continuous at a.

If a function f is defined on a closed, bounded interval $[a, b]$, we usually say that f is continuous in $[a, b]$ if it is continuous at each point of (a, b), and is in addition right-continuous at a and left-continuous at b. It should be obvious how to define continuity on half-open intervals. The continuity of a function at all points of an interval, including any endpoints it contains, is often a minimum requirement we impose when speaking about "well-behaved" functions.

Problems

1. Which of the following functions are likely to be continuous functions of time?
   a. The price in the Zürich gold market of an ounce of gold.
   b. The height of a growing child.
   c. The height of an aeroplane above the ground.
   d. The distance traveled by a car.
2. Consider the functions defined by the six graphs shown in Fig. 6.10.
   a. Are any of these functions continuous at a?
   b. For which of the functions will $f(x)$ tend to a limit as x tends to a?
   c. Determine the limits of $f(x)$ as $x \to a^-$ and $x \to a^+$ in each case.
   d. Which of the functions are left-continuous at a, and which of them are right-continuous at a?
   e. What seems to be the limit of $f(x)$ as $x \to \infty$ in graphs (v) and (vi) of the figure?

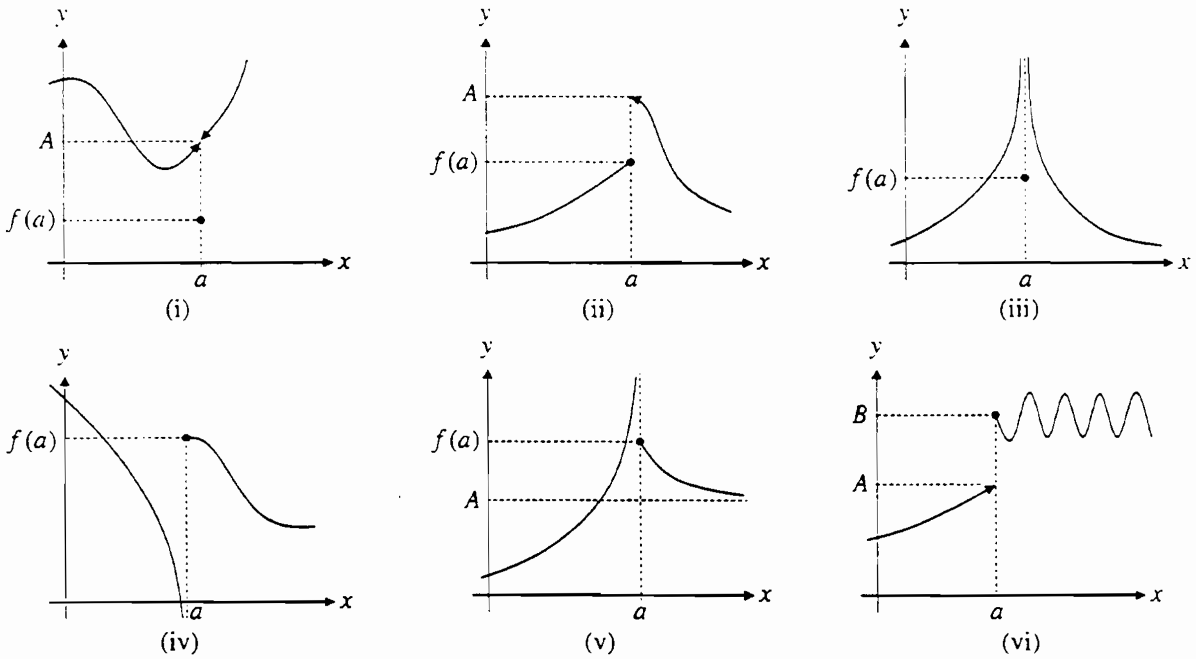


FIGURE 6.10

3. Let f and g be defined for all x by

$$f(x) = \begin{cases} x^2 - 1, & \text{for } x \leq 0 \\ -x^2, & \text{for } x > 0 \end{cases} \quad \text{and } g(x) = \begin{cases} 3x - 2, & \text{for } x \leq 2 \\ -x + 6, & \text{for } x > 2 \end{cases}$$

Draw a graph of each function. Is f continuous at $x = 0$? Is g continuous at $x = 2$?

4. Determine the values of x at which each of the following functions is continuous:

a. $f(x) = x^5 + 4x$ **b.** $f(x) = \dfrac{x}{1 - x}$ **c.** $f(x) = \dfrac{1}{\sqrt{2 - x}}$

d. $\dfrac{x}{x^2 + 1}$ **e.** $\dfrac{x^8 - 3x^2 + 1}{x^2 + 2x - 2}$ **f.** $\left(\dfrac{x + 1}{x - 1}\right)^{1/2}$

g. $\dfrac{\sqrt{x} + 1/x}{x^2 + 2x + 2}$ **h.** $|x| + \dfrac{1}{|x|}$ **i.** $\dfrac{1}{\sqrt{x}} + x^7(x + 2)^{-3/2}$

5. For what value of a is the following function continuous for all x?

$$f(x) = \begin{cases} ax - 1, & \text{for } x \leq 1 \\ 3x^2 + 1, & \text{for } x > 1 \end{cases}$$

6. Draw the graph of y as a function of x if y depends on x as indicated in Fig. 6.11—that is, y is the height of the aeroplane above the point on the ground vertically below. Is y a continuous function of x? Suppose $d(x)$ is the distance from the aeroplane to the *nearest* point on the ground. Is d a continuous function of x?

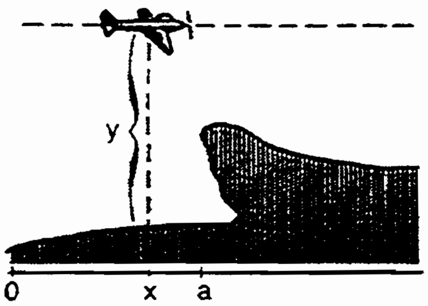


FIGURE 6.11

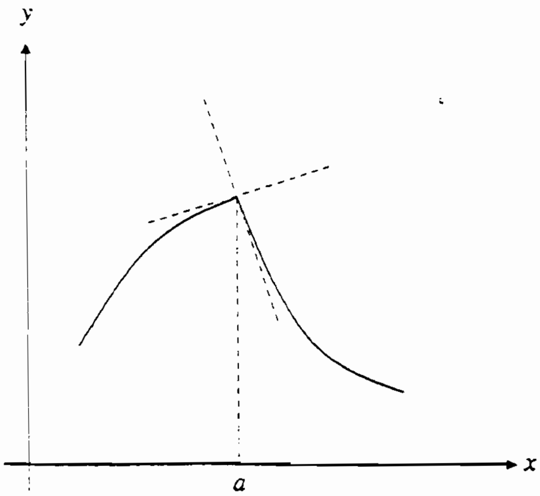
7. Functions f and g are discontinuous at $x = a$. Are $f+g$ and $f \cdot g$ necessarily discontinuous at a? If not, supply examples.
8. Let f be defined by $f(x) = x^2 - 2$ for $x < 0$, and $f(x) = -3x^2 + 15$ for $x > 2$. Can you define $f(x)$ as a linear function on $[0, 2]$ so that f is continuous for all x?

6.3 Continuity and Differentiability

Consider the function f graphed in Fig. 6.12. At point $(a, f(a))$, the graph does not have (a unique) tangent. Thus f has no derivative at $x = a$, but f is continuous at $x = a$. So a function can be continuous at a point without being differentiable at that point. (For a standard example, see Problem 2.) On the other hand, it is easy to see that differentiability implies continuity:

> If f is differentiable at $x = a$, then f is continuous at $x = a$. [6.7]

FIGURE 6.12 ***f* is continuous, but not differentiable at *x* = *a*.**



Proof Function f is continuous at $x = a$ provided that $f(a+h) - f(a)$ tends to 0 as $h \to 0$. Now, for $h \neq 0$,

$$f(a+h) - f(a) = \frac{f(a+h) - f(a)}{h} \cdot h \qquad [*]$$

If f is differentiable at $x = a$, the Newton quotient $\left[f(a+h) - f(a)\right]/h$ tends to the number $f'(a)$ as $h \to 0$. So the right-hand side of [*] tends to $f'(a) \cdot 0 = 0$ as $h \to 0$. Thus, f is continuous at $x = a$.

Suppose that f is some function whose Newton quotient $\left[f(a+h) - f(a)\right]/h$ tends to a limit as h tends to 0 through positive values. Then the limit is called the **right derivative** of f at a, and we use the notation

$$f'(a^+) = \lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h} \qquad [6.8]$$

The **left derivative** of f at a is defined similarly:

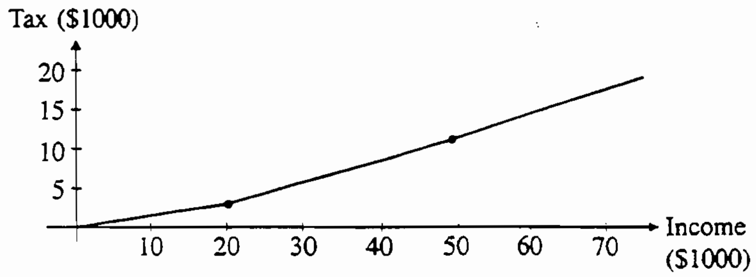
$$f'(a^-) = \lim_{h \to 0^-} \frac{f(a+h) - f(a)}{h} \qquad [6.9]$$

if the one-sided limit exists.

If f is continuous at a, and if $f'(a^+) = \alpha$ and $f'(a^-) = \beta$ with $\alpha \neq \beta$, then we say that the graph of f has a **corner** (or **kink**) at $(a, f(a))$. Then f is not differentiable at a. Thus, the function in Fig. 6.12 has a corner at $(a, f(a))$. If f is continuous at a and $\alpha = \beta$, then the corner gets smoothed out and f is seen to be differentiable at a.

Example 6.9 (U.S. Federal Income Taxes (1991) for single persons)
This income tax function was discussed in Example 2.10 of Section 2.4. Figure 6.13 reproduces Fig. 2.21.[1] If $t(x)$ denotes the tax paid at income x,

FIGURE 6.13 U.S. Federal income taxes (1991) (for single persons).



[1]Of course, Fig. 6.13 is an idealization. The true income tax function is defined only for integral numbers of dollars—or, more precisely, it is a discontinuous "step function" which jumps up slightly whenever income rises by another dollar.

its graph has corners at $x = 20,250$ and at $x = 49,300$. We see, for instance, that $t'(20,250^-) = 0.15$ because on the last dollar you earn before reaching \$20,250, you pay 15 cents. Also $t'(20,250^+) = 0.28$ because on the first dollar you earn above \$20,250, you pay 28 cents in tax. Because $t'(20,250^-) \neq t'(20,250^+)$, the tax function t is not differentiable at $x = 20,250$. Check that $t'(49,300^+) = 0.31$.

Problems

1. Graph the function f defined by $f(x) = 0$ for $x \leq 0$, and $f(x) = x$ for $x > 0$. Compute $f'(0^+)$ and $f'(0^-)$.
2. Function f is defined for all x by $f(x) = |x|$. Compute $f'(0^+)$ and $f'(0^-)$. Is f continuous and/or differentiable at $x = 0$? (The graph is shown in Fig. 9.31 of Section 9.6.)
3. The graph of a continuous function f is said to have a **cusp** at a if $f'(x) \to \infty$ as x tends to a from one side, whereas $f'(x) \to -\infty$ as x tends to a from the other side. Show that $f(x) = |\sqrt[3]{x}|$ has a cusp at $x = 0$, and draw its graph.
4. Give an algebraic definition of the tax function $t(x)$ in Example 6.9. (The function is called piecewise linear, since it is linear on each of the different income intervals.) Compute $t(22,000)$ and $t(50,000)$.

6.4 Infinite Sequences

Consider the function f defined for $n = 1, 2, 3, \ldots$ by the formula $f(n) = 1/n$. Then $f(1) = 1$, $f(2) = 1/2$, $f(3) = 1/3$, and so on. The list of numbers

$$1, \ \frac{1}{2}, \ \frac{1}{3}, \ \frac{1}{4}, \ \ldots, \ \frac{1}{n}, \ \ldots \qquad [*]$$

is called an **infinite sequence.** Its first *term* is 1, and its general (nth) term is $1/n$. In general, any function whose domain is the entire set of positive integers is called an **infinite sequence**. Similarly $s_n = 100 \cdot 1.08^{n-1} (n = 1, 2, \ldots)$ determines an infinite sequence whose first terms are

$$100, \ 100 \cdot 1.08, \ 100 \cdot 1.08^2, \ 100 \cdot 1.08^3, \ \ldots \qquad [**]$$

If s is an infinite sequence, its terms $s(1), s(2), s(3), \ldots, s(n), \ldots$ are usually denoted by using subscripts: $s_1, s_2, s_3, \ldots, s_n, \ldots$. We use the notation $\{s_n\}_{n=1}^{\infty}$, or simply $\{s_n\}$, for an arbitrary infinite sequence.

Consider the previous sequence [*]. If we choose n large enough, the terms can be made as small as we like. We say that the sequence *converges* to 0. In general, we introduce the following definition:

A sequence $\{s_n\}$ is said to **converge** *to a number s if s_n is arbitrarily close to s for all n sufficiently large.* We write

$$\lim_{n \to \infty} s_n = s \qquad \text{or} \qquad s_n \to s \quad \text{as} \quad n \to \infty$$

A sequence that does not converge to any real number is said to **diverge**. For example, the sequence in [**] earlier diverges because $100 \cdot 1.08^{n-1}$ tends to ∞ as n tends to ∞.

The definition of convergence for a sequence is a special case of the previous definition that $f(x) \to A$ as $x \to \infty$. All the ordinary limit rules in Section 4.4 apply to limits of sequences.

Example 6.10

Write down the first five terms of the following sequences:

(a) $\left\{(-1)^{n-1}\frac{1}{n}\right\}$

(b) $\left\{3 + \left(\frac{1}{10}\right)^n\right\}$

(c) $\left\{\frac{n^3+1}{n^2+2}\right\}$

Then decide whether or not each converges.

Solution

(a) $\left\{(-1)^{n-1}\frac{1}{n}\right\}: \quad 1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, \ldots$

(b) $\left\{3 + \left(\frac{1}{10}\right)^n\right\}: \quad 3.1,\ 3.01,\ 3.001,\ 3.0001,\ 3.00001, \ldots$

(c) $\left\{\frac{n^3+1}{n^2+2}\right\}: \quad \frac{2}{3}, \frac{9}{6}, \frac{28}{11}, \frac{65}{18}, \frac{126}{27}, \ldots$

The sequence in (a) converges to 0, because $1/n$ tends to 0 as n tends to ∞. The sequence in (b) converges to 3, because $(1/10)^n$ tends to 0 as n tends to ∞. The sequence in (c) is divergent. To see this note that

$$s_n = \frac{n^3+1}{n^2+2} = \frac{n+1/n^2}{1+2/n^2}$$

Clearly, $s_n \to \infty$ as $n \to \infty$, so $\{s_n\}$ diverges.

Example 6.11

For $n \geq 3$ let A_n be the area of a regular n-polygon inscribed in a circle with radius 1. For $n = 3$, A_3 is the area of a triangle; for $n = 4$, A_4 is the area of a square; for $n = 5$, A_5 is the area of a pentagon; and so on (see Fig. 6.14).

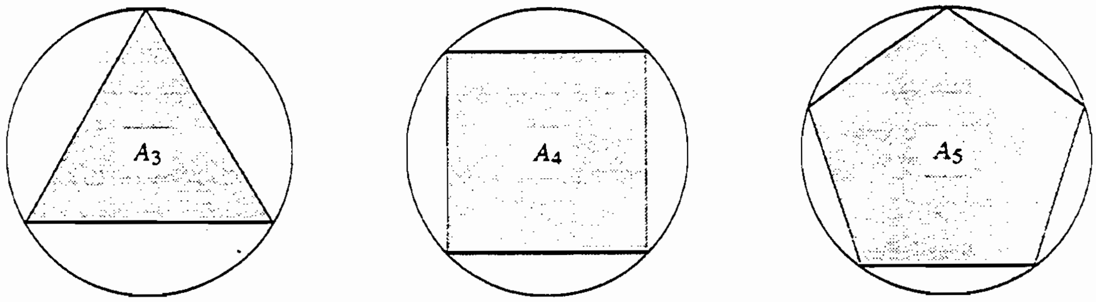


FIGURE 6.14

The larger n is, the larger is A_n, but each A_n is less than π, the area of a circle with radius 1. It seems intuitively evident that we can make the difference between A_n and π as small we wish if only n becomes sufficiently large, so that

$$A_n \to \pi \qquad \text{as} \qquad n \to \infty$$

In this example, A_1 and A_2 have no meaning, so the sequence starts with A_3.

The sequence $\{A_n\}$ in the previous example converges to the irrational number $\pi = 3.14159265\ldots$. Another sequence that converges to π starts this way: $s_1 = 3.1$, $s_2 = 3.14$, $s_3 = 3.141$, $s_4 = 3.1415$, etc. Each new number is obtained by including an additional digit in the decimal expansion for π. For this sequence, $s_n \to \pi$ as $n \to \infty$.

Consider an arbitrary irrational number r. Just as for π, the decimal expansion of r will define one particular sequence r_n of rational numbers that converges to r. Actually, each irrational number is the limit of infinitely many different sequences of rational numbers.

Example 6.12

It is often difficult to determine whether or not a sequence is convergent. For example, consider the sequence whose general term is $s_n = (1 + 1/n)^n$. Do you think that this converges? The values of s_n for some values of n are given by

n	1	2	3	5	10	100	10,000	100,000
$\left(1+\frac{1}{n}\right)^n$	2	2.25	2.37	2.49	2.59	2.70	2.7181	2.7182

This table seems to suggest that s_n tends to a number close to 2.718. One can prove that $\{s_n\}$ *does* converge by relying on the general property. Any increasing sequence of real numbers that has an upper bound is convergent. The limit of $\{s_n\}$ is an irrational number denoted by e, which is one of the most important constants in mathematics. See Section 8.1.

Problems

1. Let

$$\alpha_n = \frac{3-n}{2n-1} \quad \text{and} \quad \beta_n = \frac{n^2+2n-1}{3n^2-2} \qquad (n = 1, 2, \ldots)$$

Find the following limits:

a. $\lim_{n\to\infty} \alpha_n$ **b.** $\lim_{n\to\infty} \beta_n$ **c.** $\lim_{n\to\infty} (3\alpha_n + 4\beta_n)$

d. $\lim_{n\to\infty} \alpha_n \beta_n$ **e.** $\lim_{n\to\infty} \alpha_n/\beta_n$ **f.** $\lim_{n\to\infty} \sqrt{\beta_n - \alpha_n}$

2. Examine the convergence of the sequences whose general terms are as follows:

a. $s_n = 5 - \dfrac{2}{n}$ **b.** $s_n = \dfrac{n^2-1}{n}$ **c.** $s_n = \dfrac{3n}{\sqrt{2n^2-1}}$

6.5 Series

This section primarily studies finite and infinite geometric series. These have many applications in economics such as in calculations concerning compound interest. Some other applications are studied more closely in the next section.

Finite Geometric Series

Let us begin with an example.

Example 6.13

This year a firm has a revenue of \$100 million that it expects to increase by 16% per year throughout the next decade. How large is its expected revenue in the tenth year, and what is the total revenue expected over the whole period?

Solution The second year's expected revenue is $100(1 + 16/100) = 100 \cdot 1.16$ (in millions), and in the third year, it is $100 \cdot (1.16)^2$. In the tenth year, the expected revenue is $100 \cdot (1.16)^9$. The total revenue expected during the decade is thus

$$100 + 100 \cdot 1.16 + 100 \cdot (1.16)^2 + \cdots + 100 \cdot (1.16)^9 \qquad [*]$$

With a calculator, we find that the sum is approximately \$2,132 million.

We found the sum in [*] by adding 10 numbers on the calculator. Especially in cases where there are many terms to add, this method is troublesome. There is an easier method to find such sums, as will be explained now.

Consider n numbers $a, ak, ak^2, \ldots, ak^{n-1}$. Each term is obtained by multiplying the previous one by a constant k. We wish to find the sum

$$s_n = a + ak + ak^2 + \cdots + ak^{n-2} + ak^{n-1} \tag{1}$$

of these numbers. We call this sum a (finite) **geometric series** with **quotient** k. The sum [*] occurs in the special case when $a = 100$, $k = 1.16$, and $n = 10$.

To find the sum s_n of the series, first multiply both sides of [1] by k to obtain

$$ks_n = ak + ak^2 + ak^3 + \cdots + ak^{n-1} + ak^n \tag{2}$$

Subtracting [2] from [1] yields

$$s_n - ks_n = a - ak^n \tag{3}$$

because all the other terms, $(ak + ak^2 + \cdots + ak^{n-1}) - (ak + ak^2 + \cdots + ak^{n-1})$, cancel.

If $k = 1$, then all terms in [1] are equal to a, and the sum is equal to $s_n = an$. For $k \neq 1$, because $s_n - ks_n = (1-k)s_n$, [3] implies that

$$s_n = \frac{a - ak^n}{1-k} \tag{4}$$

In conclusion:

Summation Formula for a Finite Geometric Series

$$a + ak + ak^2 + \cdots + ak^{n-1} = a\,\frac{1-k^n}{1-k} \qquad (k \neq 1) \tag{6.10}$$

Example 6.14

For the sum [*] in Example 6.13 we have $a = 100$, $k = 1.16$, and $n = 10$. Hence, [6.10] yields

$$100 + 100 \cdot 1.16 + \cdots + 100 \cdot (1.16)^9 = 100\,\frac{1-(1.16)^{10}}{1-1.16}$$

It takes fewer operations on the calculator than in Example 6.13 to show that the sum is about 2,132.

Infinite Geometric Series

Consider the infinite sequence of numbers

$$1, \quad \frac{1}{2}, \quad \frac{1}{4}, \quad \frac{1}{8}, \quad \frac{1}{16}, \quad \frac{1}{32}, \quad \ldots$$

Each term in the sequence is formed by halving its predecessor, so that the nth term is $1/2^{n-1}$. The sum of the first n terms is a finite geometric series with quotient $k = 1/2$ and the first term $a = 1$. Hence, [6.10] gives

$$1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}} = \frac{1-(1/2)^n}{1-1/2} = 2 - \frac{1}{2^{n-1}} \qquad [*]$$

We ask now what is meant by the "infinite sum"

$$1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{n-1}} + \cdots \qquad [**]$$

Because all the terms are positive, and there are infinitely many terms, you might be inclined to think that the sum must be infinitely large. However, if we look at formula [*], we see that the sum of the n first terms is equal to $2 - 1/2^{n-1}$, and this number is never larger than 2, irrespective of our choice of n. As n increases, $1/2^{n-1}$ comes closer and closer to 0, and the sum in [*] tends to 2 in the limit. This makes it natural to *define* the infinite sum in [**] as the number 2.

An Illustration: At a birthday party, there are two identical cakes. The person having the birthday takes all of one cake. From the second cake, the first guest is given one-half, the second guest is given one-quarter, and so on. Each successive guest is given half what is left. The sum in [*] shows how much has been taken after $n - 1$ guests have received their allocation. (The person having the birthday is not regarded as a guest.) Thus, we see that infinitely many guests can be invited to this party. (However, even if each cake were worth as much as \$100, the thirteenth guest would get only slightly more than 1 cent's worth of cake.)

In general, we ask what meaning can be given to the "infinite sum"

$$a + ak + ak^2 + \cdots + ak^{n-1} + \cdots \qquad [6.11]$$

We use the same idea as in [**], and consider the sum s_n of the n first terms in [6.11]. According to [6.10],

$$s_n = a\,\frac{1-k^n}{1-k} \qquad (k \neq 1)$$

What happens to this expression as n tends to infinity? The answer evidently depends on k^n, because only this term depends on n. In fact, k^n tends to 0 if

$-1 < k < 1$, whereas k^n does not tend to any limit if $k > 1$ or $k \leq -1$. (If you are not yet convinced that this claim is true, study the cases $k = -2$, $k = -1$, $k = -1/2$, $k = 1/2$, and $k = 2$.) Hence, it follows that if $|k| < 1$, then the sum s_n of the n first terms in [6.11] will tend to the limit $a/(1-k)$ as n tends to infinity. We let this limit be the *definition* of the sum in [6.11], and we say that the infinite series [6.11] **converges** in this case. To summarize:

Summation Formula for an Infinite Geometric Series:

$$a + ak + ak^2 + \cdots + ak^{n-1} + \cdots = \frac{a}{1-k} \qquad (\text{if } |k| < 1) \tag{6.12}$$

Using summation notation as in Sec. B1 of Appendix B, [6.12] becomes:

$$\sum_{n=1}^{\infty} ak^{n-1} = \frac{a}{1-k} \qquad (\text{if } |k| < 1) \tag{6.13}$$

If $|k| \geq 1$, we say that the infinite series [6.11] **diverges**. A divergent series has no (finite) sum. Divergence is obvious if $|k| > 1$. When $k = 1$, then $s_n = na$, which tends to $+\infty$ if $a > 0$ or to $-\infty$ if $a < 0$. When $k = -1$, then s_n is a when n is odd, but 0 when a is even; again there is no limit as $n \to \infty$.

Geometric series appear in many economic applications. Let us look at a somewhat contrived example.

Example 6.15

A rough estimate of the total oil and gas reserves in the Norwegian continental shelf at the beginning of 1981 was 12 billion ($12 \cdot 10^9$) tons. Production that year was approximately 50 million ($50 \cdot 10^6$) tons.

(a) When will the reserves be exhausted if production is kept at the same level?

(b) Suppose that production is reduced each year by 1% per year beginning in 1982. How long will the reserves last in this case?

Solution

(a) The number of years the reserves will last is given by

$$\frac{12 \cdot 10^9}{5 \cdot 10^7} = 2.4 \cdot 10^2 = 240$$

The reserves will be exhausted around the year 2220.

(b) In 1981, production was $a = 5 \cdot 10^7$. In 1982, it becomes $a - a/100 = a \cdot 0.99$. In 1983, it becomes $a \cdot 0.99^2$, and so on. If this continues forever, the total amount extracted will be

$$a + a \cdot 0.99 + a \cdot (0.99)^2 + \cdots + a \cdot (0.99)^{n-1} + \cdots$$

This is a geometric series with quotient $k = 0.99$. According to [6.12], the sum is

$$s = \frac{a}{1 - 0.99} = 100a$$

Because $a = 5 \cdot 10^7$, we get $s = 5 \cdot 10^9$, which is less than $12 \cdot 10^9$. The extraction, therefore, may be continued indefinitely, and there will never be less than 7 billion tons left.

General Series (Optional)

The determination of $\sum 1/n$ occupied Leibniz all his life
but the solution never came within his grasp.
—*H. H. Goldstine* (1977)

We briefly consider general infinite series that are not necessarily geometric,

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots \qquad [6.14]$$

What does it mean to say that this infinite series converges? By analogy with the definition for geometric series, we form the "partial" sum s_n of the n first terms:

$$s_n = a_1 + a_2 + \cdots + a_n \qquad [6.15]$$

In particular, $s_1 = a_1$, $s_2 = a_1 + a_2$, $s_3 = a_1 + a_2 + a_3$, and so on. As n increases, these partial sums include more and more terms of the series. Hence, if s_n tends toward a limit s as n tends to ∞, it is reasonable to consider s as the sum of *all* the terms in the series. Then we say that the infinite series is **convergent** with sum s. If s_n does not tend to a finite limit as n tends to infinity, we say that the series is **divergent**. The series then has no sum. (As with limits of functions, if $s_n \to \pm\infty$ as $n \to \infty$, this is not regarded as a limit.)

For geometric series, it was easy to determine when there is convergence because we found a simple expression for s_n. Usually, it will not be possible to find such a simple formula for the sum of the n first terms in a series, and the problem of determining whether a given series converges or diverges can be very difficult. No general method exists that will reveal whether or not any given series is convergent. However, there are a number of standard tests, so called *convergence* and *divergence criteria*, that will give the answer in many cases. These criteria are seldom used directly in economics.

Let us make a general observation: If the series [6.14] converges, then the nth term must tend to 0 as n tends to infinity. The argument is simple: If the series is convergent, then s_n in [6.15] will tend to a limit s as n tends to infinity. Now $a_n = s_n - s_{n-1}$, and by the definition of convergence, s_{n-1} will also tend to s as n tends to infinity. It follows that $a_n = s_n - s_{n-1}$ must tend to $s - s = 0$ as n tends to infinity. Expressed briefly,

$$a_1 + a_2 + \cdots + a_n + \cdots \text{ converges} \implies \lim_{n\to\infty} a_n = 0 \qquad [6.16]$$

The condition in [6.16] is necessary for convergence, but not sufficient. That is, a series may satisfy the condition $\lim_{n\to\infty} a_n = 0$ and yet diverge. This is shown by the following standard example (which gave Leibniz infinite trouble!).

Example 6.16

The series

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \cdots + \tfrac{1}{n} + \cdots \qquad [6.17]$$

is called the **harmonic series**. The nth term is $1/n$, which tends to 0. But the series is still divergent. To see this, we group the terms together in the following way:

$$1 + \tfrac{1}{2} + \left(\tfrac{1}{3} + \tfrac{1}{4}\right) + \left(\tfrac{1}{5} + \cdots + \tfrac{1}{8}\right) + \left(\tfrac{1}{9} + \cdots + \tfrac{1}{16}\right) + \left(\tfrac{1}{17} + \cdots + \tfrac{1}{32}\right) + \cdots \qquad [*]$$

Between the first pair of parentheses there are two terms, one greater than $1/4$ and the other equal to $1/4$, so their sum is greater than $2/4 = 1/2$. Between the second pair of parentheses there are four terms, three greater than $1/8$ and the last equal to $1/8$, so their sum is greater than $4/8 = 1/2$. Between the third pair of parentheses there are eight terms, seven greater than $1/16$ and the last equal to $1/16$, so their sum is greater than $8/16 = 1/2$. Between the fourth pair of parentheses there are sixteen terms, fifteen greater than $1/32$ and the last equal to $1/32$, so their sum is greater than $16/32 = 1/2$. This pattern repeats itself infinitely often. Between the nth pair of parentheses there will be 2^n terms, of which $2^n - 1$ are greater than 2^{-n-1} whereas the last is equal to 2^{-n-1}, so their sum is greater than $2^n \cdot 2^{-n-1} = 1/2$. We conclude that the series in [$*$] must diverge because its sum is larger than that of an infinite number of $1/2$'s.

An illustration: If you plan a birthday party with infinitely many guests where the person having the birthday takes 1 cake, the best friend is given half a cake, the next person a third, and so on, then you must bake infinitely many cakes!

One can prove in general (see Problem 11 in Section 11.3) that

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ is convergent} \iff p > 1 \qquad [6.18]$$

Problems

1. Find the sum s_n of the finite geometric series

$$1 + \frac{1}{3} + \frac{1}{3^2} + \cdots + \frac{1}{3^{n-1}}$$

What limit does s_n tend to as n tends to infinity? Evaluate

$$\sum_{n=1}^{\infty} \frac{1}{3^{n-1}}$$

2. Determine whether the following series are geometric, and find the sums of those geometric series that do converge:

a. $8 + 1 + 1/8 + 1/64 + \cdots$ **b.** $-2 + 6 - 18 + 54 - \cdots$

c. $2^{1/3} + 1 + 2^{-1/3} + 2^{-2/3} + \cdots$ **d.** $1 - 1/2 + 1/3 - 1/4 + \cdots$

3. Examine the convergence of the following geometric series and find the sums when they exist:

a. $\frac{1}{p} + \frac{1}{p^2} + \frac{1}{p^3} + \cdots$ **b.** $x + \sqrt{x} + 1 + 1/\sqrt{x} + \cdots$

c. $\sum_{n=1}^{\infty} x^{2n}$ **d.** $1 + \frac{1}{1+x} + \frac{1}{(1+x)^2} + \cdots$

4. Find the sum

$$\sum_{k=0}^{\infty} b\left(1 + \frac{p}{100}\right)^{-k} \qquad (p > 0)$$

5. Total world consumption of iron in 1971 was approximately 794 million tons. If consumption increases by 5% each year and the world's total resources of iron are 249 billion tons, how long will these resources last?

6. Show that the following series diverge:

a. $\sum_{n=1}^{\infty} \frac{n}{1+n}$ **b.** $\sum_{n=1}^{\infty} (101/100)^n$ **c.** $\sum_{n=1}^{\infty} \frac{1}{(1+1/n)^n}$

7. Examine the convergence or divergence of the following series:

a. $\sum_{n=1}^{\infty} (100/101)^n$ **b.** $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ **c.** $\sum_{n=1}^{\infty} \frac{1}{n^{1.00000001}}$

d. $\sum_{n=1}^{\infty} \frac{1+n}{4n-3}$ **e.** $\sum_{n=1}^{\infty} \left(-\frac{1}{2}\right)^n$ **f.** $\sum_{n=1}^{\infty} (\sqrt{3})^{1-n}$

8. Let

$$s_n = \sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n(n+1)}$$

By using the identity

$$\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$$

prove that $s_n = n/(n+1)$, and then find the sum of the infinite series

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$$

6.6 Present Discounted Values and Investment

The sum of \$1000 in your hand today is worth more than \$1000 received at some future date. One important reason is that you can invest the \$1000.[2] If the interest rate is 11% per year, then after 6 years, the original \$1000 will have grown to an amount $1000(1 + 11/100)^6 = 1000 \cdot (1.1)^6 \approx \$1870$. (See Section A.1 of Appendix A.) So if the amount \$1870 is due for payment 6 years from now and the interest rate is 11% per year, then the *present value* of this amount is \$1000. Because \$1000 is less than \$1870, we often speak of \$1000 as the *present discounted value* (or PDV) of \$1870. The ratio \$1000/\$1870 is called the *discount factor*. The interest rate, 11% per year in this case, is often called the *discount rate* as well.

Suppose three payments are to be made, with the amount \$1000 being paid after 1 year, \$1500 after 2 years, and \$2000 after 3 years. How much must be deposited in an account today in order to have enough savings to cover these three payments, given that the interest rate is 11% per year? We call this total amount the *present value* of the three payments.

In order to have \$1000 after 1 year, the amount x_1 we must deposit today is given by

$$x_1 \cdot \left(1 + \frac{11}{100}\right) = 1000, \qquad \text{that is,} \qquad x_1 = \frac{1000}{1 + 11/100} = \frac{1000}{1.11}$$

In order to have \$1500 after 2 years, we must deposit an amount x_2 today, where

$$x_2 \cdot \left(1 + \frac{11}{100}\right)^2 = 1500, \qquad \text{that is,} \qquad x_2 = \frac{1500}{(1 + 11/100)^2} = \frac{1500}{(1.11)^2}$$

[2] If prices are expected to increase, another reason for preferring \$1000 today is inflation, because \$1000 to be paid at some future date will buy less then than \$1000 does today.

Finally, to have \$2000 after 3 years, we must deposit an amount x_3 today, where

$$x_3 \cdot \left(1 + \frac{11}{100}\right)^3 = 2000, \qquad \text{that is,} \qquad x_3 = \frac{2000}{(1 + 11/100)^3} = \frac{2000}{(1.11)^3}$$

So the total present value of the three payments, which is the total amount A that must be deposited today in order to cover all three payments, is given by

$$A = \frac{1000}{1.11} + \frac{1500}{(1.11)^2} + \frac{2000}{(1.11)^3}$$

The total is approximately $900.90 + 1217.43 + 1462.38 = 3580.71$.

Suppose now that n successive payments $a_1, \ldots, a_n$ are to be made, with a_1 being paid after 1 year, a_2 after 2 years, and so on. How much must be deposited into an account today in order to have enough savings to cover all these future payments, given that the interest rate is $p\%$ per year? In other words, what is the *present value* of all these payments? Let $r = p/100$ represent the *interest factor*.

In order to have a_1 after 1 year, we must deposit $a_1/(1+r)$ today, to have a_2 after 2 years we must deposit $a_2/(1+r)^2$ today, and so on. The total amount A_n that must be deposited today in order to cover all n payments is therefore

$$A_n = \frac{a_1}{1+r} + \frac{a_2}{(1+r)^2} + \cdots + \frac{a_n}{(1+r)^n} \qquad [6.19]$$

In other words:

The **present value** of the n installments $a_1, a_2, \ldots, a_n$, where the first amount a_1 has to be paid after 1 year and the remaining amounts at intervals of 1 year, and with the interest rate $p\%$ per year, is given by

$$A_n = \sum_{i=1}^{n} \frac{a_i}{(1+r)^i} \qquad \text{where } r = p/100 \qquad [6.20]$$

Often, the annual payments are equal, so that $a_1 = a_2 = \cdots = a_n = a$. Then [6.19] is a finite geometric series with n terms. The first term is $a/(1+r)$ and the quotient is $1/(1+r)$. According to formula [6.10] with $k = (1+r)^{-1}$, the sum is

$$A_n = \frac{a}{1+r}\frac{1-(1+r)^{-n}}{1-(1+r)^{-1}} = \frac{a}{r}\left[1 - \frac{1}{(1+r)^n}\right]$$

(where the second equality holds because the denominator of the middle expression reduces to r). Hence, we have the following:

The present value of n installments of \$$a$ each, where the first amount has to be paid 1 year from now and the remaining amounts at intervals of 1 year, with the interest rate at $p\%$ per year, is given by

$$A_n = \frac{a}{1+r} + \cdots + \frac{a}{(1+r)^n} = \frac{a}{r}\left[1 - \frac{1}{(1+r)^n}\right] \qquad [6.21]$$

where $r = p/100$.

Example 6.17

What is the present value of 10 annual deposits of \$1000 if the first payment occurs after 1 year and the interest rate is 14% per year?

Solution Using [6.21] with $a = 1000$, $n = 10$, and $r = 14/100 = 0.14$ yields

$$A_{10} = \frac{1000}{0.14}\left[1 - \frac{1}{(1.14)^{10}}\right] \approx 5216.12$$

Example 6.18

A house loan valued at \$50,000 today is to be repaid in equal annual amounts over 15 years, the first repayment starting 1 year from now. The interest rate is 8%. What are the annual amounts?

Solution We can use [6.21] again. This time $A_{15} = 50{,}000$, $r = 0.08$, and $n = 15$. Hence, we obtain the following equation for determining the annual amount a:

$$50{,}000 = \frac{a}{0.08}\left[1 - \frac{1}{(1.08)^{15}}\right]$$

We find $50.000 = a \cdot 8.55948$, so that $a \approx 5841$.

If n tends to infinity in [6.21] and if $r > 0$, then $(1+r)^n$ will tend to infinity, and thus A_n will tend to $A = a/r$:

$$A = \frac{a}{1+r} + \frac{a}{(1+r)^2} + \cdots = \frac{a}{r} \qquad (r > 0) \qquad [6.22]$$

Thus, $a = rA$. This corresponds to the case where an investment of \$$A$ pays \$$a$ per year in perpetuity when the interest rate is r.

Investment Projects

Consider n numbers $a_0, a_1, \ldots, a_{n-1}$ that represent the returns in successive years to an investment project. Negative numbers represent losses, positive numbers represent profits, and we think of a_i as associated with year i, whereas a_0 is associated with the present period. In most investment projects, a_0 is a big negative number, because a large expense precedes any returns. If we consider an interest rate of $p\%$ per year and let $r = p/100$, then the net present value of the profits accruing from the project is given by

$$A = a_0 + \frac{a_1}{1+r} + \frac{a_2}{(1+r)^2} + \cdots + \frac{a_{n-1}}{(1+r)^{n-1}}$$

Several different criteria are used to compare alternative investment projects. One is simply this: Choose the project whose profit stream has the largest net present value A. The interest rate to use could be an accepted rate for capital investments. This rule is the natural extension to many periods of static profit maximization, with the discount factors $(1+r)^{-1}$, $(1+r)^{-2}$, ... attached to future profits like the prices of future money (which is less valuable than present money).

A different criterion is based on the **internal rate of return**, defined as an interest rate that makes the present value of all payments equal to 0. For the investment project yielding returns $a_0, a_1, \ldots, a_{n-1}$, the internal rate of return is thus a number r such that

$$a_0 + \frac{a_1}{1+r} + \frac{a_2}{(1+r)^2} + \cdots + \frac{a_{n-1}}{(1+r)^{n-1}} = 0 \qquad [6.23]$$

If two investment projects both have a unique internal rate of return, then a criterion for choosing between them is to prefer the project that has the higher internal rate of return. Note that [6.23] is a polynomial equation of degree $n-1$ in the discount factor $(1+r)^{-1}$. In general, this equation does not have a unique positive solution r. Nevertheless, Problem 7 asks you to show that there exists a unique positive internal rate of return in an important special case.

Problems

1. What is the present value of 15 annual deposits of \$3500 each when the first deposit is 1 year from now and the interest rate is 12% per year?
2. An author is to be paid a royalty for a book. Two alternative offers are made:
   (a) The author can be paid \$21,000 immediately,
   (b) There can be five equal annual payments of \$4600, the first being paid at once.

   Which of these offers will be more valuable if the interest rate is 6% per annum?

3. A is obliged to pay B the amount \$1000 yearly for 5 years, the first payment in 1 year's time. B sells this claim to C for \$4340 in cash. Find an equation that determines the rate of return p that C obtains from this investment. Can you prove that it is a little less than 5%?

4. A construction firm wants to buy a building site and has the choice between three different payment schedules:
   (a) Pay \$67,000 in cash.
   (b) Pay \$12,000 per year for 8 years, where the first installment is to be paid at once.
   (c) Pay \$22,000 in cash and thereafter \$7000 per year for 12 years, where the first installment is to be paid after 1 year.

   Determine which schedule is least expensive if the interest rate is 11.5% and the firm has at least \$67,000 available to spend in cash. What happens if the firm can only afford \$22,000 as an immediate payment? Or if the interest rate is 12.5%?

5. Suppose that in [6.23] we have $a_0 < 0$ and $a_i = a$ for $i = 1, 2, \ldots$. If n is very large, find an approximate expression for the internal rate of return.

6. The present discounted value of a payment D growing at a constant rate g when the discount rate is r is given by

$$\frac{D}{1+r} + \frac{D(1+g)}{(1+r)^2} + \frac{D(1+g)^2}{(1+r)^3} + \cdots$$

   where r and g are positive. What is the condition for convergence? Show that if the series converges with sum P_0, then $P_0 = D/(r-g)$.

7. Consider an investment project with an initial loss, so that $a_0 < 0$, and thereafter no losses. Suppose too that the sum of the later profits is larger than the initial loss. Prove that there exists a unique positive internal rate of return. (*Hint:* Define $f(r)$ as the expression on the left side of [6.23]. Then study $f(r)$ and $f'(r)$ on the interval $(0, \infty)$.)

6.7 A Rigorous Approach to Limits (Optional)

Our preliminary definition of the limit concept [4.14] of in Section 4.4 was this:

$\lim_{x\to a} f(x) = A$ means that $f(x)$ in as close to A as we want, for all x sufficiently close (but not equal) to a [1]

The closeness or, more generally, the distance between two numbers can be measured by the absolute value of the difference between them. Let us briefly consider some examples of the use of absolute values before proceeding further.

Example 6.19

Use both absolute values and double inequalities to answer the following questions:

(a) Which numbers x have a distance from 5 that is less than 0.1?

(b) Which numbers x have a distance from a that is less than δ?

Solution

(a) The distance between x and 5 is $|x-5|$, so the requirement is that $|x-5| < 0.1$. Using [1.4] in Section 1.4, we have alternatively $-0.1 < x-5 < 0.1$. Adding 5 to each side gives $4.9 < x < 5.1$. (The result is obvious: The numbers x that differ from 5 by less than 0.1 are those lying between 4.9 and 5.1.)

(b) Here $|x-a| < \delta$ or $-\delta < x-a < \delta$. Adding a to each side yields $a-\delta < x < a+\delta$. We can also write $x \in (a-\delta, a+\delta)$.

Absolute values can be used to reformulate [1] as follows:

$\lim_{x\to a} f(x) = A$ means that $|f(x)-A|$ is as small as we want for all $x \neq a$ with $|x-a|$ sufficiently small. [2]

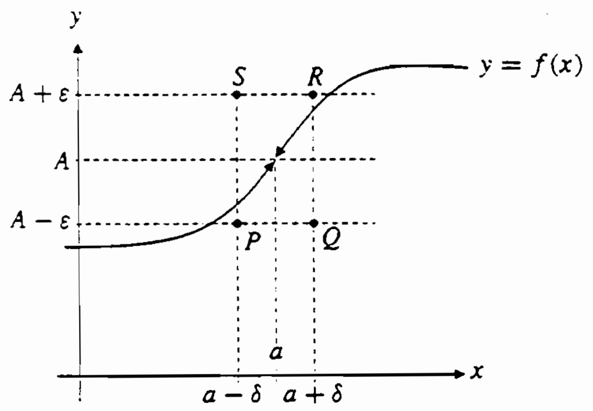
Note that the condition $x \neq a$ is equivalent to $0 < |x-a|$.

The German mathematician Heine was the first to realize (in 1872) that this formulation could be made precise with the following $\varepsilon\delta$ definition:

> We say that $f(x)$ tends to A in the limit as x tends to a, and write $\lim_{x\to a} f(x) = A$, provided that for each number $\varepsilon > 0$ there exists a number $\delta > 0$ such that
>
> $$|f(x) - A| < \varepsilon \qquad \text{whenever} \qquad 0 < |x-a| < \delta$$ [6.24]

Definition [6.24] is illustrated in Fig. 6.15. Note that the tolerance for the deviation in $f(x)$, which is ε, is marked off along the y-axis, and the corresponding deviation in x, which

FIGURE 6.15 For every ε, there is a δ, so $\lim_{x\to a} f(x) = A$.



is δ, is marked off along the x-axis. Geometrically, $f(x)A$ as $x \to a$ means that the graph must not only intersect box $PQRS$, but also "come out of" its vertical sides. Note how $\delta > 0$ must be chosen so that if $x \neq a$ and $x \in (a - \delta, a + \delta)$, then $f(x)$ belongs to the interval $(A-\varepsilon, A+\varepsilon)$. If $\varepsilon > 0$ is chosen smaller, then δ usually has to be chosen smaller as well. Our choice of δ must therefore, in general, depend on the choice of ε. *This interplay between ε and δ is the whole point of the definition:* However small we choose $\varepsilon > 0$, it must be possible to find a $\delta > 0$ so small that whenever x is closer to a than δ (and $x \neq a$), then $f(x)$ is closer to A than ε.

It must be regarded as a part of one's general mathematical education *to have seen* this $\varepsilon\delta$ definition of a limit. However, if you have difficulties with this definition and with arguments based on it, you are in very good company indeed. Hundreds of thousands of mathematics students all over the world struggle with this definition every year. Furthermore, many of the world's best mathematicians in the nineteenth century were unable to solve some important problems for want of a precise definition of limits, so the concept did not come easily to them either.

Example 6.20

Use [6.24] to show that

$$\lim_{x \to 3} (3x - 2) = 7 \tag{1}$$

Solution In this case, $f(x) = 3x - 2$, $a = 3$, and $A = 7$. Hence,

$$\begin{aligned} |f(x) - A| &= |(3x - 2) - 7| \\ &= |3x - 9| = 3|x - 3| \end{aligned} \tag{2}$$

Let $\varepsilon > 0$ be given. We see from [2] that $|f(x) - A| = 3|x - 3| < \varepsilon$ provided that $0 < |x - 3| < \varepsilon/3$. So $|f(x) - A| < \varepsilon$ if $|x - 3| < \delta$, where $\delta = \varepsilon/3$. According to definition [6.24], we conclude that [1] is correct.

Note that the value of δ in definition [6.24] is not unique. Having found *one* value of δ, any smaller value of δ will work as well. In Example 6.20, we chose $\delta = \varepsilon/3$: we could also have chosen any $\delta \leq \varepsilon/3$, but not $\delta = \varepsilon/2$.

The proof in Example 6.20 is about as easy as a limit proof can get. Usually, a little more ingenuity is required. Let us consider a more typical example.

Example 6.21

Show by using the $\varepsilon\delta$ definition that if $a > 0$, then

$$\lim_{x \to a} \sqrt{x} = \sqrt{a} \tag{1}$$

Solution Here $f(x) = \sqrt{x}$ and $A = \sqrt{a}$. Given any $\varepsilon > 0$, we must find a $\delta > 0$ such that

$$|f(x) - A| = |\sqrt{x} - \sqrt{a}\,| < \varepsilon \qquad \text{whenever} \qquad 0 < |x - a| < \delta \tag{2}$$

It seems a good idea to try to express $|\sqrt{x}-\sqrt{a}|$ in terms of $|x-a|$. We use a common algebraic trick:

$$|\sqrt{x}-\sqrt{a}| = \left|\frac{(\sqrt{x}-\sqrt{a})(\sqrt{x}+\sqrt{a})}{\sqrt{x}+\sqrt{a}}\right| = \frac{|x-a|}{\sqrt{x}+\sqrt{a}} \qquad [3]$$

Because $\sqrt{x}+\sqrt{a} \geq \sqrt{a}$ whatever the value of $x \geq 0$, we obtain from [3] that

$$|\sqrt{x}-\sqrt{a}| = \frac{|x-a|}{\sqrt{x}+\sqrt{a}} \leq \frac{1}{\sqrt{a}}|x-a|$$

Thus, we see that if $|x-a|$ is small, then $|\sqrt{x}-\sqrt{a}|$ is small as well. *More precisely:*

$$|\sqrt{x}-\sqrt{a}| \leq \frac{1}{\sqrt{a}} \cdot |x-a| < \varepsilon$$

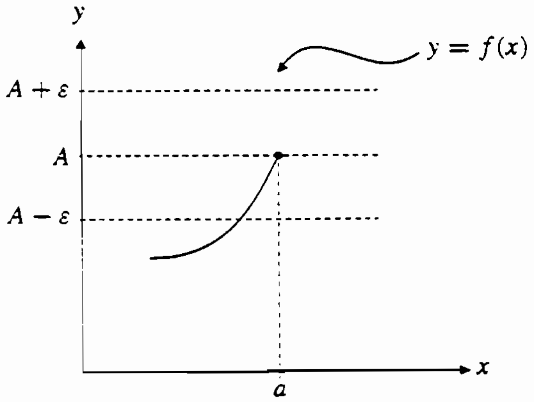
provided that $0 < |x-a| < \delta = \varepsilon\sqrt{a}$.

So far we have concentrated on cases in which the limit exists. What does it mean to say that $f(x)$ does *not* tend to the number A as x tends to a? Negating statement [6.24], we have (compare Problem 9(d) in Section 1.5):

> $f(x)$ does not tend to A as a limit as x tends to a if we can find an $\varepsilon > 0$ such that for all $\delta > 0$, there exists a number x satisfying $0 < |x-a| < \delta$ and $|f(x)-A| \geq \varepsilon$. [6.25]

Definition [6.25] is illustrated in Fig. 6.16. If we choose ε as in the figure, we see that if x is slightly larger than a, then the distance $|f(x)-A|$ is larger than ε. For every $\delta > 0$, there exists a number x satisfying $0 < |x-a| < \delta$ and $|f(x)-A| \geq \varepsilon$. This shows that $f(x)$ does not tend to A as a limit as x tends to a.

FIGURE 6.16



Extensions of the Limit Concept

In Section 6.1 we extended the limit concept heuristically in several different ways. All these definitions can be made precise in the same way as [6.24]. We include only the following definition:

$\lim_{x\to\infty} f(x) = A$ means that for each $\varepsilon > 0$, there exists a number N such that $|f(x) - A| < \varepsilon$ for all $x > N$ [6.26]

Illustrate this definition in connection with Fig. 6.4 of Section 6.1.

The following "geometrically obvious" theorem is quite useful:

The Squeezing Rule for Limits

Suppose that $f(x) \le g(x) \le h(x)$ for all x in an interval around a, but not necessarily at a. If there exists a number M such that $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = M$, then $\lim_{x\to a} g(x) = M$. [6.27]

The theorem is illustrated in Fig. 6.17. Because $g(x)$ is "squeezed" between two functions that both tend to M as $x \to a$, $g(x)$ must also tend to M as $x \to a$. One can prove this theorem by using definition [6.24], but we skip the proof. Ambitious readers may want to try it for themselves.

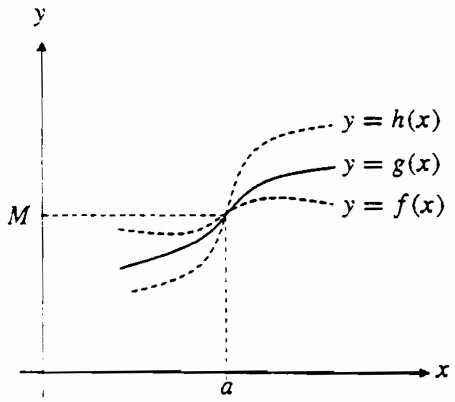


FIGURE 6.17

An $\varepsilon\delta$ Definition of Continuity

In [6.2] of Section 6.2 we defined continuity in terms of the limit concept. The precise definition [6.24], leads to the following $\varepsilon\delta$ definition of continuity:

f is **continuous** at $x = a$ if for every $\varepsilon > 0$, there is a $\delta > 0$ such that $|f(x) - f(a)| < \varepsilon$ whenever $|x - a| < \delta$. [6.28]

Note that $0 < |x - a|$ is unnecessary because if $|x - a| = 0$, then $x = a$, and so $|f(x) - f(a)| = 0$.

Problems

1. Use [6.24] to show that $\lim_{x \to -1} (5x + 2) = -3$.
2. Prove that for $|x| \leq 1$, one has $|(x + 1)^3 - 1| \leq 7|x|$. Use this and the definition of limits to show that $\lim_{x \to 0} (x + 1)^3 = 1$. Is $f(x) = (x + 1)^3$ continuous at $x = 0$?
3. Let $f(x) = 2 - \frac{1}{2}x^2$ and $h(x) = 2 + x^2$. Suppose that the only thing we know about the function g is that $f(x) \leq g(x) \leq h(x)$ for all x. What is $\lim_{x \to 0} g(x)$?
4. Show by using the definition of limits that:

   a. $$\lim_{x \to 5} \frac{4x^2 - 100}{x - 5} = 40$$

   b. $$\lim_{x \to -\pi} \frac{x^2 - \pi^2}{x + \pi} = -2\pi$$

   Hint: Try to simplify the fractions.

8

Exponential and Logarithmic Functions

Then you ought to have invested my money with the bankers, and at my coming I should have received what was mine with interest.
—Matthew 25:27

Exponential functions of the form a^x were briefly considered in Sections 3.5. They were shown to be well suited to describing certain economic phenomena such as growth and compound interest. This chapter shows how such functions can be differentiated. And it introduces logarithms, which are inverses of exponential functions. Logarithms also feature in an alternative definition of elasticity.

8.1 The Natural Exponential Function

Recall that an exponential function with base a is

$$f(x) = a^x$$

where a is the factor by which $f(x)$ changes when x increases by 1. Each base a gives a different exponential function. In mathematics, one particular value of a gives an exponential function that is far more important than the others. One might guess that $a = 2$ or $a = 10$ would be this special base. Strangely enough, it turns out that an irrational number a little larger than 2.7 is the most important base for an exponential function.

In order to explain why, we must study the derivative of $f(x) = a^x$. Earlier rules of differentiation cannot help here. So we rely on the definition of the derivative and consider the Newton quotient of $f(x) = a^x$, which is

$$\frac{f(x+h) - f(x)}{h} = \frac{a^{x+h} - a^x}{h} \qquad [*]$$

If this fraction tends to a limit as h tends to 0, then $f(x) = a^x$ is differentiable and $f'(x)$ is precisely equal to this limit.

Substituting $x = 0$ in [∗] and letting $h \to 0$ yields in particular

$$f'(0) = \lim_{h \to 0} \frac{a^h - 1}{h} \qquad [8.1]$$

(provided the limit exists).

The fraction in [∗] can be simplified if we use the rule $a^{x+h} = a^x \cdot a^h$. Then we have $a^{x+h} - a^x = a^x(a^h - 1)$, so that

$$\frac{f(x+h) - f(x)}{h} = a^x \cdot \frac{a^h - 1}{h}$$

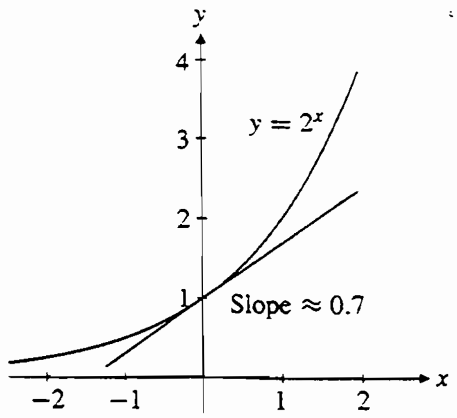
When taking the limit of this last expression as h tends to 0, the term a^x is a constant, whereas according to [8.1], the fraction $(a^h - 1)/h$ tends to $f'(0)$. Hence,

$$f(x) = a^x \quad \Longrightarrow \quad f'(x) = a^x f'(0) \qquad [8.2]$$

We have thus shown that if $f(x) = a^x$ has a derivative at 0 (in the sense that the limit in [8.1] exists), then f is differentiable for every x, and $f'(x) = a^x f'(0)$.
Note: Observe that $f'(0)$ is a function of a. For each $a > 0$, the number $f'(0)$ is defined as the limit of $(a^h - 1)/h$ as h tends to 0. One can prove that this limit exists for every $a > 0$. Later we shall see that $f'(0) = \ln a$, the natural logarithm of a.

Geometrically, $f'(0)$ may be interpreted as the slope of the tangent to the graph of $y = a^x$ at (0, 1). In Figs. 8.1 and 8.2 we have measured these slopes for

FIGURE 8.1



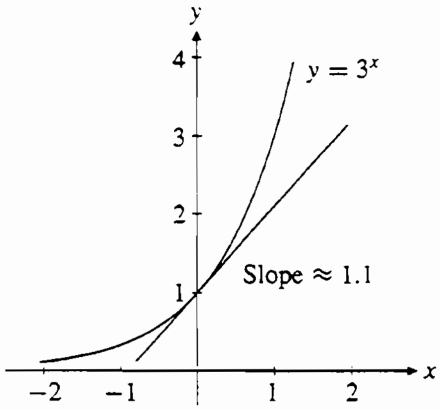


FIGURE 8.2

2^x and 3^x, and they are respectively ≈ 0.7 and ≈ 1.1. (Accordingly, [8.2] implies that $f(x) = 2^x \Rightarrow f'(x) \approx 0.7 \cdot 2^x$, and $f(x) = 3^x \Rightarrow f'(x) \approx 1.1 \cdot 3^x$.)

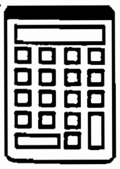
It is reasonable to assume that, as a increases from 2 to 3, so $f'(0)$ increases from ≈ 0.7 to ≈ 1.1 without skipping any intermediate values. For some value a between 2 and 3, we ought then to have $f'(0) = 1$ in particular. This value of a is a fundamental constant in mathematical analysis. It is an irrational number so distinguished that it is usually denoted by the single letter e, and is given by

$$e = 2.7\,1828\,1828\,459045\ldots$$

Because $a = e$ is precisely the choice of a that gives $f'(0) = 1$ in [8.2], we obtain

$$f(x) = e^x \implies f'(x) = e^x \qquad [8.3]$$

The **natural exponential function** $f(x) = e^x$, *therefore has the remarkable property that its derivative is equal to the function itself.* This is the main reason why the function appears so often in mathematics and applications. Observe also that $f''(x) = e^x$. Because $e^x > 0$ for all x, both $f'(x)$ and $f''(x)$ are positive. Hence, both f and f' are strictly increasing. This confirms the shape of the graph in Fig. 8.3.



Powers with e as their base are difficult to compute by hand—even $e^1 = e$. A scientific calculator with an $\boxed{e^x}$ function key can do this immediately, however. For instance, one finds that $e^{0.5} \approx 1.6487$, $e^{-\pi} \approx 0.0432$.

By combining [8.3] with other rules of differentiation, we can differentiate complicated expressions involving the exponential function e^x. Before looking at some special examples, let us consider general functions of the form $y = e^{g(x)}$. To

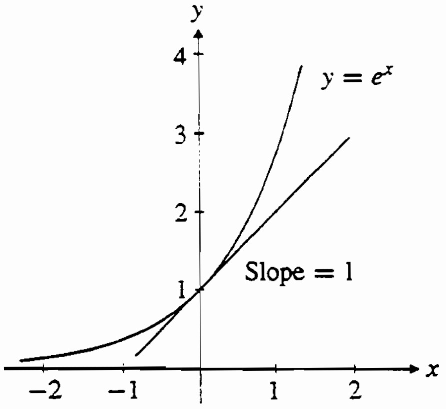


FIGURE 8.3 The natural exponential function.

differentiate these we apply the chain rule $dy/dx = dy/du \cdot du/dx$ with $y = e^u$ and $u = g(x)$. Thus, $y' = e^u u' = e^{g(x)} g'(x)$, and so

$$y = e^{g(x)} \implies y' = e^{g(x)} g'(x) \qquad [8.4]$$

Example 8.1

Differentiate the following:

(a) $y = e^{3x}$ (b) $y = e^x/x$ (c) $y = \sqrt{e^{2x} + x}$

Solution

(a) Use [8.4] with $g(x) = 3x$. Then $g'(x) = 3$ so $y = e^{3x} \Rightarrow y' = e^{3x} \cdot 3 = 3e^{3x}$.

(b) Using the quotient rule yields

$$y = \frac{e^x}{x} \Longrightarrow y' = \frac{e^x x - e^x \cdot 1}{x^2} = \frac{e^x(x-1)}{x^2}$$

(c) Here $y = \sqrt{e^{2x} + x} = \sqrt{u}$, with $u = e^{2x} + x$, and so $u' = 2e^{2x} + 1$, where we used the chain rule. Using the chain rule again yields

$$y = \sqrt{e^{2x} + x} = \sqrt{u} \Longrightarrow y' = \frac{1}{2\sqrt{u}} \cdot u' = \frac{2e^{2x} + 1}{2\sqrt{e^{2x} + x}}$$

Example 8.2

Find the derivative of

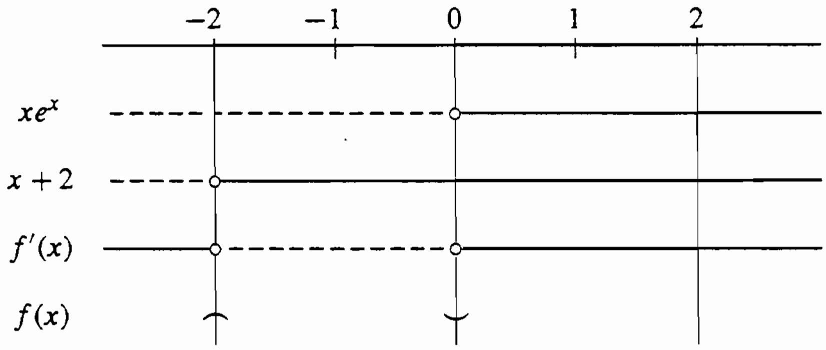
$$f(x) = x^2 e^x$$

Where is $f(x)$ increasing? (Its graph is drawn in Fig. 9.23 of Sec. 9.5.)

Solution Differentiating using the product rule yields

$$f'(x) = 2xe^x + x^2e^x = xe^x(2+x)$$

We see that $f'(x) = 0$ for $x = 0$ and for $x = -2$. The accompanying sign diagram tells us that f is increasing in the intervals $(-\infty, -2]$ and $[0, \infty)$ (but decreasing in $[-2, 0]$).



Note 1: A common error when differentiating exponential functions is to believe that the derivative of e^x is "xe^{x-1}". This is due to confusing the exponential function with a power function.
Note 2: Sometimes the notation $\exp(u)$ is used in place of e^u. If u is a complicated expression like $x^3 + x\sqrt{x - 1/x}$, it is easier (typographically) to read and write $\exp(x^3 + x\sqrt{x - 1/x})$ instead of $e^{x^3+x\sqrt{x-1/x}}$.

A Survey of the Properties of e^x

The natural exponential function

$$f(x) = e^x \qquad (e = 2.71828\ldots)$$

is differentiable and strictly increasing for all real numbers x. In fact, [8.5]

$$f(x) = e^x \Longrightarrow f'(x) = f(x) = e^x$$

The following properties hold for all exponents s and t:

(a) $e^se^t = e^{s+t}$ (b) $e^s/e^t = e^{s-t}$ (c) $(e^s)^t = e^{st}$

Problems

1. Differentiate the following functions, using the chain rule:

 a. $y = e^{-3x}$ **b.** $y = 2e^{x^3}$ **c.** $y = e^{1/x}$ **d.** $y = 5e^{2x^2-3x+1}$

2. Find the following:

a. $\frac{d}{dx}\left(e^{e^x}\right)$

b. $\frac{d}{dt}\left(e^{t/2}+e^{-t/2}\right)$

c. $\frac{d}{dt}\left(\frac{1}{e^t+e^{-t}}\right)$

d. $\frac{d}{dz}\left(e^{z^3}-1\right)^{1/3}$

3. Consider the function f defined for all x by $f(x) = xe^x$.
   a. Compute $f'(x)$ and $f''(x)$. Find the intervals on which f is increasing.
   b. Draw the graph of f.

4. In an economic model, the number of families with income $\leq x$ is given by

$$p(x) = a + k(1 - e^{-cx}) \qquad (a,\ k,\ \text{and } c \text{ are positive constants})$$

Determine $p'(x)$ and $p''(x)$, and then draw the graph of p.

5. Let $f(x) = (x^2 - 2x - 3)e^x$. Draw the graph of f for $-4 \leq x \leq 3$.

Harder Problems

6. The expressions $\frac{1}{2}(e^x - e^{-x})$ and $\frac{1}{2}(e^x + e^{-x})$ occur so often that they have been given the special symbols

$$\sinh x = \frac{e^x - e^{-x}}{2}, \qquad \cosh x = \frac{e^x + e^{-x}}{2}$$

indicating the *hyperbolic sine* and *hyperbolic cosine* respectively. Draw the graphs of the two functions, and show that the following formulas hold for all x:

a. $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$
b. $\cosh 2x = (\cosh x)^2 + (\sinh x)^2$
c. $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$
d. $\sinh 2x = 2 \sinh x \cosh x$
e. $(\cosh x)^2 - (\sinh x)^2 = 1$
f. $\sinh^2 x = \frac{1}{2}(\cosh 2x - 1)$
g. $\frac{d}{dx}(\sinh x) = \cosh x$
h. $\frac{d}{dx}(\cosh x) = \sinh x$

7. Show by induction that the nth derivative of xe^x is $(x+n)e^x$.

8. Let $f(x) = a^x$. Show that

$$f(z+x) = f(z)f(x) \qquad (\text{for all } x \text{ and } z) \qquad [*]$$

Assume that f is differentiable. Differentiate [*] with respect to z (holding x fixed), and then put $z = 0$. Explain why this gives an alternative justification for [8.2].

8.2 The Natural Logarithmic Function

In Section 3.5, the doubling time of an exponential function $f(t) = a^t$, with $a > 1$, was defined as the time it takes for $f(t)$ to become twice as large. In order to find the doubling time t^*, we must solve the equation $a^{t^*} = 2$ for t^*. In economics, we often need to solve similar problems:

1. At the present rate of inflation, how long will it take the price level to triple?
2. If the national debt of the U.S. continues to grow at the present proportional rate, how long will it take to reach \$10 trillion?
3. If \$1000 is invested in a savings account bearing interest at the rate of 8% per annum, how long does it take for the account to reach \$10,000?

All these questions involve solving equations of the form $a^x = b$ for x. For instance, problem 3 is to find which x solves the equation $1000(1.08)^x = 10{,}000$, or $(1.08)^x = 10$.

We begin with equations in which the base of the exponentials is e. Here are some examples:

$$e^x = 4 \tag{1}$$

$$5e^{-3x} = 16 \tag{2}$$

$$A\alpha e^{-\alpha x} = k \tag{3}$$

In all these equations, the unknown occurs as an exponent. We therefore introduce the following useful definition. If $e^u = a$, we call u the **natural logarithm** of a, and we write $u = \ln a$. Hence, we have the following definition of the symbol $\ln a$:

$$e^{\ln a} = a \qquad (a \text{ is any positive number}) \tag{8.6}$$

Thus, $\ln a$ is the power of e you need to get a.

Because e^u is a strictly increasing function of u, it follows that $\ln a$ is uniquely determined by the definition [8.6]. You should memorize this definition. It is the foundation for everything in this section, and for a good part of what comes later. In the following, we practice applying this definition.

Example 8.3

Find the following:

(a) $\ln 1$ **(b)** $\ln e$ **(c)** $\ln(1/e)$ **(d)** $\ln 4$ **(e)** $\ln(-6)$

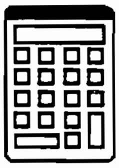
Solution

(a) $\ln 1 = 0$, because $e^0 = 1$ and so 0 is the power of e that you need to get 1.

(b) $\ln e = 1$, because $e^1 = e$ and so 1 is the power of e that you need to get e.

(c) $\ln(1/e) = \ln e^{-1} = -1$, because -1 is the power of e that you need to get $1/e$.

(d) $\ln 4$ is the power of e you need to get 4. Because we have $e^1 \approx 2.7$ and $e^2 = e^1 \cdot e^1 \approx 7.3$, the number $\ln 4$ must lie between 1 and 2.



By experimenting with the $\boxed{e^x}$ key on a scientific calculator, you should able to find a good approximation to $\ln 4$ by trial and error. However, it is easier to press 4 and the $\boxed{\ln x}$ key. Then you find that $\ln 4 \approx 1.386$. Thus, $e^{1.386} \approx 4$.

(e) $\ln(-6)$ would be the power of e you need to get -6. Because e^x is positive for all x, it is obvious that $\ln(-6)$ must be undefined.

Box [8.7] collects some useful rules for natural logarithms. All are simple implications of the rules for powers.

Useful Rules for ln

$$\ln(xy) = \ln x + \ln y \qquad (x \text{ and } y \text{ are positive}) \qquad \text{(a)}$$

(The logarithm of a product is equal to the *sum* of the logarithms of each of the factors.)

$$\ln \frac{x}{y} = \ln x - \ln y \qquad (x \text{ and } y \text{ are positive}) \qquad \text{(b)}$$

[8.7]

(The logarithm of a quotient is equal to the *difference* between the logarithms of its numerator and denominator.)

$$\ln x^p = p \ln x \qquad (x \text{ is positive}) \qquad \text{(c)}$$

(The logarithm of a power is equal to the exponent multiplied by the logarithm of the base.)

$$\ln 1 = 0, \qquad \ln e = 1, \qquad x = e^{\ln x} \qquad \text{and} \qquad \ln e^x = x \qquad \text{(d)}$$

To show (a), observe first that the definition of $\ln(xy)$ implies that $e^{\ln(xy)} = xy$. Furthermore, $x = e^{\ln x}$ and $y = e^{\ln y}$, so

$$e^{\ln(xy)} = xy = e^{\ln x} e^{\ln y} = e^{\ln x + \ln y} \qquad [*]$$

where we have used property [8.5](a). In general, $e^u = e^v$ implies $u = v$, so we conclude from [*] that $\ln(xy) = \ln x + \ln y$. The proofs of (b) and (c) are based on properties [8.5](b) and (c), respectively, and are left to the reader. Finally, [8.7](d) displays some important properties for convenient reference.

Warning: There are no simple rules for the logarithms of sums and differences. It *is* tempting to replace $\ln(x+y)$ by $\ln x + \ln y$, for instance, but this is quite wrong. In fact, $\ln x + \ln y$ is equal to $\ln(xy)$, not to $\ln(x+y)$.

There are no simple formulas for $\ln(x+y)$ and $\ln(x-y)$

Here are some examples that apply the previous rules.

Example 8.4

Express each of (a) $\ln 4$, (b) $\ln \sqrt[3]{2^5}$, and (c) $\ln(1/16)$ in terms of $\ln 2$.

Solution

(a) $\ln 4 = \ln(2 \cdot 2) = \ln 2 + \ln 2 = 2 \ln 2$. (Or $\ln 4 = \ln 2^2 = 2 \ln 2$.)

(b) We have $\sqrt[3]{2^5} = 2^{5/3}$. Therefore, $\ln \sqrt[3]{2^5} = \ln 2^{5/3} = (5/3) \ln 2$.

(c) $\ln(1/16) = \ln 1 - \ln 16 = 0 - \ln 2^4 = -4 \ln 2$. (Or $\ln(1/16) = \ln 2^{-4} = -4 \ln 2$.)

Example 8.5

Solve the following equations for x:

(a) $5e^{-3x} = 16$ **(b)** $A\alpha e^{-\alpha x} = k$

(c) $(1.08)^x = 10$ **(d)** $e^x + e^{-x} = 2$

Solution

(a) Take ln of each side of the equation to obtain $\ln(5e^{-3x}) = \ln 16$. The product rule gives $\ln(5e^{-3x}) = \ln 5 + \ln e^{-3x}$. Here $\ln e^{-3x} = -3x \ln e = -3x$, because $\ln e = 1$. Hence, $\ln 5 - 3x = \ln 16$, which gives

$$x = \tfrac{1}{3}(\ln 5 - \ln 16) = \tfrac{1}{3} \ln \tfrac{5}{16}$$

(b) We argue as in (a) and obtain $\ln(A\alpha e^{-\alpha x}) = \ln k$, or $\ln(A\alpha) + \ln e^{-\alpha x} = \ln k$, so $\ln(A\alpha) - \alpha x = \ln k$. Finally, therefore,

$$x = \frac{1}{\alpha}[\ln(A\alpha) - \ln k] = \frac{1}{\alpha} \ln \frac{A\alpha}{k}$$

(c) Again we take the ln of each side of the equation and obtain $x \ln 1.08 = \ln 10$. So the solution is $x = \ln 10 / \ln 1.08$, which is ≈ 29.9. Thus, it takes just short of 30 years for \$1 to increase to \$10 when the interest rate is 8%.

(d) It is very tempting to begin with $\ln(e^x + e^{-x}) = \ln 2$, but this leads nowhere, because $\ln(e^x + e^{-x})$ cannot be further evaluated. Instead, we argue like this: Putting $u = e^x$ gives $e^{-x} = 1/e^x = 1/u$, so the equation is $u + 1/u = 2$, or $u^2 + 1 = 2u$. Solving this quadratic equation for u yields $u = 1$ as the only solution. Hence, $e^x = 1$, and so $x = 0$. (Check this solution. Consider also the graph of $\cosh x$ in Problem 6 of Section 8.1.)

The Function $g(x) = \ln x$

For each positive number x, the number $\ln x$ is defined by $e^{\ln x} = x$. We call the function

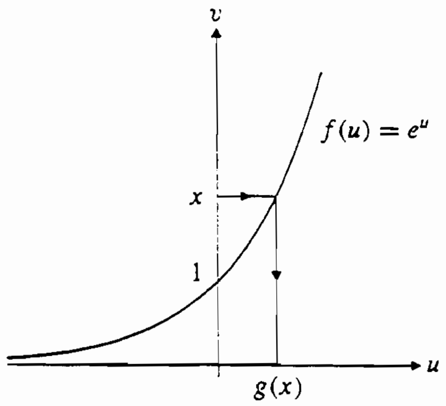
$$g(x) = \ln x \qquad (x > 0) \tag{8.8}$$

the **natural logarithmic function**. This definition is illustrated in Fig. 8.4. Think of x as a point moving upwards on the vertical axis from the origin. As x increases from values less than 1 to values greater than 1, so $g(x)$ increases from negative to positive values. In fact, because $f(u) = e^u$ is strictly increasing with range $(0, \infty)$, it follows from Theorem 7.9 of Section 7.6 that f *has* an inverse function g that is also strictly increasing with domain $(0, \infty)$. Because f has the domain $(-\infty, \infty)$, we know that g has the range $(-\infty, \infty)$. Thus, the exponential function $f(x) = e^x$ and the natural logarithm function $g(x) = \ln x$ are inverses of each other. In particular, we have (see (7.20)):

$$\ln e^x = x \qquad \text{for all } x$$

$$e^{\ln y} = y \qquad \text{for all } y > 0$$

FIGURE 8.4 Illustration of the definition of $g(x) = \ln x$.



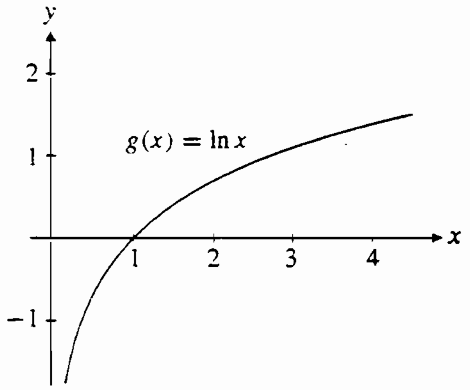


FIGURE 8.5 The graph of the natural logarithmic function $g(x) = \ln x$.

In Fig. 8.5 we have drawn the graph of $g(x) = \ln x$. The shape of this graph ought to be remembered. According to Example 8.3, we have $g(1/e) = -1$, $g(1) = 0$, and $g(e) = 1$. Observe that this corresponds well with the graph.

Differentiation of Logarithmic Functions

If we assume that $g(x) = \ln x$ has a derivative for all $x > 0$, then this derivative can be easily found. Differentiate implicitly the equation

$$e^{g(x)} = x \qquad [*]$$

with respect to x, using the result in [8.4]. This gives

$$e^{g(x)} g'(x) = 1$$

Because $e^{g(x)} = x$, so $xg'(x) = 1$. Hence:

$$g(x) = \ln x \implies g'(x) = \frac{1}{x} \qquad [8.9]$$

Thus, the derivative of $\ln x$ at point x is simply the number $1/x$. For $x > 0$, we have $g'(x) > 0$, so that $g(x)$ is *strictly* increasing. Note moreover that $g''(x) = -1/x^2$, which is less than 0 for all $x > 0$, so that $g'(x)$ is *strictly* decreasing. This confirms the shape of the graph in Fig. 8.5. In fact, the growth of $\ln x$ is quite slow. For example, $\ln x$ first attains the value 10 when $x > 22,026$, because $\ln x = 10$ gives $x = e^{10} \approx 22,026.5$.

Note: We derived [8.9] *assuming* that $g(x) = \ln x$ was differentiable. In fact, by Theorem 7.9 in Section 7.6, the logarithmic function g *is* differentiable. Because the derivative of $f(x) = e^x$ is e^x, applying (7.24) to $y_0 = e^{x_0}$ tells us that $g'(y_0) = 1/e^{x_0} = 1/y_0$. This is the same as [8.9], except that the symbol y_0 has replaced x.

Often, we need to consider composite functions involving natural logarithms. Because $\ln u$ is defined only when $u > 0$, a composite function of the form $y = \ln h(x)$ will only be defined for values of x satisfying $h(x) > 0$.

Combining the rule for differentiating $\ln x$ with the chain rule allows us to differentiate many different types of function. Suppose, for instance, that $y = \ln h(x)$, where $h(x)$ is differentiable and positive. By the chain rule, $y = \ln u$ with $u = h(x)$ implies that $y' = (1/u)u' = [1/h(x)]h'(x)$, so:

$$y = \ln h(x) \implies y' = \frac{h'(x)}{h(x)} \qquad [8.10]$$

Example 8.6

Find the domains of the following functions and compute their derivatives:

(a) $y = \ln(1 - x)$ (b) $y = \ln(4 - x^2)$

(c) $y = \ln[(x - 1)/(x + 1)] - \frac{1}{4}x$

Solution

(a) $\ln(1-x)$ is defined if $1-x > 0$, that is, if $x < 1$. To find the derivative, we use [8.10] with $h(x) = 1 - x$. Then $h'(x) = -1$, so by [8.10],

$$y' = \frac{-1}{1 - x}$$

(b) $\ln(4 - x^2)$ is defined if $4 - x^2 > 0$, that is, if $(2 - x)(2 + x) > 0$. This is true if $-2 < x < 2$. Formula [8.10] yields

$$y' = \frac{-2x}{4 - x^2}$$

(c) We require that $(x - 1)/(x + 1) > 0$. A sign diagram shows this to be satisfied if $x < -1$ or $x > 1$. We have $y = \ln u - \frac{1}{4}x$, where $u = (x - 1)/(x + 1)$. Using [8.10], we find that

$$f'(x) = \frac{u'}{u} - \frac{1}{4}$$

where

$$u' = \frac{1 \cdot (x + 1) - 1 \cdot (x - 1)}{(x + 1)^2} = \frac{2}{(x + 1)^2}$$

Thus,

$$f'(x) = \frac{2(x+1)}{(x+1)^2(x-1)} - \frac{1}{4} = \frac{9-x^2}{4x^2-4} = \frac{(3-x)(3+x)}{4(x-1)(x+1)}$$

Note: If we apply the quotient rule [8.7](b) for ln to the formula in (c) of Example 8.6, we obtain

$$f(x) = \ln(x-1) - \ln(x+1) - \tfrac{1}{4}x \qquad [*]$$

By differentiating this expression, it is easier to derive a correct formula for $f'(x)$. But note that the expression in [*] is only defined when $x > 1$, whereas the formula in (c) is also defined for $x < -1$. The point is that the formula $\ln(p/q) = \ln p - \ln q$ is correct only when p and q are both positive, whereas $\ln(p/q)$ also is meaningful when p and q are both negative. Then $\ln(p/q) = \ln(-p) - \ln(-q)$.

Logarithmic Differentiation

When differentiating an expression containing products, quotients, roots, powers, and combinations of these, it is often an advantage to use **logarithmic differentiation**. The method is illustrated by the following example:

Example 8.7

Find the derivative of

$$y = A\frac{x^p(ax+b)^q}{(cx+d)^r} \qquad [1]$$

Solution First, take the natural logarithm of each side to obtain

$$\ln y = \ln A + p\ln x + q\ln(ax+b) - r\ln(cx+d)$$

Differentiation with respect to x yields

$$\frac{y'}{y} = p\frac{1}{x} + q\frac{1}{ax+b}a - r\frac{1}{cx+d}c$$

Multiplying by y, which is given by [1], yields

$$y' = A\frac{x^p(ax+b)^q}{(cx+d)^r}\cdot\left(\frac{p}{x} + \frac{aq}{ax+b} - \frac{cr}{cx+d}\right)$$

A Survey of the Properties of ln

The natural logarithmic function

$$g(x) = \ln x$$

is differentiable and strictly increasing for all $x > 0$. In fact,

$$g'(x) = 1/x$$

The following properties hold for all $x > 0$, $y > 0$:

(a) $\ln(xy) = \ln x + \ln y$ (b) $\ln(x/y) = \ln x - \ln y$ (c) $\ln x^p = p \ln x$

Moreover, $\ln e^x = x$ for all real x, and

$$\ln x \to -\infty \text{ as } x \to 0^+, \qquad \ln x \to \infty \text{ as } x \to \infty$$

Problems

1. Express the following in terms of $\ln 3$:

   a. $\ln 9$ b. $\ln \sqrt{3}$ c. $\ln \sqrt[5]{3^2}$ d. $\ln \dfrac{1}{81}$

2. Solve the following equations for x:

   a. $3^x = 8$ b. $\ln x = 3$ c. $\ln(x^2 - 4x + 5) = 0$

   d. $\ln[x(x-2)] = 0$ e. $\dfrac{x \ln(x+3)}{x^2+1} = 0$ f. $\ln(\sqrt{x} - 5) = 0$

3. Solve the following equations for x:

   a. $3^x 4^{x+2} = 8$ b. $3 \ln x + 2 \ln x^2 = 6$ c. $4^x - 4^{x-1} = 3^{x+1} - 3^x$

4. Solve the following equations for t:

   a. $x = e^{at+b}$ b. $e^{-at} = 1/2$ c. $\dfrac{1}{\sqrt{2\pi}} e^{-t^2} = \dfrac{1}{8}$

5. Prove the following equalities (with appropriate restrictions on the variables):

   a. $\ln x - 2 = \ln(x/e^2)$
   b. $\ln x - \ln y + \ln z = \ln(xz/y)$
   c. $3 + 2 \ln x = \ln(e^3 x^2)$
   d. $\dfrac{1}{2} \ln x - \dfrac{3}{2} \ln \dfrac{1}{x} - \ln(x+1) = \ln \dfrac{x^2}{x+1}$
   e. $-p_1 \ln p_1 - p_2 \ln p_2 - \cdots - p_n \ln p_n = \sum_{i=1}^{n} p_i \ln(1/p_i)$

6. True or false: (a) $\pi^e < e^\pi$ and (b) $\sqrt[e]{e} > \sqrt[\pi]{\pi}$?

7. Decide whether the following formulas are always correct or sometimes wrong (all variables are positive):

   a. $(\ln A)^4 = 4 \ln A$ b. $\ln B = 2 \ln \sqrt{B}$ c. $\ln A^{10} - \ln A^4 = 3 \ln A^2$

8. Decide whether the following formulas are always correct or sometimes wrong (all variables are positive):

a. $\ln \dfrac{A+B}{C} = \ln A + \ln B - \ln C$ **b.** $\ln \dfrac{A+B}{C} = \ln(A+B) - \ln C$

c. $\ln \dfrac{A}{B} + \ln \dfrac{B}{A} = 0$ **d.** $p \ln(\ln A) = \ln(\ln A^p)$

e. $p \ln(\ln A) = \ln(\ln A)^p$ **f.** $\dfrac{\ln A}{\ln B + \ln C} = \ln A(BC)^{-1}$

9. Determine the domains of the functions given by the following:

a. $y = \ln(x+1)$ **b.** $y = \ln \dfrac{3x-1}{1-x}$ **c.** $y = \ln |x|$

d. $y = \ln(x^2 - 1)$ **e.** $y = \ln(\ln x)$ **f.** $y = \dfrac{1}{\ln(\ln x) - 1}$

10. Find the derivatives of the functions defined by the following:

a. $\ln(x+1)$ **b.** $\ln x + 1$ **c.** $x \ln x$ **d.** $\dfrac{x}{\ln x}$

11. Find the derivatives of the functions defined by the following:

a. $\ln(\ln x)$ **b.** $\ln \sqrt{(1-x^2)}$ **c.** $e^x \ln x$

d. $e^{x^3} \ln x^2$ **e.** $\ln(e^x + 1)$ **f.** $\ln(x^2 + 3x - 1)$

12. Find the equation of the tangent line for the following:

a. $y = \ln x$ at the point with the x-coordinate: (i) 1; (ii) $\frac{1}{2}$; and (iii) e.

b. $y = xe^x$ at the point with the x-coordinate: (i) 0; (ii) 1; and (iii) -2.

13. Use logarithmic differentiation to find the derivatives of the following:

a. $f(x) = \left(\dfrac{x+1}{x-1}\right)^{1/3}$ **b.** $f(x) = x^x$

c. $f(x) = \sqrt{x-2}\,(x^2+1)(x^4+6)$

14. If $f(x) = e^x - 1 - x$, then $f(0) = 0$ and $f'(x) = e^x - 1 > 0$ for all $x > 0$. Hence, $f(x)$ is strictly increasing and $f(x) > 0$ for all $x > 0$, so $e^x > 1 + x$ for all $x > 0$. Prove the following inequalities using the same method.

a. $e^x > 1 + x + x^2/2$ for $x > 0$

b. $\frac{1}{2}x < \ln(1+x) < x$ for $0 < x < 1$

c. $\ln\left(\dfrac{1+t}{1-t}\right) > 2t$ for $0 < t < 1$

15. Consider the function f defined for all x by

$$f(x) = e^{x-1} - x$$

a. Show that $f(x) \geq 0$ for all x. (*Hint:* Study the sign of $f'(x)$. Draw the graph.)

b. Show that the equation $e^{x-1} - x = 1$ has precisely two solutions.

c. Define the function g by the formula

$$g(x) = \frac{1}{\ln(e^{x-1} - x)}$$

For which x is g defined? Examine $g(x)$ as $x \to \infty$ and $x \to -\infty$.

d. Draw the graph of g.

16. Simplify the following expressions:

a. $\exp\left[\ln(x)\right] - \ln\left[\exp(x)\right]$ b. $\ln\left[x^4 \exp(-x)\right]$

c. $\exp\left[\ln(x^2) - 2\ln y\right]$

17. The *extreme-value distribution* in statistics is given by

$$F(x) = \exp\left[-\exp(-x)\right]$$

a. Write $F(x)$ in standard form.

b. Compute $f(x) = F'(x)$, and write the result in two ways.

c. Function f is called the *density function* associated with F. Compute $f'(x)$.

18. The elasticity of $y = f(x)$ with respect to x is defined in Section 5.6 as

$$\mathrm{El}_x y = \frac{x}{y} y'$$

Use this definition to find the elasticities of the following:

a. $y = e^x$ b. $y = \ln x$ c. $y = a^x$

19. Compute the elasticities of the following functions (where a and δ are constants, $\delta \neq 0$):

a. $y = e^{ax}$ b. $y = x^3 e^{2x}$ c. $y = x\ln(x+1)$ d. $y = (x^{-\delta} + 1)^{-1/\delta}$

20. Differentiate the following functions using logarithmic differentiation:

a. $x^{\sqrt{x}}$ b. $\left(\sqrt{x}\right)^x$ c. $x^{(x^x)}$

21. Show by logarithmic differentiation that if u and v are differentiable functions of x, and if $u > 0$, then

$$y = u^v \Longrightarrow y' = u^v\left(v'\ln u + \frac{vu'}{u}\right)$$

22. In an article on production theory, the function

$$F(\alpha) = a\left(\frac{N^\alpha K^\alpha}{N^\alpha + bK^\alpha}\right)^{v/\alpha} \quad (a,\ b,\ v,\ N, \text{ and } K \text{ are positive constants})$$

was studied. Find an expression for $F'(\alpha)$.

23. Find the inverse of $y = \sinh x = \frac{1}{2}(e^x - e^{-x})$. (See Problem 6 in Section 8.1.) (*Hint:* You will have to solve a quadratic equation for $u = e^x$.)

8.3 Generalizations

Every positive number a can be written in the form $a = e^{\ln a}$, so using the general property $(e^r)^s = e^{rs}$, we have the formula

$$a^x = (e^{\ln a})^x = e^{x \ln a}$$

In problems where function a^x occurs, we can just as easily work with the special exponential function e^{bx}, where b is a constant equal to $\ln a$. In particular, we can differentiate a^x by differentiating $e^{x \ln a}$. Letting $g(x) = x \ln a$ and applying the chain rule [8.4], we obtain the following:

$$y = a^x \Longrightarrow y' = a^x \ln a \qquad [8.11]$$

If $a = 10$, for example, then $y = 10^x \Rightarrow y' = 10^x \ln 10$. Whereas if $a = e$, we obtain $y = e^x \Rightarrow y' = e^x$, because $\ln e = 1$.

Note: Comparing [8.11] with [8.2] in Section 8.1, we see that $f'(0) = \ln a$. From the definition of $f'(0)$ in [8.1], it follows that $(a^h - 1)/h \to \ln a$ as $h \to 0$. Replacing a by x, we have

$$\lim_{h \to 0} \frac{x^h - 1}{h} = \ln x \qquad (x > 0)$$

Let us take a closer look at this limit. For any $h > 0$, define the function g_h by[1]

$$g_h(x) = \frac{x^h - 1}{h}$$

for all $x > 0$. Then

$$\lim_{h \to 0} g_h(x) = \lim_{h \to 0} \frac{x^h - 1}{h} = \ln x$$

In fact, $\ln x$ is bounded above by each of the functions $g_h(x)$ ($h > 0$). To see this, consider for each $h > 0$ the function $F_h(x) = g_h(x) - \ln x = (x^h - 1)/h - \ln x$

[1] Function g_h and its limit as h approaches 0 are related to the well known Box–Cox transformation in statistics.

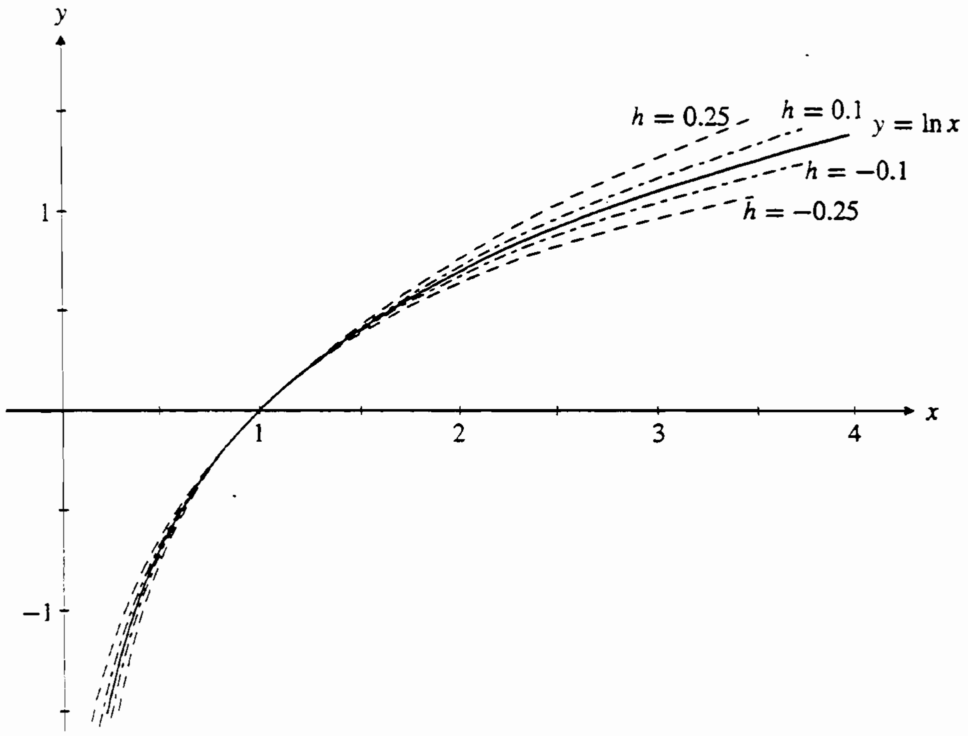


FIGURE 8.6 $y = (x^h - 1)/h$ $(h = \pm 0.25, \pm 0.1)$ and $y = \ln x$.

defined for $x > 0$. Then $F_h(1) = 0$ and

$$F_h'(x) = \frac{hx^{h-1}}{h} - \frac{1}{x} = \frac{x^h - 1}{x} \begin{cases} < 0, & \text{if } 0 < x < 1 \\ > 0, & \text{if } x > 1 \end{cases}$$

Thus, $F_h(x)$ decreases from positive values to 0 when $0 < x < 1$, but increases from 0 to positive values when $x > 1$. It follows that $F_h(x) > 0$ for all $x > 0$, except at $x = 1$, and so

$$g_h(x) = \frac{x^h - 1}{h} > \ln x \qquad (\text{for all } x > 0,\ x \neq 1)$$

Figure 8.6 illustrates how $g_h(x)$ tends to $\ln x$ as h tends to 0.

Logarithms with Bases Other Than e

Recall that we defined $\ln x$ as the exponent to which we must raise the base e in order to obtain x. From time to time, it is useful to have logarithms based on numbers other than e. For many years, until the use of mechanical and then electronic calculators became widespread, tables of logarithms to the base 10 were frequently used to simplify complicated calculations involving multiplication, division, square roots, and so on.

Suppose that a is a fixed positive number (usually chosen > 1). If $a^u = x$, then we call u the **logarithm of x to base a** and write $u = \log_a x$. The symbol $\log_a x$ is then defined for every positive number x by the following:

$$a^{\log_a x} = x \tag{8.12}$$

For instance, $\log_2 32 = 5$, because $2^5 = 32$, whereas $\log_{10}(1/100) = -2$, because $10^{-2} = 1/100$. By taking the ln of each side of [8.12], we obtain

$$\log_a x \cdot \ln a = \ln x$$

so that

$$\log_a x = \frac{1}{\ln a} \ln x \tag{8.13}$$

This reveals that the logarithm of x in the system with base a is proportional to $\ln x$, with a proportionality factor $1/\ln a$. It follows immediately that $\log_a$ obeys the same rules as ln (compare [8.7] of Section 8.2):

Rules for $\log_a$

[8.14]

$$\log_a(xy) = \log_a x + \log_a y \qquad \text{(a)}$$

$$\log_a \frac{x}{y} = \log_a x - \log_a y \qquad \text{(b)}$$

$$\log_a x^p = p \log_a x \qquad \text{(c)}$$

$$\log_a 1 = 0 \quad \text{and} \quad \log_a a = 1 \qquad \text{(d)}$$

For example, 8.14(a) follows from the corresponding rule [8.7](a) for ln:

$$\begin{aligned}\log_a(xy) &= \frac{1}{\ln a} \ln(xy) = \frac{1}{\ln a}(\ln x + \ln y) \\ &= \frac{1}{\ln a} \ln x + \frac{1}{\ln a} \ln y = \log_a x + \log_a y\end{aligned}$$

From [8.13] and [8.9], we obtain

$$y = \log_a x \implies y' = \frac{1}{\ln a} \frac{1}{x} \tag{8.15}$$

A Characterization of the Number e

In Section 8.2, we showed by implicit differentiation that if $g(x) = \ln x$ is differentiable, then $g'(x) = 1/x$. More specifically, $g'(1) = 1$. If we use the *definition* of $g'(1)$ and [8.7](c), together with the fact that $\ln 1 = 0$, we obtain

$$1 = g'(1) = \lim_{h\to 0} \frac{\ln(1+h) - \ln 1}{h} = \lim_{h\to 0} \frac{1}{h}\ln(1+h) = \lim_{h\to 0} \ln(1+h)^{1/h}$$

Because $\ln(1+h)^{1/h}$ tends to 1 as h tends to 0, it follows that $(1+h)^{1/h}$ itself must tend to e, and so

$$e = \lim_{h\to 0}(1+h)^{1/h} \qquad [8.16]$$

TABLE 8.1 *Values of* $(1+h)^{1/h}$

h	1	1/2	1/10	1/1000	1/100000	1/1000000
$(1+h)^{1/h}$	2.00	2.25	2.5937...	2.7169...	2.71825...	2.718281828...

Table 8.1 has been computed using a scientific calculator. The results seem to confirm that the decimal expansion we gave for e is correct. From the table, we can see that a closer and closer approximation to e is obtained by choosing h smaller and smaller. If we let $h = 1/n$, where the natural number n becomes larger and larger, we obtain the following:

$$e = \lim_{n\to\infty}(1+1/n)^n \qquad [8.17]$$

Another Important Limit

If a is an arbitrary number greater than 1, then $a^x \to \infty$ as $x \to \infty$. For example, $(1.0001)^x \to \infty$ as $x \to \infty$. Furthermore, if p is an arbitrary positive number, then $x^p \to \infty$ as $x \to \infty$. If we compare $(1.0001)^x$ and x^{1000}, it is clear that the former increases quite slowly at first, whereas the latter increases very quickly. Nevertheless, $(1.0001)^x$ eventually "overcomes" x^{1000}. In general, we claim the following:

$$\lim_{x\to\infty} \frac{x^p}{a^x} = 0 \qquad (a > 1,\ p \text{ is a fixed number}) \qquad [8.18]$$

For example x^2/e^x and $x^{10}/(1.1)^x$ will both tend to 0 as x tends to ∞. The result [8.18] is actually quite remarkable. It can be expressed briefly by saying that, for an arbitrary base greater than 1, *the exponential function increases faster than any power of* x. Even more succinctly: *"Exponentials drown powers."*

To prove [8.18], it suffices to prove that $\ln(x^p/a^x) \to -\infty$ as $x \to \infty$, for then $x^p/a^x \to 0$ as $x \to \infty$ (see Fig. 8.5). In fact,

$$\ln \frac{x^p}{a^x} = p \ln x - x \ln a = x \left(p \frac{\ln x}{x} - \ln a \right)$$

Because $a > 1$, we have $\ln a > 0$. If we could show that

$$\frac{\ln x}{x} \to 0 \quad \text{as} \quad x \to \infty \tag{8.19}$$

then we would be able to infer that $p(\ln x/x) - \ln a \to -\ln a$, and so the proof would be complete. But [8.19] is an easy consequence of l'Hôpital's rule for the "$\pm\infty/\pm\infty$" case. In fact,

$$\lim_{x\to\infty} \frac{\ln x}{x} = \text{"}\frac{\infty}{\infty}\text{"} = \lim_{x\to\infty} \frac{1/x}{1} = 0$$

The General Power Function

In Section 4.5 we claimed that, for all real numbers a,

$$f(x) = x^a \Longrightarrow f'(x) = ax^{a-1} \tag{$*$}$$

Actually, however, we did not even define x^a for irrational values of a. Now we can give such a definition for all $x > 0$. Because $x = e^{\ln x}$, we can define

$$x^a = (e^{\ln x})^a = e^{a \ln x}$$

Using the chain rule, we obtain

$$\frac{d}{dx}(x^a) = \frac{d}{dx}(e^{a \ln x}) = e^{a \ln x} \cdot \frac{a}{x} = x^a \frac{a}{x} = ax^{a-1}$$

In this way, the differentiation rule [$*$] is proved also when a is an irrational number.

Taylor's Formula for e^x

If $f(x) = e^x$, then all the derivatives of f are equal to e^x, and so the kth derivative of f at $x = 0$ is 1—that is, $f^{(k)}(0) = 1$ for $k = 1, 2, \ldots, n$. Therefore, Taylor's theorem [7.10] in Section 7.4 says that, for some number c between 0 and x, one has

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \frac{x^{n+1}}{(n+1)!}e^c \qquad [8.20]$$

One can show that for every fixed number x, the remainder in [8.20] approaches 0 as n approaches infinity. So [8.20] allows one to compute the value of e^x for any x to an arbitrary degree of accuracy. However, if $|x|$ is large, we must be prepared to use a large number of terms in order to obtain a good approximation, because the remainder in this case approaches 0 very slowly as n approaches infinity—indeed, the early terms will grow bigger quite fast before eventually starting to decline.

Let us see what estimate of $e^{0.1} = \sqrt[10]{e}$ emerges when $n = 3$. Putting $x = 0.1$ and $n = 3$ in [8.20] yields

$$e^{0.1} = 1 + \frac{1}{10} + \frac{1}{200} + \frac{1}{6000} + \frac{(0.1)^4}{24}e^c \qquad [*]$$

for some c in the interval $(0, 0.1)$. Because $c < 0.1$, so $e^c < e^{0.1} < 1.2$, where the last inequality holds because $(e^{0.1})^{10} = e < (1.2)^{10} \approx 6.2$. Hence,

$$0 < \frac{(0.1)^4}{24}e^c < \frac{1}{240000}1.2 = 0.000005$$

So when we drop the remainder in [*], the error that results is less than 0.000005. The approximation $e^{0.1} \approx 1 + 0.1 + 0.005 + 0.00017 = 1.10517$ is accurate to five decimal places.

Problems

1. Compute the following:

   a. $\log_5 25$ **b.** $\log_5 \sqrt{125}$ **c.** $\log_5 1/25$ **d.** $\log_{10} 100^{-3}$

2. Find x for the following:

   a. $\log_2 x = 2$ **b.** $\log_x e^2 = 2$ **c.** $\log_3 x = -3$ **d.** $\log_{10} x^2 = 100$

3. Differentiate the functions given by the following:

   a. $y = 5 \cdot 3^x$ **b.** $y = 2^x \ln x$ **c.** $y = x \log_2 x$ **d.** $y = \log_2 \sqrt{1 + x^2}$

4. Solve for x:

   a. $\dfrac{e^{x+1}}{e^{4/x}} = e$ **b.** $\left[\ln(x+e)\right]^3 - \left[\ln(x+e)^2\right]^2 = \ln(x+e) - 4$

5. Solve the following inequalities:

   a. $\ln x \le -1$ **b.** $\ln(x^2 - x - 1) \ge 0$ **c.** $\ln x + \ln(x-3) \le \ln 4$

6. By using l'Hôpital's rule (Theorem 7.8 in Section 7.5), or otherwise, determine the following limits:

a. $\lim_{x\to 0} \frac{e^x - 1}{x}$ b. $\lim_{t\to 0} \frac{e^{t+1} - e^{t-1}}{t^2}$ c. $\lim_{x\to 2} \frac{\ln(x-1)}{\sqrt{2+x} - \sqrt{8-x^2}}$

d. $\lim_{x\to\infty} x^{1/x}$ e. $\lim_{x\to 0^+} x \ln x$ f. $\lim_{x\to 0^+} x^x$

7. Evaluate the limit

$$\lim_{\lambda\to 0^+} \frac{x^\lambda - y^\lambda}{\lambda}$$

where x and y are positive constants.

8. For the following functions, find Taylor approximations of order 3 about $x = 0$ by using [5.11] in Section 5.5. (You can partly check the results by using [8.20].)

a. xe^x b. e^{2x} c. $x^2 + e^{x/2}$ d. $\sqrt{e^x + 1}$

9. Use Taylor polynomials of degree 3 to find approximate solutions of the equation

$$\tfrac{1}{3}x^3 + x(e^x + e^{-x}) - (e^x - e^{-x}) - x = 0$$

Harder Problems

10. For the function $f(x) = e^{-1/x^2}$ $(x \neq 0)$, $f(0) = 0$, verify that $f^{(k)}(x) = x^{-3k} p_k(x) e^{-1/x^2}$ $(x \neq 0)$, where $p_k(x)$ denotes some polynomial whose degree is $2k - 2$. Hence, show that $f^{(k)}(0) = 0$ for all positive integers k. (For this function, *all* Taylor polynomials at the origin are identically equal to 0, but the function itself is 0 only at the origin. The lesson of this example is that in order to be certain that the Taylor polynomials of a function give a good approximation to the function, one *must* estimate the size of the remainder.)

8.4 Applications of Exponentials and Logarithms

Suppose that $f(t)$ denotes the stock of some quantity at time t. The ratio $f'(t)/f(t)$ is the *relative* or *proportional rate of increase* of the stock at time t. In many applications, the relative rate of increase is a constant r. Then

$$f'(t) = rf(t) \qquad \text{(for all } t\text{)} \tag{8.21}$$

Which functions have a constant relative rate of increase? Functions of the type $f(t) = Ae^{rt}$ certainly have, because $f'(t) = Are^{rt} = rf(t)$. We claim that there are no other functions having this property. Suppose that g is *any* function satisfying $g'(t) = rg(t)$, for all t. Define a new function h by $h(t) = g(t)e^{-rt}$. Then $h'(t) = g'(t)e^{-rt} + g(t)(-r)e^{-rt} = e^{-rt}\left[g'(t) - rg(t)\right]$, which is 0 for all t. Thus, $h(t) = A$ for some constant A, so that $g(t) = Ae^{rt}$. Hence, we have proved that

$$f'(t) = rf(t) \quad \text{for all } t \iff f(t) = Ae^{rt} \quad \text{for a constant } A \qquad [8.22]$$

We now consider some applications where [8.22] is important.

Ecology

Suppose that $f(t)$ denotes the number of individuals in a population at time t. The population could be, for instance, a particular colony of bacteria, or the polar bears in the Arctic. We call $f'(t)/f(t)$ the *per capita rate of increase* of the population. If there is neither immigration nor emigration, then the per capita rate of increase of the population will be equal to the difference between the per capita birth and death rates. These rates will depend on many factors such as food supply, age distribution, available living space, predators, disease, and parasites, among other things.

Equation [8.21] specifies a very simple model of population increase. The result [8.22] implies that the population must grow exponentially. In reality, of course, exponential growth can go on only for a limited time. Instead of assuming that the relative rate of increase is constant, it is more realistic to assume that once the population is above a certain quantity K (called the population's *carrying capacity*), the per capita rate of increase is negative. A special form of this assumption is expressed by the equation

$$f'(t) = rf(t)\left(1 - \frac{f(t)}{K}\right) \qquad [8.23]$$

Observe that when the population $f(t)$ is small in proportion to K, so that $f(t)/K$ is small, then $f'(t) \approx rf(t)$, and $f(t)$ increases (approximately) exponentially. As $f(t)$ becomes larger, however, the factor $1 - f(t)/K$ increases in significance. In general, one can show (see Problem 8) that if $f(t)$ satisfies [8.23] (and is not identically equal to 0), then $f(t)$ must have the form

$$f(t) = \frac{K}{1 + Ae^{-rt}} \qquad \text{(for some constant } A\text{)} \qquad [8.24]$$

If there are N_0 individuals at time $t = 0$, then $f(0) = N_0$, and [8.24] gives $N_0 = K/(1 + A)$, so that $A = (K - N_0)/N_0$. Provided that $N_0 < K$, then $A > 0$. So it follows from [8.24] that $f(t)$ is strictly increasing, and $f(t) \to K$ as $t \to \infty$ (assuming that $r > 0$). The graph of $f(t)$ is shown in Fig. 8.7.

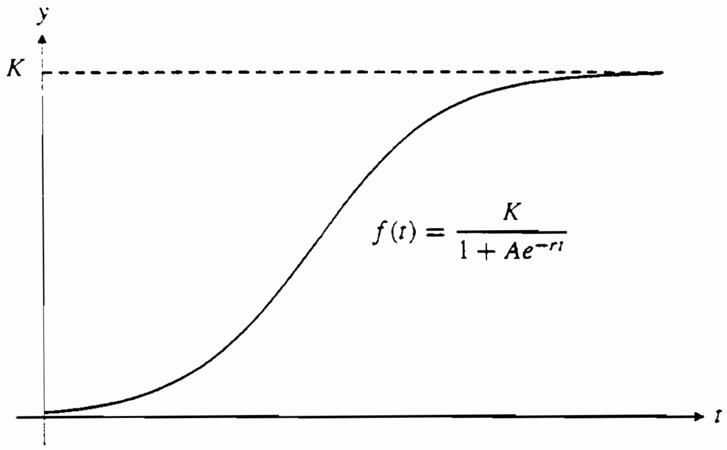


FIGURE 8.7 Logistic growth towards the level K.

Equations like [8.23] with solutions of the form [8.24] appear in numerous models—see, for instance, Problems 5 and 6. Function f defined by [8.24] is called a **logistic function.**

Log-Linear Relations

Suppose that two variables x and y are related by the equation

$$y = Ax^a \qquad (x,\ y,\ \text{and } A \text{ are positive}) \tag{1}$$

Let log denote the logarithm to any base. By taking the log of each side of [1] while applying the rules [8.14], we find that [1] is equivalent to the equation

$$\log y = \log A + a \log x \tag{2}$$

From [2], we see that $\log y$ is a linear function of $\log x$, and so we say that [1] is a *log-linear* relation between x and y. The transformation from [1] to [2] is often seen in economic models, usually with natural logarithms. (See Problems 9 and 10, for example.)

Suppose that the relation between the two positive variables x and y is set out in a table. Make a new table for the relation between $\ln x$ and $\ln y$. Plot the results in a new coordinate system where $\ln x$ and $\ln y$ are measured along the two axes. If, to a good approximation, the resulting points are all on a straight line, then the relation between x and y will be approximately of the form $y = Ax^a$. (In Section 15.7 we show how to find a straight line that, in a certain precise sense, fits the data as well as possible.)

Example 8.8

Table 8.2 is from *Consumer Survey 1980–1982* (published by the Norwegian Central Bureau of Statistics, 1984). It gives the relationship between y, consumption expenditure on health care, and x, total consumption expenditure, for married couples without children whose total consumption expenditure was below 150,000 Norwegian crowns. For this purpose, the population has been divided into four different consumption expenditure groups.

TABLE 8.2 *From the Consumer Survey 1980–1982*

x	28,316	49,412	77,906	122,085
y	664	1028	1501	2010

(a) Make a table for the relation between $\ln x$ and $\ln y$, and plot this data in a coordinate system where $\ln x$ and $\ln y$ are measured along the two axes.

(b) Roughly fit a straight line through the point pairs in the resulting diagram, and construct an empirical formula for y as a function of x.

Solution

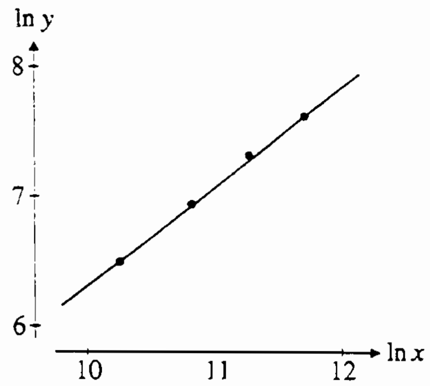
(a) We construct Table 8.3.

TABLE 8.3

$\ln x$	10.25	10.81	11.26	11.71
$\ln y$	6.50	6.94	7.31	7.61

(b) The straight line we have drawn through the two extreme points in Fig. 8.8 seems to give a good approximation to all the numbers in the table. The equation for the line is of the form $\ln y = \ln A + a \ln x$.

FIGURE 8.8



Using the extreme points (10.25, 6.50) and (11.71, 7.61), we find that the slope of the line is

$$a = \frac{7.61 - 6.5}{11.71 - 10.25} = \frac{1.11}{1.46} \approx 0.76$$

If we require the line to pass through (11.71, 7.61), then $7.61 = \ln A + 0.76 \cdot 11.71$. Hence, $\ln A = 7.61 - 0.76 \cdot 11.71 = -1.2896$, so that $A = e^{-1.2896} \approx 0.275$. The relation between y and x is then

$$y = 0.275\, x^{0.76}$$

Suppose that y is an exponential function

$$y = Aa^x \qquad (a \text{ and } A \text{ are positive})$$

Taking the log of each side gives

$$\log y = \log A + x \log a \tag{*}$$

We see that in this case $\log y$ becomes *a linear function of* x. In a coordinate system where there is an ordinary (linear) scale on the horizontal axis and a $\log y$ scale along the vertical axis, [*] represents a straight line with slope $\log a$.

Elasticities and Logarithmic Differentiation

In Example 5.20 of Section 5.6, we considered the demand function $D(p) = 8000p^{-1.5}$ and showed that the elasticity $\mathrm{El}_p D = (p/D)\,(dD/dp)$ is equal to the exponent -1.5. Taking natural logarithms of this demand relation gives

$$\ln D(p) = \ln 8000 - 1.5 \ln p$$

So $\mathrm{El}_p D$ is also equal to the (double) logarithmic derivative $d \ln D(p)/d \ln p$, which is the constant slope of this *log-linear* relationship.

This example illustrates the general rule that elasticities are equal to such logarithmic derivatives. In fact, whenever x and y are both positive variables, with y a differentiable function of x, one has

$$\mathrm{El}_x y = \frac{x}{y}\frac{dy}{dx} = \frac{d \ln y}{d \ln x} = \frac{d \log_a y}{d \log_a x} \tag{8.25}$$

where a is any positive base for constructing logarithms. The first equality just repeats the definition of elasticity. To see why the second must hold, note that

$\ln y$ is a differentiable function of y, whereas y is assumed to be a differentiable function of x, and $x = e^{\ln x}$ is a differentiable function of $\ln x$. So the chain rule can be applied twice to give

$$\frac{d \ln y}{d \ln x} = \frac{d \ln y}{dy} \cdot \frac{dy}{dx} \cdot \frac{dx}{d \ln x}$$

But

$$\frac{d \ln y}{dy} = \frac{1}{y}, \qquad \frac{dx}{d \ln x} = \frac{d e^{\ln x}}{d \ln x} = e^{\ln x} = x$$

Substituting these values into the previous expression for $d \ln y / d \ln x$ implies that

$$\frac{d \ln y}{d \ln x} = \frac{1}{y} \cdot \frac{dy}{dx} \cdot x = \frac{x}{y}\frac{dy}{dx} = \mathrm{El}_x y$$

Finally, from [8.13], it follows that $\log_a z$ is proportional to $\ln z$, so the third equality in [8.25] is easy to verify.

Problems

1. Compute the relative rate of increase $\dot{x}/x$ for the following:

   a. $x = 5t + 10$ **b.** $x = \ln(t + 1)$ **c.** $x = 5e^t$

   d. $x = -3 \cdot 2^t$ **e.** $x = e^{t^2}$ **f.** $x = e^t + e^{-t}$

   Which of these functions have a constant relative rate of increase? Compare your findings with the result in [8.22].

2. In a stable market where no sales promotion is carried out, the decrease in $S(t)$, sales per unit of time of a commodity, has shown a tendency to be proportional to the quantity of sales, so that

   $$S'(t) = -aS(t)$$

   a. Find an expression for $S(t)$ when sales at time 0 are S_0. (*Hint:* Use [8.22].)

   b. Solve the equation $S_0 e^{-at} = \frac{1}{2} S_0$ for t. Interpret the answer.

3. The world's population in 1975 was almost 4 billion, and increasing by nearly 2% per year. If we assume that the population increases exponentially at this relative rate, then t years after the year 1975 the population in billions will be equal to

   $$P(t) = 4e^{0.02t}$$

   a. Estimate the world's population by the year 2000 ($t = 25$).

b. How long will it take before the world's population doubles, if its growth continues at the same rate?

4. Let $P(t)$ denote Europe's population in millions t years after 1960. According to Example 2.17 of Section 2.5, we have $P(0) = 641$ and $P(10) = 705$. Suppose that $P(t)$ grows exponentially, with $P(t) = 641e^{kt}$. Compute k and then find $P(15)$ and $P(40)$ (estimates of the population in 1975 and in 2000). Compare with the UN numbers in Table 2.4.

5. The number $N(t)$ of persons who develop influenza t days after a group of 1000 persons has been in contact with a carrier of infection is given by

$$N(t) = \frac{1000}{1 + 999e^{-0.39t}}$$

a. How many develop influenza after 20 days?
b. How many days does it take until 800 are sick?
c. Will everyone eventually get influenza?

6. A study of tractors in British agriculture from 1950 onwards estimated that the number y in use (measured in 1000 tractors), as a function of t (measured in years, so that $t = 0$ corresponds to 1950), is given by

$$y = 250.9 + \frac{228.46}{1 + 8.11625e^{-0.340416t}}$$

a. Find the number of tractors in 1950. How many tractors were added in the decade up to 1960?
b. Find the limit for y as $t \to \infty$, and draw the graph.

7. After the big flood catastrophe in Holland in 1953, a research project was initiated to determine the optimal height of the dikes. One of the (simpler) models involved finding the value of x that minimizes the function

$$f(x) = I_0 + kx + Ae^{-\alpha x} \qquad (x \geq 0)$$

where x denotes the number of meters that should be added to the dikes, $I_0 + kx$ is the construction cost, and $Ae^{-\alpha x}$ is an estimate of the losses caused by flooding. I_0, k, A, and α are all positive constants.

a. Suppose that $f(x)$ has minimum for some $x_0 > 0$. Find x_0.
b. What condition must we put on α, A, and k for x_0 to be positive? Show that if the condition is satisfied, then x_0 solves the minimization problem.
c. Constant A is given by the formula

$$A = \frac{100}{\delta} p_0 V \left(1 + \frac{\delta}{100}\right)$$

where p_0 is the probability that the dikes will be flooded if they are not rebuilt, V is an estimate of the cost of flood damage, and δ is an interest rate. Show that x_0 may be written in the form

$$x_0 = \frac{1}{\alpha} \ln \frac{100\alpha p_0 V (1 + \delta/100)}{k\delta}$$

Examine what happens to x_0 when one of the variables p_0, V, δ, or k increases. Comment on the reasonableness of the results.[2]

8. Suppose that $f(t)$ is a function satisfying [8.23], and define a new function h by $h(t) = -1 + K/f(t)$. Prove that $h'(t) = -rh(t)$ for a constant A, so (using [8.22]) one has $h(t) = Ae^{-rt}$ for some constant A. Then what can be said about $f(t)$?

9. Voorhees and colleagues studied the transportation systems in 37 American cities and estimated the average travel time to work, m (in minutes), as a function of the number of inhabitants, N. They found that

$$m = e^{-0.02} N^{0.19}$$

Write the relation in ln-linear form. What is the value of m when $N =$ 480,000?

10. The following data are taken from a survey of persons who in 1933 migrated to Tartu in Estonia from the surrounding countryside. Here y is the number of persons that moved per 100,000 rural inhabitants, and x is the distance moved (measured in kilometers and rounded to the nearest whole number divisible by 20).

x	20	40	60	80	100	120	140	160	180	200
y	1700	550	230	120	75	60	45	35	25	20

a. Construct a table of the relationship between $\ln x$ and $\ln y$, and plot the data in a coordinate system where $\ln x$ and $\ln y$ are plotted along the two axes.

b. Roughly fit a straight line to the pairs of points in the diagram plotted for part (a), and derive an empirical formula for y as a function of x.

11. Write the relation $z = 694{,}500p^{-0.3}$ in ln-linear form (see Example 3.8 in Section 3.4). In addition, find p expressed in terms of z.

12. a. Determine the constants A and a such that the graph of $y = Ax^a$ passes through the points $(x, y) = (2, 5)$ and $(3, 7)$. (*Hint:* Use the ln-linear form.)

[2] The problem is discussed in D. van Dantzig, "Economic Decision Problems for Flood Prevention," *Econometrica*, 24 (1956): 276–287.

b. Repeat part (a) when the graph goes through points (x_1, y_1) and (x_2, y_2), where $x_1 \neq x_2$.

13. The effect on chicken embryos of cooling the eggs has been studied. The following table shows results from an experiment where the pulse of a chicken embryo was measured at different temperatures.

Temperature T (°C)	36.3	35.0	33.9	32.4	24.7	24.2
Pulse n (heartbeats/minute)	154	133	110	94	38	36

a. Prepare a table showing the relationship between T and $\ln n$, and plot the pairs of numbers $(T, \ln n)$ in a coordinate system with $\ln n$ on the vertical axis and T on the horizontal axis. Fit a straight line to these points.

b. We want to find an empirical function $f(T) = ce^{aT}$ that approximates the pulse rate as a function of the temperature T. Using the line obtained in part (a), determine a and c.

c. By how many degrees does the temperature have to fall in order to halve the pulse rate?

Harder Problems

14. All organic material contains stable carbon 12 and some (very little) of the radioactive isotope carbon 14. The proportion between the quantities of carbon 14 and of stable carbon in living organisms is constant, and seems to have been constant for thousands of years. When an organism dies, carbon 14 decays according to the law

$$f(t) = f(t_0)e^{-1.25 \cdot 10^{-4}(t-t_0)}$$

where $f(t_0)$ is the quantity of carbon at the moment of death t_0, and $f(t)$ is the quantity that is left at time t. Show that t_0 is given by

$$t_0 = t + 8000 \ln \frac{f(t)}{f(t_0)}$$

(This formula is the basis for "radioactive dating." In 1960, the American W. F. Libby received the Nobel prize in chemistry for the discovery of radioactive dating.)

15. Helge and Anne Stine Ingstad found several Viking tools on old settlements in Newfoundland. The charcoal from the fireplaces was analyzed in 1972, and the percentage of carbon 14 in the charcoal (compared with the content of carbon 14 in fresh wood) was 88.6%. Use the result from Problem 14 to determine when the Viking settlers lived in Newfoundland.

8.5 Compound Interest and Present Discounted Values

Equation [8.21], $f'(t) = rf(t)$ for all t, has a particularly important application to economics. After t years, a deposit of $\$K$ earning interest at the rate $p\%$ per year will increase to

$$K(1+r)^t \quad (\text{where} \quad r = p/100) \tag{1}$$

(see Section A.1, Appendix A). Each year the principal increases by the factor $1+r$.

Formula [1] assumes that the interest is added to the principal at the end of each year. Suppose instead that payment of interest is offered each half year, but at an interest rate $p/2$. Then the principal after 1/2 year will have increased to

$$K + K\frac{p/2}{100} = K\left(1+\frac{r}{2}\right)$$

Therefore, the principal increases by the factor $1+r/2$ each half year. After 1 year, the principal will have increased up to $K(1+r/2)^2$, and after t years it will be

$$K\left(1+\frac{r}{2}\right)^{2t} \tag{2}$$

It is clear that a biannual interest payment at the rate $\frac{1}{2}p\%$ is better for a lender than an annual interest payment at the rate $p\%$. This is easily seen also from the fact that $(1+r/2)^2 = 1+r+r^2/4 > 1+r$.

More generally, suppose that interest at the rate $p/n\%$ is added to the principal at n different times distributed evenly over the year. Then the principal will be multiplied by a factor $(1+r/n)^n$ each year. After t years, the principal is

$$K\left(1+\frac{r}{n}\right)^{nt} \tag{3}$$

The greater is n, the more profitable is the investment for the lender. See Problem 3.

In practice, there is a limit to how frequently interest can be added to savings accounts. However, let us examine what happens to the expression in [3] as the annual frequency n tends to infinity. We put $r/n = 1/m$. Then $n = mr$ and so

$$K\left(1+\frac{r}{n}\right)^{nt} = K\left(1+\frac{1}{m}\right)^{mrt} = K\left[\left(1+\frac{1}{m}\right)^{m}\right]^{rt} \tag{4}$$

As $n \to \infty$ (with r fixed), so $m = n/r \to \infty$, and according to [8.17], we have $(1+1/m)^m \to e$. Hence, the expression in [4] approaches Ke^{rt} as n tends to infinity. When we let n approach infinity, the accumulation of interest happens more and more frequently. In the limit, we talk about **continuous compounding** of interest. After t years, an initial amount $\$K$ will have increased to

$$K(t) = Ke^{rt} \qquad \textbf{(continuous compounding)} \tag{8.26}$$

The number r is often referred to as the **rate of interest**. By differentiating [8.26], we have the following important fact.

> With continuous compounding of interest at rate r, the principal increases at the constant relative rate r, so that $K'(t)/K(t) = r$.

From [8.26], we infer that $K(1) = Ke^r$, so that the principal increases by the factor e^r during the first year. In general, $K(t+1) = Ke^{r(t+1)} = Ke^{rt}e^r = K(t)e^r$, so that with continuous compounding of interest, the principal increases *each* year by the fixed factor e^r.

Comparing Different Forms of Interest

At an interest rate of $p\%$ ($= 100r$) per year, continuous compounding of interest is best for the lender. (See Problem 3.) For comparatively low interest rates, however, the difference between annual and continuous compounding of interest is quite small.

Example 8.9

Find the amount by which \$1 increases in the course of a year when the interest rate is 8% per year and interest is added:

(a) only at the end of the year
(b) at the end of each half year
(c) continuously

Solution In this case, $r = 8/100 = 0.08$, so we obtain the following:

(a) $K = (1 + 0.08) = 1.08$
(b) $K = (1 + 0.08/2)^2 = 1.0816$
(c) $K = e^{0.08} \approx 1.08329$

If we increase the interest rate or increase the number of years over which interest accumulates, then the difference between yearly and continuous compounding of interest increases.

Note: A consumer who wants to take out a loan may be faced with several offers from financial institutions. It is therefore of considerable importance to compare the various offers. The concept of **effective interest rate** is often used in making such comparisons. Imagine an offer that implies a yearly interest rate $p\%$ with interest p/n added n times during the year. A principal amount of K will then have increased after 1 year to $K(1+r/n)^n$, where $r = p/100$. Define the *effective interest rate* P as the annual percentage interest rate that, when compounding is continuous, gives the same total interest over the year. If $R = P/100$, then after 1 year, the initial amount K increases to Ke^R. Hence, R is defined by the equation

$$Ke^R = K(1+r/n)^n$$

Canceling K and then taking ln of both sides gives

$$R = n\ln(1+r/n) \qquad [8.27]$$

If $r = 0.08$ and $n = 1$, for example, then $R = \ln(1+0.08) \approx 0.077$. Thus, a yearly interest rate of 8% corresponds to an effective interest rate (with continuous compounding) of about 7.7%.

The Present Value of a Future Claim

Suppose that an amount K is due for payment t years after the present date. What is the *present value* of this amount when the interest rate is $p\%$ per year? Equivalently, how much must be deposited today earning $p\%$ annual interest in order to have the amount K after t years?

If interest is paid annually, the amount A will have increased to $A(1+p/100)^t$ after t years, so that we need $A(1+p/100)^t = K$. Thus, $A = K(1+p/100)^{-t} = K(1+r)^{-t}$, where $r = p/100$. If interest is compounded continuously, however, then the amount A will have increased to Ae^{rt} after t years. Hence, $Ae^{rt} = K$, or $A = Ke^{-rt}$. Altogether, we have the following:

> If the interest rate is $p\%$ per year and $r = p/100$, an amount K that is payable in t years has the present value:
>
> $K(1+r)^{-t}$, with yearly interest payments
>
> Ke^{-rt}, with continuous compounding of interest

[8.28]

Problems

1. An amount \$1000 earns interest at 5% per year. What will this amount have grown to after (a) 10 years, and (b) 50 years, when interest is compounded (i) yearly, (ii) monthly, (iii) continuously?

2. Suppose that the price of a commodity after x years is given by $f(x) = Ae^{kx}$, where A and k are constants.
   a. Find A and k when $f(0) = 4$ and $f'(0) = 1$. In this case, what is the price after 5 years?
   b. We assume now that $A = 4$ and $k = 0.25$. When the price has increased to 18, it becomes controlled so that the annual price increase is limited to 10%. When are price controls first needed? What length of time is needed for the price to double before and after price controls are introduced?

Harder Problems

3. We showed (in the discussion following Equation [4]) that $(1 + r/n)^n \to e^r$ as $n \to \infty$. For each fixed $r > 0$, we claim that $(1 + r/n)^n$ is strictly increasing in n, so that

$$\left(1 + \frac{r}{n}\right)^n < \lim_{n\to\infty}\left(1 + \frac{r}{n}\right)^n = e^r \qquad \text{(for } n = 1, 2, \ldots) \qquad [*]$$

This shows that continuous interest at interest rate p% per year, with $r = p/100$, is more profitable for the lender than interest payments n times a year at interest rate $p/100n$.

To confirm this, define the function g for all $x > 0$ by

$$g(x) = \left(1 + \frac{r}{x}\right)^x \qquad (r \text{ is a positive constant})$$

Show that

$$g'(x) = g(x)\left[\ln\left(1 + \frac{r}{x}\right) - \frac{r/x}{1 + r/x}\right]$$

(Use logarithmic differentiation.) Put $h(u) = \ln(1 + u) - u/(1 + u)$. Then $h(0) = 0$. Show that $h'(u) > 0$ for $u > 0$, and hence that $g'(x) > 0$ for all $x > 0$. What conclusion can you draw?

9

Single-Variable Optimization

If you want literal realism, look at the world around you; if you want understanding, look at theories.
—*R. Dorfman* (1964)

Looking for the best way of pursuing a certain goal involves what are called **optimization problems**. Examples can be drawn from almost all areas of human activity. A manager seeks those combinations of inputs (such as capital and labor) that maximize profits or minimize costs. A doctor might want to know when the concentration of a drug in the bloodstream is at its greatest. A farmer might want to know what amount of fertilizer per square yard will maximize profits. An oil company may wish to find the optimal rate of extraction from one of its wells.

Studying an optimization problem of this sort using mathematical methods requires us to construct a mathematical model for the problem. This is usually not easy, and only in simple cases will the model lead to the problem of maximizing or minimizing a function of a single variable—the main topic of this chapter.

In general, no mathematical methods have more important applications in economics than those designed to solve optimization problems. Though economic optimization problems usually involve several variables, the examples of quadratic optimization in Section 3.2 indicate how useful economic insights can be gained even from simple one-variable optimization.

9.1 Some Basic Definitions

Recall from Section 7.2 that, if $f(x)$ has domain D, then

$$c \in D \text{ is a **maximum point** for } f \iff f(x) \le f(c), \text{ for all } x \in D \quad [9.1]$$

$$d \in D \text{ is a **minimum point** for } f \iff f(x) \ge f(d), \text{ for all } x \in D \quad [9.2]$$

In [9.1], we call $f(c)$ the **maximum value**, and in [9.2], we call $f(d)$ the **minimum value**. If the value of f at c is strictly larger than at any other point in D, then c is a **strict** maximum point. Similarly, d is a **strict** minimum point if $f(x) > f(d)$ for all $x \in D$, $x \neq d$. As collective names, we use **optimal points** and **values**, or **extreme points** and **values.**

If f is any function with domain D, then $-f$ is defined in D by $(-f)(x) = -f(x)$. Note that $f(x) \le f(c)$ for all $x \in D$, if and only if $-f(x) \ge -f(c)$ for all $x \in D$. Thus, c maximizes f in D if and only if c minimizes $-f$ in D. This simple observation, which is illustrated in Fig. 9.1, can be used to convert maximization problems to minimization problems and vice versa.

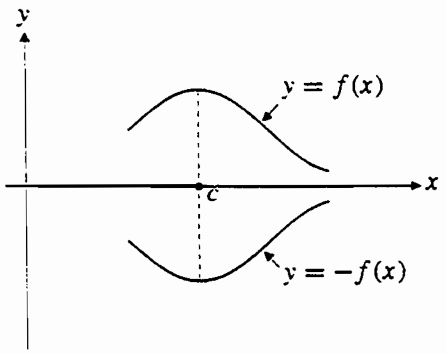
Our main task in this chapter is to study how to determine the possible maximum and minimum points of a function. In this connection, the following definition is crucial:

x_0 is a **stationary** point for f if $f'(x_0) = 0$	[9.3]

Geometrically, stationary points occur where the tangent to the graph of the function is parallel to the x-axis.

Before starting to explore systematically the properties of maxima and minima, we provide some geometric examples based on the graph of the function. They will indicate for us the role played by the stationary points of a function in the theory of optimization.

FIGURE 9.1 Point c is a maximum point for $f(x)$ and a minimum point for $-f(x)$.



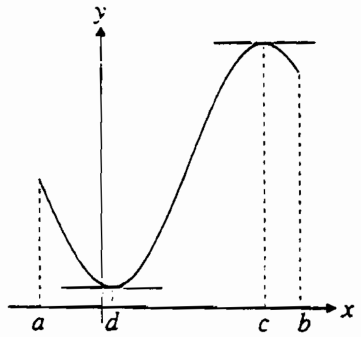


FIGURE 9.2

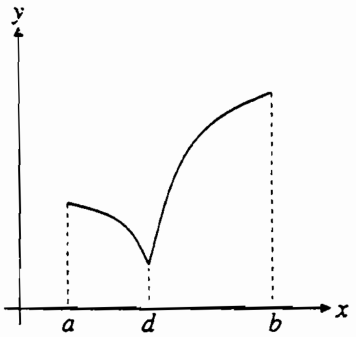


FIGURE 9.3

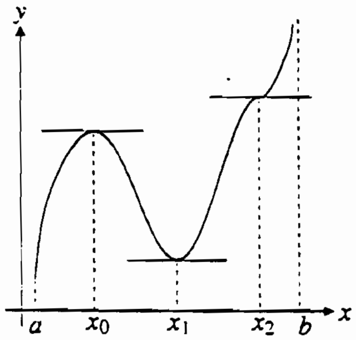


FIGURE 9.4

Figure 9.2 is the graph of a function f having two stationary points, c and d. At c, there is a maximum; at d, there is a minimum.

In Fig. 9.3, the function has no stationary points. There is a maximum at the end point b and a minimum at d. At d, the function is not differentiable. At b, the derivative (the left-hand derivative) is not 0.

Finally, the function f whose graph is shown in Fig. 9.4 has three stationary points, x_0, x_1, and x_2. At end point a, there is a minimum, whereas f does not have any maximum value because it approaches ∞ as x tends to b. At the critical point x_0, function f has a local maximum in the sense that its value at that point is higher than at all neighboring points. Similarly, at x_1, it has a local minimum, whereas at x_2 there is a stationary point that is neither a local maximum nor a local minimum. We call x_2 an *inflection point.*

The three figures represent the most important properties of single-variable optimization problems. Because the theory is so important in practical applications, we must not simply rely on geometric insights, but must rather develop a firmer analytical foundation for optimization theory.

9.2 A First-Derivative Test for Extreme Points

In many important cases, we can find maximum or minimum values for a function just by studying the sign of its first derivative. Suppose $f(x)$ is differentiable on an interval I and suppose $f(x)$ has only one stationary point, $x = c$. If

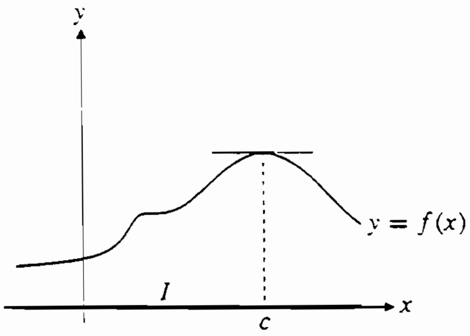


FIGURE 9.5 Point $x = c$ is a maximum point.

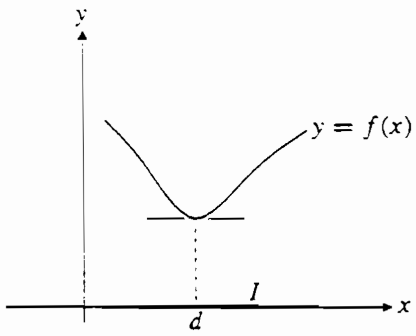


FIGURE 9.6 Point $x = d$ is a minimum point.

$f'(x) \geq 0$ for all $x \in I$ such that $x \leq c$, whereas $f'(x) \leq 0$ for all $x \in I$ such that $x \geq c$, then $f(x)$ is increasing to the left of c and decreasing to the right of c. It follows that $f(x) \leq f(c)$ for all $x \leq c$ and $f(c) \geq f(x)$ for all $x \geq c$. Hence, $x = c$ is a maximum point for f in I, as illustrated in Fig. 9.5.

With obvious modification, a similar result holds for minimum points, as illustrated in Fig. 9.6. Briefly stated:[1]

A First-Derivative Test for Max/Min

If $f'(x) \geq 0$ for $x \leq c$, and $f'(x) \leq 0$ for $x \geq c$, then $x = c$ is a maximum point for f.

If $f'(x) \leq 0$ for $x \leq c$, and $f'(x) \geq 0$ for $x \geq c$, then $x = c$ is a minimum point for f. [9.4]

Example 9.1

Measured in milligrams per liter, the concentration of a drug in the bloodstream t hours after injection is given by the formula

$$c(t) = \frac{t}{t^2 + 4}$$

Find the time of maximum concentration.

Solution Differentiation with respect to t yields

$$c'(t) = \frac{1 \cdot (t^2 + 4) - t \cdot 2t}{(t^2 + 4)^2} = \frac{4 - t^2}{(t^2 + 4)^2} = \frac{(2 + t)(2 - t)}{(t^2 + 4)^2}$$

[1]Many books in mathematics for economists instruct students always to check so-called second-order conditions, even in cases where the first-derivative test [9.4] is much easier to check.

For $t \geq 0$, the term $(2-t)$ alone determines the algebraic sign of the fraction, because the other terms are positive. In fact, if $t \leq 2$, then $c'(t) \geq 0$, whereas if $t \geq 2$, then $c'(t) \leq 0$. From [9.4], we conclude that $t = 2$ maximizes $c(t)$. Thus, the concentration of the drug is highest 2 hours after injection. Because $c(2) = 0.25$, the maximum concentration is 0.25 milligrams per liter.

Example 9.2

Suppose $Y(N)$ bushels of wheat are harvested per acre of land when N pounds of fertilizer per acre are used. If P is the dollar price per bushel of wheat and q is the dollar price per pound of fertilizer, then profits in dollars per acre are

$$\pi(N) = PY(N) - qN \qquad (N \geq 0) \tag{1}$$

Suppose that for some N^*, $\pi'(N) \geq 0$ for $N \leq N^*$, and $\pi'(N) \leq 0$ for $N \geq N^*$. Then N^* maximizes profits, and $\pi'(N^*) = 0$, that is, $PY'(N^*) - q = 0$, so

$$PY'(N^*) = q \tag{2}$$

Let us give an economic interpretation to this condition. Suppose N^* units of fertilizer are used and we contemplate increasing N^* by one unit. What do we gain? If N^* increases by one unit, then $Y(N^* + 1) - Y(N^*)$ more bushels are produced. Now $Y(N^*+1) - Y(N^*) \approx Y'(N^*)$. For each of these bushels, we get P dollars, so

by increasing N^* by one unit, we gain $\approx PY'(N^*)$ dollars

On the other hand,

by increasing N^* by one unit, we lose q dollars

because this is the cost of one unit of fertilizer. Hence, we can interpret [2] as follows: In order to maximize profits, you should increase the amount of fertilizer to the level N^* at which an additional pound of fertilizer equates your gains and losses.

In a certain study from Iowa in 1952, the yield function $Y(N)$ was estimated as

$$Y(N) = -13.62 + 0.984N - 0.05N^{3/2}$$

If the price of wheat is \$1.40 per bushel and the price of fertilizer is \$0.18 per pound, find the amount of fertilizer that maximizes profits.

Solution In this case,

$$\pi(N) = 1.4(-13.62 + 0.984N - 0.05N^{3/2}) - 0.18N,$$

so

$$\pi'(N) = 1.4\left[0.984 - (3/2)\cdot 0.05 \cdot N^{1/2}\right] - 0.18 \qquad [3]$$

Thus, $\pi'(N^*) = 0$ provided that

$$1.4 \cdot (3/2) \cdot 0.05(N^*)^{1/2} = 1.4 \cdot 0.984 - 0.18$$

This implies that

$$(N^*)^{1/2} = \frac{1.4 \cdot 0.984 - 0.18}{1.4(3/2)0.05} = \frac{1.1976}{0.105} \approx 11.406$$

Hence,

$$N^* \approx (11.406)^2 \approx 130$$

Looking at [3], we see that $\pi'(N) \geq 0$ for $N \leq N^*$, and $\pi'(N) \leq 0$ for $N \geq N^*$. Hence, $N^* \approx 130$ maximizes profits.

Example 9.3 **("Neither a Borrower nor a Lender Be")**[2]

A student has current income y_1 and expects future income y_2. She plans current consumption c_1 and future consumption c_2 in order to maximize the utility function

$$\ln c_1 + \frac{1}{1+\delta}\ln c_2$$

where δ is her discount rate. If she borrows now, so that $c_1 > y_1$, then future consumption, after repaying the loan amount $c_1 - y_1$ with interest charged at rate r, will be

$$c_2 = y_2 - (1+r)(c_1 - y_1)$$

Alternatively, if she saves now, so that $c_1 < y_1$, then future consumption will be

$$c_2 = y_2 + (1+r)(y_1 - c_1)$$

after receiving interest at the same rate on her savings. Find the optimal borrowing or saving plan.

[2]According to Shakespeare, Polonius' advice to Hamlet was "Neither a borrower nor a lender be."

Solution Whether the student borrows or saves, second period consumption will be given by

$$c_2 = y_2 - (1+r)(c_1 - y_1)$$

in either case. So the student will want to maximize

$$U = \ln c_1 + \frac{1}{1+\delta}\ln[y_2 - (1+r)(c_1 - y_1)]$$

Differentiating this function with respect to c_1 gives

$$\frac{dU}{dc_1} = \frac{1}{c_1} - \frac{1+r}{1+\delta}\cdot\frac{1}{y_2 - (1+r)(c_1 - y_1)}$$

Rewriting the fractions so that they have a common denominator yields

$$\frac{dU}{dc_1} = \frac{(1+\delta)[y_2 - (1+r)(c_1 - y_1)] - (1+r)c_1}{c_1(1+\delta)[y_2 - (1+r)(c_1 - y_1)]} \qquad [1]$$

Rearranging the numerator and equating the derivative to 0, we have

$$\frac{dU}{dc_1} = \frac{(1+\delta)[(1+r)y_1 + y_2] - (2+\delta)(1+r)c_1}{c_1(1+\delta)[y_2 - (1+r)(c_1 - y_1)]} = 0 \qquad [2]$$

The unique solution of this equation is

$$c_1^* = \frac{(1+\delta)[(1+r)y_1 + y_2]}{(2+\delta)(1+r)} = y_1 + \frac{(1+\delta)y_2 - (1+r)y_1}{(2+\delta)(1+r)} \qquad [3]$$

From [2], we see that for $c_1 > c_1^*$ one has $dU/dc_1 < 0$, whereas for $c_1 < c_1^*$ one has $dU/dc_1 > 0$. We conclude that c_1^* indeed maximizes U. Moreover, the student lends if and only if $(1+\delta)y_2 < (1+r)y_1$. In the more likely case when $(1+\delta)y_2 > (1+r)y_1$ because future income is considerably higher than present income, she will borrow. Only if by some chance $(1+\delta)y_2$ is exactly equal to $(1+r)y_1$ will she be neither a borrower nor a lender. However, this discussion has neglected the difference between borrowing and lending rates of interest that one always observes in reality.

Problems

1. Let y denote the total weekly weight of pigs slaughtered by the butcheries of Chicago during 1948 (in millions of pounds) and let x be total weekly work

effort (in thousands of hours). Nichols estimated the relation

$$y = -2.05 + 1.06x - 0.04x^2$$

Determine the value of x that maximizes y by studying the sign variation of y'.

2. Find the derivative of the function h defined for all x by

$$h(x) = \frac{8x}{3x^2 + 4}$$

Use the sign variation of $h'(x)$ to find the maximum/minimum value of $h(x)$.

3. Consider the function V defined by

$$V(x) = 4x(9 - x)^2 = 4x^3 - 72x^2 + 324x \qquad (x \in [0, 9])$$

(See Problem 1(e) in Section 1.3 for an interpretation of V.)

a. Compute $V'(x)$ and show that V is increasing in $(0, 3)$ and decreasing in $(3, 9)$. Find the maximum point of V in $[0, 9]$.

b. Explain what the result in part (a) implies for Problem 1(e) in Section 1.3.

c. Also solve the problem by logarithmic differentiation of $V(x)$, for $x \in (0, 9)$. Which method do you prefer?

4. **a.** Show that

$$f(x) = \frac{2x^2}{x^4 + 1} \implies f'(x) = \frac{4x(1 + x^2)(1 + x)(1 - x)}{(x^4 + 1)^2} \qquad [*]$$

b. Use [*] to find the maximum value of f on $[0, \infty)$. Show that $f(-x) = f(x)$, for all x. What are the maximum points for f on $(-\infty, \infty)$?

5. Occasionally, one can find maximum/minimum points of a function just by studying the formula. For example, consider $f(x) = \sqrt{x - 5} - 100$, defined for $x \geq 5$. Because $\sqrt{x - 5}$ is ≥ 0 for all $x \geq 5$, so $f(x) \geq -100$ for all $x \geq 5$. Because $f(5) = -100$, we conclude that $x = 5$ is a minimum point. Use similar direct arguments to find maximum/or minimum points for the following:

a. $f(x) = \dfrac{8}{3x^2 + 4}$ **b.** $g(x) = 3 - (x - 2)^2$

c. $h(x) = 5(x + 2)^4 - 3$ **d.** $F(x) = \dfrac{-2}{2 + x^2}$

e. $G(x) = 2 - \sqrt{1 - x}$ **f.** $H(x) = \dfrac{1}{1 + x^4}$ $(x \in [-1, 1])$

6. Study the sign variation of the derivative of each function in Problem 5 and confirm the conclusions obtained there.

Harder Problems

7. If the tax T a person pays on gross income Y is $T = a(bY + c)^p + kY$, where a, b, and c are positive constants, then the average tax rate is

$$\overline{T}(Y) = \frac{T}{Y} = a\frac{(bY+c)^p}{Y} + k \qquad (p > 1)$$

Find the value of Y that maximizes the average tax.

8. Given n numbers $a_1, a_2, \ldots, a_n$, find the number x that approximates these numbers best, in the sense that

$$d(x) = (x - a_1)^2 + (x - a_2)^2 + \cdots + (x - a_n)^2$$

is as small as possible. What do you call this x value?

9.3 Alternative Ways of Finding Extreme Points

Sometimes it is awkward or impossible to locate extreme points by considering how the sign of the first derivative varies. Other ways of characterizing extreme points are often more useful, as this section demonstrates.

We begin by examining precisely the role played by stationary points of a function in locating extreme points. Suppose we know that a function f has a maximum at a point c in an interval I. That maximum might very well occur at an end point of the interval, as is the case in Fig. 9.3. However, when c is not an end point, and if f is differentiable, it seems geometrically obvious that the tangent to the graph at c must be horizontal. In other words, c must be a stationary point. The same conclusion applies to a minimum point. A formal statement and a proof of this important result were given in Theorem 7.4 of Section 7.2. Thus, the condition $f'(c) = 0$ is a *necessary* condition for an interior point c at which f' exists to be an optimal point. The condition is not sufficient, however. In Fig. 9.4 of Section 9.1, points x_0, x_1, and x_2 are all stationary points, but none is an optimal point. (In fact, x_0 is a local maximum point, x_1 a local minimum point, and x_2 an inflection point.)

How to Search for Maxima/Minima

Suppose we know that a function f has a maximum and/or a minimum in a bounded interval I. The optimum must occur either at an interior point of I or at one of the end points. If it occurs at an interior point (inside the interval I) and f is differentiable, then by Theorem 7.4 in Section 7.2 the derivative f' is zero at that point. In addition, there is the possibility that the optimum occurs at a point where f is not differentiable. Hence, extreme points can be only one of the following three types:

1. interior points in I where $f'(x) = 0$
2. end points of I
3. points in I where f' does not exist

A typical example showing that a minimum can occur at a point of type 3 is suggested in Fig. 9.3 of Section 9.1. However, the functions economists study are usually differentiable everywhere. The following recipe, therefore, covers most problems of interest.

Problem:

Find the maximum and the minimum values of a differentiable function f defined on a closed, bounded interval $[a, b]$.

Solution

[9.5]

(a) Find all stationary points of f in (a, b)—that is, find all points $x \in (a, b)$ that satisfy the equation $f'(x) = 0$.
(b) Evaluate f at the end points a and b of the interval and at all stationary points found in (a).
(c) The largest function value in (b) is the maximum value of f in $[a, b]$.
(d) The smallest function value in (b) is the minimum value of f in $[a, b]$.

A differentiable function is continuous, so the extreme-value theorem (Theorem 7.3 of Section 7.2) assures us that maximum and minimum points do exist. Following the procedure just given, we can, in principle, find these extreme points. (In very special examples, there could be an infinite number of stationary points. Such "pathological" functions almost never appear in applied problems.)

Example 9.4

Find the maximum and minimum values of

$$f(x) = \tfrac{1}{9}x^3 - \tfrac{1}{6}x^2 - \tfrac{2}{3}x + 1, \qquad (x \in [-3, 3])$$

Solution The function is differentiable everywhere, and

$$f'(x) = \tfrac{1}{3}x^2 - \tfrac{1}{3}x - \tfrac{2}{3} = \tfrac{1}{3}(x^2 - x - 2) = \tfrac{1}{3}(x + 1)(x - 2)$$

Thus, there are two points in the interval $(-3, 3)$ where $f'(x) = 0$, namely, $x = -1$ and $x = 2$. Evaluating f at these points and the end points, we have

$$f(-3) = -3/2, \qquad f(-1) = 25/18, \qquad f(2) = -1/9, \qquad f(3) = 1/2$$

The minimum value is $-3/2$ at $x = -3$, and the maximum value is $25/18$ at $x = -1$.

Example 9.5

A firm is producing some commodity and wants to maximize its profits. The total revenue generated in a certain period by producing and selling Q units is $R(Q)$ dollars, whereas $C(Q)$ denotes the associated total dollar cost. The profit obtained as a result of producing and selling Q units is then

$$\pi(Q) = R(Q) - C(Q) \qquad [1]$$

Because of technical limitations, suppose there is a maximum quantity $\bar{Q}$ that can be produced by the firm in a given period. Assume that R and C are differentiable functions of Q in the interval $[0, \bar{Q}]$. The profit function π is then differentiable, so continuous, and consequently π *has* a maximum value. In special cases, that maximum might occur at $Q = 0$ or at $Q = \bar{Q}$. If not, the maximum production level Q^* satisfies $\pi'(Q^*) = 0$, and so

$$R'(Q^*) = C'(Q^*) \qquad [2]$$

Hence, *production should be adjusted to a point where the marginal revenue is equal to the marginal cost.*

Let us assume that the firm gets a fixed price P per unit sold. Then $R(Q) = PQ$, and [2] takes the form

$$P = C'(Q^*) \qquad [3]$$

Thus, in the case in which the firm has no control over the price, *production should be adjusted to a level at which the marginal cost is equal to the price per unit of the commodity* (assuming that π does not have a maximum at 0 or at $\bar{Q}$).

For special choices of $R(Q)$ and $C(Q)$, it might happen that [2] has several solutions. If so, the maximum profit occurs at that point among the solutions of [2] that gives the highest value to $\pi(Q)$.

An interpretation of [2] in line with that given for the corresponding optimality condition in the wheat example in Example 9.2 of Section 9.2 is as follows. Suppose we contemplate increasing production from the level Q^* by one unit. Revenue will increase by: $R(Q^* + 1) - R(Q^*) \approx R'(Q^*)$. We would lose the amount $C(Q^*+1)-C(Q^*) \approx C'(Q^*)$, because this is the cost increase by increasing production by one unit. Equation [2] equates $R'(Q^*)$ and $C'(Q^*)$, so that marginal revenue of selling an extra unit is exactly offset by the marginal cost of producing that unit.

Suppose a tax of t dollars per unit is imposed on the production of the commodity. Then the profit function becomes

$$\pi(Q) = R(Q) - C(Q) - tQ \tag{4}$$

because selling Q units incurs a total additional cost of tQ. Assuming again that the maximum profit is not at $Q = 0$ or $Q = \bar{Q}$, it can only occur at a level Q^* where $\pi'(Q^*) = 0$. Now, $\pi'(Q) = R'(Q) - C'(Q) - t$, so the condition for maximum profit is

$$R'(Q^*) = C'(Q^*) + t \tag{5}$$

What we gain by increasing production by one unit from the level Q^* is still (approximately) $R'(Q^*)$. What we lose is $C'(Q^*) + t$, because we have to pay t dollars in tax for the extra unit of output.

In the previous examples that involved explicit functions, we had no trouble in finding the solutions to the equation $f'(x) = 0$. However, in some cases, finding all the solutions to $f'(x) = 0$ might constitute a formidable problem. For instance, the continuous function

$$f(x) = x^{26} - 32x^{23} - 11x^5 - 2x^3 - x + 28 \qquad (x \in [-1, 5])$$

does have a maximum and a minimum in $[-1, 5]$, but it is impossible to find the exact solutions to the equation $f'(x) = 0$.

Difficulties of this kind are often encountered in practical optimization problems. In fact, only in very special cases can the equation $f'(x) = 0$ be solved exactly. Fortunately, there are standard numerical methods available for use on a computer that in most cases will find points arbitrarily close to the actual solutions of such equations.

Note: Suppose f is differentiable in an interval $[a, b]$ and let x_0 be a maximum point for f in $[a, b]$. If $x_0 = a$, then $f'(a)$ cannot be positive because then there would exist points x to the right of a where f had a higher value than at a. Arguing analogously, if $x_0 = b$ is a maximum point of f in $[a, b]$, then $f'(b)$ cannot be negative. If $x_0 \in (a, b)$, then $f'(x_0) = 0$. Figure 9.7 illustrates the three cases.

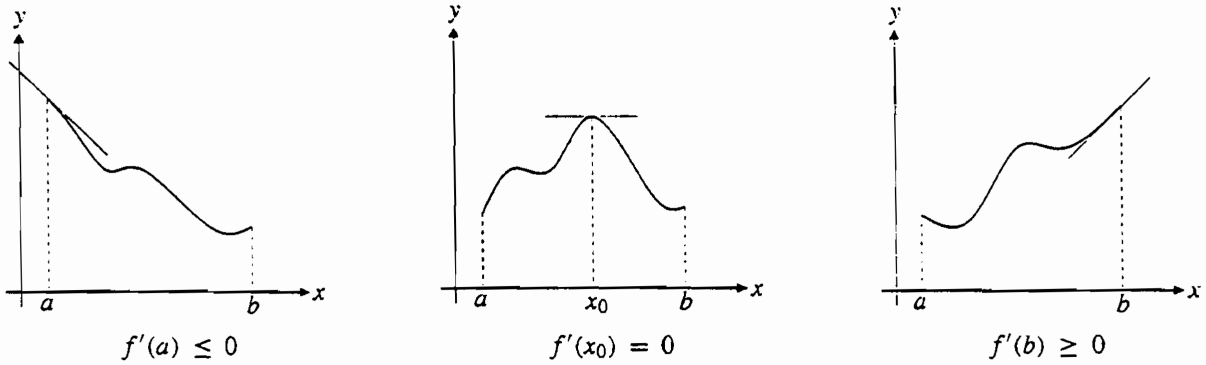


FIGURE 9.7 Maximum at a, x_0, or b.

Problems

1. Find the maximum and minimum of

$$f(x) = 4x^2 - 40x + 80, \qquad x \in [0, 8]$$

Draw the graph of f over $[0, 8]$.

2. Find the maximum and minimum of each function over the indicated interval:
   a. $f(x) = -2x - 1, \quad [0, 3]$ b. $f(x) = x^3 - 3x + 8, \quad [-1, 2]$
   c. $f(x) = \dfrac{x^2 + 1}{x}, \quad [\frac{1}{2}, 2]$ d. $f(x) = x^5 - 5x^3, \quad [-1, \sqrt{5}]$
   e. $f(x) = x^3 - 4500x^2 + 6 \cdot 10^6 x, \quad [0, 3000]$
3. Find two positive numbers whose sum is 16 and whose product is as large as possible.
4. A sports club plans to charter a plane. The charge for 60 passengers is \$800 each. For each additional person above 60, all travelers get a discount of \$10. The plane can take at most 80 passengers.
   a. What is the total cost when there are 61, 70, and 80 passengers?
   b. If $60 + x$ passengers fly, what is the total cost?
   c. Find the number of passengers that maximizes the total amount of airfares paid out by the sports club members.
5. Consider Example 9.5 and let $R(Q) = pQ$ and $C(Q) = \beta Q + \gamma Q^2$.
   a. Find the solution Q^* to Equation [2] in this case.
   b. Which value of Q maximizes profits in the following cases, assuming that $Q \in [0, 500]$?
      (i) $R(Q) = 1840Q$ and $C(Q) = 2Q^2 + 40Q + 5000$
      (ii) $R(Q) = 2240Q$ and $C(Q) = 2Q^2 + 40Q + 5000$
      (iii) $R(Q) = 1840Q$ and $C(Q) = 2Q^2 + 1940Q + 5000$

6. The height of a plant after t months is given by

$$h(t) = \sqrt{t} - \tfrac{1}{2}t \qquad (t \in [0, 3])$$

At what time is the plant at its highest?

7. Find the maximum of $y = x^2 e^{-x}$ on $[0, 4]$.

8. Let $C(Q)$ be the total cost function for a firm in producing Q units of some commodity. $A(Q) = C(Q)/Q$ is then called the *average cost function*. If $C(Q)$ is differentiable, prove that $A(Q)$ has a stationary point at $Q_0 > 0$ if and only if the marginal cost and the average cost are equal at Q_0. ($C'(Q_0) = A(Q_0)$.)

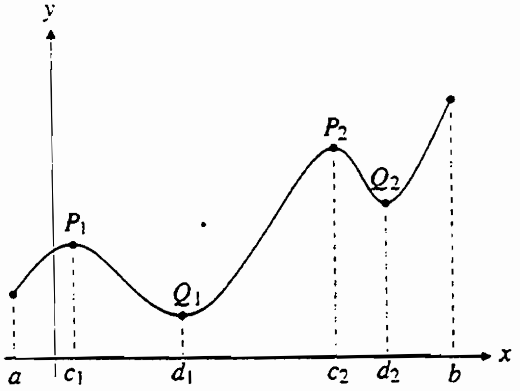
9. With reference to the previous problem, let $C(Q) = aQ^3 + bQ^2 + cQ + d$, where $a > 0$, $b \geq 0$, $c > 0$ and $d > 0$. Prove that $A(Q) = C(Q)/Q$ has a minimum in the interval $(0, \infty)$. Let $b = 0$ and find the minimum point in this case.

10. With reference to Problem 8, let $C(Q) = aQ^b + c$, for $a > 0$, $b > 1$, and $c \geq 0$. Prove that the average cost function has a minimum on $(0, \infty)$, and find it.

9.4 Local Maxima and Minima

So far in this chapter we have studied what are often referred to as *global* optimization problems. The reason for this terminology is that we have been seeking the absolutely largest or smallest values of a function, when we compare the function values at *all* points in the domain. In applied optimization problems, it is usually these global maxima and minima that are of interest. However, sometimes one is interested in the local maxima and minima of a function. In this case, we compare the function value at the point in question only with alternative function values at nearby points. For example, considering Fig. 9.8 and thinking of the graph as representing the profile of a landscape, mountain tops P_1 and P_2 represent local maxima, whereas valley bottoms Q_1 and Q_2 represent local minima.

FIGURE 9.8 Points c_1 and c_2 are local maxima; d_1 and d_2 are local minima.



If $f(x)$ is defined on domain A, the precise definitions are as follows:

Function f has a **local maximum** at c if there is an interval (α, β) about c such that $f(x) \leq f(c)$ for all those x in A that also lie in (α, β).	[9.6]

Function f has a **local minimum** at d if there is an interval (α, β) about d such that $f(x) \geq f(d)$ for all those x in A that also lie in (α, β).	[9.7]

Note: These definitions imply that point a in Fig. 9.8 is a local minimum point and b is a local (and global) maximum point. Some authors restrict the definition of local maximum/minimum points only to *interior* points of the domain of the function. According to this definition, a global maximum that is not an interior point of the domain is not a local maximum point. We want a global maximum/minimum point always to be a local maximum/minimum point, so we stick to definitions [9.6] and [9.7].

It is obvious what we mean by local maximum/minimum values of a function, and the collective names are **local extreme points** and **values.**

In searching for maximum and minimum points, Theorem 7.4 of Section 7.2 is very useful. Actually, the same result is valid for local extreme points: *At a local extreme point in the interior of the domain of a differentiable function, the derivative must be zero.* This is clear if we recall that the proof of Theorem 7.4 was concerned only with the behavior of the function in a small interval about the optimal point. Consequently, in order to find possible local maxima and minima for a function f defined in an interval I, we can again search among the following types of points:

1. interior points in I where $f'(x) = 0$
2. end points of I
3. points in I where f' does not exist

We have thus established *necessary* conditions for a function f defined in an interval I to have a local extreme point. But how do we decide whether a point satisfying the necessary conditions is a local maximum, a local minimum, or neither? In contrast to global extreme points, it does not help to calculate the function value at the different points. To see why, consider again the function whose graph is given in Fig. 9.8. Point c_1 is a local maximum point and d_2 is a local minimum point, but the function value at c_1 is *smaller* than the function value at d_2.

The First-Derivative Test

There are two main ways of determining whether a given stationary point is a local maximum, a local minimum, or neither. One of them is based on studying the sign of the first derivative about the stationary point, and is an easy modification of [9.4] in Section 9.2.

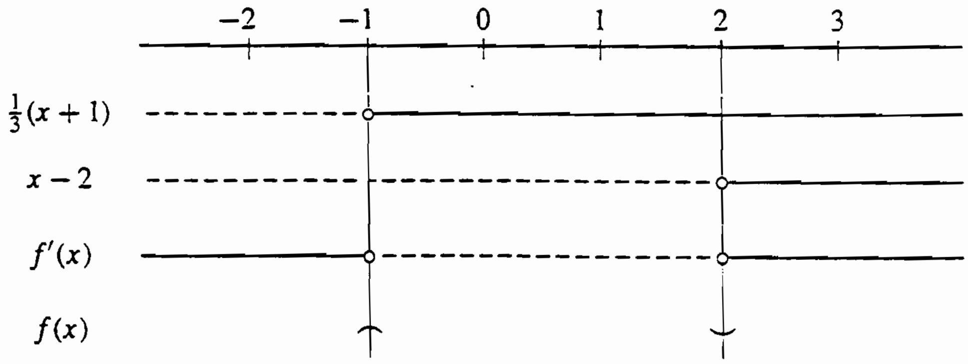
> **Theorem 9.1 (The first-derivative test for local extrema)**
> Suppose c is a stationary point for $y = f(x)$.
>
> **(a)** If $f'(x) \geq 0$ throughout some interval (a, c) to the left of c and $f'(x) \leq 0$ throughout some interval (c, b) to the right of c, then $x = c$ is a local maximum point for f.
>
> **(b)** If $f'(x) \leq 0$ throughout some interval (a, c) to the left of c and $f'(x) \geq 0$ throughout some interval (c, b) to the right of c, then $x = c$ is a local minimum point for f.
>
> **(c)** If $f'(x) > 0$ both throughout some interval (a, c) to the left of c and throughout some interval (c, b) to the right of c, then $x = c$ is not a local extreme point for f. The same conclusion holds if $f'(x) < 0$ on both sides of c.

Only case (c) is not already covered by [9.4] in Section 9.2. In fact, if $f'(x) > 0$ in (a, c) and in (c, b), then $f(x)$ is strictly increasing in $(a, c]$ as well as in $[c, b)$. Then $x = c$ cannot be a local extreme point.

Example 9.6

Classify the stationary points of $f(x) = \frac{1}{9}x^3 - \frac{1}{6}x^2 - \frac{2}{3}x + 1$.

Solution In this case (see Example 9.4), we have $f'(x) = \frac{1}{3}(x+1)(x-2)$, so $x = -1$ and $x = 2$ are the stationary points. The sign diagram for $f'(x)$ is:



We conclude from this sign diagram that $x = -1$ is a local maximum point whereas $x = 2$ is a local minimum point.

Example 9.7

Classify the stationary points of

$$f(x) = \frac{6x^3}{x^4 + x^2 + 2}$$

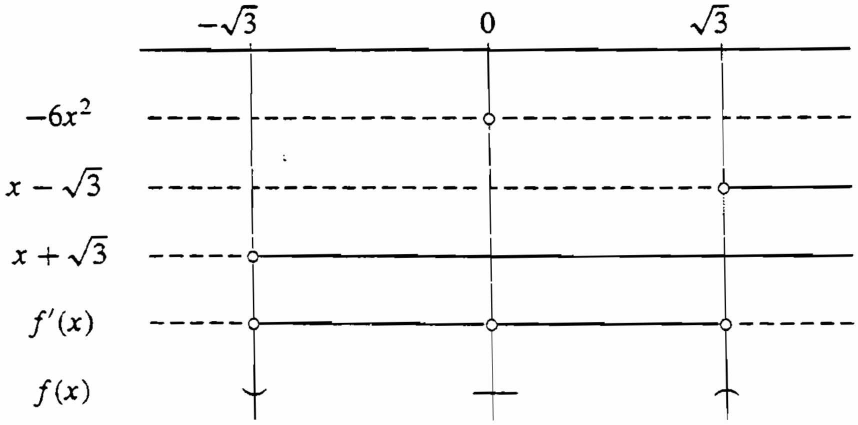
Solution Because $x^4 + x^2 + 2$ is ≥ 2 for all x, the denominator is never 0, so $f(x)$ is defined for all x. Differentiation of $f(x)$ yields

$$f'(x) = \frac{-6x^6 + 6x^4 + 36x^2}{(x^4 + x^2 + 2)^2} = \frac{-6x^2(x^4 - x^2 - 6)}{(x^4 + x^2 + 2)^2}$$

In order to study the sign variation of $f'(x)$, we must factorize $x^4 - x^2 - 6$. In fact, we have $x^4 - x^2 - 6 = (x^2)^2 - (x^2) - 6 = (x^2 - 3)(x^2 + 2) = (x - \sqrt{3})(x + \sqrt{3})(x^2 + 2)$. Hence,

$$f'(x) = \frac{-6x^2(x - \sqrt{3})(x + \sqrt{3})(x^2 + 2)}{(x^4 + x^2 + 2)^2}$$

Both the denominator and the factor $(x^2 + 2)$ in the numerator are always positive. Hence, the sign variation of $f'(x)$ is determined by the other factors in the numerator, as in the following sign diagram. Studying it we conclude from (a) in Theorem 9.1 that $x = \sqrt{3}$ is a local maximum point, and from (b) that $x = -\sqrt{3}$ is a local minimum point. According to (c), $x = 0$ is neither a local maximum, nor a local minimum point, because $f'(x) > 0$ in $(-\sqrt{3}, 0)$ and in $(0, \sqrt{3})$.



The graph is shown in Fig. 9.9. Note that $f(-x) = -f(x)$ for all x, so the graph of f is symmetric about the origin.

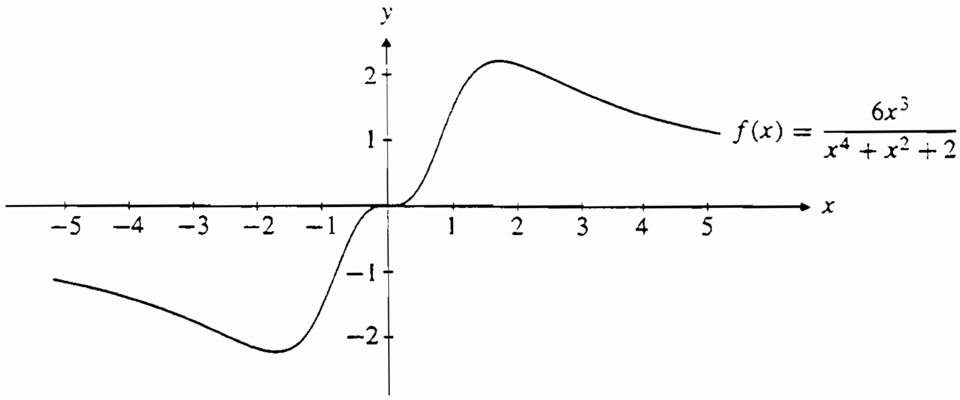


FIGURE 9.9

The Second-Derivative Test

For most problems of practical interest in which an explicit function is specified, Theorem 9.1 will determine whether a stationary point is a local maximum, a local minimum, or neither. Note that the theorem requires the knowledge of $f'(x)$ at points in a neighborhood of the given stationary point. In the next sufficiency theorem, we need only properties of the function at the stationary point.

> **Theorem 9.2 (The Second-Derivative Test)** Let f be a twice differentiable function in an interval I. Suppose c is an interior point of I. Then:
>
> **(a)** $f'(c) = 0$ and $f''(c) < 0 \Longrightarrow c$ is a strict local maximum point.
> **(b)** $f'(c) = 0$ and $f''(c) > 0 \Longrightarrow c$ is a strict local minimum point.
> **(c)** $f'(c) = 0$ and $f''(c) = 0 \Longrightarrow$?

Proof To prove part (a), assume $f'(c) = 0$ and $f''(c) < 0$. By definition of $f''(c)$ as the derivative of $f'(x)$ at c.

$$f''(c) = \lim_{h\to 0} \frac{f'(c+h) - f'(c)}{h} = \lim_{h\to 0} \frac{f'(c+h)}{h} \qquad [*]$$

Because $f''(c) < 0$, it follows from [*] that $f'(c+h)/h < 0$ if $|h|$ is sufficiently small. In particular, if h is a small positive number, then $f'(c+h) < 0$, so f' is negative in an interval to the right of c. In the same way, we see that f' is positive in some interval to the left of c. But then c is a strict local maximum point for f. Part (b) can be proved in the same way; for the inconclusive part (c), see the comments that follow.

Theorem 9.2 leaves unsettled case (c) when $f'(c) = f''(c) = 0$. Then "anything" can happen. Each of three functions $f(x) = x^4$, $f(x) = -x^4$, and $f(x) = x^3$ satisfies $f'(0) = f''(0) = 0$. At $x = 0$, they have, respectively, a (local)

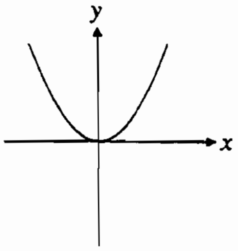


FIGURE 9.10
$f'(0) = f''(0) = 0$.
0 is a minimum point.

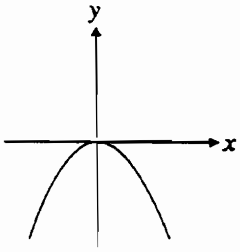


FIGURE 9.11
$f'(0) = f''(0) = 0$.
0 is a maximum point.

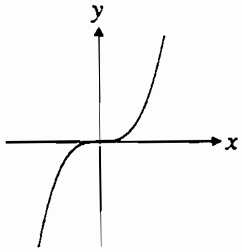


FIGURE 9.12
$f'(0) = f''(0) = 0$.
0 is an inflection point.

minimum, a (local) maximum, and a point of inflection, as shown in Figs. 9.10 to 9.12. Usually (as here), Theorem 9.1 can be used to classify stationary points at which $f'(c) = f''(c) = 0$. (For the definition of an inflection point, see [9.11] in Section 9.5.)

Theorem 9.2 can be used to obtain a useful necessary condition for local extrema. Suppose f is differentiable in the interval I and suppose that c is an interior point of I that is a local maximum point. Then $f'(c) = 0$. Moreover, $f''(c) > 0$ is impossible, because by Theorem 9.2 (b) this inequality would imply that c is a strict local minimum. Hence, $f''(c)$ has to be ≤ 0. In the same way, we see that $f''(c) \geq 0$ is a necessary condition for local minimum. Briefly formulated:

$$c \text{ is a local maximum for } f \Longrightarrow f''(c) \leq 0 \qquad [9.8]$$

$$c \text{ is a local minimum for } f \Longrightarrow f''(c) \geq 0 \qquad [9.9]$$

The function studied in Example 9.7 is a typical example of when it is convenient to study the sign variation of the first derivative in order to classify the stationary points. (Using Theorem 9.2 requires finding $f''(x)$, which is a rather involved expression.)

In theoretical economic models, it is more common to restrict the signs of second derivatives than to postulate a certain behavior in the sign variation of first derivatives. We consider a typical example.

Example 9.8

If a firm producing some commodity has revenue function $R(Q)$, cost function $C(Q)$, and there is a sales tax of t dollars per unit, then $Q^* > 0$ can only maximize profits provided that

$$R'(Q^*) = C'(Q^*) + t \qquad [*]$$

(See Example 9.5 of Section 9.3, Equation [5].) Suppose $R''(Q^*) < 0$ and $C''(Q^*) > 0$. Equation [*] implicitly defines Q^* as a differentiable function of t. Find an expression for dQ^*/dt and discuss its sign. Also compute the derivative with respect to t of the optimal value $\pi(Q^*)$ of the profit function, and show that $d\pi(Q^*)/dt = -Q^*$.

Solution Differentiating [∗] totally with respect to t yields

$$R''(Q^*)\frac{dQ^*}{dt} = C''(Q^*)\frac{dQ^*}{dt} + 1$$

Solving for dQ^*/dt gives

$$\frac{dQ^*}{dt} = \frac{1}{R''(Q^*) - C''(Q^*)} \qquad [**]$$

The sign assumptions on R'' and C'' imply that $dQ^*/dt < 0$. Thus, the optimal number of units produced will decline if the tax rate t increases.

The optimal value of the profit is $\pi(Q^*) = R(Q^*) - C(Q^*) - tQ^*$. Taking into account the dependence of Q^* on t, we get

$$\begin{aligned}\frac{d\pi^*(Q^*)}{dt} &= R'(Q^*)\frac{dQ^*}{dt} - C'(Q^*)\frac{dQ^*}{dt} - Q^* - t\frac{dQ^*}{dt} \\ &= \left[R'(Q^*) - C'(Q^*)\right]\frac{dQ^*}{dt} - Q^* - t\frac{dQ^*}{dt} = -Q^*\end{aligned}$$

where we used [∗]. Thus, we see that by increasing the tax rate by one unit, the optimal profit will decline by Q^* units. Note how the terms in dQ^*/dt disappear from this last expression because of the first-order condition [∗]. This is an instance of the "envelope theorem," which will be discussed in Section 18.7.

Example 9.9 (When to Harvest a Tree?)

Consider a tree that is planted at time $t = 0$, and let $P(t)$ be its current market value at time t, where $P(t)$ is differentiable. When should this tree be cut down in order to maximize its present discounted value? Assume that the interest rate is $100r\%$ per year, compounded continuously.

Solution By using [8.28] in Section 8.5, the present value is

$$f(t) = P(t)e^{-rt} \qquad [1]$$

whose derivative is

$$f'(t) = P'(t)e^{-rt} + P(t)(-r)e^{-rt} = e^{-rt}\left[P'(t) - rP(t)\right] \qquad [2]$$

A necessary condition for $t^* > 0$ to maximize $f(t)$ is that $f'(t^*) = 0$. We see from [2] that this occurs when

$$P'(t^*) = rP(t^*) \qquad [3]$$

The tree, therefore, should be cut down precisely at time t^* when the increase in the value of the tree over time interval $(t^*, t^* + 1)$ ($\approx P'(t^*)$) is equal to the interest one would obtain over this time interval by investing amount $P(t^*)$ at interest rate r ($\approx rP(t^*)$).

Let us look at the second-order condition. From [2], we find that

$$f''(t) = -re^{-rt}\left[P'(t) - rP(t)\right] + e^{-rt}\left[P''(t) - rP'(t)\right]$$

Evaluating $f''(t)$ at t^* and using [3] yields

$$f''(t^*) = e^{-rt^*}\left[P''(t^*) - rP'(t^*)\right] \tag{4}$$

Assuming $P(t^*) > 0$ and $P''(t^*) < 0$, from [3] we have $P'(t^*) > 0$. Then [4] gives $f''(t^*) < 0$, so t^* defined by [3] is a local maximum point. An example is given in Problem 4.

In this example, we did not consider how the ground the tree grows on may be used after cutting—for example, by planting a new tree. See Problem 5.

Note: In accepting maximization of present discounted value as a reasonable criterion for when a tree ought to be felled, one automatically dismisses the naïve solution to the problem: Cut down the tree at the time when its current market value is greatest. Instead, the tree is typically cut down a bit sooner, because of "impatience" associated with discounting.

Problems

1. Consider the function f defined for all x by

$$f(x) = x^3 - 12x$$

Find the two stationary points of f and classify them both by using the first- and second-derivative tests.

2. Determine all local extreme points and corresponding extreme values for the functions given by the following formulas:

a. $f(x) = -2x - 1$ **b.** $f(x) = x^3 - 3x + 8$

c. $f(x) = x + 1/x$ **d.** $f(x) = x^5 - 5x^3$

e. $f(x) = \frac{1}{2}x^2 - 3x + 5$ **f.** $f(x) = x^3 + 3x^2 - 2$

3. A function f is given by the formula

$$f(x) = (1 + 2/x)\sqrt{x+6}$$

a. Find the domain of f, the zeros of f, and the intervals where $f(x)$ is positive.

b. Find possible local extreme points and values.

c. Examine $f(x)$ as $x \to 0^-$, $x \to 0^+$, and $x \to \infty$. Also determine the limit of $f'(x)$ as $x \to \infty$. Has f a maximum or a minimum in the domain?

4. With reference to the tree-cutting problem of Example 9.9, consider the case where

$$f(t) = (t^2 + 10t + 25)e^{-0.05t} \qquad (t \geq 0)$$

a. Find the value of t that maximizes $f(t)$. Prove that the maximum point has been found.

b. Find $\lim_{t\to\infty} f(t)$ and draw the graph of f.

5. Consider Example 9.9. Assume now that immediately after a tree is felled, a new tree of the same type is planted. If we assume that a new tree is planted at times t, $2t$, $3t$, etc., then the present value of all the trees will be

$$f(t) = P(t)e^{-rt} + P(t)e^{-2rt} + \cdots$$

a. Find the sum of this infinite geometric series.

b. Prove that if $f(t)$ has a maximum for some $t^* > 0$, then

$$P'(t^*) = r\,\frac{P(t^*)}{1 - e^{-rt^*}}$$

and compare this condition to condition [3] in Example 9.9.

6. What requirements must be imposed on constants a, b, and c in order that

$$f(x) = x^3 + ax^2 + bx + c$$

a. will have a local minimum at $x = 0$?

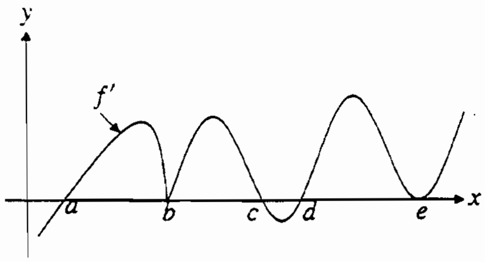
b. will have stationary points at $x = 1$ and $x = 3$?

7. Figure 9.13 graphs the *derivative* of a function f. Which of points a, b, c, d, and e are local maximum or minimum points for f?

8. Let function f be defined by

$$f(x) = \frac{x}{x^2 + 3x + 2}$$

FIGURE 9.13



a. Find $f'(x)$ and $f''(x)$, and find the local extreme points of f.
b. Find the global extreme points, and draw the graph of f.
c. Use the previous results to find global extreme points for the function g defined for all x by $g(x) = f(e^x)$.

9. Consider the function

$$f(x) = \frac{3}{x^4 - x^2 + 1}$$

a. Compute $f'(x)$ and find all local maximum and minimum points for f. Has f any global extreme points?
b. Draw the graph of f.

Harder Problems

10. Discuss local extreme points for the function $f(x) = x^3 + ax + b$. Use the result to show that the equation $f(x) = 0$ has three different real roots if and only if $4a^3 + 27b^2 < 0$.
11. Let f be defined for all x by $f(x) = (x^2 - 1)^{2/3}$.
a. Compute $f'(x)$ and $f''(x)$.
b. Find the local extreme points of f, and draw the graph of f.

9.5 Convex and Concave Functions and Inflection Points

What can be learnt from the sign of the second derivative? Recall how the sign of the first derivative determines whether a function is increasing or decreasing:

$$f'(x) \geq 0 \text{ on } (a, b) \iff f(x) \text{ is increasing on } (a, b) \qquad [1]$$

$$f'(x) \leq 0 \text{ on } (a, b) \iff f(x) \text{ is decreasing on } (a, b) \qquad [2]$$

The second derivative $f''(x)$ is the derivative of $f'(x)$. Hence:

$$f''(x) \geq 0 \text{ on } (a, b) \iff f'(x) \text{ is increasing on } (a, b) \qquad [3]$$

$$f''(x) \leq 0 \text{ on } (a, b) \iff f'(x) \text{ is decreasing on } (a, b) \qquad [4]$$

The equivalence in [3] is illustrated in Fig. 9.14. The slope of the tangent, $f'(x)$, is increasing as x increases. On the other hand, the slope of the tangent to the graph in Fig. 9.15 is decreasing as x increases. (Place a ruler as a tangent to the graph of the function. As the ruler slides along the curve from left to right, the tangent rotates counterclockwise in Fig. 9.14, clockwise in Fig. 9.15.)

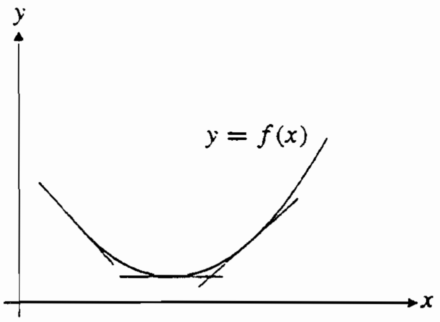


FIGURE 9.14 **The slope of the tangent increases as x increases. $f'(x)$ is increasing.**

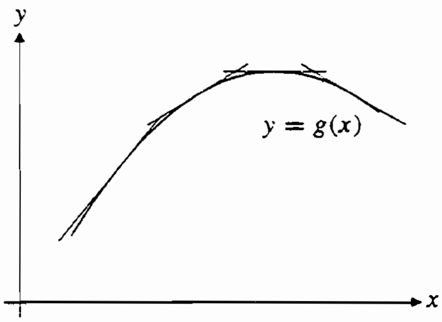


FIGURE 9.15 **The slope of the tangent decreases as x increases. $g'(x)$ is decreasing.**

We introduce the following definitions, assuming that f is continuous in the interval I and twice differentiable in the interior of I, denoted by I^0:

$$f \text{ is } \textbf{convex} \text{ on } I \iff f''(x) \geq 0 \text{ for all } x \text{ in } I^0$$
$$f \text{ is } \textbf{concave} \text{ on } I \iff f''(x) \leq 0 \text{ for all } x \text{ in } I^0 \qquad [9.10]$$

The distinction between convexity and concavity of a function is absolutely crucial in many economic models. Study carefully the cases illustrated in Fig. 9.16.

Example 9.10

Check the convexity/concavity of the following:

(a) $f(x) = x^2 - 2x + 2$ and **(b)** $f(x) = ax^2 + bx + c$

Solution

(a) Here $f'(x) = 2x - 2$ so $f''(x) = 2$. Because $f''(x) > 0$ for all x, f is convex.

(b) Here $f'(x) = 2ax + b$, so $f''(x) = 2a$. If $a = 0$, then f is linear and f is convex as well as concave. If $a > 0$, then $f''(x) > 0$, so f is convex. If $a < 0$, then $f''(x) < 0$, so f is concave. Compare with the graphs in Fig. 3.1 in Section 3.1.

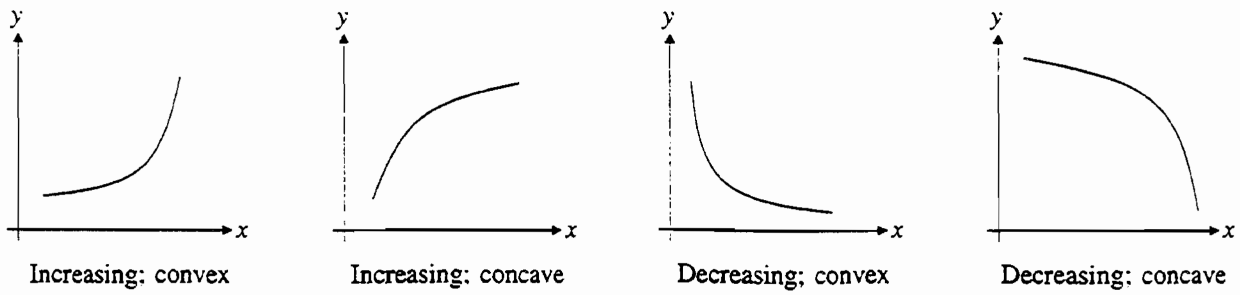


FIGURE 9.16

Some Typical Examples

We consider two typical examples of convex and concave functions. In Fig. 9.17 we have drawn roughly the graph of function P, where

$$P(t) = \text{world population (in 1000 millions) in year } t$$

It appears from the figure that not only is $P(t)$ increasing, but the rate of increase increases. (Each year the *increase* becomes larger.) So $P(t)$ is convex.

The graph in Fig. 9.18 shows the crop of wheat $Y(N)$ when N pounds of fertilizer per acre are used, based on fertilizer experiments in Iowa during 1952 (see Example 9.2 in Section 9.2). The function has a maximum at $N = N_0 \approx 172$. Increasing the amount of fertilizer beyond N_0 will cause wheat production to decline. Moreover, $Y(N)$ is concave. If $N < N_0$, increasing N by one unit will lead to less *increase* in $Y(N)$ the larger is N. On the other hand, if $N > N_0$, increasing N by one unit will lead to a larger *decrease* in $Y(N)$ the larger is N.

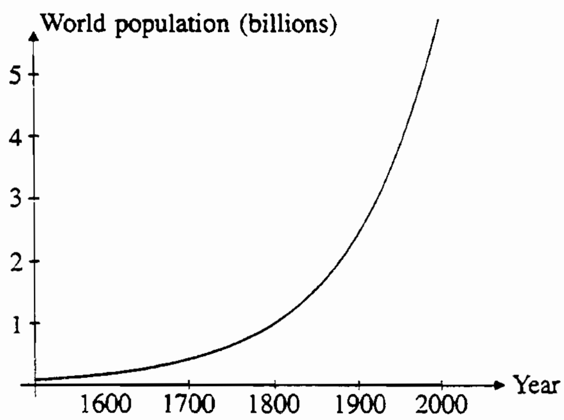


FIGURE 9.17

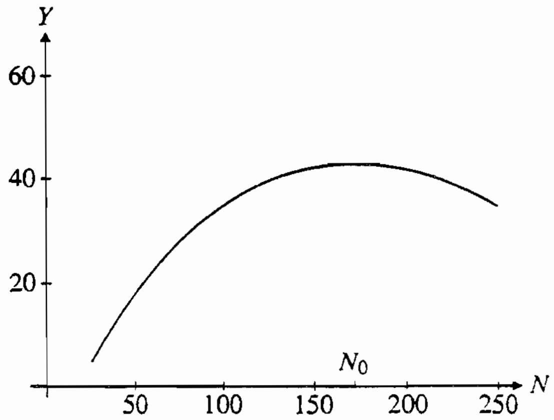


FIGURE 9.18

Example 9.11
Examine the concavity/convexity of the production function

$$Y = AK^a \qquad (A > 0, \quad 0 < a < 1)$$

defined for all $K > 0$.

Solution Differentiating Y twice with respect to K yields

$$Y'' = Aa(a-1)K^{a-2}$$

Because $a \in (0, 1)$, coefficient $Aa(a-1) < 0$, so that $Y'' < 0$ for all $K > 0$. Hence, the function is concave. The graph of $Y = AK^a$, for $0 < a < 1$, is

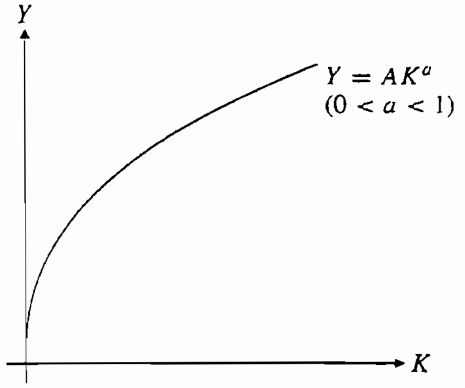


FIGURE 9.19 $Y = AK^a$, $A > 0$, $a \in (0, 1)$.

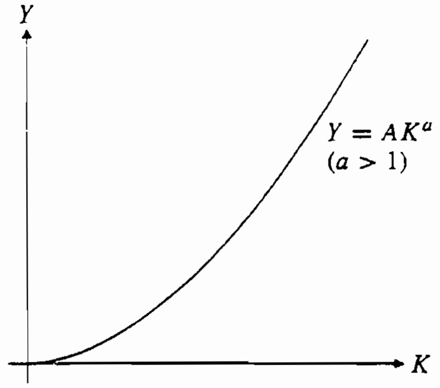


FIGURE 9.20 $Y = AK^a$, $A > 0$, $a > 1$.

shown in in Fig. 9.19. If $a > 1$, then $Y'' > 0$ and Y is a convex function of K, as shown in Fig. 9.20.

Example 9.12

Suppose that functions U and g are both increasing and concave, so that $U' \geq 0$, $U'' \leq 0$, $g' \geq 0$, and $g'' \leq 0$. Prove that the composite function

$$f(x) = g\big(U(x)\big)$$

is also increasing and concave.

Solution Using the chain rule yields

$$f'(x) = g'\big(U(x)\big) \cdot U'(x) \qquad [*]$$

Because g' and U' are both ≥ 0, so $f'(x) \geq 0$. Hence, f is increasing. (*An increasing transformation of an increasing function is increasing.*)

In order to compute $f''(x)$, we must differentiate the product of the two functions $g'\big(U(x)\big)$ and $U'(x)$. According to the chain rule, the derivative of $g'\big(U(x)\big)$ is equal to $g''\big(U(x)\big) \cdot U'(x)$. Hence,

$$f''(x) = g''\big[U(x)\big] \cdot \big(U'(x)\big)^2 + g'\big(U(x)\big) \cdot U''(x) \qquad [**]$$

Because $g'' \leq 0$, $g' \geq 0$, and $U'' \leq 0$, it follows that $f''(x) \leq 0$. (*An increasing concave transformation of a concave function is concave.*)

Inflection Points

Functions we study in economics are often convex in some parts of the domain but concave in others. Points at which a function changes from being convex to being concave, or vice versa, are called inflection points.

Inflection Points

Point c is an **inflection point** for a twice differentiable function f if there is an interval (a, b) containing c such that either of the following two conditions holds: [9.11]

(a) $f''(x) \geq 0$ if $a < x < c$ and $f''(x) \leq 0$ if $c < x < b$

or

(b) $f''(x) \leq 0$ if $a < x < c$ and $f''(x) \geq 0$ if $c < x < b$

Briefly, $x = c$ is an inflection point if $f''(x)$ changes sign at c. We also refer to the point $(c, f(c))$ as an inflection point on the graph. An example is given in Fig. 9.21. Figure 9.22 shows the profile of a ski jump. Point P, where the hill is steepest, is an inflection point.

Theorem 9.3 (Test for Inflection Points) Let f be a function with a continuous second derivative in an interval I, and suppose that c is an interior point of I.

(a) If c is an inflection point for f, then $f''(c) = 0$.

(b) If $f''(c) = 0$ and f'' changes sign at c, then c is an inflection point for f.

FIGURE 9.21 Point *P* is an inflection point on the graph ($x = c$ is an inflection point for the function).

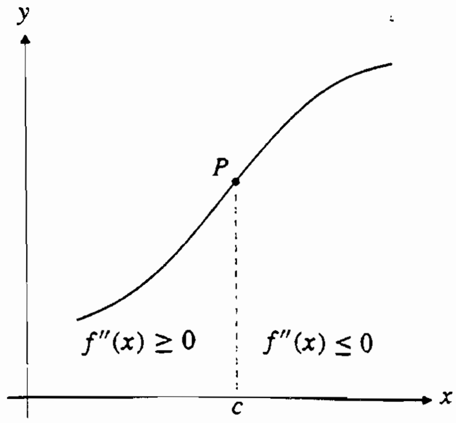
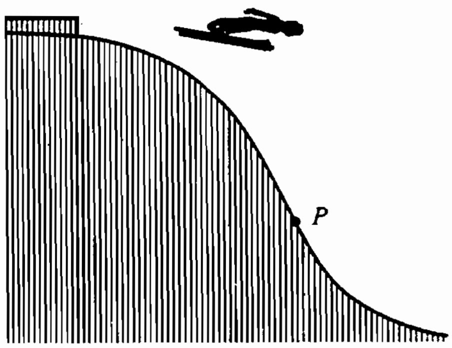


FIGURE 9.22 Point *P*, where the slope is steepest, is an inflection point.



Proof

(a) Because $f''(x) \le 0$ on one side of c and $f''(x) \ge 0$ on the other, $f''(c) = 0$.

(b) If f'' changes sign about point c, then c is an inflection point for f according to [9.11].

According to Theorem 9.3 (a), the condition $f''(c) = 0$ is a *necessary* condition for c to be an inflection point. It is not a sufficient condition, however, because $f''(c) = 0$ does not imply that f'' changes sign at $x = c$. A typical case is given in the next example.

Example 9.13

Show that $f(x) = x^4$ does not have an inflection point at $x = 0$, even though $f''(0) = 0$.

Solution Here $f'(x) = 4x^3$ and $f''(x) = 12x^2$, so that $f''(0) = 0$. But $f''(x) > 0$ for all $x \neq 0$, and hence f'' does not change sign at $x = 0$. Hence, $x = 0$ is not an inflection point. (In fact, it is a global minimum, of course, as shown in Fig. 9.10 in Section 9.4.)

Example 9.14

Find possible inflection points for $f(x) = \frac{1}{9}x^3 - \frac{1}{6}x^2 - \frac{2}{3}x + 1$.

Solution We find the first and second derivatives to be

$$f'(x) = \tfrac{1}{3}x^2 - \tfrac{1}{3}x - \tfrac{2}{3} \qquad \text{and} \qquad f''(x) = \tfrac{2}{3}x - \tfrac{1}{3} = \tfrac{2}{3}\left(x - \tfrac{1}{2}\right)$$

Hence, $f''(x) < 0$ for $x < 1/2$, whereas $f''(1/2) = 0$ and $f''(x) > 0$ for $x > 1/2$. According to Theorem 9.3(b), $x = 1/2$ is an inflection point for f.

Example 9.15

Find possible inflection points for $f(x) = x^2e^x$. Draw its graph. (See Example 8.2, Section 8.1.)

Solution The first derivative of f is $f'(x) = 2xe^x + x^2e^x$, so the second derivative is

$$f''(x) = 2e^x + 2xe^x + 2xe^x + x^2e^x = e^x(x^2 + 4x + 2) = e^x(x - x_1)(x - x_2)$$

where $x_1 = -2 - \sqrt{2} \approx -3.41$ and $x_2 = -2 + \sqrt{2} \approx -0.59$ are the two roots of the quadratic equation $x^2 + 4x + 2 = 0$. The sign diagram associated with $f''(x)$ is shown below. From this diagram we see that f has inflection points at $x = x_1$ and at $x = x_2$. The graph is convex in the intervals $(-\infty, x_1]$ and $[x_2, \infty)$, and it is concave in $[x_1, x_2]$. See

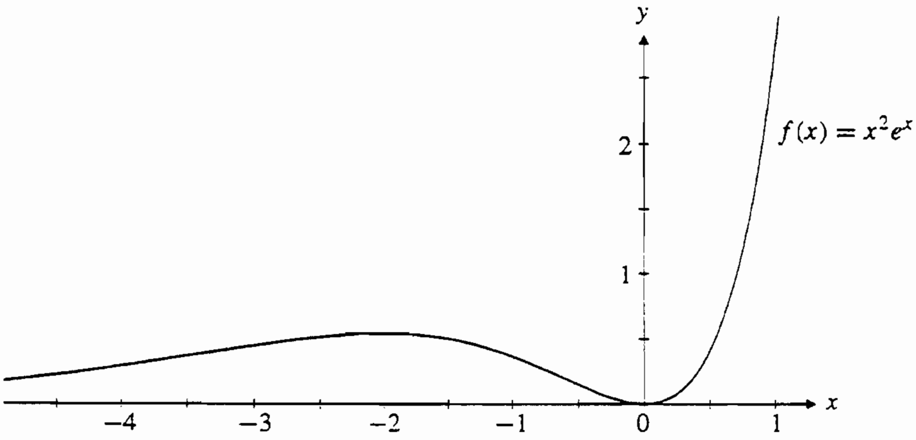
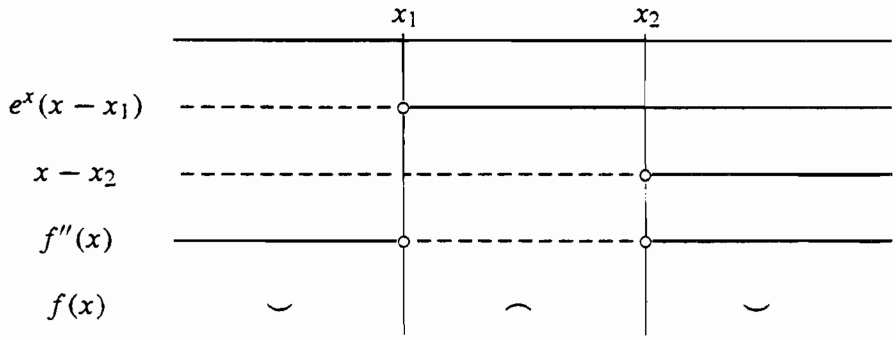


FIGURE 9.23 $f(x) = x^2e^x$

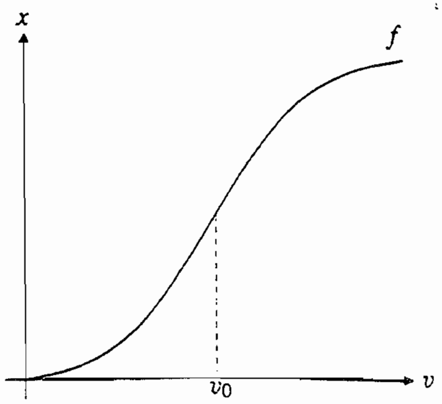
Fig. 9.23 in which we have also taken advantage of the results of Example 8.2.



Example 9.16

A firm produces a commodity using only one input. Let $x = f(v)$, $v \geq 0$, be the maximum production obtainable when v units of the input are used. Then f is called a **production function**. It is often assumed that the marginal product $f'(v)$ is increasing up to a certain production level v_0, and then decreasing. Such a production function is indicated in Fig. 9.24. If f is

FIGURE 9.24 f **is a production function.** v_0 **is an inflection point.**



twice differentiable, then $f''(v)$ is ≥ 0 in $[0, v_0)$ and ≤ 0 in (v_0, ∞). Hence, f is first convex and then concave, with v_0 as an inflection point. An example of such a function is given in Problem 9.

A Useful Result

Suppose that $f''(x) \leq 0$ for all x in an interval I. Then $f'(x)$ is decreasing in I. So if $f'(c) = 0$ for an interior point c in I, then $f'(x)$ must be ≥ 0 to the left of c, whereas $f'(x) \leq 0$ to the right of c. This implies that the function itself is increasing to the left of c, and decreasing to the right of c. We conclude that $x = c$ is a maximum point for f in I. This important observation is illustrated in Fig. 9.25. We have a corresponding result for the minimum of a convex function.

Theorem 9.4 (Maximum/Minimum for Concave/Convex Functions) Suppose f is a concave (convex) function in an interval I. If c is a stationary point for f in the interior of I, then c is a maximum point (minimum point) for f in I. Briefly stated, when c is an interior point of I, then

$$f''(x) \leq 0 \text{ for all } x \in I, \text{ and } f'(c) = 0 \Longrightarrow \\ x = c \text{ is a maximum point for } f \text{ in } I \tag{9.12}$$

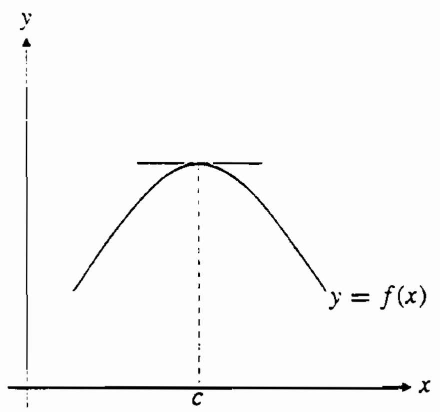
$$f''(x) \geq 0 \text{ for all } x \in I, \text{ and } f'(c) = 0 \Longrightarrow \\ x = c \text{ is a minimum point for } f \text{ in } I \tag{9.13}$$

Example 9.17

Let the total cost of producing Q units of a commodity be

$$C(Q) = aQ^2 + bQ + c, \qquad (Q > 0)$$

FIGURE 9.25 ***f*** **is concave,** $f'(c) = 0$**, and** ***c*** **is a maximum point.**



where a, b, and c are positive constants. Prove that the average cost function $A(Q) = aQ + b + c/Q$ has a minimum at $Q^* = \sqrt{c/a}$. (See also Problem 8 in Section 9.3.)

Solution The first-order derivative of $A(Q)$ is

$$A'(Q) = a - c/Q^2$$

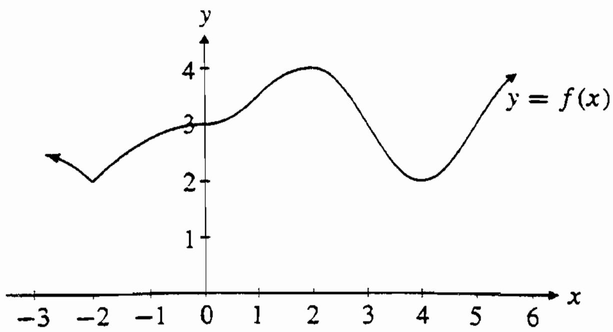
and the only stationary point is $Q^* = \sqrt{c/a}$. Because $A''(Q) = 2c/Q^3 > 0$ for all $Q > 0$, $A(Q)$ is convex, and by Theorem 9.4, $Q^* = \sqrt{c/a}$ is the minimum point.

Problems

1. Determine the concavity/convexity of $f(x) = -\frac{1}{3}x^2 + 8x - 3$.
2. Let f be defined for all x by $f(x) = x^3 + \frac{3}{2}x^2 - 6x + 10$.
   a. Find $f'(x)$ and $f''(x)$.
   b. Find the stationary points of f and the intervals where f is increasing.
   c. Find the inflection points of f and the intervals of concavity/convexity.
3. A competitive firm receives a price p for each unit of its output, pays a price w for each unit of its only variable input, and incurs fixed costs of F. Its output from using x units of variable input is $f(x) = \sqrt{x}$.
   a. Write the firm's revenue, cost, and profit functions.
   b. Write the first-order condition for profit maximization, and give it an economic interpretation.
   c. Check whether profit really is maximized at a point satisfying the first-order condition.
   d. Explain how your answers would change if $f(x) = x^2$.
4. What are the extreme points and the inflection points of function f whose graph is given in Fig. 9.26?
5. Decide where the following functions are convex and determine possible inflection points:

   a. $f(x) = \dfrac{x}{1+x^2}$ b. $g(x) = \dfrac{1-x}{1+x}$ c. $h(x) = xe^x$

FIGURE 9.26



6. Find numbers a and b such that the graph of

$$f(x) = ax^3 + bx^2$$

passes through $(-1, 1)$ and has an inflection point at $x = 1/2$.

7. Find the intervals where the following cubic cost function is convex and where it is concave, and find the unique inflection point:

$$C(Q) = aQ^3 + bQ^2 + cQ + d, \qquad (a > 0, \quad b < 0, \quad c > 0, \quad d > 0)$$

8. With reference to Example 9.5, let $R(Q) = PQ$ and $C(Q) = aQ^b + c$, where P, a, b, and c are positive constants with $b > 1$. Find the value of Q that maximizes profits $\pi(Q) = PQ - (aQ^b + c)$. (Use Theorem 9.4.)

Harder Problems

9. With reference to Example 9.16, let $f(v) = (v - 1)^{1/3} + 1$ for $v \geq 0$.
   - **a.** Show that f is an increasing function of v and that $f''(v) > 0$ in $[0, 1)$, $f''(v) < 0$ in $(1, \infty)$. Draw the graph of f.
   - **b.** Suppose that the price per unit of the commodity is 1 and that the price the firm must pay per unit of the input is p. The profit is then $\pi(v) = f(v) - pv$. Suppose that $v_m > 0$ maximizes $\pi(v)$ for the given value of $p > 0$. Find v_m expressed in terms of p.
   - **c.** Draw the graph of π for the case $p = 1$. Use the same diagram as in part (a).
   - **d.** Find the nonnegative roots of the equation $\pi(v) = 0$. For which values of p are there three real roots?
   - **e.** For all values of p, find the solution of the problem

$$\text{maximize } \pi(v) \text{ subject to } v \geq 0$$

9.6 More on Concave and Convex Functions

So far convexity and concavity have been defined only for functions that are twice differentiable. An alternative geometric characterization of convexity and concavity suggests a more general definition that is valid even for functions that are not differentiable. It is also easier to extend this new generalized definition to functions of several variables.

> Function f is called **concave (convex)** if the line segment joining any two points on the graph is never above (below) the graph. [9.14]

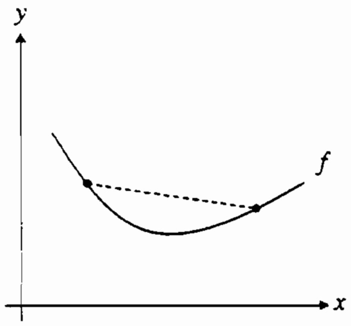


FIGURE 9.27 *f* **is convex.**

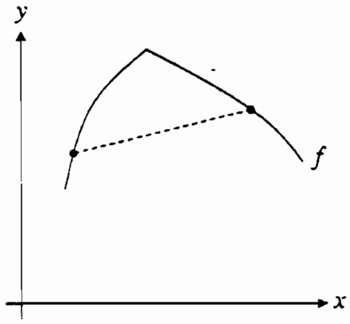


FIGURE 9.28 *f* **is concave.**

These definitions are illustrated in the Figs. 9.27 and 9.28. For twice differentiable functions, one can prove that the definition in [9.14] is equivalent to the definitions of convexity/concavity in terms of the sign of the second derivative.

In order to use [9.14] to examine convexity/concavity of a given function, we must have an algebraic formulation of this definition. To this end, note that an arbitrary point x in the interval $[a, b]$ (with $a < b$) can be written as

$$x = (1-\lambda)a + \lambda b = a + \lambda(b-a) \qquad (\text{for some } \lambda \in [0, 1])$$

For if $b > a$ and $0 \le \lambda \le 1$, then $a \le a + \lambda(b-a) \le b$. Conversely, if $x \in [a, b]$ and we put $\lambda = (x-a)/(b-a)$, then $0 \le \lambda \le 1$ and

$$(1-\lambda)a + \lambda b = \left(1 - \frac{x-a}{b-a}\right)a + \frac{x-a}{b-a}b = \frac{ba - a^2 - xa + a^2 + xb - ab}{b-a} = x$$

(Here $\lambda = (x-a)/(b-a)$ is the ratio between the distance from x to a and the total distance from a to b.)

Consider Fig. 9.29. We want to calculate the number s. According to the point–point formula (2.7) of Section 2.5, the line through $(a, f(a))$ and $(b, f(b))$ has the equation

$$y - f(a) = \frac{f(b) - f(a)}{b-a}(x-a)$$

Let $x = (1-\lambda)a + \lambda b$. Then $y = s$, and so

$$s - f(a) = \frac{f(b) - f(a)}{b-a}\left[(1-\lambda)a + \lambda b - a\right] = \lambda\left[f(b) - f(a)\right]$$

implying that $s = (1-\lambda)f(a) + \lambda f(b)$. Now, as λ takes on all values in $[0, 1]$, so the number $(1-\lambda)a + \lambda b$ will take on all values in $[a, b]$. The requirement that the line segment joining $(a, f(a))$ and $(b, f(b))$ always lies below (or on) the

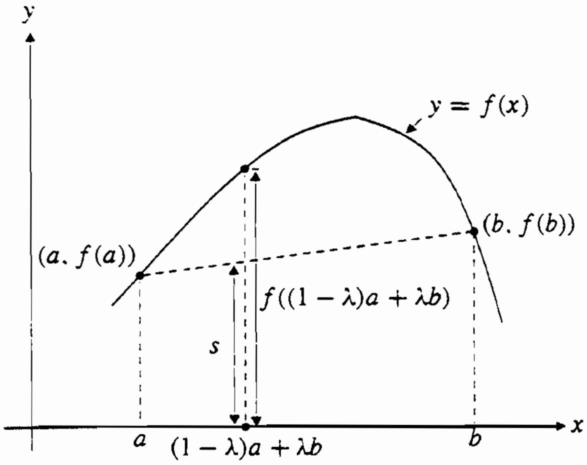


FIGURE 9.29

graph of f is therefore equivalent to the requirement that $s \leq f((1-\lambda)a+\lambda b)$ for all $\lambda \in [0, 1]$. The following definitions should now be quite understandable.

> Function f is **concave** in the interval I if for all $a, b \in I$ and all $\lambda \in (0, 1)$,
>
> $$f((1-\lambda)a+\lambda b) \geq (1-\lambda)f(a)+\lambda f(b)$$ [9.15]

Function f is convex if $-f$ is concave (see Fig. 9.1). So the following holds:

> Function f is **convex** in the interval I if for all $a, b \in I$ and all $\lambda \in (0, 1)$,
>
> $$f((1-\lambda)a+\lambda b) \leq (1-\lambda)f(a)+\lambda f(b)$$ [9.16]

Note that these definitions can be applied to functions that are not even differentiable.

In definition [9.15], if we require that the inequality is strict for $a \neq b$, then f is called **strictly concave**; the graph of f will always be strictly above the line segment joining any two points on the graph. For instance, the function graphed in Fig. 9.29 is strictly concave. Fig. 9.30 shows a typical case in which the function is concave, but not strictly concave. Function f is **strictly convex** if $-f$ is strictly concave.

Example 9.18

Prove that $f(x) = |x|$ is convex in $(-\infty, \infty)$. (See the graph of f in Fig. 9.31.)

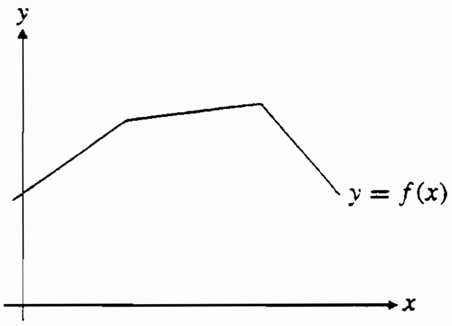


FIGURE 9.30 Concave; not strictly concave.

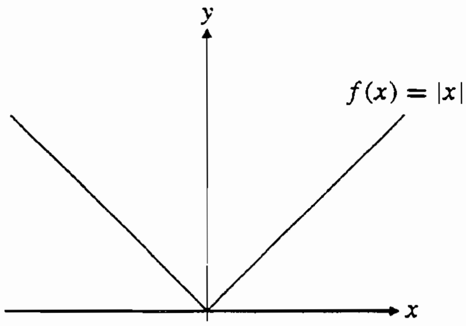


FIGURE 9.31 Convex; not differentiable at $x = 0$.

Solution Let a and b be arbitrary numbers, and let $\lambda \in [0, 1]$. We have to show that the difference D between the left and the right side of the inequality in [9.16] is always ≤ 0. Because $|xy| = |x||y|$ and $|x + y| \leq |x| + |y|$ for all real x, y, it follows that

$$\begin{aligned} f((1-\lambda)a + \lambda b) - [(1-\lambda)f(a) + \lambda f(b)] &= |(1-\lambda)a + \lambda b| - (1-\lambda)|a| - \lambda|b| \\ &\leq (1-\lambda)|a| + \lambda|b| - (1-\lambda)|a| - \lambda|b| \\ &= 0 \end{aligned}$$

Thus, $f(x) = |x|$ is convex.

For twice differentiable functions, it is usually much easier to decide concavity/convexity by checking the sign of the second derivative than by using definitions [9.15] and [9.16]. However, in theoretical arguments, the latter definitions are often very useful, and they generalize easily to functions of several variables.

Example 9.19

Suppose $U(x)$ is a concave function defined in an interval I. Let g be an increasing concave function defined in an interval containing the range of U, and define $f(x) = g(U(x))$. Prove that $f(x)$ is concave in I. (In Example 9.12 in Section 9.5 we proved this result with "unnecessary" differentiability assumptions.)

Solution Let a and b belong to I, with $a < b$, and let $\lambda \in [0, 1]$. By definition of f,

$$f((1-\lambda)a + \lambda b) = g(U((1-\lambda)a + \lambda b)) \qquad [1]$$

Because U is concave,

$$U((1-\lambda)a + \lambda b) \geq (1-\lambda)U(a) + \lambda U(b) \qquad [2]$$

Because g is increasing, $r \geq s$ implies $g(r) \geq g(s)$. Hence, applying g to each side of [2] yields

$$g(U((1-\lambda)a+\lambda b)) \geq g\big((1-\lambda)U(a)+\lambda U(b)\big) \qquad [3]$$

By the concavity of g,

$$\begin{aligned} g((1-\lambda)U(a)+\lambda U(b)) &\geq (1-\lambda)g(U(a))+\lambda g(U(b)) \\ &= (1-\lambda)f(a)+\lambda f(b) \end{aligned} \qquad [4]$$

From [1], [3], and [4], we see that $f((1-\lambda)a+\lambda b) \geq (1-\lambda)f(a)+\lambda f(b)$, so f *is* concave.

An easy test for strict concavity/convexity is the following, which we present without proof:

$$\begin{aligned} f''(x) < 0 \text{ for all } x \in (a,b) &\Longrightarrow f(x) \text{ is strictly concave in } (a,b) \\ f''(x) > 0 \text{ for all } x \in (a,b) &\Longrightarrow f(x) \text{ is strictly convex in } (a,b) \end{aligned} \qquad [9.17]$$

The reverse implications are not correct. For instance, one can prove that $f(x) = x^4$ is strictly convex in the interval $(-\infty, \infty)$, but $f''(x)$ is not > 0 everywhere, because $f''(0) = 0$.

Note: Here are some of the most commonly used functions that are concave (convex) in their domains:

Concave: ax^2+bx+c $(a \leq 0)$, x^a $(0 \leq a \leq 1,\ x > 0)$, $\ln x$ $(x > 0)$ [1]

Convex: ax^2+bx+c $(a \geq 0)$, x^a $(a \geq 1,\ x > 0)$, e^{ax}, $|x|$ [2]

It follows immediately from definitions [9.15 $f(x)$, $g(x)$] and [9.16] that nonnegative linear combinations $af(x)+bg(x)(a, b \geq a)$ of concave (convex) functions are concave (convex). Using these facts and [1] and [2], we can often quite easily decide concavity/convexity. In Sections 17.7 and 17.8, we shall prove many other properties that will help us decide concavity (convexity).

Jensen's Inequality

Looking at definition [9.15] of a concave function, suppose we put $a = x_1$, $b = x_2$, $1-\lambda = \lambda_1$, and $\lambda = \lambda_2$. Then definition [9.15] would read: $f(x)$ is concave on I if for all x_1 and x_2 in I, and for all $\lambda_1 \geq 0$ and $\lambda_2 \geq 0$ with $\lambda_1 + \lambda_2 = 1$,

$$f(\lambda_1 x_1 + \lambda_2 x_2) \geq \lambda_1 f(x_1) + \lambda_2 f(x_2)$$

Jensen's inequality is a generalization of this inequality.

Jensen's Inequality

A function f is concave in the interval I if and only if the following inequality is satisfied for all $x_1, \ldots, x_n$ in I, and for all $\lambda_1 \geq 0, \ldots, \lambda_n \geq 0$ with $\lambda_1 + \cdots + \lambda_n = 1$:

$$f(\lambda_1 x_1 + \cdots + \lambda_n x_n) \geq \lambda_1 f(x_1) + \cdots + \lambda_n f(x_n) \qquad [9.18]$$

The corresponding result for the case where f is convex is obtained by reversing the inequality in [9.18]. The more general vector version of this result is given in Section 17.6.

Example 9.20 (Production Smoothing)

Consider a manufacturing firm producing a single commodity. The cost of maintaining an output level y per year for a fraction λ of a year is $\lambda C(y)$, where $C'(y) > 0$ and $C''(y) \geq 0$ for all $y \geq 0$. In fact, the firm's output level can fluctuate over the year. Show that, given the total output Y that the firm produces over the whole year, the firm's total cost per year is minimized by choosing a constant flow of output.

Solution Suppose the firm chooses different output levels $y_1, \ldots, y_n$ per year for fractions of the year $\lambda_1, \ldots, \lambda_n$, respectively. Then the total output is $\sum_{i=1}^{n} \lambda_i y_i = Y$ produced at total cost $\sum_{i=1}^{n} \lambda_i C(y_i)$. Applying Jensen's inequality to the convex function C gives

$$\sum_{i=1}^{n} \lambda_i C(y_i) \geq C\left(\sum_{i=1}^{n} \lambda_i y_i\right) = C(Y)$$

The right-hand side is the cost of maintaining the constant output level Y over the whole year, and this is the minimum cost.

Problems

1. Suppose $f(x) = 1 - x^2$.
   a. Show that $D = f((1-\lambda)a + \lambda b) - (1-\lambda)f(a) - \lambda f(b)$ can be written in the form

   $$D = \lambda(1-\lambda)(a^2 - 2ab + b^2) = \lambda(1-\lambda)(a-b)^2$$

   b. If $\lambda \in (0, 1)$, what is the sign of D? Is f concave, convex, or neither?
   c. Is f strictly concave/convex?
   d. Check the result in part (c) by using [9.17].
2. Suppose that a function f is concave. What restrictions on a and b will guarantee that $g(x) = af(x) + b$ is also concave?

3. Are the following functions concave/convex (assuming that $x > 0$ in parts (b) and (c))?

   a. $\frac{1}{2}e^x + \frac{1}{2}e^{-x}$ b. $2x - 3 + 4\ln x$ c. $5x^{0.5} - 10x^{1.5}$

   d. $3x^2 - 2x + 1 + e^{-x-3}$

Harder Problems

4. A consumer is planning to choose a lifetime consumption stream $c_1, \dots, c_T$ to maximize $(1/T)\sum_{t=1}^{T} u(c_t)$ subject to the budget constraint $(1/T)\sum_{t=1}^{T} c_t \le (1/T)\sum_{t=1}^{T} y_t$. Here y_t is the income stream, and the utility function satisfies $u'(c) > 0$ and $u''(c) < 0$.

   a. Use Jensen's inequality to show that the optimal consumption is constant and equal to the mean lifetime income.

   b. Replace $(1/T)\sum_{t=1}^{T} u(c_t)$ by $\sum_{t=1}^{T}(1+r)^{-t}u(c_t)$, with the new budget constraint

$$\sum_{t=1}^{T}(1+r)^{-t}(c_t - y_t) \le 0$$

   where $r > -1$ is the rate of interest. What is the new optimal consumption stream?

5. Prove that if f and g are both concave, then

$$h(x) = \min\{f(x), g(x)\}$$

   is concave. Illustrate. (Note that for each given x, $h(x)$ is the smaller of the two numbers $f(x)$ and $g(x)$.)

3. Are the following functions concave/convex (assuming that $x > 0$ in parts (b) and (c))?

   a. $\frac{1}{2}e^x + \frac{1}{2}e^{-x}$ b. $2x - 3 + 4\ln x$ c. $5x^{0.5} - 10x^{1.5}$

   d. $3x^2 - 2x + 1 + e^{-x-3}$

Harder Problems

4. A consumer is planning to choose a lifetime consumption stream $c_1, \ldots, c_T$ to maximize $(1/T)\sum_{t=1}^{T} u(c_t)$ subject to the budget constraint $(1/T)\sum_{t=1}^{T} c_t \leq (1/T)\sum_{t=1}^{T} y_t$. Here y_t is the income stream, and the utility function satisfies $u'(c) > 0$ and $u''(c) < 0$.
   a. Use Jensen's inequality to show that the optimal consumption is constant and equal to the mean lifetime income.
   b. Replace $(1/T)\sum_{t=1}^{T} u(c_t)$ by $\sum_{t=1}^{T}(1+r)^{-t}u(c_t)$, with the new budget constraint

$$\sum_{t=1}^{T}(1+r)^{-t}(c_t - y_t) \leq 0$$

   where $r > -1$ is the rate of interest. What is the new optimal consumption stream?

5. Prove that if f and g are both concave, then

$$h(x) = \min\{f(x), g(x)\}$$

   is concave. Illustrate. (Note that for each given x, $h(x)$ is the smaller of the two numbers $f(x)$ and $g(x)$.)

10

Integration

Indeed, models basically play the same role in economics as in fashion. They provide an articulated frame on which to show off your material to advantage, . . . a useful role, but fraught with the dangers that the designer may get carried away by his personal inclination for the model, while the customer may forget that the model is more streamlined than reality.
—*J. H. Drèze* (1984)

The geometric problem of finding the steepness of a curve at a point leads to the concept of the derivative of a function. The derivative turns out to have important interpretations apart from the geometric one. Particularly important in economics is the fact that the derivative represents the rate of change of a function.

The main concept to be discussed in this chapter can also be introduced geometrically. In fact, we begin with the problem of measuring the areas of certain plane regions that are bounded not only by straight lines. Solving this problem will involve the concept of the definite integral of a function over an interval. This concept also has a number of important interpretations in addition to the geometric one.

As early as about 360 B.C., the Greek mathematician Eudoxos developed a general method for determining the areas of plane regions, known as the *method of exhaustion*. The idea was to inscribe and circumscribe the region (say, a circular disk) by simpler geometric regions such as rectangles, triangles, or general polygonal regions—whose area we already know how to measure. Now, if the area of the inscribed region and the area of the circumscribed region tend to the same limit as more and more refined polygons are chosen, this limit is defined as the *area* of the region.

The method of exhaustion was used by Eudoxos and Archimedes to determine the areas of a number of specific plane regions. Similar methods were developed to determine the lengths of curves and the volumes of solids. However, the method of exhaustion turned out to work only in a limited number of cases, partly because of the algebraic problems encountered. Nearly 1900 years passed after Archimedes before anyone else made significant progress in measuring areas of plane regions. In the seventeenth century, a new method of finding areas was devised, called integration, that is closely related to differential calculus. Demonstrating the precise relationship between differentiation and integration is one of the main achievements of mathematical analysis. It has even been argued that this discovery is the single most important in all of science. Barrow, who was Newton's teacher, and Newton and Leibniz in particular, are the mathematicians associated with this discovery.

After these introductory comments, we begin by solving the geometric problem of finding the areas of certain specific plane regions. We then develop the theory of integration based on this foundation.

10.1 Areas under Curves

The problem to be considered in this section is illustrated in Fig. 10.1. It can be formulated as follows: How do we compute the area A under the graph of f from a to b, assuming that $f(x)$ is positive and continuous?

To answer this question, we first introduce the function $A(x)$ that measures the area under the curve $y = f(x)$ over the interval $[a, x]$, as shown in Fig. 10.2. Clearly, $A(a) = 0$, because there is no area from a to a, and the area in Fig. 10.1 is $A = A(b)$.

It is obvious from Fig. 10.2 that because f is always positive, $A(x)$ increases as x increases. Suppose we increase x by a positive amount Δx. Then $A(x + \Delta x)$

FIGURE 10.1

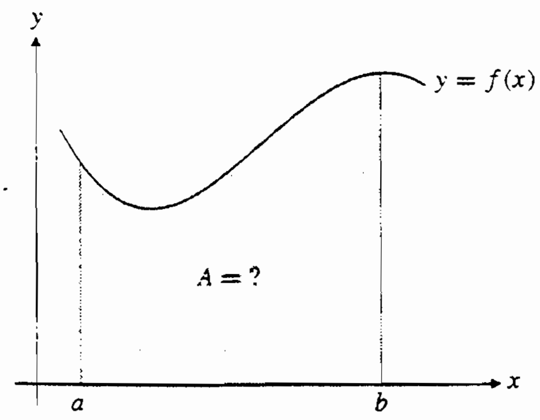
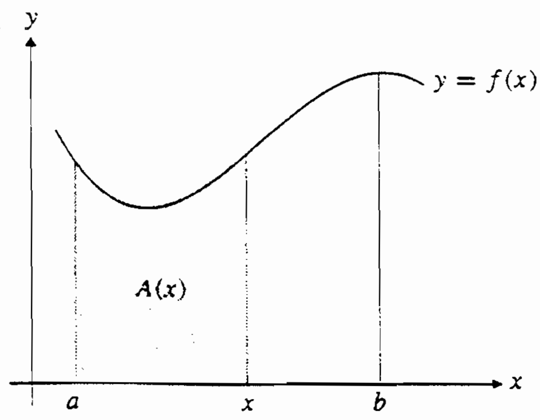


FIGURE 10.2



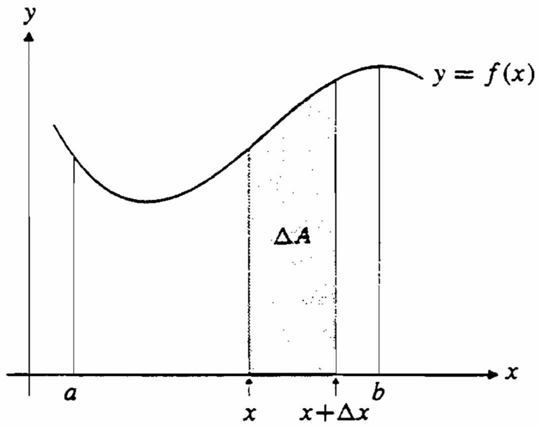


FIGURE 10.3

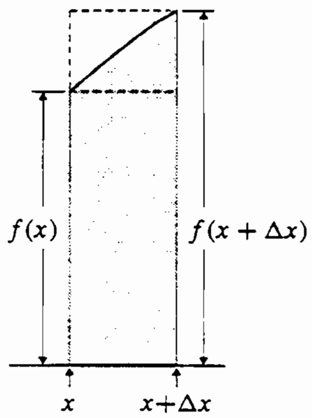


FIGURE 10.4

is the area under the curve $y = f(x)$ over the interval $[a, x + \Delta x]$. Hence, $A(x + \Delta x) - A(x)$ is the area ΔA under the curve over the interval $[x, x + \Delta x]$, as shown in Fig. 10.3.

In Fig. 10.4, area ΔA is magnified. It cannot be larger than the area of the rectangle with edges Δx and $f(x + \Delta x)$, nor smaller than the area of the rectangle with edges Δx and $f(x)$. Hence, for all $\Delta x > 0$,

$$f(x)\,\Delta x \le A(x + \Delta x) - A(x) \le f(x + \Delta x)\,\Delta x \qquad [*]$$

But then

$$f(x) \le \frac{A(x + \Delta x) - A(x)}{\Delta x} \le f(x + \Delta x) \qquad [**]$$

(If $\Delta x < 0$, the inequalities in [*] are reversed, whereas the inequalities in [**] are preserved. The following argument is equally valid when $\Delta x < 0$.) Let us consider what happens to [**] as $\Delta x \to 0$. The interval $[x, x + \Delta x]$ shrinks to the single point x, and by continuity of f, the value $f(x + \Delta x)$ approaches $f(x)$. The Newton quotient $[A(x + \Delta x) - A(x)]/\Delta x$, squeezed between $f(x)$ and a quantity that approaches $f(x)$, must therefore approach $f(x)$ as $\Delta x \to 0$.[1] So we arrive at the remarkable conclusion that the function $A(x)$, which measures the area under the graph of f over the interval $[a, x]$, is differentiable, with derivative given by

$$A'(x) = f(x) \qquad (\text{for all } x \in (a, b))$$

This proves that *the derivative of the area function $A(x)$ is the curve's "height" function $f(x)$.*

[1]The function f in the figures is increasing in the interval $[x, x + \Delta x]$. It is easy to see that the same conclusion is obtained whatever the behavior of f in the interval $[x, x + \Delta x]$. On the left-hand side of [*], just replace $f(x)$ by $f(c)$, where c is the minimum point of the continuous function f in the interval; and on the right-hand side, replace $f(x + \Delta x)$ by $f(d)$, where d is the maximum point of f in $[x, x + \Delta x]$.

Suppose that $F(x)$ is another continuous function with $f(x)$ as its derivative, so that $F'(x) = A'(x) = f(x)$ for all $x \in (a, b)$. Because $(d/dx)[A(x) - F(x)] = A'(x) - F'(x) = 0$, it must be true that $A(x) = F(x) + C$ for some constant C (see Theorem 7.7 of Section 7.3). Recall that $A(a) = 0$. Hence, $0 = A(a) = F(a) + C$, so $C = -F(a)$. Therefore,

$$A(x) = F(x) - F(a) \quad \text{when} \quad F'(x) = f(x) \tag{10.1}$$

This leads to the following.

Method for finding the area below the curve $y = f(x)$ and above the x-axis from $x = a$ to $x = b$:

1. Find an arbitrary function F that is continuous on $[a, b]$ such that $F'(x) = f(x)$ for all $x \in (a, b)$.
2. The required area is then $F(b) - F(a)$.

[10.2]

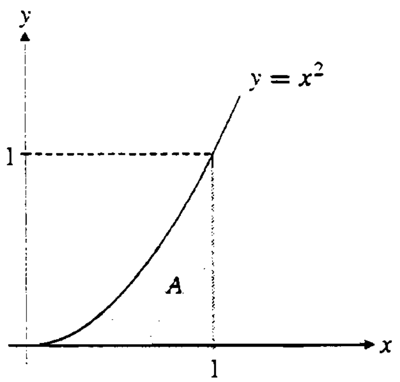
A function F with the property that $F'(x) = f(x)$ for all x in some open interval, is often called an **antiderivative** of f. Note that there are always many such antiderivatives because $(d/dx)[F(x) + C] = F'(x) = f(x)$ whenever C is any real constant.

Example 10.1

Calculate the area under the parabola $f(x) = x^2$ over the interval [0, 1].

Solution The area in question is the shaded region A in Fig. 10.5. According to step 1 of [10.2], we must find a function having x^2 as its derivative. We look for a power function. Indeed $(d/dx)ax^n = anx^{n-1} = x^2$ when $n = 3$ and $a = 1/3$. So we put $F(x) = \frac{1}{3}x^3$ and then $F'(x) = x^2$. Thus, the

FIGURE 10.5



required area is

$$A = F(1) - F(0) = \tfrac{1}{3} \cdot 1^3 - \tfrac{1}{3} \cdot 0^3 = \tfrac{1}{3}$$

Figure 10.5 suggests that this answer is reasonable, because the shaded region appears to have roughly 1/3 the area of a square whose side is of length 1.

Note: If you tried seriously to use the method of exhaustion for determining the area in Fig. 10.5, you would appreciate the extreme simplicity of the method based on [10.2].

Example 10.2

Find the area A under the straight line $f(x) = cx + d$ over the interval $[a, b]$. (We assume that the constants c and d are chosen so that $f(x) \geq 0$ in $[a, b]$.)

Solution The area is shown in Fig. 10.6. If we put $F(x) = \frac{1}{2}cx^2 + dx$, then $F'(x) = cx + d$, and so

$$\begin{aligned} A = F(b) - F(a) &= (\tfrac{1}{2}cb^2 + db) - \left(\tfrac{1}{2}ca^2 + da\right) \\ &= \tfrac{1}{2}c(b^2 - a^2) + d(b - a) \end{aligned}$$

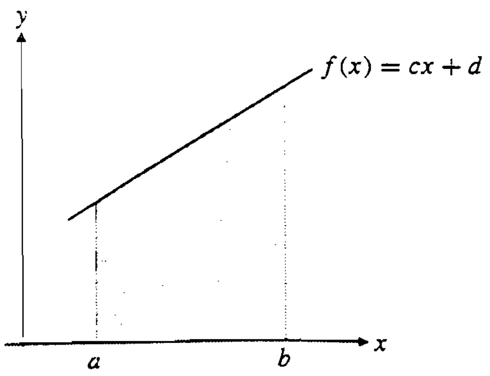
Compute the same area in another way and check that you get the same answer.

The argument leading to [10.2] was based on rather intuitive considerations. However, the concept of area that emerges agrees with the usual concept for regions bounded by straight lines. Example 10.2 is a case in point.

Formally, we choose to *define* the area under the graph of a continuous and nonnegative function f over the interval $[a, b]$ as the number $F(b) - F(a)$, where $F'(x) = f(x)$. Suppose $G(x)$ is any other function with $G'(x) = f(x)$ for $x \in (a, b)$. Then $G(x) = F(x) + C$, for some constant C. Hence,

$$G(b) - G(a) = F(b) + C - [F(a) + C] = F(b) - F(a)$$

FIGURE 10.6



This argument tells us that the area we compute using [10.2] is independent of which antiderivative of f we choose. Moreover, according to Theorem 10.1 of Section 10.3, *any* continuous function f in $[a, b]$ has an antiderivative.

What Happens if $f(x)$ Has Negative Values in $[a, b]$?

We assumed before that f was continuous and positive-valued. Let us consider the case in which f is a function defined and continuous in $[a, b]$, with $f(x) \leq 0$ for all $x \in [a, b]$. The graph of f, the x-axis, and the lines $x = a$ and $x = b$ still enclose an area. If $F'(x) = f(x)$, we define the area to be $-[F(b) - F(a)]$. We choose this definition because we want the area of a region always to be positive.

Example 10.3

Compute the area shaded in Fig. 10.7. It is the area between the x-axis and the graph of $f(x) = e^{x/3} - 3$, over the interval $[0, 3\ln 3]$.

Solution We need to find a function $F(x)$ whose derivative is $e^{x/3} - 3$. Trial and error leads to the suggestion $F(x) = 3e^{x/3} - 3x$. (Check that $F'(x) = e^{x/3} - 3$.) The area is therefore equal to

$$
\begin{aligned}
-\left[F(3\ln 3) - F(0)\right] &= -(3e^{\ln 3} - 3 \cdot 3\ln 3 - 3e^0) \\
&= -(9 - 9\ln 3 - 3) = 9\ln 3 - 6 \approx 3.89
\end{aligned}
$$

Is the answer reasonable? (Yes, because the shaded set in Fig. 10.7 seems to be a little less than 4 units in area.)

Suppose f is defined and continuous in $[a, b]$, positive in some subintervals, and negative in others, as is the case in Fig. 10.8. The total area bounded

FIGURE 10.7

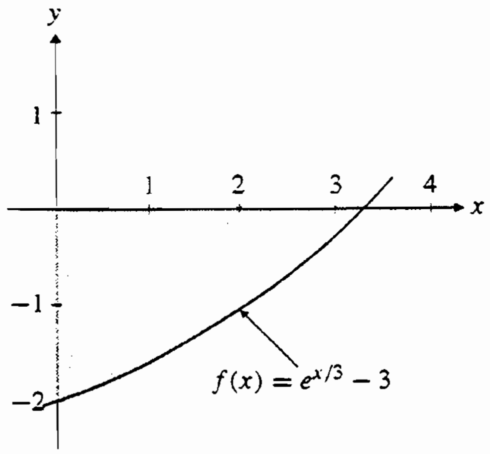
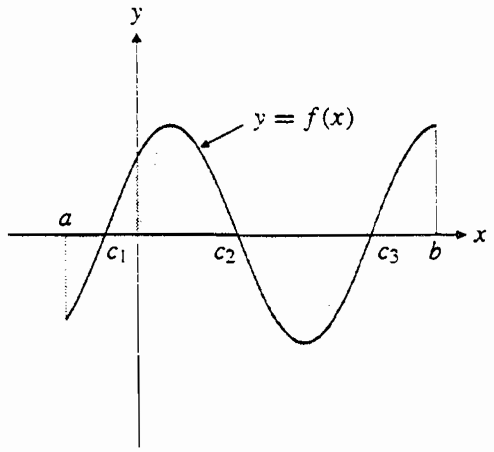


FIGURE 10.8



by the graph of f, the x-axis, and the lines $x = a$ and $x = b$ is then calculated by computing the positive areas in each subinterval $[a, c_1]$, $[c_1, c_2]$, $[c_2, c_3]$, and $[c_3, b]$ in turn according to the previous definitions, and then adding these areas.

Problems

1. Compute the area under the graph of $f(x) = x^3$ over $[0, 1]$ by using [10.2].
2. For each of the following cases, draw a rough graph of f and indicate (by shading) the area of the set bounded by the x-axis, the lines $x = a$ and $x = b$, and the graph of f. Also calculate the area in question.
   a. $f(x) = 3x^2$ in $[0, 2]$ **b.** $f(x) = x^6$ in $[0, 1]$
   c. $f(x) = e^x$ in $[-1, 1]$ **d.** $f(x) = 1/x^2$ in $[1, 10]$
3. Compute the area A bounded by the graph of $f(x) = 1/x^3$, the x-axis, and the lines $x = -2$ and $x = -1$. (Make a drawing.)
4. Compute the area A bounded by the graph of $f(x) = \frac{1}{2}(e^x + e^{-x})$, the x-axis, and the lines $x = -1$ and $x = 1$.

10.2 Indefinite Integrals

The problem of computing areas under the graph of a function f leads to the problem of finding an *antiderivative* of f—that is, a function F whose derivative is f.

Although the name antiderivative is very appropriate, we shall follow the usual practice and call F an **indefinite integral** of f. As a symbol for an indefinite integral of f, we use $\int f(x)\,dx$. Two functions having the same derivative throughout an interval must differ by a constant, so we write

$$\int f(x)\,dx = F(x) + C \quad \text{when} \quad F'(x) = f(x) \tag{10.3}$$

For instance,

$$\int x^3\,dx = \tfrac{1}{4}x^4 + C \quad \text{because} \quad \left(\tfrac{1}{4}x^4\right)' = x^3$$

where ( )′ denotes differentiation. The symbol $\int$ is the **integral sign**, the function $f(x)$ appearing in [10.3] is the **integrand**, and C is the **constant of integration**. The dx part of the integral notation indicates that x is the **variable of integration**.

Let a be a fixed number $\neq -1$. Because the derivative of $x^{a+1}/(a+1)$ is x^a,

$$\int x^a\,dx = \frac{1}{a+1}x^{a+1} + C \qquad (a \neq -1) \tag{10.4}$$

This very important integration result states that the indefinite integral of any power of x (except x^{-1}) is obtained by increasing the exponent of x by 1, dividing by the new exponent, and then adding the constant of integration. Here are some examples:

(a) $\int x\,dx = \int x^1\,dx = \frac{1}{1+1}x^{1+1} + C = \frac{1}{2}x^2 + C$

(b) $\int \frac{1}{x^3}\,dx = \int x^{-3}\,dx = \frac{1}{-3+1}x^{-3+1} + C = -\frac{1}{2x^2} + C$

(c) $\int \sqrt{x}\,dx = \int x^{1/2}\,dx = \frac{1}{1/2+1}x^{1/2+1} + C = \frac{2}{3}x^{3/2} + C$

When $a = -1$, the formula in [10.4] is not valid, because the right-hand side involves division by zero and so becomes meaningless. The integrand is then $1/x$, and the problem is thus to find a function having $1/x$ as its derivative. Now $\ln x$ has this property, but it is only defined for $x > 0$. Note, however, that $\ln(-x)$ is defined for $x < 0$, and according to the chain rule, its derivative is $[1/(-x)](-1) = 1/x$. Recall that $|x| = x$ when $x \geq 0$ and $|x| = -x$ when $x < 0$. Thus, whether we integrate over an interval where $x > 0$ or $x < 0$, we have

$$\int \frac{1}{x}\,dx = \ln|x| + C \tag{10.5}$$

Consider next the exponential function. The derivative of e^x is e^x. Thus, $\int e^x\,dx = e^x + C$. More generally,

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C \qquad (a \neq 0) \tag{10.6}$$

because the derivative of $(1/a)e^{ax}$ is e^{ax}.

For $a > 0$ we can write $a^x = e^{x \ln a}$. As an application of [10.6], for $\ln a \neq 0$ (that is, for $a \neq 1$), we have

$$\int a^x \, dx = \frac{1}{\ln a} a^x + C \qquad (a > 0 \text{ and } a \neq 1) \qquad [10.7]$$

Some General Rules

Two rules of differentiation are $(aF(x))' = aF'(x)$ and $(F(x) + G(x))' = F'(x) + G'(x)$. They immediately imply the following integration rules:

Constant Multiple Property

$$\int af(x) \, dx = a \int f(x) \, dx \qquad (a \text{ is a real constant}) \qquad [10.8]$$

The integral of a sum is the sum of the integrals

$$\int [f(x) + g(x)] \, dx = \int f(x) \, dx + \int g(x) \, dx \qquad [10.9]$$

Repeated use of these two properties yields the general rule

$$\int [a_1 f_1(x) + \cdots + a_n f_n(x)] \, dx = a_1 \int f_1(x) \, dx + \cdots + a_n \int f_n(x) \, dx \qquad [10.10]$$

for the indefinite integral of any linear combination of continuous functions.

Example 10.4
Find the integral $\int (3x^4 + 5x^2 - 2) \, dx$.

Solution

$$\begin{aligned}\int (3x^4 + 5x^2 - 2)\,dx &= 3\int x^4\,dx + 5\int x^2\,dx - 2\int 1\,dx \\ &= 3\left(\tfrac{1}{5}x^5 + C_1\right) + 5\left(\tfrac{1}{3}x^3 + C_2\right) - 2\,(x + C_3) \\ &= \tfrac{3}{5}x^5 + \tfrac{5}{3}x^3 - 2x + 3C_1 + 5C_2 - 2C_3 \\ &= \tfrac{3}{5}x^5 + \tfrac{5}{3}x^3 - 2x + C\end{aligned}$$

Because C_1, C_2, and C_3 are arbitrary constants, $3C_1 + 5C_2 - 2C_3$ is also an arbitrary constant. So in the last line we have replaced it by C for simplicity.

It is not necessary to write all the intermediate steps when integrating in this way. More simply, we write

$$\begin{aligned}\int (3x^4 + 5x^2 - 2)\,dx &= 3\int x^4\,dx + 5\int x^2\,dx - 2\int 1\,dx \\ &= \frac{3}{5}x^5 + \frac{5}{3}x^3 - 2x + C\end{aligned}$$

By systematically using the proper rules, we can *differentiate* very complicated functions. On the other hand, finding the indefinite integral of even quite simple functions can be very difficult, or impossible. Note, however, that it is usually quite easy to check whether a proposed indefinite integral is correct. We simply differentiate the proposed function to see if its derivative is equal to the integrand.

Example 10.5

Verify that (in an interval where $ax + b > 0$)

$$\int \frac{x}{\sqrt{ax+b}}\,dx = \frac{2}{3a^2}(ax - 2b)\sqrt{ax+b} + C$$

Solution We put $F(x) = (2/3a^2)(ax - 2b)\sqrt{ax+b} = (2/3a^2)u \cdot v$, where $u = ax - 2b$ and $v = \sqrt{ax+b}$. Now

$$F'(x) = \frac{2}{3a^2}(u'v + uv')$$

where, after introducing the new variable $w = ax + b$, one has

$$u' = a, \qquad v = \sqrt{ax+b} = \sqrt{w} \quad \Longrightarrow \quad v' = \frac{1}{2\sqrt{w}}\,w' = \frac{a}{2\sqrt{ax+b}}$$

Hence,

$$F'(x) = \frac{2}{3a^2}\left[a\sqrt{ax+b} + (ax-2b)\frac{a}{2\sqrt{ax+b}}\right]$$

$$= \frac{2}{3a^2}\left[\frac{2a(ax+b)+(ax-2b)a}{2\sqrt{ax+b}}\right] = \frac{2}{3a^2}\frac{2a^2x+2ab+a^2x-2ab}{2\sqrt{ax+b}}$$

$$= \frac{2}{3a^2}\frac{3a^2x}{2\sqrt{ax+b}} = \frac{x}{\sqrt{ax+b}}$$

which shows that the integral formula is correct.

Initial-Value Problems

As was seen before, there are infinitely many "antiderivatives," or indefinite integral functions, having a given function as their common derivative. For instance, the derivative of $\frac{1}{5}x^5 + C$ is x^4 for all choices of the constant C. The graphs of these functions are all translates of each other in the direction of the y-axis. Given any point (x_0, y_0), there is one and only one of these curves that passes through (x_0, y_0).

Example 10.6

Find all functions $F(x)$ such that

$$F'(x) = -(x-1)^2 \qquad [1]$$

and draw some of the graphs in the xy-plane. Find in particular the function whose graph passes through the point $(x_0, y_0) = (1, 1)$.

Solution Equation [1] implies that

$$F(x) = \int -(x-1)^2\,dx = -\tfrac{1}{3}(x-1)^3 + C$$

All values of C are possible. Some of the associated graphs are drawn in Fig. 10.9. The curve that passes through (1, 1) is found by solving the equation $F(1) = 1$, or

$$-\tfrac{1}{3}(1-1)^3 + C = 1$$

This gives $C = 1$, so the required function is

$$F(x) = 1 - \tfrac{1}{3}(x-1)^3$$

The last part of Example 10.6 can be formulated this way: Find the unique function $F(x)$ such that $F'(x) = -(x-1)^2$ and $F(1) = 1$. This is called an **initial-value problem** and the requirement that $F(1) = 1$ is called an **initial condition**.

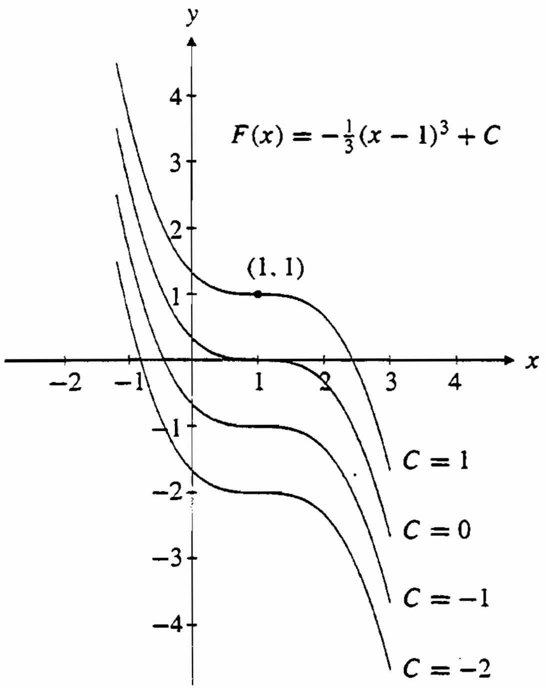


FIGURE 10.9 $F(x) = -\frac{1}{3}(x-1)^3 + C$.

Example 10.7

The marginal cost of producing x units of some commodity is $1 + x + 3x^2$ and fixed costs are 150. Find the total cost function.

Solution Denoting the total cost function by $c(x)$, we have

$$[1] \quad c'(x) = 1 + x + 3x^2 \quad \text{and} \quad [2] \quad c(0) = 150$$

because $c(0)$ is the cost incurred even if nothing is produced. Integrating [1] yields

$$c(x) = x + \tfrac{1}{2}x^2 + x^3 + C \tag{3}$$

Substituting $x = 0$ in [3] gives $c(0) = C$, and so $C = 150$ because of [2]. Hence, the required total cost function is

$$c(x) = x + \tfrac{1}{2}x^2 + x^3 + 150$$

So far, we have always used x as the variable of integration. In economics, the variables often have other labels.

Example 10.8

Find the following:

(a) $\int \frac{B}{r^{2.5}}\, dr$

(b) $\int (a + bq + cq^2)\, dq$

Solution

(a) Writing $B/r^{2.5}$ as $Br^{-2.5}$, we see that formula [10.4] applies, and so

$$\int \frac{B}{r^{2.5}}\,dr = B\int r^{-2.5}\,dr = B\frac{1}{-2.5+1}r^{-2.5+1} + C = -\frac{B}{1.5}r^{-1.5} + C$$

(b) $\int (a + bq + cq^2)\,dq = aq + \frac{1}{2}bq^2 + \frac{1}{3}cq^3 + C$

Problems

1. Find the following integrals using [10.4]:

a. $\int x^{13}\,dx$ b. $\int x\sqrt{x}\,dx$ c. $\int \frac{1}{\sqrt{x}}\,dx$ d. $\int \sqrt{x\sqrt{x\sqrt{x}}}\,dx$

2. Find the following integrals:

a. $\int (t^3 + 2t - 3)\,dt$ b. $\int (x-1)^2\,dx$ c. $\int (x-1)(x+2)\,dx$

d. $\int (x+2)^3\,dx$ e. $\int (e^{3x} - e^{2x} + e^x)\,dx$ f. $\int \frac{x^3 - 3x + 4}{x}\,dx$

3. Find the following integrals:

a. $\int \frac{(y-2)^2}{\sqrt{y}}\,dy$ b. $\int \frac{x^3}{x+1}\,dx$ c. $\int x(1+x^2)^{15}\,dx$

(*Hint:* In part (a), first expand $(y-2)^2$ and then divide each term by $\sqrt{y}$. In part (b), do long division. In part (c), what is the derivative of $(1+x^2)^{16}$?)

4. **a.** Show that

$$\int (ax+b)^p\,dx = \frac{1}{a(p+1)}(ax+b)^{p+1} + C \qquad (a \neq 0,\ p \neq -1)$$

b. Find the following:

(i) $\int (2x+1)^4\,dx$ (ii) $\int \sqrt{x+2}\,dx$ (iii) $\int \frac{1}{\sqrt{4-x}}\,dx$

5. Show that

$$\int x\sqrt{ax+b}\,dx = \frac{2}{15a^2}(3ax - 2b)(ax+b)^{3/2} + C$$

6. Solve the following initial-value problems:
   a. Find $F(x)$ if $F'(x) = \frac{1}{2} - 2x$ and $F(0) = 1/2$.
   b. Find $F(x)$ if $F'(x) = x(1-x^2)$ and $F(1) = 5/12$.

7. In the manufacture of a product, the marginal cost of producing x units is $c'(x) = 3x + 4$. If fixed costs are 40, find the total cost function $c(x)$.

8. Find the general form of a function f whose second derivative is x^2. If we require in addition that $f(0) = 1$ and $f'(0) = -1$, what is $f(x)$?

9. a. Suppose that $f''(x) = 2$ for all x, and $f(0) = 2$, $f'(0) = 1$. First find $f'(x)$ and then $f(x)$.
   b. Similarly, suppose that $f''(x) = 1/x^2 + x^3 + 2$ for $x > 0$, and $f(1) = 0$, $f'(1) = 1/4$. Find $f(x)$.

10.3 The Definite Integral

Let f be a continuous function defined in the interval $[a, b]$. Suppose that the function F is continuous in $[a, b]$ and has a derivative satisfying $F'(x) = f(x)$ for every $x \in (a, b)$. Then the difference $F(b) - F(a)$ is called the **definite integral** of f over $[a, b]$. As observed in Section 10.1, this difference does not depend on which of the infinitely many indefinite integrals of f we choose as F. The definite integral of f over $[a, b]$ is therefore a number that depends only on the function f and the numbers a and b. We denote it by

$$\int_a^b f(x)\,dx \tag{10.11}$$

This notation makes explicit the function $f(x)$ we integrate, which is called the **integrand**, and the interval of integration $[a, b]$. The numbers a and b are called, respectively, the **lower** and **upper limits of integration**. The letter x is a *dummy variable* in the sense that the integral is independent of its label. For instance,

$$\int_a^b f(x)\,dx = \int_a^b f(y)\,dy = \int_a^b f(\xi)\,d\xi$$

In many other mathematical writings, the difference $F(b) - F(a)$ is often denoted by $F(x)\Big|_a^b$, or by $[F(x)]_a^b$. But $\Big|_a^b F(x)$ is also common, and this is the notation we shall use. Thus:

Definition of the Definite Integral

$$\int_a^b f(x)\,dx = \Big|_a^b F(x) = F(b) - F(a) \tag{10.12}$$

where $F'(x) = f(x)$ for all $x \in (a, b)$.

Definition [10.12] does not necessarily require $a < b$. However, if $a > b$ and $f(x)$ is positive throughout the interval $[b, a]$, then $\int_a^b f(x)\,dx$ is a negative number.

Note that we have defined the definite integral without necessarily giving it a geometric interpretation. In fact, depending on the context, it can have different interpretations. For instance, if $f(r)$ is an income density function, then $\int_a^b f(r)\,dr$ is the proportion of people with income between a and b. (See the next section.)

With the new notation, the results in Examples 10.1 and 10.2 can be written as

$$\int_0^1 x^2\,dx = \Big|_0^1 \tfrac{1}{3}x^3 = \tfrac{1}{3}$$

$$\int_a^b (cx+d)\,dx = \Big|_a^b \left(\tfrac{1}{2}cx^2 + dx\right)$$

$$= \left(\tfrac{1}{2}cb^2 + db\right) - \left(\tfrac{1}{2}ca^2 + da\right) = \tfrac{1}{2}c(b^2 - a^2) + d(b-a)$$

Although the notations for definite and for indefinite integrals are similar, they are entirely different concepts. In fact, $\int_a^b f(x)\,dx$ denotes a single number, whereas $\int f(x)\,dx$ represents any one of the infinite set of functions all having $f(x)$ as their derivative. The relationship between the two is that $\int f(x)\,dx = F(x) + C$ over an interval I, if and only if $\int_a^b f(x)\,dx = F(b) - F(a)$ for all a and b in I.

Properties of the Definite Integral

From the definition of the definite integral in [10.12], a number of properties can be derived. If f is a continuous function in an interval that contains a, b, and c, then

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx \qquad [10.13]$$

$$\int_a^a f(x)\,dx = 0 \qquad [10.14]$$

$$\int_a^b \alpha f(x)\,dx = \alpha\int_a^b f(x)\,dx \qquad (\alpha \text{ is an arbitrary number}) \qquad [10.15]$$

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \qquad [10.16]$$

All these rules follow easily from [10.12]. For example, [10.16] can be proved as follows. Let F be continuous in $[a, b]$, and suppose that $F'(x) = f(x)$ for all x

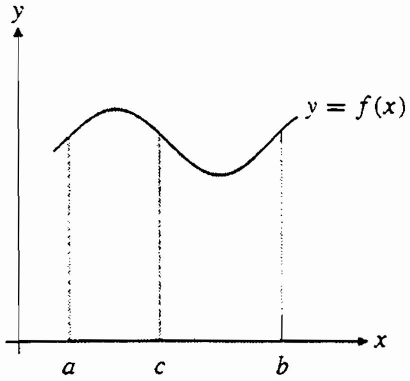


FIGURE 10.10 $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$.

in the interior of an interval long enough to include a, b, and c. Then

$$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \left[F(c) - F(a)\right] + \left[F(b) - F(c)\right]$$
$$= F(b) - F(a) = \int_a^b f(x)\,dx$$

When the definite integral is interpreted as an area, [10.16] is the additivity property of areas, as illustrated in Fig. 10.10. Of course, [10.16] easily generalizes to the case in which we partition the interval $[a, b]$ into an arbitrary finite number of subintervals.

The constant multiple property [10.8] and the summation property [10.9] are also valid for definite integrals. In fact, if f and g are continuous in $[a, b]$, and if α and β are real numbers, then

$$\int_a^b \left[\alpha f(x) + \beta g(x)\right] dx = \alpha \int_a^b f(x)\,dx + \beta \int_a^b g(x)\,dx \qquad [10.17]$$

The proof is simple. Let $F'(x) = f(x)$ and $G'(x) = g(x)$ for all $x \in (a, b)$. Then $[(\alpha F(x) + \beta G(x)]' = \alpha F'(x) + \beta G'(x) = \alpha f(x) + \beta g(x)$. Hence,

$$\int_a^b \left[\alpha f(x) + \beta g(x)\right] dx = \Big|_a^b \left[\alpha F(x) + \beta G(x)\right]$$
$$= \left[\alpha F(b) + \beta G(b)\right] - \left[\alpha F(a) + \beta G(a)\right]$$
$$= \alpha\left[F(b) - F(a)\right] + \beta\left[G(b) - G(a)\right]$$
$$= \alpha \int_a^b f(x)\,dx + \beta \int_a^b g(x)\,dx$$

The rule in [10.17] can obviously be extended to more than two functions.

Some Important Observations

It follows directly from the definition of the indefinite integral that the derivative of the integral is equal to the integrand:

$$\frac{d}{dx}\int f(x)\,dx = f(x) \tag{10.18}$$

Also

$$\int F'(x)\,dx = F(x) + C \tag{10.19}$$

Moreover,

$$\int_a^t f(x)\,dx = \Big|_a^t F(x) = F(t) - F(a)$$

So, differentiating w.r.t. t with a fixed, it follows that

$$\frac{d}{dt}\int_a^t f(x)\,dx = F'(t) = f(t) \tag{10.20}$$

In other words: *The derivative of the definite integral w.r.t. the upper limit of integration is equal to the integrand as a function evaluated at that limit.*

Correspondingly,

$$\int_t^a f(x)\,dx = \Big|_t^a F(x) = F(a) - F(t)$$

so that

$$\frac{d}{dt}\int_t^a f(x)\,dx = -F'(t) = -f(t) \tag{10.21}$$

In other words: *The derivative of the definite integral w.r.t. the lower limit of integration is equal to minus the integrand as a function evaluated at that limit.*

The results in [10.20] and [10.21] can be generalized. In fact, if $a(t)$ and $b(t)$ are differentiable and $f(x)$ is continuous, then

$$\frac{d}{dt}\int_{a(t)}^{b(t)} f(x)\,dx = f(b(t))\,b'(t) - f(a(t))\,a'(t) \tag{10.22}$$

To prove this formula, let $F'(x) = f(x)$. Then $\int_u^v f(x)\,dx = F(v) - F(u)$, so in particular,

$$\int_{a(t)}^{b(t)} f(x)\,dx = F(b(t)) - F(a(t))$$

Using the chain rule to differentiate the right-hand side of this equation w.r.t. t, we obtain $F'(b(t))b'(t) - F'(a(t))a'(t)$. But $F'(b(t)) = f(b(t))$ and $F'(a(t)) = f(a(t))$, so [10.22] results. (Formula [10.22] is a special case of Leibniz's formula discussed in Section 16.2.)

Continuous Functions Are Integrable

Suppose $f(x)$ is a continuous function in $[a, b]$. Then we defined $\int_a^b f(x)\,dx$ as the number $F(b) - F(a)$, provided that $F(x)$ is some function whose derivative is $f(x)$. In some cases, we are able to find an explicit expression for $F(x)$. For instance, we can evaluate $\int_0^1 x^5\,dx$ as $1/6$ because $(1/6)x^6$ has x^5 as its derivative. On the other hand, for the integral

$$\int_0^2 e^{-x^2}\,dx$$

(closely related to the "normal distribution" in statistics), there is no standard function whose derivative is e^{-x^2}.[2] Still, the integrand function is continuous in $[0, 2]$ and there should be an area under the graph from 0 to 2.

In fact, one can prove that any continuous function has an antiderivative:

> **Theorem 10.1** If f is a continuous function in $[a, b]$, then there exists a continuous function $F(x)$ in $[a, b]$ such that $F'(x) = f(x)$, for all $x \in (a, b)$.

A sketch of a proof: Let $x \in (a, b)$. Subdivide the interval $[a, x]$ into n equal parts so that the points of subdivision are $a+(x-a)/n$, $a+2(x-a)/n$, $\ldots$, $a+(n-1)(x-a)/n$. For each natural number n, define the new function F_n as an approximation to F using the formula

$$F_n(x) = \frac{x-a}{n}\left[f(a) + f\left(a + \frac{x-a}{n}\right) + f\left(a + 2\frac{x-a}{n}\right) + \cdots + f\left(a + (n-1)\frac{x-a}{n}\right)\right]$$

(Try to illustrate this definition of $F_n(x)$.) Define $F(x) = \lim_{n\to\infty} F_n(x)$. It is possible (but not easy) to show that this limit exists for each $x \in [a, b]$, that F is continuous in $[a, b]$, and finally that $F(x)$ has $f(x)$ as its derivative in (a, b).

The Riemann Integral

The kind of integral discussed so far, which is based on the antiderivative, is called the *Newton–Leibniz* (N–L) *integral*. Several other kinds of integral are considered by mathematicians. For continuous functions, they all give the same result as the N–L integral.

[2]See (11.4) in Section 11.2 for other examples of "unsolvable integrals."

We briefly sketch the so-called *Riemann integral*. The idea behind the definition is closely related to the exhaustion method that was described in the introduction to this chapter.

Let f be a *bounded* function in the interval $[a, b]$, and let n be a natural number. Subdivide $[a, b]$ into n parts by choosing points $a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$. Put $\Delta x_i = x_{i+1} - x_i$, $i = 0, 1, \ldots, n-1$, and choose an arbitrary number ξ_i in each interval $[x_i, x_{i+1}]$ (draw a figure). The sum

$$f(\xi_0)\Delta x_0 + f(\xi_1)\Delta x_1 + \cdots + f(\xi_{n-1})\Delta x_{n-1}$$

is called a *Riemann sum* associated with the function f. This sum will depend on f as well as on the subdivision and on the choice of the ξ_i's. Suppose that, when n approaches infinity and simultaneously the largest of the numbers $\Delta x_0, \Delta x_1, \ldots, \Delta x_{n-1}$ approaches 0, the limit of the sum exists. Then f is called *Riemann integrable* (R-integrable) in the interval $[a, b]$, and we put

$$\int_a^b f(x)\,dx = \lim \sum_{i=0}^{n-1} f(\xi_i)\,\Delta x_i$$

The value of the integral is independent of the choice of the ξ_i's. One can show that every continuous function is R-integrable, and that the R integral in this case can be evaluated using [10.12]. The N–L integral and the R integral thus coincide for continuous functions.

Problems

1. Evaluate the following integrals by using [10.12]:

 a. $\displaystyle\int_0^1 x\,dx$ **b.** $\displaystyle\int_1^2 (2x + x^2)\,dx$ **c.** $\displaystyle\int_{-2}^3 \left(\tfrac{1}{2}x^2 - \tfrac{1}{3}x^3\right) dx$

2. Evaluate the following integrals:

 a. $\displaystyle\int_0^2 (t^3 - t^4)\,dt$ **b.** $\displaystyle\int_1^2 \left(2t^5 - \frac{1}{t^2}\right) dt$ **c.** $\displaystyle\int_2^3 \left(\frac{1}{t-1} + t\right) dx$

3. The profit of a firm as a function of its output x $(x > 0)$ is

$$f(x) = 4000 - x - \frac{3000000}{x}$$

 a. Find the output that maximizes profit. Draw the graph of f.

 b. The actual output varies between 1000 and 3000 units. Compute the average profit

$$I = \frac{1}{2000}\int_{1000}^{3000} f(x)\,dx$$

4. Evaluate the integrals:

a. $\int_1^3 \frac{3x}{10}\,dx$ **b.** $\int_{-3}^{-1} \xi^2\,d\xi$

c. $\int_0^1 \alpha e^{\beta\tau}\,d\tau \qquad (\beta \neq 0)$ **d.** $\int_{-2}^{-1} \frac{1}{y}\,dy$

5. By using [10.22] or otherwise, evaluate the following:

a. $\frac{d}{dt}\int_0^t x^2\,dx$ **b.** $\frac{d}{dt}\int_t^3 e^{-x^2}\,dx$ **c.** $\frac{d}{dt}\int_{-t}^t e^{-x^2}\,dx$

d. $\frac{d}{dt}\int_{\sqrt{t}}^t \ln x\,dx$ **e.** $\frac{d}{dt}\int_{t^{1/6}}^{t^{1/3}} x^6\,dx$ **f.** $\frac{d}{dt}\int_{-t}^t \frac{1}{\sqrt{x^4+1}}\,dx$

6. Compute $\int_0^2 2x^2(2-x)^2\,dx$. Give a rough check of the answer by drawing the graph of $f(x) = 2x^2(2-x)^2$ over $[0, 2]$.

7. Find the area between the two parabolas defined by the equations $y + 1 = (x - 1)^2$ and $3x = y^2$. (The points of intersection have integer coordinates.)

8. Compute the following:

a. $\int_0^1 (x + \sqrt{x} + \sqrt[4]{x})\,dx$ **b.** $\int_1^b \left(A\frac{x+b}{x+c} + \frac{d}{x}\right) dx$

c. $\int_0^1 \frac{x^2 + x + \sqrt{x+1}}{x+1}\,dx$

Harder Problems

9. A theory of investment has used a function W defined for all $T > 0$ by

$$W(T) = \frac{K}{T}\int_0^T e^{-\varrho t}\,dt \qquad (K \text{ and } \varrho \text{ are positive constants})$$

Evaluate the integral, and prove that $W(T)$ takes values in the interval $(0, K)$ and is strictly decreasing.

10. **a.** Show that if f is continuous in $[a, b]$, then there exists a number $x^* \in [a, b]$ such that

$$f(x^*) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

This is called the **mean-value theorem for integrals**, and $f(x^*)$ is called the *mean value* of f in $[a, b]$. (*Hint:* Put $F(x) = \int_a^x f(t)\,dt$, and use Theorem 7.5 of Section 7.3.)

b. Find the mean value of $f(x) = \sqrt{x}$ in $[0, 4]$, and illustrate.

10.4 Economic Applications of Integration

We motivated the definite integral as a tool for computing the area under a curve. However, the integral has many other important interpretations. For instance, we are led to a definite integral when we want to find the volume of a solid of revolution or the length of a curve. Several of the most important concepts in statistics are also expressed by integrals of continuous probability distributions. This section presents some examples showing more directly the importance of integrals in economics.

Extraction from an Oil Well

Assume that at time $t = 0$ we start extracting oil from a well that contains K barrels of oil. Let us define

$$x(t) = \text{amount of oil in barrels that is left at time } t$$

In particular, $x(0) = K$. If we assume that we cannot pump oil back into the well, then $x(t)$ is a decreasing function of t. The amount of oil that is extracted in a time interval $[t, t + \Delta t]$ (where $\Delta t > 0$) is $x(t) - x(t + \Delta t)$. Extraction per unit of time is, therefore,

$$\frac{x(t) - x(t + \Delta t)}{\Delta t} = -\frac{x(t + \Delta t) - x(t)}{\Delta t} \qquad [*]$$

If we assume that $x(t)$ is differentiable, then the limit as Δt approaches zero of the fraction [*] is equal to $-\dot{x}(t)$. Letting $u(t)$ denote the rate of extraction at time t, we have

$$\dot{x}(t) = -u(t) \qquad \text{with} \qquad x(0) = K \qquad [10.23]$$

The solution to the initial-value problem [10.23] is

$$x(t) = K - \int_0^t u(\tau)\, d\tau \qquad [10.24]$$

Indeed, we check [10.24] as follows. First, setting $t = 0$ gives $x(0) = K$. Moreover, differentiating [10.24] w.r.t. t according to rule [10.20] in Section 10.3 yields $\dot{x}(t) = -u(t)$. The result [10.24] may be interpreted as follows: The amount of oil left at time t is equal to the initial amount K, minus the total amount that has been extracted during the time span $[0, t]$, namely $\int_0^t u(\tau)\, d\tau$.

If the rate of extraction is constant, with $u(t) = \bar{u}$, then [10.24] yields

$$x(t) = K - \int_0^t \bar{u}\, d\tau = K - \Big|_0^t \bar{u}\tau = K - \bar{u}t$$

In particular, we see that the well will be empty when $K - \bar{u}t = 0$, or when

$t = K/\bar{u}$. (Of course, this particular answer could have been found more directly, without recourse to integration.)

The example illustrates two concepts that are important to distinguish in many economic arguments. The quantity $x(t)$ is a *stock*, measured in barrels. On the other hand, $u(t)$ is a *flow*, measured in barrels *per unit of time*.

A Country's Foreign Exchange Reserves

Let $F(t)$ denote a country's foreign exchange reserves at time t. Assuming that F is differentiable, the rate of change in the foreign exchange reserves per unit of time will be

$$f(t) = F'(t) \qquad [10.25]$$

If $f(t) > 0$, this means that there is a net flow of foreign exchange into the country at time t, whereas $f(t) < 0$ means that foreign exchange is flowing out. From the definition of the definite integral, it follows that

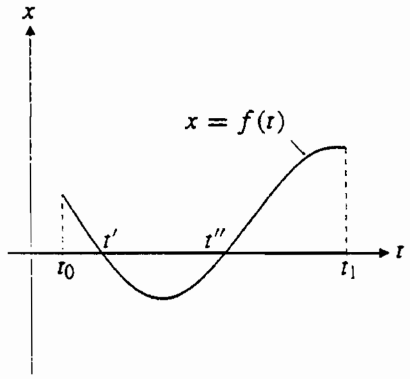
$$F(t_1) - F(t_0) = \int_{t_0}^{t_1} f(t)\,dt \qquad [10.26]$$

We see that this expression measures the change in the foreign exchange reserves over the time interval $[t_0, t_1]$. An example is illustrated in Fig. 10.11. Here there is a net flow of foreign exchange into the country from t_0 to t', then a net flow out of the country from t' to t'', and, finally, there is a net flow into the country from t'' to t_1. (Note that $\int_{t_0}^{t_1} f(t)\,dt$ does not denote the total area bounded by the graph, the x-axis, and the lines $t = t_0$ and $t = t_1$ in this case. See the end of Section 10.1.)

Income Distribution

In many countries, anonymous data from income tax authorities can be used to reveal some properties of the income distribution within a given year, as well as how the distribution changes from year to year.

FIGURE 10.11 The rate of change of foreign exchange.



We measure income in dollars and let $F(r)$ denote the proportion of individuals who receive no more than r dollars. Thus, if there are n individuals in the population, $nF(r)$ is the number of individuals with income no greater than r. If r_0 is the lowest and r_1 is the highest (registered) income in the group, we are interested in the function F in the interval $[r_0, r_1]$. By definition, F is not continuous and therefore also not differentiable in $[r_0, r_1]$ because r has to be a multiple of \$0.01 and $F(r)$ has to be a multiple of $1/n$. However, if the population consists of a large number of individuals, then it is usually possible to find a "smooth" function that gives a good approximation to the true income distribution. Assume, therefore, that F is a function with a continuous derivative denoted by f, so that

$$f(r) = F'(r) \qquad (\text{for all } r \in (r_0, r_1))$$

According to the definition of the derivative, we have

$$f(r)\,\Delta r \approx F(r + \Delta r) - F(r)$$

for all small Δr. Thus, $f(r)\,\Delta r$ is approximately equal to the proportion of individuals who earn between r and $r + \Delta r$. The function f is called an **income density function**, and F is the associated **cumulative distribution function**.[3]

Suppose that f is a continuous income distribution for a certain population with incomes in the interval $[r_0, r_1]$. If $r_0 \le a \le b \le r_1$, then the previous discussion and the definition of the definite integral imply that $\int_a^b f(r)\,dr$ is the proportion of individuals with incomes in $[a, b]$. Thus,

$$n\int_a^b f(r)\,dr = \begin{cases} \text{the } \textbf{number of individuals} \\ \text{with incomes in the interval } [a, b] \end{cases} \qquad [10.27]$$

We will now find an expression for the combined income of those who earn between a and b dollars. Let $M(r)$ denote the total income of those who earn no more than r dollars, and consider the income interval $[r, r + \Delta r]$. There are approximately $nf(r)\,\Delta r$ individuals with incomes in this interval. Each of them has an income approximately equal to r, so that the total income of these individuals, $M(r + \Delta r) - M(r)$, is approximately equal to $nrf(r)\,\Delta r$. So we have

$$\frac{M(r + \Delta r) - M(r)}{\Delta r} \approx nrf(r)$$

[3]Readers who know some elementary statistics will see the analogy with probability density functions and with cumulative (probability) distribution functions.

The approximation improves (in general) as Δr decreases, and by taking the limit as $\Delta r \to 0$, we obtain $M'(r) = nrf(r)$, so $n\int_a^b rf(r)\,dr = M(b) - M(a)$. Hence,

$$n\int_a^b rf(r)\,dr = \left\{\begin{array}{l}\text{the **total income** of individuals}\\ \text{with incomes in the interval } [a, b]\end{array}\right. \qquad [10.28]$$

The argument that leads to [10.28] can be made more exact: $M(r+\Delta r) - M(r)$ is the total income of those who have income in the interval $[r, r+\Delta r]$, when $\Delta r > 0$. In this income interval, there are $n[F(r+\Delta r) - F(r)]$ individuals each of whom earns at most $r + \Delta r$ and at least r. Thus,

$$nr\big[F(r+\Delta r) - F(r)\big] \le M(r+\Delta r) - M(r) \le n(r+\Delta r)\big[F(r+\Delta r) - F(r)\big] \qquad [1]$$

If $\Delta r > 0$, division by Δr yields

$$nr\,\frac{F(r+\Delta r) - F(r)}{\Delta r} \le \frac{M(r+\Delta r) - M(r)}{\Delta r} \le n(r+\Delta r)\frac{F(r+\Delta r) - F(r)}{\Delta r} \qquad [2]$$

(If $\Delta r < 0$, then the inequalities in [1] are left unchanged, whereas those in [2] are reversed.) Letting $\Delta r \to 0$ gives $nrF'(r) \le M'(r) \le nrF'(r)$, so that

$$M'(r) = nrF'(r) = nrf(r) \qquad [3]$$

The ratio between the total income and the number of individuals belonging to a certain income interval $[a, b]$, is called the mean income for the individuals in this income interval. We have, therefore,

$$\left.\begin{array}{l}\text{The **mean income** of individuals}\\ \text{with incomes in the interval } [a, b]\end{array}\right\}: \quad m = \frac{\int_a^b rf(r)\,dr}{\int_a^b f(r)\,dr} \qquad [10.29]$$

An income distribution function that approximates actual income distributions quite well, particularly for large incomes, is the **Pareto distribution**. In this case, the proportion of individuals who earn at most r dollars is given by

$$f(r) = Br^{-\beta} \qquad [10.30]$$

Here B and β are positive constants. Empirical estimates of β are usually in the range $2.4 < \beta < 2.6$. For values of r close to 0, the formula is of no use when $\beta \ge 1$, because $\int_a^b f(r)\,dr \to \infty$ as $r \to 0$ (See Section 11.3).

Example 10.9

In a population with incomes between a and b, suppose the income distribution is given by

$$f(r) = Br^{-2.5} \qquad (B \text{ a positive constant}) \tag{1}$$

Determine the mean income in this group.

Solution Here

$$\int_a^b f(r)\,dr = \int_a^b Br^{-2.5}\,dr = B \Big|_a^b \left(-\tfrac{2}{3}r^{-1.5}\right) = \tfrac{2}{3}B\left(a^{-1.5} - b^{-1.5}\right)$$

Also

$$\int_a^b rf(r)\,dr = \int_a^b rBr^{-2.5}\,dr = B \Big|_a^b r^{-1.5}\,dr$$

$$= -2B \Big|_a^b r^{-0.5} = 2B\left(a^{-0.5} - b^{-0.5}\right)$$

So the mean income of the group is

$$m = \frac{2B\left(a^{-0.5} - b^{-0.5}\right)}{(2/3)B\left(a^{-1.5} - b^{-1.5}\right)} = 3\frac{a^{-0.5} - b^{-0.5}}{a^{-1.5} - b^{-1.5}} \tag{2}$$

Suppose that b is very large. Then $b^{-0.5}$ and $b^{-1.5}$ are both close to 0, and so [2] implies that $m \approx 3a$. The mean income of those who earn at least a is therefore approximately $3a$.

The Influence of Income Distribution on Demand

Assume that the individuals in a population are offered a commodity for which demand depends only on the price p and the income r of each individual. Let $D(p, r)$ be a continuous function that denotes the number of commodity units demanded by an individual with income r when the price per unit is p. If the incomes in the group vary between a and b, and the income distribution is $f(r)$, what is the total demand for the commodity when the price is p?

Let the price p be fixed, and denote by $T(r)$ the total demand for the commodity by all individuals who earn less than or equal to r. Consider the income interval $[r, r + \Delta r]$. There are approximately $nf(r)\,\Delta r$ individuals with incomes in this interval. Because each of them demands approximately $D(p, r)$ units of the commodity, the total demand of these individuals will be approximately $nD(p, r)f(r)\,\Delta r$. However, the actual total demand of individuals with incomes in the interval $[r, r + \Delta r]$ is given by $T(r + \Delta r) - T(r)$. So we must have $T(r+\Delta r) - T(r) \approx nD(p, r)f(r)\,\Delta r$, and thus

$$\frac{T(r+\Delta r) - T(r)}{\Delta r} \approx nD(p, r)f(r)$$

The approximation improves (in general) as Δr decreases, and by taking the limit as $\Delta r \to 0$, we obtain $T'(r) = nD(p, r)f(r)$. By definition of the definite integral, $T(b) - T(a) = n\int_a^b D(p, r)f(r)\,dr$. But $T(b) - T(a)$ is the desired measure of total demand for the commodity by all the individuals in the group. This will naturally depend on the price p. So we denote it by $x(p)$, and thus have

$$x(p) = \int_a^b nD(p, r)f(r)\,dr \qquad \textbf{(total demand)} \qquad [10.31]$$

Example 10.10

Let the income distribution function be that of Example 10.9, and let $D(p, r) = Ap^{-1.5}r^{2.08}$. (This function describes the demand for milk in Norway during the period 1925–1935. See Example 15.2.) Compute the total demand.

Solution Using [10.31] gives

$$x(p) = \int_a^b nAp^{-1.5}r^{2.08}Br^{-2.5}\,dr = nABp^{-1.5}\int_a^b r^{-0.42}\,dr$$

Hence,

$$x(p) = nABp^{-1.5}\Big|_a^b \frac{1}{0.58}r^{0.58} = \frac{nAB}{0.58}p^{-1.5}(b^{0.58} - a^{0.58})$$

Present Discounted Value of a Continuous Future Income Stream

Section 6.6 discussed the present value of a series of future payments made at specific discrete moments in time. It is often more natural to consider revenue as accruing continuously, such as the proceeds from a large growing forest.

Suppose that income is to be received continuously from time $t = 0$ to time $t = T$ at the rate of $f(t)$ dollars per year at time t. We assume that interest is compounded continuously at rate r. Let $P(t)$ denote the present discounted value of all payments made over the time interval $[0, t]$. This means that $P(t)$ represents the amount of money you would have to deposit at time $t = 0$ in order to match what results from (continuously) depositing the income stream $f(t)$ over the time interval $[0, T]$. If dt is any number, the present value of the income received in the interval $[t, t+dt]$ is $P(t+dt) - P(t)$. If dt is a small number, the income received in this interval is approximately $f(t)\,dt$, and the present discounted value (PDV) of this amount is approximately $f(t)e^{-rt}\,dt$. Thus, $P(t+dt) - P(t) \approx f(t)e^{-rt}\,dt$ and so

$$\frac{P(t+dt) - P(t)}{dt} \approx f(t)e^{-rt}$$

This approximation gets better the smaller is dt, and in the limit as $dt \to 0$, we have

$$P'(t) = f(t)e^{-rt}$$

By the definition of the definite integral, $P(T) - P(0) = \int_0^T f(t)e^{-rt}\,dt$. Because $P(0) = 0$, we have the following:

The **present discounted value** (at time 0) of a continuous income stream at the rate of $f(t)$ dollars per year over the time interval $[0, T]$, with continuously compounded interest at rate r, is given by

$$\text{PDV} = \int_0^T f(t)e^{-rt}\,dt \qquad [10.32]$$

Equation [10.32] gives the value at time 0 of income stream $f(t)$ received during time interval $[0, T]$. The value of this amount at time T, with continuously compounded interest at rate r, is $e^{rT}\int_0^T f(t)e^{-rt}\,dt$. Because the number e^{rT} is a constant, we can rewrite the integral as $\int_0^T f(t)e^{r(T-t)}\,dt$. This is called the future discounted value (FDV) of the income stream:

The **future discounted value** (at time T) of a continuous income stream at the rate of $f(t)$ dollars per year over the time interval $[0, T]$, with continuously compounded interest at rate r, is given by

$$\text{FDV} = \int_0^T f(t)e^{r(T-t)}\,dt \qquad [10.33]$$

An easy modification of [10.32] will give us the discounted value (DV) at time $s \in [0, T]$ of an income stream $f(t)$ received during time interval $[s, T]$. In fact, the DV at time s of income $f(t)$ received in the small time interval $[t, t+dt]$ is $f(t)e^{-r(t-s)}\,dt$. So we have the following:

The **discounted value** at time s of a continuous income stream at the rate of $f(t)$ dollars per year over the time interval $[s, T]$, with continuously compounded interest at rate r, is given by

$$\text{DV} = \int_{t=s}^T f(t)e^{-r(t-s)}\,dt \qquad [10.34]$$

Example 10.11

Find the PDV and the FDV of a constant income stream of \$1000 per year over the next 10 years, assuming an interest rate of $r = 8\% = 0.08$ annually, compounded continuously.

Solution

$$\text{PDV} = \int_0^{10} 1000e^{-0.08t}\,dt = \Big|_0^{10} 1000\left(-\frac{e^{-0.08t}}{0.08}\right) = \frac{1000}{0.08}(1 - e^{-0.8}) \approx 6883.39$$

$$\text{FDV} = e^{0.08\cdot 10}\text{PDV} \approx e^{0.8} \cdot 6883.39 \approx 15{,}319.27$$

Problems

1. Assume that the rate of extraction $u(t)$ from an oil well decreases exponentially over time, with $u(t) = \bar{u}e^{-at}$, where a is a positive constant. Given the initial stock $x(0) = x_0$, find an expression $x(t)$ for the remaining amount of oil at time t. Under what condition will the well never be exhausted?

2. **a.** Follow the pattern in Example 10.9 and find the mean income m over the interval $[b, 2b]$ when $f(r) = Br^{-2}$.
   b. Assume that the individual's demand function is $D(p, r) = Ap^{\gamma}r^{\delta}$, $A > 0$, $\gamma < 0$, $\delta > 0$, $\delta \neq 1$. Compute the total demand $x(p)$ by using formula [10.31], assuming that there are n individuals in the population.

3. Let $K(t)$ denote the capital stock of an economy at time t. Then **net investment** at time t, denoted by $I(t)$, is given by the rate of increase $\dot{K}(t)$ of $K(t)$.
   a. If $I(t) = 3t^2 + 2t + 5$ $(t \geq 0)$, what is the total increase in the capital stock during the interval from $t = 0$ to $t = 5$?
   b. If $K(t_0) = K_0$, find an expression for the total increase in the capital stock from time $t = t_0$ to $t = T$ when the investment function $I(t)$ is as in part (a).

4. Find the present and future values of a constant income stream of \$500 per year over the next 15 years, assuming an interest rate of $r = 6\% = 0.06$ annually, compounded continuously.

5. **a.** Find the present discounted value (PDV) of a constant income stream of a dollars per year over the next T years, assuming an interest rate of r annually, compounded continuously.
   b. What is the limit of the PDV as $T \to \infty$? Compare this result with (6.22) in Section 6.6.

11

Further Topics in Integration

The true mathematician is not a juggler of numbers, but of concepts.
—*I. Stewart* (1975)

This chapter continues the study of integration started in Chapter 10. In particular, it presents some methods of integration that are used quite often in economics and even more often in statistics. These include integration by parts and by substitution, integrals of discontinuous functions, and integrals over infinite intervals. The last part of this chapter considers Lorenz curves, which can be a useful way of visualizing income distributions and some of their properties.

11.1 Integration by Parts

We often need to evaluate integrals such as $\int x^2 e^{2x}\, dx$ whose integrand is a product of two functions. We know that $\frac{1}{3}x^3$ has x^2 as its derivative and that $\frac{1}{2}e^{2x}$ has e^{2x} as its derivative, but $(\frac{1}{3}x^3)(\frac{1}{2}e^{2x})$ certainly does not have x^2e^{2x} as its derivative. In general, because the derivative of a product is *not* the product of the derivatives, the integral of a product is not the product of the integrals.

The correct rule for differentiating a product allows us to derive an important and useful rule for intergrating products. In fact,

$$\big(f(x)g(x)\big)' = f'(x)g(x) + f(x)g'(x) \qquad [*]$$

Taking the indefinite integral of each side and using the rule for integrating a sum gives

$$f(x)g(x) = \int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx$$

where the constants of integration are implicit in the indefinite integrals on the right-hand side of this equation. Rearranging this last equation yields:

Formula for Integration by Parts

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx \qquad [11.1]$$

At first sight, this formula does not look at all helpful. Yet the examples that follow show how this impression is quite wrong, once one has learned to use it properly.

Suppose we are asked to integrate a function $H(x)$ that can be written in the form $f(x)g'(x)$. By using [11.1], the problem can then be transformed into that of integrating $f'(x)g(x)$. Usually, a function $H(x)$ can be written as $f(x)g'(x)$ in several different ways. The point is, therefore, to choose f and g so that it is easier to find $\int f'(x)g(x)\,dx$ than it is to find $\int f(x)g'(x)\,dx$. Sometimes the method works not by producing a simpler integral, but one that is similar. See Example 11.2(a).

Example 11.1

Use integration by parts to evaluate $\int xe^x\,dx$.

Solution In order to use [11.1], we must write the integrand in the form $f(x)g'(x)$. Let $f(x) = x$ and $g(x) = e^x$. Then $f(x)g'(x) = xe^x$, and so

$$\int \underset{f(x)}{\underset{\downarrow}{x}} \cdot \underset{g'(x)}{\underset{\downarrow}{e^x}}\,dx = \underset{f(x)}{\underset{\downarrow}{x}} \cdot \underset{g(x)}{\underset{\downarrow}{e^x}} - \int \underset{f'(x)}{\underset{\downarrow}{1}} \cdot \underset{g(x)}{\underset{\downarrow}{e^x}}\,dx = xe^x - \int e^x\,dx = xe^x - e^x + C$$

The derivative of $xe^x - e^x + C$ is $e^x + xe^x - e^x = xe^x$, so the integration has been carried out correctly.

The right choice of f and g enabled us to evaluate the integral. Let us see what happens if we try $f(x) = e^x$ and $g(x) = \frac{1}{2}x^2$ instead. Again $f(x)g'(x) = e^x x = xe^x$, and by [11.1]:

$$\int \underset{f(x)}{\underset{\downarrow}{e^x}} \cdot \underset{g'(x)}{\underset{\downarrow}{x}}\,dx = \underset{f(x)}{\underset{\downarrow}{e^x}} \cdot \underset{g(x)}{\underset{\downarrow}{\tfrac{1}{2}x^2}} - \int \underset{f'(x)}{\underset{\downarrow}{e^x}} \cdot \underset{g(x)}{\underset{\downarrow}{\tfrac{1}{2}x^2}}\,dx$$

In this case, the integral on the right-hand side is more complicated than the original one. Thus, this second choice of f and g does not simplify the integral.

The example illustrates that we must be careful how we split the integrand. Insights into making a good choice, if any, will come only with practice. Often, even experienced "integrators" must resort to trial and error.

Example 11.2

Evaluate the following:

(a) $I = \int \frac{1}{x} \ln x \, dx$

(b) $J = \int e^{2x} x^3 \, dx$

Solution

(a) Choosing $f(x) = 1/x$ and $g'(x) = \ln x$ does not work well because it is difficult to find $g(x)$. Choosing $f(x) = \ln x$ and $g'(x) = 1/x$ works better:

$$I = \int \frac{1}{x} \ln x \, dx = \int \underset{\substack{\downarrow \\ f(x)}}{\ln x} \; \underset{\substack{\downarrow \\ g'(x)}}{\frac{1}{x}} \, dx = \underset{\substack{\downarrow \\ f(x)}}{\ln x} \; \underset{\substack{\downarrow \\ g(x)}}{\ln x} - \int \underset{\substack{\downarrow \\ f'(x)}}{\frac{1}{x}} \; \underset{\substack{\downarrow \\ g(x)}}{\ln x} \, dx$$

In this case, the last integral is exactly the one we started with, namely I. So it must be true that $I = (\ln x)^2 - I$, implying that $I = \frac{1}{2}(\ln x)^2$. Adding an arbitrary constant, we conclude that

$$\int \frac{1}{x} \ln x \, dx = \frac{1}{2}(\ln x)^2 + C$$

(b) We begin by arguing rather loosely as follows. Differentiation makes x^3 simpler by reducing the power in the derivative $3x^2$ from 3 to 2. On the other hand, e^{2x} is about equally simple whether we differentiate or integrate it. Therefore, we choose $f(x) = x^3$ and $g'(x) = e^{2x}$ so that we differentiate f and integrate g'. This yields $f'(x) = 3x^2$ and we can choose $g(x) = \frac{1}{2}e^{2x}$. Therefore,

$$J = \int x^3 e^{2x} \, dx = x^3 \left(\tfrac{1}{2}e^{2x}\right) - \int \left(3x^2\right)\left(\tfrac{1}{2}e^{2x}\right) dx$$

$$= \tfrac{1}{2}x^3 e^{2x} - \tfrac{3}{2} \int x^2 e^{2x} \, dx \qquad [1]$$

The last integral *is* somewhat simpler than the one we started with because the power of x has been reduced. Integrating by parts once more yields

$$\int x^2 e^{2x}\,dx = x^2\left(\tfrac{1}{2}e^{2x}\right) - \int (2x)\left(\tfrac{1}{2}e^{2x}\right)dx$$
$$= \tfrac{1}{2}x^2e^{2x} - \int xe^{2x}\,dx \qquad [2]$$

Using integration by parts a third and final time gives

$$\int xe^{2x}\,dx = x\left(\tfrac{1}{2}e^{2x}\right) - \int \tfrac{1}{2}e^{2x}\,dx = \tfrac{1}{2}xe^{2x} - \tfrac{1}{4}e^{2x} + C \qquad [3]$$

Successively inserting the results of [3] and [2] into [1] yields (with $3C/2 = c$):

$$J = \tfrac{1}{2}x^3e^{2x} - \tfrac{3}{4}x^2e^{2x} + \tfrac{3}{4}xe^{2x} - \tfrac{3}{8}e^{2x} + c$$

It is a good idea to double-check your work by verifying that $dJ/dx = x^3e^{2x}$.

There is a corresponding result for definite integrals. From the definition of the definite integral and [∗] (the product rule for differentiation), we have

$$\int_a^b \left[f'(x)g(x) + f(x)g'(x)\right]dx = \int_a^b \frac{d}{dx}\left[f(x)g(x)\right]dx = \Big|_a^b f(x)g(x)$$

implying that

$$\int_a^b f(x)g'(x)\,dx = \Big|_a^b f(x)g(x) - \int_a^b f'(x)g(x)\,dx \qquad [11.2]$$

Example 11.3

Evaluate $\int_0^3 x\sqrt{1+x}\,dx$.

Solution We must write the integrand in the form $f(x)g'(x)$. If we let $f(x) = x$ and $g'(x) = \sqrt{1+x} = (1+x)^{1/2}$, then what is g? A certain amount of reflection should suggest choosing $g(x) = \tfrac{2}{3}(1+x)^{3/2}$. Using

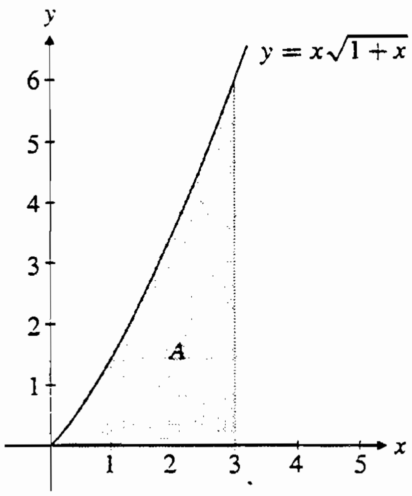


FIGURE 11.1

[11.2] then gives

$$\int_0^3 x\sqrt{1+x}\,dx = \Big|_0^3 x \cdot \tfrac{2}{3}(1+x)^{3/2} - \int_0^3 1 \cdot \tfrac{2}{3}(1+x)^{3/2}\,dx$$

$$= 3 \cdot \tfrac{2}{3} \cdot 4^{3/2} - \tfrac{2}{3} \Big|_0^3 \tfrac{2}{5}(1+x)^{5/2}$$

$$= 16 - \tfrac{4}{15}(4^{5/2} - 1) = 16 - \tfrac{4}{15} \cdot 31 = 7\tfrac{11}{15}$$

Alternatively, we could have found the indefinite integral of $x\sqrt{1+x}$ first, and then evaluated the definite integral by using definition [10.12] of the definite integral. Figure 11.1 shows the area under the graph of $y = x\sqrt{1+x}$ over the interval $[0, 3]$, and you should ask yourself if $7\frac{11}{15}$ is a reasonable estimate of area A.

Problems

1. Use integration by parts to find the following:

   a. $\int xe^{-x}\,dx$ **b.** $\int 3xe^{4x}\,dx$ **c.** $\int (1+x^2)e^{-x}\,dx$ **d.** $\int x\ln x\,dx$

2. Evaluate the following: (a) $\int_{-1}^{1} x\ln(x+2)\,dx$ (b) $\int_0^2 x2^x\,dx$ (c) $\int_0^1 x^2e^x\,dx$

3. Of course, $f(x) = 1 \cdot f(x)$ for any function $f(x)$. Use this fact to prove that

$$\int f(x)\,dx = xf(x) - \int xf'(x)\,dx$$

Apply this formula to the case when $f(x) = \ln x$.

4. Suppose $\mu(t_0) = \mu(t_1) = 0$. Show that, with appropriate requirements on F and μ,

$$\int_{t_0}^{t_1} F(t)\dot{\mu}(t)\,dt = -\int_{t_0}^{t_1} \dot{F}(t)\mu(t)\,dt$$

(Recall that the dot notation $\dot{\mu}(t)$ and $\dot{F}(t)$ means differentiation w.r.t. t.)

5. Show that

$$\int x^\rho \ln x\,dx = \frac{x^{\rho+1}}{\rho+1}\ln x - \frac{x^{\rho+1}}{(\rho+1)^2} + C \qquad (\rho \neq -1)$$

6. With appropriate requirements on the functions involved, show that if $U\big(C(0)\big) = 0$, then

$$\int_0^T U\big(C(t)\big)e^{-rt}\,dt = \frac{1}{r}\left(\int_0^T U'\big(C(t)\big)C'(t)e^{-rt}\,dt - U\big(C(T)\big)e^{-rT}\right)$$

Harder Problems

7. Compute the following integral when $\gamma > c$:

$$T^* = k\int_0^{\bar{u}} u^2(\bar{u}-u)^{\gamma-1}\,du$$

11.2 Integration by Substitution

In this section, we shall see how the chain rule for differentiation leads to an important method for evaluating many complicated integrals. We start with a simple example,

$$\int (x^2+10)^{50}2x\,dx \tag{1}$$

One way of integrating this would be to write out all 51 terms of $(x^2+10)^{50}$, and then integrate term by term. But this would be extremely cumbersome.[1] Instead, let us introduce x^2+10 as a new variable. We pretend that the symbol dx in [1] denotes the differential of x, and argue as follows: If we let $u = x^2+10$, then $du = 2x\,dx$, and using this in [1] yields

$$\int u^{50}\,du$$

[1]The expression $(x^2+10)^{50}$ can be evaluated using the Newton binomial formula (7.16) in Section 7.4.

This integral is easy, $\int u^{50}\,du = \frac{1}{51}u^{51} + C$. Because $u = x^2 + 10$, it appears that

$$\int (x^2 + 10)^{50} 2x\,dx = \tfrac{1}{51}(x^2 + 10)^{51} + C \qquad [2]$$

By the chain rule, the derivative of $\frac{1}{51}(x^2 + 10)^{51} + C$ is precisely $(x^2 + 10)^{50}2x$, so the result in [2] *is* confirmed.

Let us try this method on another example, namely

$$\int \frac{e^x\,dx}{\sqrt[3]{1 + e^x}} \qquad [3]$$

This time we introduce $u = 1 + e^x$ as a new variable. Then $du = e^x\,dx$, and so the integral reduces to

$$\int \frac{du}{\sqrt[3]{u}} = \int u^{-1/3}\,du$$

This integral is equal to $\frac{3}{2}u^{2/3} + C$. Because $u = 1 + e^x$, it appears that

$$\int \frac{e^x\,dx}{\sqrt[3]{1 + e^x}} = \frac{3}{2}(1 + e^x)^{2/3} + C \qquad [4]$$

Again, using the chain rule, we can quickly confirm that [4] is correct, because the derivative of $\frac{3}{2}(1 + e^x)^{2/3}$ is $(1 + e^x)^{-1/3}e^x = e^x/\sqrt[3]{1 + e^x}$. (Actually, the substitution $u = \sqrt[3]{1 + e^x}$ works even better.)

In both of these examples, the integrand could be written in the form $f(u)u'$, where $u = g(x)$. (In [1], put $f(u) = u^{50}$ and $u = g(x) = x^2 + 10$. In [3], put $f(u) = 1/\sqrt[3]{u}$ and $u = g(x) = 1 + e^x$.)

Let us try the same method on the more general integral

$$\int f\big(g(x)\big)g'(x)\,dx \qquad [5]$$

If we put $u = g(x)$, then $du = g'(x)\,dx$, and so (5) reduces to

$$\int f(u)\,du$$

Suppose we could find an antiderivative function $F(u)$ such that $F'(u) = f(u)$. Then

$$\int f(u)\,du = F(u) + C$$

which implies that

$$\int f(g(x))g'(x)\,dx = F(g(x)) + C \qquad [6]$$

Does this purely formal method always give the right result? To convince you that it does, we use the chain rule to differentiate $F(g(x)) + C$ w.r.t. x. The derivative is $F'(g(x))g'(x)$, which is precisely equal to $f(g(x))g'(x)$, thus confirming [6]. We frame this result for further reference:

Integration by Substitution

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du \qquad (u = g(x)) \qquad [11.3]$$

Note: Precise assumptions for this formula to be valid are as follows: g is continuously differentiable, and $f(u)$ is continuous whenever u belongs to the range of g.

It is quite easy to integrate by substitution when the integrand is directly of the form $f(g(x))g'(x)$, as in the previous examples. Sometimes we need to make some preliminary adjustments.

Example 11.4

Integrate the following:

$$\int 8x^2(3x^3 - 1)^{16}\,dx$$

Solution We substitute $u = 3x^3 - 1$. Then $du = 9x^2\,dx$ and so $8x^2\,dx = (8/9)9x^2\,dx = (8/9)\,du$. Thus,

$$\begin{aligned}\int 8x^2(3x^3 - 1)^{16}\,dx &= (8/9)\int u^{16}\,du \\ &= (8/9)\cdot(1/17)u^{17} + C \\ &= (8/153)(3x^3 - 1)^{17} + C\end{aligned}$$

Check your understanding of this method by doing Problems 1 and 2 right now.

More Complicated Cases

The examples of integration by substitution considered so far were rather simple. More challenging applications of this integration method are to cases where it is difficult to see how the integrand can be expressed in the form $f(g(x))g'(x)$.

Example 11.5

Try to evaluate the integral

$$\int \frac{x - \sqrt{x}}{x + \sqrt{x}}\, dx \qquad (x > 0)$$

Solution Because $\sqrt{x}$ occurs in the numerator as well as in the denominator, it might be a good idea to try to simplify the integral by substituting $u = \sqrt{x} = g(x)$. Then $du = g'(x)\, dx = dx/2\sqrt{x}$. This last expression does not occur in the given integral. However, we can remedy this problem by multiplying the integrand by $2\sqrt{x}/2\sqrt{x}$, obtaining

$$\int \frac{x - \sqrt{x}}{x + \sqrt{x}}\, dx = \int \frac{x - \sqrt{x}}{x + \sqrt{x}} 2\sqrt{x}\, \frac{1}{2\sqrt{x}}\, dx \qquad [*]$$

Now, if we replace $\sqrt{x}$ by u and hence x by u^2, and also replace $dx/2\sqrt{x}$ by du, then the integral becomes

$$\begin{aligned}\int \frac{u^2 - u}{u^2 + u} 2u\, du &= 2 \int \frac{u^2 - u}{u + 1}\, du \\ &= 2 \int \left(u - 2 + \frac{2}{u + 1}\right) du \\ &= u^2 - 4u + 4 \ln |u + 1| + C\end{aligned}$$

where we have performed the division $(u^2 - u) \div (u + 1)$ in order to derive the second equality. Replacing u by $\sqrt{x}$ yields the result

$$\int \frac{x - \sqrt{x}}{x + \sqrt{x}}\, dx = x - 4\sqrt{x} + 4 \ln \left(\sqrt{x} + 1\right) + C$$

Actually, the trick used in [*] is unnecessary. If $u = \sqrt{x}$, then $x = u^2$ and $dx = 2u\, du$, so we get immediately

$$\begin{aligned}\int \frac{x - \sqrt{x}}{x + \sqrt{x}}\, dx &= \int \frac{u^2 - u}{u^2 + u} 2u\, du \\ &= 2 \int \frac{u^2 - u}{u + 1}\, du\end{aligned}$$

The last method in the previous example is the one used most frequently. We can summarize it as follows:

Method for Finding a Complicated Integral $\int G(x)\,dx$:

1. Pick out a "part" of $G(x)$ and introduce this "part" as a new variable, $u = g(x)$.
2. Compute $du = g'(x)\,dx$.
3. Using the substitution $u = g(x)$, $du = g'(x)\,dx$, transform (if possible) $\int G(x)\,dx$ to an integral of the form $\int f(u)\,du$.
4. Find (if possible) $\int f(u)\,du = F(u) + C$.
5. Replace u by $g(x)$. The final answer is then

$$\int G(x)\,dx = F\big(g(x)\big) + C$$

At step 3 of this procedure, it is crucial that after substituting you are integrating a function that only contains u (and du), without any x's. Probably the most common error when integrating by substitution is to replace dx by du, rather than use the correct formula $du = g'(x)\,dx$. If one particular substitution does not work, one can try another. *Note:* There is always the possibility (assumed much too quickly by some students) that no substitution works because the integral is "insoluble." Here are some quite common integrals that really are impossible to "solve," except by introducing special new functions:

$$\int e^{x^2}\,dx, \quad \int e^{-x^2}\,dx, \quad \int \frac{e^x}{x}\,dx, \quad \int \frac{1}{\ln x}\,dx, \quad \int \frac{dx}{\sqrt{x^4+1}}\,dx \qquad [11.4]$$

Example 11.6

Find the following:

(a) $\int x^3\sqrt{1+x^2}\,dx$

(b) $\int_0^1 x^3\sqrt{1+x^2}\,dx$

Solution

(a) We follow previous steps 1 to 5:

1. We pick a "part" of $x^3\sqrt{1+x^2}$ as a new variable. Let us try $u = \sqrt{1+x^2}$.

2. When $u = \sqrt{1+x^2}$, then $u^2 = 1 + x^2$ and so $2u\,du = 2x\,dx$, implying that $u\,du = x\,dx$. (Note that this is easier than differentiating u directly.)

3. $$\int x^3\sqrt{1+x^2}\,dx = \int x^2\sqrt{1+x^2}\cdot x\,dx = \int (u^2-1)u\cdot u\,du$$
$$= \int (u^4 - u^2)\,du$$

4. $$\int (u^4 - u^2)\,du = \tfrac{1}{5}u^5 - \tfrac{1}{3}u^3 + C$$

5. $$\int x^3\sqrt{1+x^2}\,dx = \tfrac{1}{5}\left(\sqrt{1+x^2}\right)^5 - \tfrac{1}{3}\left(\sqrt{1+x^2}\right)^3 + C$$

(b) Using the result in part (a),

$$\int_0^1 x^3\sqrt{1+x^2}\,dx = \Big|_0^1 \left[\tfrac{1}{5}\left(\sqrt{1+x^2}\right)^5 - \tfrac{1}{3}\left(\sqrt{1+x^2}\right)^3\right]$$
$$= \tfrac{2}{15}(\sqrt{2}+1)$$

Note 1: In this example, show that the substitution $u = 1 + x^2$ also works.
Note 2: One is inclined to think that an integral like $\int x^2\sqrt{1+x^2}\,dx$ should be even easier to find than the one considered in Example 11.6. However, the substitution $u = \sqrt{1+x^2}$ leads to the integral $\int xu^2\,du = \int \pm\sqrt{u^2-1}\,u^2\,du$, which is *not* very encouraging. (Actually, one has to introduce a rather bizarre substitution in order to find this integral. The substitution suggested in Problem 11 works.)

The definite integral in the previous example can also be evaluated by "carrying over the limits of integration" as follows. We substituted $u = \sqrt{1+x^2}$. As x varies from 0 to 1, so u varies from 1 to $\sqrt{2}$, and the right answer is obtained as follows:

$$\int_0^1 x^3\sqrt{1+x^2}\,dx = \int_1^{\sqrt{2}} (u^4 - u^2)\,du$$
$$= \Big|_1^{\sqrt{2}} \left(\tfrac{1}{5}u^5 - \tfrac{1}{3}u^3\right) = \tfrac{2}{15}(\sqrt{2}+1)$$

This method of carrying over the limits of integration works in general. Under the same assumptions as in the note to [11.3], we obtain

$$\int_a^b f\big(g(x)\big)g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du \qquad (u = g(x)) \qquad [11.5]$$

The reason is simple. If $F'(u) = f(u)$, then

$$\int_a^b f(g(x))g'(x)\,dx = \Big|_a^b F(g(x)) = F(g(b)) - F(g(a)) = \int_{g(a)}^{g(b)} f(u)\,du$$

Problems

1. Find the following integrals by using [11.3]:

   a. $\int (x^2+1)^8\, 2x\,dx$ b. $\int (x+2)^{10}\,dx$ c. $\int \frac{2x-1}{x^2-x+8}\,dx$

2. Find the following integrals by means of an appropriate substitution:

   a. $\int x(2x^2+3)^5\,dx$ b. $\int x^2 e^{x^3+2}\,dx$ c. $\int \frac{\ln(x+2)}{2x+4}\,dx$

   d. $\int x\sqrt{1+x}\,dx$ e. $\int \frac{x^3}{(1+x^2)^3}\,dx$ f. $\int x^5\sqrt{4-x^3}\,dx$

3. Find the following integrals:

   a. $\int_0^1 x\sqrt{1+x^2}\,dx$ b. $\int_1^e \frac{\ln y}{y}\,dy$ c. $\int_1^3 \frac{1}{x^2}e^{2/x}\,dx$

4. Solve the following equation for x:

$$\int_3^x \frac{2t-2}{t^2-2t}\,dt = \ln\left(\tfrac{2}{3}x - 1\right)$$

5. Find the following integrals:

   a. $\int_0^1 (x^4-x^9)(x^5-1)^{12}\,dx$ b. $\int \frac{\ln x}{\sqrt{x}}\,dx$ c. $\int_0^4 \frac{dx}{\sqrt{1+\sqrt{x}}}$

6. Show that

$$\int_{t_0}^{t_1} S'(x(t))\dot{x}(t)\,dt = S(x(t_1)) - S(x(t_0))$$

7. **a.** Show that if $a \neq b$, then for all $x \neq a$ and $x \neq b$,

$$\frac{cx+d}{(x-a)(x-b)} = \frac{1}{a-b}\left(\frac{ac+d}{x-a} - \frac{bc+d}{x-b}\right)$$

   b. Use the identity in part (a) to compute:

   (i) $\int_3^4 \frac{x\,dx}{x^2-3x+2}$ (ii) $\int_4^5 \frac{2x+3}{x^2-5x+6}\,dx$

8. Show that if f is continuous in the interval $[a, b]$, and λ is a constant $\neq 0$, then

   a. $\int_a^b f(x)\,dx = \int_{a+\lambda}^{b+\lambda} f(x-\lambda)\,dx$

   b. $\int_a^b f(x)\,dx = \frac{1}{\lambda}\int_{\lambda a}^{\lambda b} f\left(\frac{x}{\lambda}\right)\,dx$

Harder Problems

9. In a model of optimal macroeconomic stabilization, A. J. Preston makes investment I a function of the time that is given by the integral

$$I = \int_0^t \frac{A(1 - De^{\beta\tau})}{1 + CDe^{\beta\tau}} \, d\tau$$

All constants are positive. Find I by using the substitution $x = CDe^{\beta\tau}$ and also the identity in Problem 7(a).

10. Find the following:

$$I = \int \frac{x^{1/2}}{1 - x^{1/3}} \, dx$$

(*Hint:* How can you eliminate the fractional exponents in $x^{1/2}$ and $x^{1/3}$ *simultaneously* using only one substitution?)

11. Sometimes the change of variable formula [11.3] is used the other way around in the following sense: To evaluate $\int f(x)\,dx$, we introduce $x = g(t)$, $dx = g'(t)\,dt$, and try to solve the new integral expressed in terms of t. Finally, we use $t = g^{-1}(x)$ to get the answer in terms of x. (This requires g to have an inverse.) Apply this method to

$$\text{(a)} \int \frac{dx}{\sqrt{x^2 + 1}} \qquad \text{(b)} \int \sqrt{x^2 + 1}\, dx$$

(*Hint:* Introduce the substitution $x = \frac{1}{2}(e^t - e^{-t})$. This might strike you as rather odd, but it works. You will need the answers to Problem 6 of Section 8.1 and Problem 23 of Section 8.2.)

11.3 Extending the Concept of the Integral

In this section, we extend the concept of the integral in several directions. Again, each of these extensions is useful in economics and/or statistics.

Integrals of Certain Discontinuous Functions

So far we have only been integrating continuous functions. It is useful to extend the definition to certain discontinuous functions. A function f is called *piecewise continuous* over the interval from a to b if it has at most a finite number of discontinuity points in the interval, with one-sided limits on both sides at each discontinuity point.

A typical graph of a piecewise continuous function is shown in Fig. 11.2, where the discontinuity points are at $x = c$ and at $x = d$. Suppose we replace

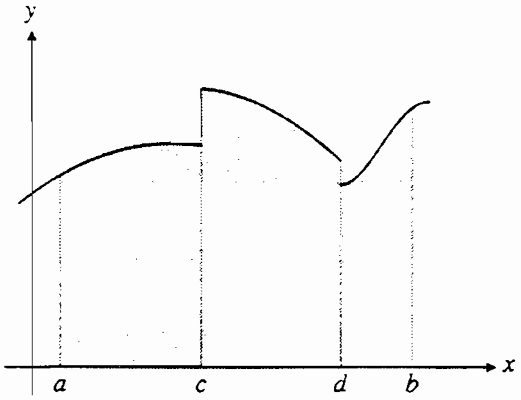


FIGURE 11.2 $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^d f(x)\,dx + \int_d^b f(x)\,dx$.

$f(x)$ in $[a, c]$ by the continuous function $f_1(x)$ that is equal to $f(x)$ throughout $[a, c)$ and has the value $f_1(c) = \lim_{x\to c^-} f(x)$ at $x = c$. Then $\int_a^c f_1(x)\,dx$ is well defined and it is reasonable to define $\int_a^c f(x)\,dx = \int_a^c f_1(x)\,dx$. By a similar trick, we define $\int_c^d f(x)\,dx$ and $\int_d^b f(x)\,dx$ by considering continuous functions in the intervals $[c, d]$ and $[d, b]$, respectively, that are equal to f except at one or both of the end points. The only sensible definition now is

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^d f(x)\,dx + \int_d^b f(x)\,dx$$

Then the interpretation of $\int_a^b f(x)\,dx$ is simply the sum of the three areas in Fig. 11.2. This should make clear how $\int_a^b f(x)\,dx$ can be defined for all functions $f(x)$ that are piecewise continuous on $[a, b]$.

Infinite Intervals of Integration

Suppose f is a function that is continuous for all $x \geq a$. Then $\int_a^b f(x)\,dx$ is defined for each $b \geq a$. If the limit of this integral as $b \to \infty$ exists (and is finite), then we say that f is *integrable over* $[a, \infty)$, and define

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx \qquad [11.6]$$

The *improper integral* $\int_a^\infty f(x)\,dx$ is then said to *converge*. If the limit does *not* exist, the improper integral is said to *diverge*. If $f(x) \geq 0$ in $[a, \infty)$, we interpret the integral [11.6] as the *area* below the graph of f over the interval $[a, \infty)$.

Analogously, we define

$$\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty} \int_a^b f(x)\,dx \qquad [11.7]$$

when f is continuous in $(-\infty, b]$. If this limit exists, the improper integral is said to converge. Otherwise, it diverges.

Example 11.7

The *exponential distribution* in statistics is defined by

$$f(x) = \lambda e^{-\lambda x} \qquad (x \geq 0;\ \ \lambda \text{ is a positive constant})$$

Show that the area below the graph of f over $[0, \infty)$ is equal to 1. (See Fig. 11.3.)

Solution For $b > 0$, the area below the graph of f over $[0, b]$ is equal to

$$\int_0^b \lambda e^{-\lambda x}\, dx = \Big|_0^b \left(-e^{-\lambda x}\right) = -e^{-\lambda b} + 1$$

As $b \to \infty$, so $-e^{-\lambda b} + 1$ approaches 1. Therefore,

$$\int_0^\infty \lambda e^{-\lambda x}\, dx = \lim_{b\to\infty} \int_0^b \lambda e^{-\lambda x}\, dx = \lim_{b\to\infty} \left(-e^{-\lambda b} + 1\right) = 1$$

Example 11.8

Show that

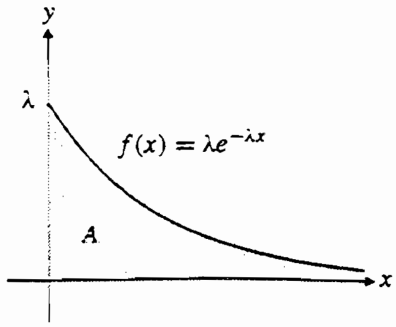
$$\int_1^\infty \frac{1}{x^a}\, dx = \frac{1}{a-1} \qquad (\text{for } a > 1) \qquad [1]$$

Then study the case $a \leq 1$.

Solution For $a \neq 1$ and $b > 1$,

$$\int_1^b \frac{1}{x^a}\, dx = \int_1^b x^{-a}\, dx = \Big|_1^b \frac{1}{1-a} x^{1-a} = \frac{1}{1-a}(b^{1-a} - 1) \qquad [2]$$

FIGURE 11.3 **Area A has an unbounded base, but the height decreases to 0 so rapidly that the total area is 1.**



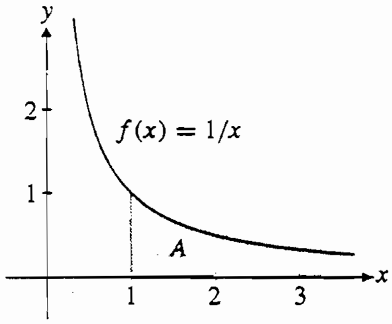


FIGURE 11.4 **"$A = \int_1^{\infty}(1/x)\,dx = \infty$." $1/x$ does not approach 0 fast enough, so the improper integral diverges.**

For $a > 1$, one has $b^{1-a} = 1/b^{a-1} \to 0$ as $b \to \infty$. Hence, [1] follows from [2] by letting $b \to \infty$.

For $a = 1$, we have $\int_1^b (1/x)\,dx = \ln b - \ln 1 = \ln b$, which tends to ∞ as b tends to ∞, so $\int_1^{\infty}(1/x)\,dx$ diverges. See Fig. 11.4.

For $a < 1$, the last expression in [2] tends to ∞ as b tends to ∞. Hence, the integral diverges in this case.

If both limits of integration are infinite, the improper integral of a continuous function f on $(-\infty, \infty)$ is defined by

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{\infty} f(x)\,dx \qquad [11.8]$$

If *both* integrals on the right-hand side converge, the improper integral $\int_{-\infty}^{\infty} f(x)\,dx$ is said to *converge*; otherwise, it *diverges*. Instead of using 0 as the point of subdivision, one could use an arbitrary fixed real number c. The value assigned to the integral will always be the same, provided that the integral does converge.

It is important to note that definition [11.8] requires both integrals on the right-hand side to converge. Note in particular that

$$\lim_{b\to\infty} \int_{-b}^{b} f(x)\,dx \qquad [*]$$

is *not* the definition of $\int_{-\infty}^{+\infty} f(x)\,dx$. Problem 4 provides an example in which [*] exists, yet the integral in [11.8] diverges. So [*] is not an acceptable definition, whereas [11.8] is.

Example 11.9

For $c > 0$, examine the convergence of

$$\int_{-\infty}^{+\infty} x e^{-cx^2}\,dx$$

Solution Let us begin with the indefinite integral $\int xe^{-cx^2}\,dx$. Making the substitution $u = -cx^2$, we have $du = -2cx\,dx$ and so

$$\int xe^{-cx^2}\,dx = -\frac{1}{2c}\int e^u\,du = -\frac{1}{2c}e^u + C = -\frac{1}{2c}e^{-cx^2} + C$$

According to [11.8], provided both integrals on the right side exist, one has

$$\int_{-\infty}^{\infty} xe^{-cx^2}\,dx = \int_{-\infty}^{0} xe^{-cx^2}\,dx + \int_{0}^{\infty} xe^{-cx^2}\,dx \qquad [*]$$

But now

$$\int_{-\infty}^{0} xe^{-cx^2}\,dx = \lim_{a\to-\infty}\int_{a}^{0} xe^{-cx^2}\,dx = \lim_{a\to-\infty}\Big|_a^0 -\frac{1}{2c}e^{-cx^2} = -\frac{1}{2c}$$

In the same way, we see that the second integral in [*] is $1/2c$, so

$$\int_{-\infty}^{\infty} xe^{-cx^2}\,dx = -\frac{1}{2c} + \frac{1}{2c} = 0 \qquad (c > 0) \qquad [**]$$

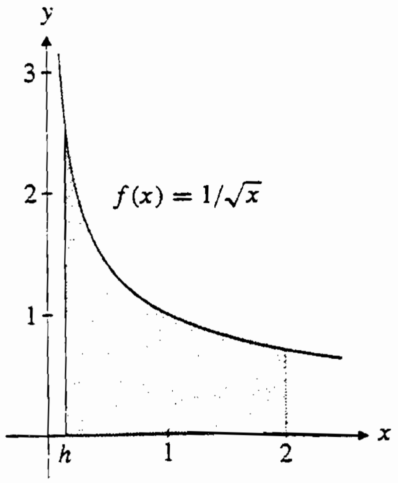
(This result is very important in statistics. See Problem 13.)

Integrals of Unbounded Functions

We turn next to improper integrals where the *integrand* is not bounded.

Consider first the function $f(x) = 1/\sqrt{x}$, with $x \in (0, 2]$. (See Fig. 11.5.) Note that $f(x) \to \infty$ as $x \to 0^+$. The function f is continuous in the interval $[h, 2]$

FIGURE 11.5 The "height" of the domain is unbounded, but $y = 1/\sqrt{x}$ approaches the y-axis so quickly that the total area is finite.



for an arbitrary fixed number h in $(0, 2)$. Therefore, the definite integral of f over the interval $[h, 2]$ exists, and

$$\int_h^2 \frac{1}{\sqrt{x}}\,dx = \Big|_h^2 2\sqrt{x} = 2\sqrt{2} - 2\sqrt{h}$$

The limit of this expression as $h \to 0^+$ is $2\sqrt{2}$. Then, by definition,

$$\int_0^2 \frac{1}{\sqrt{x}}\,dx = 2\sqrt{2}$$

The improper integral is said to converge in this case, and the area below the graph of f over the interval $(0, 2]$ is $2\sqrt{2}$. The area of $1/\sqrt{x}$ over the interval $(h, 2]$ is shown in Fig. 11.5.

More generally, suppose that f is a continuous function in the interval $(a, b]$, but $f(x)$ is not defined at $x = a$. Then we define

$$\int_a^b f(x)\,dx = \lim_{h \to 0^+} \int_{a+h}^b f(x)\,dx \tag{11.9}$$

if the limit exists, and the improper integral of f is said to converge in this case. If $f(x) \geq 0$ in $(a, b]$, we identify the integral as the *area under the graph* of f over the interval $(a, b]$. In the same way,

$$\int_a^b f(x)\,dx = \lim_{h \to 0^+} \int_a^{b-h} f(x)\,dx \tag{11.10}$$

if the limit exists, in which case the improper integral of f is said to converge.

Suppose f is continuous in (a, b). We may not even have f defined at a or b. For instance, suppose $f(x) \to -\infty$ as $x \to a^+$ and $f(x) \to +\infty$ as $x \to b^-$. In this case, f is said to be *integrable* in (a, b), and we can define

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \tag{11.11}$$

provided that both integrals on the right-hand side of [11.11] converge. Here c is an arbitrary fixed number in (a, b), and neither the convergence of the integral nor its value depends on the choice of c. If either of the integrals on the right-hand side of [11.11] does not converge, the left-hand side is not well defined.

Suppose that S is a union of a finite number of intervals of the form

$$S = (a_1, b_1) \cup (a_2, b_2) \cup \cdots \cup (a_n, b_n)$$

where $a_1 < b_1 \leq a_2 < b_2 \leq \cdots \leq a_n < b_n$, and $a_1 = -\infty$ and/or $b_n = \infty$ are allowed. Provided that f is integrable in each of the intervals $(a_1, b_1), \ldots, (a_n, b_n)$

according to the earlier definitions in this section, then f is said to be *integrable over* S. We define the *integral* over S as:

$$\int_S f(x)\,dx = \sum_{k=1}^{n} \int_{a_k}^{b_k} f(x)\,dx \qquad [11.12]$$

A Comparison Test for Convergence

The following convergence test for integrals is frequently useful because it does not require evaluation of the integral.

Theorem 11.1 (A Comparison Test for Convergence)
Suppose that f and g are continuous for all $x \geq a$ and

$$|f(x)| \leq g(x) \qquad \text{(for all } x \geq a)$$

If $\int_a^\infty g(x)\,dx$ converges, then $\int_a^\infty f(x)\,dx$ converges, and

$$\left| \int_a^\infty f(x)\,dx \right| \leq \int_a^\infty g(x)\,dx$$

Considering the case in which $f(x) \geq 0$, Theorem 11.1 can be interpreted as follows: If the area below the graph of g is finite, then the area below the graph of f is finite as well, because at no point in $[a, \infty)$ does the graph of f lie above the graph of g. (Draw a figure.) This result seems quite plausible and we shall not give an analytical proof. A corresponding theorem holds for the case where the lower limit of integration is $-\infty$. Also, similar comparison tests can be proved for unbounded functions defined on bounded intervals.

Example 11.10
Integrals of the form

$$\int_{t_0}^{\infty} U\big(c(t)\big)e^{-\alpha t}\,dt \qquad [1]$$

often appear in economic growth theory. Here $c(t)$ denotes consumption at time t, U is an instantaneous utility function, and α is a positive discount rate. Suppose that there exist numbers M and β, with $\beta < \alpha$, such that

$$|U\big(c(t)\big)| \leq Me^{\beta t} \qquad [2]$$

for all $t \geq t_0$ and for each possible consumption level $c(t)$ at time t. Thus, the absolute value of the utility of consumption is growing at a rate less than the discount rate α. Prove that then [1] converges.

Solution From [2],

$$|U(c(t))e^{-\alpha t}| \leq Me^{-(\alpha-\beta)t} \qquad \text{(for all } t \geq t_0)$$

Moreover,

$$\int_{t_0}^{T} Me^{-(\alpha-\beta)t}\,dt = \Big|_{t_0}^{T} \frac{-M}{\alpha-\beta}e^{-(\alpha-\beta)t} = \frac{M}{\alpha-\beta}\left[e^{-(\alpha-\beta)t_0} - e^{-(\alpha-\beta)T}\right]$$

Because $\alpha - \beta > 0$, the last expression tends to $[M/(\alpha-\beta)]\,e^{-(\alpha-\beta)t_0}$ as $T \to \infty$. From Theorem 11.1, it follows that [1] converges.

Example 11.11

The function $f(x) = e^{-x^2}$ is extremely important in statistics, because it is the basis of the *Gaussian*, or *normal*, distribution. It is possible to show that the improper integral

$$\int_{-\infty}^{+\infty} e^{-x^2}\,dx \qquad [1]$$

converges. Note that according to [11.4] in Section 11.2, the function $f(x) = e^{-x^2}$ has no indefinite integral that we can find. Because $f(x) = e^{-x^2}$ is symmetric about the y-axis, it suffices to prove that $\int_0^\infty e^{-x^2}\,dx$ converges. To this end, subdivide the interval of integration so that

$$\int_0^\infty e^{-x^2}\,dx = \int_0^1 e^{-x^2}\,dx + \int_1^\infty e^{-x^2}\,dx \qquad [2]$$

Of course, $\int_0^1 e^{-x^2}\,dx$ presents no problem because it is the integral of a continuous function over a bounded interval. For $x \geq 1$, one has $0 \leq e^{-x^2} \leq e^{-x}$. Now $\int_1^\infty e^{-x}\,dx$ converges (to $1/e$), so according to Theorem 11.1, the integral $\int_1^\infty e^{-x^2}\,dx$ must also converge. From [2], it follows that $\int_0^\infty e^{-x^2}\,dx$ converges. Thus, the integral [1] does converge, but we have not found its value. In fact, more advanced techniques of integration show that

$$\int_{-\infty}^{+\infty} e^{-x^2}\,dx = \sqrt{\pi} \qquad [11.13]$$

Problems

1. Determine the following integrals, if they converge. Indicate those that diverge.

   a. $\displaystyle\int_1^\infty \frac{1}{x^3}\,dx$ **b.** $\displaystyle\int_1^\infty \frac{1}{\sqrt{x}}\,dx$

c. $\int_{-\infty}^{0} e^x \, dx$ **d.** $\int_0^a \frac{x \, dx}{\sqrt{a^2 - x^2}}$ $(a > 0)$

2. Define f for all x by $f(x) = 1/(b-a)$ for $x \in [a, b]$, $f(x) = 0$ for $x \notin [a, b]$. (In statistics, f is called the *rectangular* (or *uniform*) *distribution*). Find the following:

a. $\int_{-\infty}^{+\infty} f(x) \, dx$ **b.** $\int_{-\infty}^{+\infty} x f(x) \, dx$ **c.** $\int_{-\infty}^{+\infty} x^2 f(x) \, dx$

3. In connection with Example 11.7, find the following:

a. $\int_0^{\infty} x\lambda e^{-\lambda x} \, dx$ **b.** $\int_0^{\infty} (x - 1/\lambda)^2 \, \lambda e^{-\lambda x} \, dx$

c. $\int_0^{\infty} (x - 1/\lambda)^3 \, \lambda e^{-\lambda x} \, dx$

(The three numbers you obtain are called respectively the expectation, the variance, and the third central moment of the exponential distribution.)

4. Prove that $\int_{-\infty}^{+\infty} x/(1 + x^2) \, dx$ diverges, but that $\lim_{b \to \infty} \int_{-b}^{b} x/(1 + x^2) \, dx$ converges.

5. The function f is defined for $x > 0$ by $f(x) = (\ln x)/x^3$.

a. Find the maximum and minimum points of f, if there are any.

b. Examine the convergence of $\int_0^1 f(x) \, dx$ and $\int_1^{\infty} f(x) \, dx$.

6. Use the comparison test of Theorem 11.1 to prove the convergence of

$$\int_1^{\infty} \frac{1}{1 + x^2} \, dx$$

7. Show that

$$\int_{-2}^{3} \left(\frac{1}{\sqrt{x + 2}} + \frac{1}{\sqrt{3 - x}} \right) dx = 4\sqrt{5}$$

8. R. E. Hall and D. W. Jorgenson, in their article on "Tax Policy and Investment Behavior," use the integral

$$z = \int_0^{\infty} e^{-rs} D(s) \, ds$$

to represent the present discounted value, at interest rate r, of the time-dependent stream of depreciation allowances $D(s)$ $(0 \leq s < \infty)$. Find z as a function of τ in the following cases:

a. $D(s) = 1/\tau$ for $0 \leq s \leq \tau$, $D(s) = 0$ for $s > \tau$. (Constant depreciation over τ years.)

b. $D(s) = 2(\tau - s)/\tau^2$ for $0 \leq s \leq \tau$, $D(s) = 0$ for $s > \tau$. (Straight-line depreciation.)

9. Suppose you evaluate $\int_{-1}^{+1}(1/x^2)\,dx$ by using definition [10.12] in Section 10.3 of the definite integral without thinking. You get a negative answer even though the integrand is never negative. What has gone wrong?

10. Prove that the following integral converges and find its value:

$$\int_0^1 \frac{\ln x}{\sqrt{x}}\,dx$$

Harder Problems

11. Find the integral

$$I_k = \int_1^{\infty}\left(\frac{k}{x} - \frac{k^2}{1+kx}\right)dx \qquad (k \text{ is a positive constant})$$

Find the limit of I_k as $k \to \infty$, if it exists.

12. Use the results in Example 11.8 to prove [6.18] in Section 6.5. (*Hint:* Draw the graph of $f(x) = x^{-p}$ in $[1, \infty)$, and interpret each of the sums $\sum_{n=1}^{\infty} n^{-p}$ and $\sum_{n=2}^{\infty} n^{-p}$ geometrically as sums of an infinite number of rectangles.)

13. In statistics, the normal, or Gaussian, density function is defined by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-(x-\mu)^2/2\sigma^2}$$

in the interval $(-\infty, \infty)$.[2] Prove that

(a) $\int_{-\infty}^{+\infty} f(x)\,dx = 1$ **(b)** $\int_{-\infty}^{+\infty} xf(x)\,dx = \mu$

(c) $\int_{-\infty}^{+\infty} x^2 f(x)\,dx = \sigma^2 + \mu^2$

(*Hint:* Use the substitution $u = (x-\mu)/\sqrt{2}\sigma$, together with [11.13] and the result in Example 11.9.)

11.4 A Note on Income Distribution and Lorenz Curves

In Section 10.4 it was explained how, if $f(r)$ is the income distribution function for a population of n individuals, then $n\int_a^b f(r)\,dr$ represents the number of individuals with incomes in the interval $[a, b]$—see Equation [10.27]. In addition, $n\int_a^b r\,f(r)\,dr$ represents the total income of these individuals—see Equation [10.28].

[2]This function, its bell-shaped graph, and a portrait of its inventor Carl Friedrich Gauss (1777–1855), appear on the German 10-mark currency note issued in early 1989.

TABLE 11.1 *Shares of total income*

Income group	United States 1980	United States 1990	Netherlands 1959	Netherlands 1985	World 1989
Lowest fifth	5.2	4.6	5.0	7.8	1.4
Second fifth	11.5	10.8	11.9	13.9	1.9
Third fifth	17.5	16.6	17.4	18.1	2.3
Fourth fifth	24.3	23.8	22.7	23.4	11.7
Highest fifth	41.5	44.3	43.0	36.7	82.7

A statistical device for describing some important features of any such income distribution is the **Lorenz Curve**.[3] This curve is based on the shares of total income that accrue to different groups of individuals in the population, starting with the poorest and working up to the richest. Consider, for example, the data in Table 11.1.[4] It may be apparent already that inequality increased in the United States during the 1980s, and that it decreased in the Netherlands during the much longer period 1959–1985. The distribution in the Netherlands in 1959 is quite close to that in the United States for 1980. The reported distribution for the world as a whole is close to an extreme.

These preliminary insights are confirmed by a rather more careful analysis based on Lorenz curves. To construct these, we first cumulate the incomes of different fifths of the population so that the five new groups we consider are respectively the lowest 20%, then the lowest 40%, the lowest 60%, the lowest 80%, followed by the whole population. This gives Table 11.2.[5]

Figure 11.6 illustrates two of the resulting Lorenz curves, found by fitting smoothed curves to the data points in the second and fifth columns of Table 11.2.

The question we ask now is this: If the income distribution is really described by the continuous density function $f(r)$, as in Section 10.4, how does one find the Lorenz curve? To answer this, we first need to consider the *cumulative distribution function* $F(r)$ of Section 10.4, whose value for each income level r represents the proportion of the population having incomes $\leq r$. Thus, the value of this function

[3]Named after the American statistician Max Otto Lorenz, who introduced it as one of the "Methods for Measuring Concentration of Wealth" (rather than income) in an article published in the *Journal of the American Statistical Association*, 1905.

[4]Data for the United States are taken from the Bureau of the Census. Those for the Netherlands come originally from the Dutch Central Bureau of Statistics. World income data are taken from the UN Development Program's *Human Development Report* for 1992. Actually, data of this kind for the world as a whole do not exist. The reported figures represent what the world distribution of income would be if the gross domestic product of each nation were perfectly equally distributed as income to all the inhabitants of that nation. Nevertheless, there is no reason to think that the resulting figures seriously exaggerate the true extent of world inequality.

[5]As is often the case with data of this kind, rounding errors mean that the totals of the figures in Table 11.1 are not exactly 100% in every case.

TABLE 11.2 *Cumulative incomes*

Income group	United States 1980	United States 1990	Netherlands 1959	Netherlands 1985	World 1989
Lowest 20%	5.2	4.6	5.0	7.8	1.4
Lowest 40%	16.7	15.4	16.9	21.7	3.3
Lowest 60%	34.2	32.0	34.3	39.8	5.6
Lowest 80%	58.5	55.8	57.0	63.2	17.3
Lowest 100%	100.0	100.0	100.0	100.0	100.0

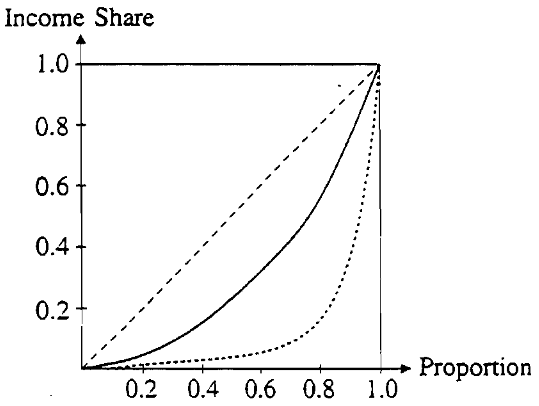


FIGURE 11.6 Approximate Lorenz curves for the U.S. in 1990 (solid curve) and the whole world in 1989 (dotted curve). The dashed curve represents perfect equality.

is given by the integral

$$F(r) = \int_0^r f(x)\,dx$$

which evidently satisfies $F'(r) = f(r)$ for all income levels r. We assume that $f(r) > 0$ at all income levels $r \geq 0$, implying that $F(r)$ is strictly increasing. Moreover, assuming that everybody has some income, even if only a little, it must be true that $F(0) = 0$. Also $F(\infty) = 1$, because everybody has a finite income, even if some individuals may have extremely large incomes. Here $F(\infty) = 1$ is a shorthand notation for $F(r) \to 1$ as $r \to \infty$.

Along the horizontal axis of the graph of the Lorenz curve, the variable is the proportion $p = F(r)$ of the population having incomes $\leq r$. Constructing the Lorenz curve requires considering the inverse of this function, $r = R(p)$, which is also strictly increasing.

The function $R(p)$ can be given an important interpretation. For each $p \in [0, 1]$, the value $R(p)$ is that income level for which the exact fraction or proportion p of the population has income $r \leq R(p)$; this must be true because, by the definition of an inverse function, $F(R(p)) = p$. When $p = 1/2$, for example, the income level $R(1/2)$ has the property that half the population has income $r \leq R(1/2)$, whereas the other half has income $r > R(1/2)$; this "middle" income level is generally called the *median* of the distribution $f(r)$. Ranges of income between different values of $R(p)$ also receive appropriate names—for example, the interval $[R(0.2), R(0.4)]$ is called the *second quintile*, $[R(0.6), R(0.7)]$ is called the *seventh decile*, and so on. The different values of $R(p)$ are generally known as *percentiles*, and also as *order statistics*.

By the usual rule for differentiating the inverse of a function (see [7.24] in Section 7.6), we have

$$R'(p) = \frac{1}{F'(r)} = \frac{1}{f(r)} = \frac{1}{f(R(p))} \tag{11.14}$$

This is valid for all $p \in (0, 1)$ because we assumed that $f(r) > 0$ at all income levels r.

The Lorenz curve is the graph of the function $L(p)$ whose value for each p represents the share of total income accruing to the poorest fraction p of the population. Now, total income is given by $n \int_0^\infty r\, f(r)\, dr$, where n is the total number of individuals in the population. Because $R(p)$ is the income level of the richest person in the poorest fraction p of the population, the total income of this group is $n \int_0^{R(p)} r\, f(r)\, dr$. Thus, we have

$$L(p) = \frac{n \int_0^{R(p)} r\, f(r)\, dr}{n \int_0^\infty r\, f(r)\, dr} = \frac{1}{m} \int_0^{R(p)} r\, f(r)\, dr \tag{11.15}$$

where m is the mean income $\int_0^\infty r\, f(r)\, dr$. Because $0 \le \int_0^{R(p)} rf(r)\, dr \le \int_0^\infty rf(r)\, dr$, Equation [11.15] implies that $0 \le L(p) \le 1$ for all $p \in [0, 1]$. The slope of the Lorenz curve can be found by making use of the differentiation rule (10.22) in Section 10.3. In fact,

$$L'(p) = \frac{1}{m} R(p) f(R(p)) R'(p) = \frac{R(p)}{m}$$

where the second equality follows from [11.14]. Thus, the slope of the Lorenz curve is equal to the ratio of the income level $R(p)$ to mean income m. This slope increases steadily from $0 = R(0)$ when $p = 0$, to "$\infty = R(1)$" when $p = 1$. In particular, differentiating a second time gives

$$L''(p) = \frac{R'(p)}{m} = \frac{1}{mf(R(p))} > 0$$

for all $p \in (0, 1)$, implying that a Lorenz curve is strictly convex. And, as Fig. 11.6 illustrates, each Lorenz curve has a horizontal tangent at $p = 0$, together with a vertical tangent at $p = 1$. Finally, $L'(p) = 1$ at the unique point where $R(p) = m$ and so for $p = F(m)$. For $0 < p < F(m)$, one has $L'(p) < 1$, so that the Lorenz curve initially rises more slowly than the 45° line. At $p = F(m)$, the horizontal distance between the Lorenz curve and the 45° line reaches a maximum. For $F(m) < p < 1$, one has $L'(p) > 1$, so that the Lorenz curve ends by rising faster than the 45° line until the two intersect once again when $p = 1$. In particular, this shows that $L(p) < p$ throughout the open interval $p \in (0, 1)$.

The Lorenz curve can also be used to define a common measure G of income inequality, generally known as the **Gini coefficient.**[6] Geometrically, G is twice the area of the set lying below the 45° line and above the Lorenz curve. But this area can be represented as the difference between the two integrals $\int_0^1 p\,dp = 1/2$ and $\int_0^1 L(p)\,dp$. So

$$G = 2\left[\frac{1}{2} - \int_0^1 L(p)dp\right] = 1 - 2\int_0^1 L(p)\,dp \qquad \textbf{(Gini coefficient)} \qquad [11.16]$$

From this it follows that $0 < G < 1$. The low extreme $G = 0$ is approached as the Lorenz curve shifts up closer to the 45° line. This occurs as income becomes distributed more equally, with each poorest fraction p of the population getting closer to receiving its full share p of the total available income. The other extreme $G = 1$ is approached as the Lorenz curve shifts down further away from the 45° line. This occurs as income becomes distributed more unequally, with each poorest fraction p of the population getting closer to a zero share of the total available income, and a decreasingly small fraction of very prosperous people getting closer to having all available income. Generally, as the Lorenz curve shifts down, the income distribution becomes more unequal, and the Gini coefficient increases.

Problems

1. Draw the Lorenz curves for the first, third, and fourth columns of Table 11.2.
2. Estimate values of the Gini coefficients for all five distributions reported in Table 11.2.

[6]This is named after the Italian Corrado Gini, who first proposed it in 1912, and apparently discovered the Lorenz curve independently of Lorenz. His definition was the double integral $G = (1/2m)\int_0^\infty \int_0^\infty |r - r'| f(r) f(r')\,dr\,dr'$, but this is equivalent to the definition given here. We do not show this because we do not consider double integrals in this book.

15

Functions of Several Variables

You know we all became mathematicians
for the same reason: We were lazy.
—*Max Rosenlicht* (1949)

So far, we have mostly studied functions of one variable—that is, functions whose domain is a set of real numbers and whose range is also a set of real numbers. Yet a realistic description of many economic phenomena requires considering a large number of variables simultaneously. For example, the demand for a good depends on consumer tastes, the price of that good, on different consumers' incomes, and on the prices of complements and substitutes, among other things. This requires a function of several variables to be considered.

Much of what economists need consists of relatively simple generalizations of functions of one variable and their properties. Most of the difficulties already arise in the transition from one variable to two variables. Therefore, it may be sensible in the following to concentrate on functions of two variables before trying to tackle the material dealing with functions of more than two variables. However, there are many interesting economic problems that can only be represented mathematically by functions of a large number of variables.

15.1 Functions of Two or More Variables

We begin with the following definition:

> A **function f of two variables x and y** with domain D is a rule that assigns a specified number $f(x, y)$ to each point (x, y) in D. [15.1]

Example 15.1

Consider the function f that, to every pair of numbers (x, y), assigns the number $2x + x^2y^3$. The function f is thus defined by

$$f(x, y) = 2x + x^2y^3$$

What are $f(1, 0)$, $f(0, 1)$, $f(-2, 3)$, and $f(a + 1, b)$?

Solution $f(1, 0) = 2 \cdot 1 + 1^2 \cdot 0^3 = 2$, $f(0, 1) = 2 \cdot 0 + 0^2 \cdot 1^3 = 0$, and $f(-2, 3) = 2(-2) + (-2)^2 \cdot 3^3 = -4 + 4 \cdot 27 = 104$. Finally, we find $f(a+1, b)$ by replacing x with $a+1$ and y with b in the formula for $f(x, y)$, giving $f(a + 1, b) = 2(a + 1) + (a + 1)^2b^3$.

Example 15.2

A study of the demand for milk by R. Frisch and T. Haavelmo found the relationship

$$x = A\frac{r^{2.08}}{p^{1.5}} \qquad (A \text{ is a positive constant}) \qquad [*]$$

where x is milk consumption, p is the relative price of milk, and r is income per family. This equation defines x as a function of p and r. Note that milk consumption goes up when income r increases, and goes down when the price of milk increases, which seems reasonable.

Example 15.3

A function of two variables appearing in many economic models is

$$F(x, y) = Ax^ay^b \quad (A,\ a,\ \text{and } b \text{ are constants}) \qquad [15.2]$$

Usually, one assumes that F is defined only for $x > 0$ and $y > 0$; sometimes for $x \geq 0$ and $y \geq 0$. Then F is generally called a **Cobb–Douglas function.**[1] Note that the function defined in [*] of Example 15.2 is a Cobb–Douglas function, because we have $x = Ap^{-1.5}r^{2.08}$.

As another example of a Cobb–Douglas function, here is an estimated production function for a certain lobster fishery:

$$F(S, E) = 2.26\, S^{0.44}E^{0.48} \qquad [**]$$

where S denotes the stock of lobster, E the harvesting effort, and $F(S, E)$ the catch.

[1]The function in [15.2] is named after two American researchers, C. W. Cobb and P. H. Douglas, who applied it (with $a+b = 1$) in a paper on the estimation of production functions that appeared in 1927. Actually, the function should properly be called a "Wicksell function," because the Swedish economist Knut Wicksell (1851–1926) introduced such production functions before 1900. See B. Sandelin, "On the origin of the Cobb–Douglas production function," *Economy and History*, 19 (1976), 117–123.

Example 15.4

For the function F given in [15.2], find an expression for $F(2x, 2y)$ and for $F(tx, ty)$, where t is an arbitrary positive number. What is $F(tS, tE)$ for the function in [**]?

Solution

$$F(2x, 2y) = A(2x)^a(2y)^b = A2^a x^a 2^b y^b = 2^a 2^b A x^a y^b = 2^{a+b} F(x, y)$$

$$F(tx, ty) = A(tx)^a(ty)^b = At^a x^a t^b y^b = t^{a+b} A x^a y^b = t^{a+b} F(x, y)$$

$$F(tS, tE) = 2.26(tS)^{0.44}(tE)^{0.48} = 2.26\, t^{0.44} S^{0.44} t^{0.48} E^{0.48} = t^{0.92} F(S, E)$$

The last calculation shows that if we multiply both S and E by the factor t, then the catch will be $t^{0.92}$ times as big. If $t = 2$, for example, then this formula shows that doubling both the stock and the harvesting effort leads to a catch that is a little less than twice as big. (It is $2^{0.92} \approx 1.89$ times as big.)

Functions of More Than Two Variables

Many of the most important functions we study in economics, such as the gross domestic product (GDP) of a country, depend in a complicated way on a large number of variables. In some abstract models, it may be sufficient to ascertain that such a connection exists without specifying the dependence more closely. In this case, we say only that the GDP is a *function* of the different variables. The function concept we use is a direct generalization of definition [15.1].

A **function f of n variables** $x_1, \dots, x_n$ with domain D is a rule that assigns a specified number $f(x_1, \dots, x_n)$ to each n-vector $(x_1, \dots, x_n)$ in D. [15.3]

Let us look at some examples of functions of several variables in economics.

Example 15.5

(a) The demand for sugar in the United States in the period 1929–1935 was estimated by T. W. Schultz, who found that it could be described approximately by the formula

$$x = 108.83 - 6.0294p + 0.164w - 0.4217t$$

Here the demand x for sugar is a function of three variables: p (the price of sugar), w (a production index), and t (the date, where $t = 0$ corresponds to 1929).

(b) R. Stone estimated the following formula for the demand for beer in England:

$$x = 1.058\, x_1^{0.136} x_2^{-0.727} x_3^{0.914} x_4^{0.816}$$

Here the quantity demanded x is a function of four variables: x_1 (the income of the individual), x_2 (the price of beer), x_3 (a general price index for all other commodities), and x_4 (the strength of the beer).

The simplest of the functions in Example 15.5 is (a). The variables p, w, and t occur here only to the first power, and they are only multiplied by constants, not by each other. Such functions are called *linear*. In general,

$$f(x_1, x_2, \dots, x_n) = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n + b \qquad [15.4]$$

(where a_1, a_2, ..., a_n and b are constants) is a **linear function**[2] in n variables. Example 15.5(b) is a special case of the general Cobb–Douglas function

$$F(x_1, x_2, \dots, x_n) = A x_1^{a_1} x_2^{a_2} \dots x_n^{a_n} \quad (A,\ a_1,\ \dots,\ a_n \text{ are constants; } A > 0) \qquad [15.5]$$

defined for $x_1 > 0, x_2 > 0, \dots, x_n > 0$. We encounter this function many times in this book.

Note: If we compare the linear function in [15.4] with the Cobb–Douglas function [15.5], the latter function is, of course, more complicated. Suppose, however, that $A > 0$ and $x_1 > 0$, ..., $x_n > 0$. Then taking the natural logarithm of each side in [15.5] gives

$$\ln F = \ln A + a_1 \ln x_1 + a_2 \ln x_2 + \cdots + a_n \ln x_n \qquad [15.6]$$

This shows that the Cobb–Douglas function is *log-linear* (or ln-linear), because $\ln F$ is a linear function of $\ln x_1$, $\ln x_2$, ..., $\ln x_n$.

Example 15.6

Suppose that the results of n observations of a quantity are given by n positive numbers x_1, x_2, ..., x_n. In statistics, several different measures for their average value are used. The most common are

$$\text{the **arithmetic** mean: } \bar{x}_A = \frac{1}{n}(x_1 + x_2 + \cdots + x_n) \qquad [1]$$

$$\text{the **geometric** mean: } \bar{x}_G = \sqrt[n]{x_1 x_2 \dots x_n} \qquad [2]$$

$$\text{the **harmonic** mean: } \bar{x}_H = \frac{1}{\frac{1}{n}\left(\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}\right)} \qquad [3]$$

[2]This is rather common terminology, although mathematicians would insist that f should really be called *affine* if $b \neq 0$, and *linear* only if $b = 0$.

Note that $\bar{x}_A$ is a linear function of $x_1, \ldots, x_n$, whereas $\bar{x}_G$ and $\bar{x}_H$ are nonlinear functions. ($\bar{x}_G$ is log-linear.)

For example, if the results of four observations are $x_1 = 1$, $x_2 = 2$, $x_3 = 3$, and $x_4 = 4$, then $\bar{x}_A = (1+2+3+4)/4 = 2.5$, $\bar{x}_G = \sqrt[4]{1 \cdot 2 \cdot 3 \cdot 4} = \sqrt[4]{24} \approx 2.21$, and $\bar{x}_H = \left[(1/1 + 1/2 + 1/3 + 1/4)/4\right]^{-1} = 48/25 = 1.92$. In this case, $\bar{x}_H \leq \bar{x}_G \leq \bar{x}_A$, and it turns out that these inequalities are valid in general:

$$\bar{x}_H \leq \bar{x}_G \leq \bar{x}_A \tag{4}$$

For $n = 2$, we showed that $\bar{x}_G \leq \bar{x}_A$ in Example 1.3 in Section 1.4. See also Problems 9 and 10 as a motivation for $\bar{x}_H$ and Problem 11 concerning a proof of the inequalities in [4].

Domains

For functions studied in economics, there are usually explicit or implicit restrictions on the domain of variation for the variables. For instance, we usually assume that the quantity x_i of a commodity is nonnegative, so $x_i \geq 0$. In economics, it is often crucially important to be clear what are the domains of the functions being used.

As for functions of one variable, we assume, unless otherwise stated, that the domain of a function defined by a formula is the largest domain in which the formula gives a meaningful and unique value.

For functions of two variables x and y, the domain is a set of points in the xy-plane. Sometimes it is helpful to draw a picture of the domain in the xy-plane. Let us look at some examples.

Example 15.7

Determine the domains of the functions given by the following formulas and draw the sets in the xy-plane.

(a) $f(x, y) = \sqrt{x-1} + \sqrt{y}$

(b) $g(x, y) = \dfrac{2}{\sqrt{x^2 + y^2 - 4}} + \sqrt{9 - (x^2 + y^2)}$

Solution

(a) We must require that $x \geq 1$ and $y \geq 0$, for only then do $\sqrt{x-1}$ and $\sqrt{y}$ have any meaning. The domain is indicated in Fig. 15.1.

(b) $\sqrt{x^2 + y^2 - 4}$ is only defined if $x^2 + y^2 \geq 4$. Moreover, we must have $x^2 + y^2 \neq 4$; otherwise, the denominator is equal to 0. Furthermore, we must require that $9 - (x^2 + y^2) \geq 0$, or $x^2 + y^2 \leq 9$. All in all, therefore, we must have $4 < x^2 + y^2 \leq 9$. Because the graph of $x^2 + y^2 = r^2$ consists of all the points on the circle with center at the origin and radius r, the domain is the set of

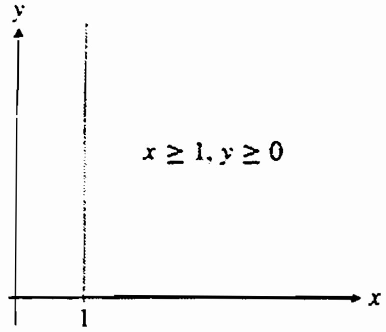


FIGURE 15.1

points (x, y) that lie outside (but not on) the circle $x^2 + y^2 = 4$, and inside or on the circle $x^2 + y^2 = 9$. This set is shown in Fig. 15.2.

Example 15.8

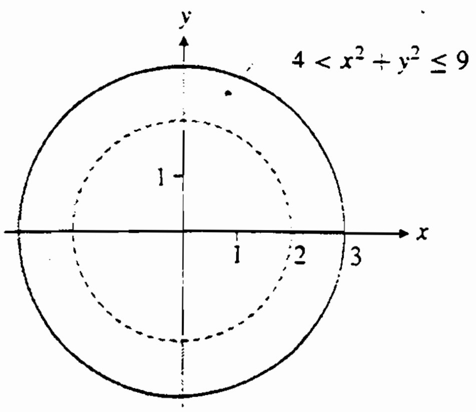
An individual must decide what quantities of n different commodities to buy during a given time period. Utility theory assumes that the individual has a utility function $U(x_1, x_2, \ldots, x_n)$ representing preferences, and that this measures the satisfaction the individual obtains by acquiring x_1 units of good no. 1, x_2 units of good no. 2, and so on. This is an important economic example of a function of n variables, to which we return several times.

Some economic models assume that

$$U(x_1, x_2, \ldots, x_n) = a_1 \ln(x_1 - c_1) + a_2 \ln(x_2 - c_2) + \cdots + a_n \ln(x_n - c_n)$$

where the parameters or constants $c_1, c_2, \ldots, c_n$ represent the minimum "subsistence" quantities that the consumer must have of the different commodities in order to survive. (Some or even many of the constants c_i could be equal to 0.) Because $\ln z$ is only defined when $z > 0$, we see that $x_1 > c_1$, $x_2 > c_2, \ldots, x_n > c_n$ is the requirement for U to be defined.

FIGURE 15.2



Problems

1. Let $f(x, y) = xy^2$. Compute $f(0, 1)$, $f(-1, 2)$, and $f(a, a)$.

2. Let $f(x, y) = 3x^2 - 2xy + y^3$. Compute $f(1, 1)$, $f(-2, 3)$, $f(1/x, 1/y)$, $[f(x+h, y) - f(x, y)]/h$, and $[f(x, y+k) - f(x, y)]/k$.

3. Let $f(x, y) = x^2 + 2xy + y^2$.
  a. Find $f(-1, 2)$, $f(a, a)$, and $f(a+h, b) - f(a, b)$.
  b. Show that $f(2x, 2y) = 2^2 f(x, y)$ and, in general, $f(tx, ty) = t^2 f(x, y)$ for all t.

4. Let $F(K, L) = 10K^{1/2}L^{1/3}$, $K \geq 0$, $L \geq 0$.
  a. Find $F(1, 1)$, $F(4, 27)$, $F(9, 1/27)$, $F(3, \sqrt{2})$, $F(100, 1000)$, and $F(2K, 2L)$.
  b. Find a constant a such that $F(tK, tL) = t^a F(K, L)$ for all $t > 0$, $K \geq 0$, and $L \geq 0$.

5. Some studies in agricultural economics employ production functions of the form $Y = F(K, L, T)$, where Y is the size of the harvest, K capital invested, L labor, and T the area of agricultural land used to grow the crop.
  a. Explain the meaning of $F(K+1, L, T) - F(K, L, T)$.
  b. Many studies assume that F is Cobb–Douglas. What form does F then have?
  c. If F is Cobb–Douglas, find $F(tK, tL, tT)$ expressed in terms of t and $F(K, L, T)$.

6. A study of milk production found that

$$y = 2.90\, x_1^{0.015} x_2^{0.250} x_3^{0.350} x_4^{0.408} x_5^{0.030}$$

where y is the output of milk, and $x_1, \ldots, x_5$ are the quantities of five different input factors. (For instance, x_1 is work effort and x_3 is grass consumption.)
  a. If all the factors of production were doubled, what would happen to y?
  b. Write the relation in log-linear form.

7. Examine for which (x, y) the functions given by the following formulas are defined and draw the domains in the xy-plane for (b) and (c).

  a. $\dfrac{x^2 + y^3}{y - x + 2}$ **b.** $\sqrt{2 - (x^2 + y^2)}$ **c.** $\sqrt{(4 - x^2 - y^2)(x^2 + y^2 - 1)}$

8. For which pairs of numbers (x, y) are the functions given by the following formulas defined?

  a. $\ln(x + y)$ **b.** $\sqrt{x^2 - y^2} + \sqrt{x^2 + y^2 - 1}$ **c.** $\sqrt{y - x^2} - \sqrt{\sqrt{x} - y}$

9. On a drive to a neighboring city center, you spend 5 minutes stopped at traffic lights at an average speed of 0 kilometers per hour, 10 minutes driving on local roads at an average speed of 30 kph, 20 minutes on an expressway at an average speed of 60 kph, and 15 minutes on a freeway driving at an average speed of 80 kph. How far do you drive, and what is your average speed for the whole journey?

Harder Problems

10. Suppose that n machines $A_1, A_2, \ldots, A_n$ produce the same product in the time span T and that the production times per unit are respectively $t_1, t_2, \ldots, t_n$. Show that if all the machines had been equally efficient and together had produced exactly the same total amount in the time span T, then each machine's production time per unit would have been precisely the harmonic mean $\bar{t}_H$ of $t_1, t_2, \ldots, t_n$.

11. In this problem, we refer to Example 15.6 and the definitions given there. Also, if $f(x)$ is concave over an interval I, and $x_1, x_2, \ldots, x_n$ belong to I, then by Jensen's inequality ([9.18] in Section 9.6),

$$f\left(\frac{1}{n}(x_1 + x_2 + \cdots + x_n)\right) \geq \frac{1}{n}f(x_1) + \frac{1}{n}f(x_2) + \cdots + \frac{1}{n}f(x_n) \qquad [*]$$

a. Show that if $x_1 = x_2 = \cdots = x_n$, then $\bar{x}_H = \bar{x}_G = \bar{x}_A$.

b. Let $f(x) = \ln x$. Then f is concave on $(0, \infty)$. Show that $\bar{x}_G \leq \bar{x}_A$ by using inequality [*].

c. In the inequality $\bar{x}_G \leq \bar{x}_A$, replace x_1 by $1/x_1$, x_2 by $1/x_2$, ..., and x_n by $1/x_n$. Prove that $\bar{x}_H \leq \bar{x}_G$.

15.2 Geometric Representations of Functions of Several Variables

This section considers how to visualize functions of several variables, in particular functions of two variables.

Surfaces in Three-Dimensional Space

An equation such as $f(x, y) = c$ in *two* variables x and y can be represented by a point set in the plane, called the **graph** of the equation. In a similar way, an equation $g(x, y, z) = c$ in *three* variables x, y, and z can be represented by a point set in 3-space, also called the **graph** of the equation. (For a discussion of 3-space, see Section 12.3.) This graph consists of all triples (x, y, z) satisfying the equation, and will usually form what can be called a **surface** in space. Three simple cases are given by the equations

$$\text{(a)} \quad x = a, \qquad \text{(b)} \quad y = b, \qquad \text{(c)} \quad z = c$$

where it is understood that there are no requirements on the variables other than those mentioned. The points (x, y, z) in space satisfying $x = a$ (with no require-

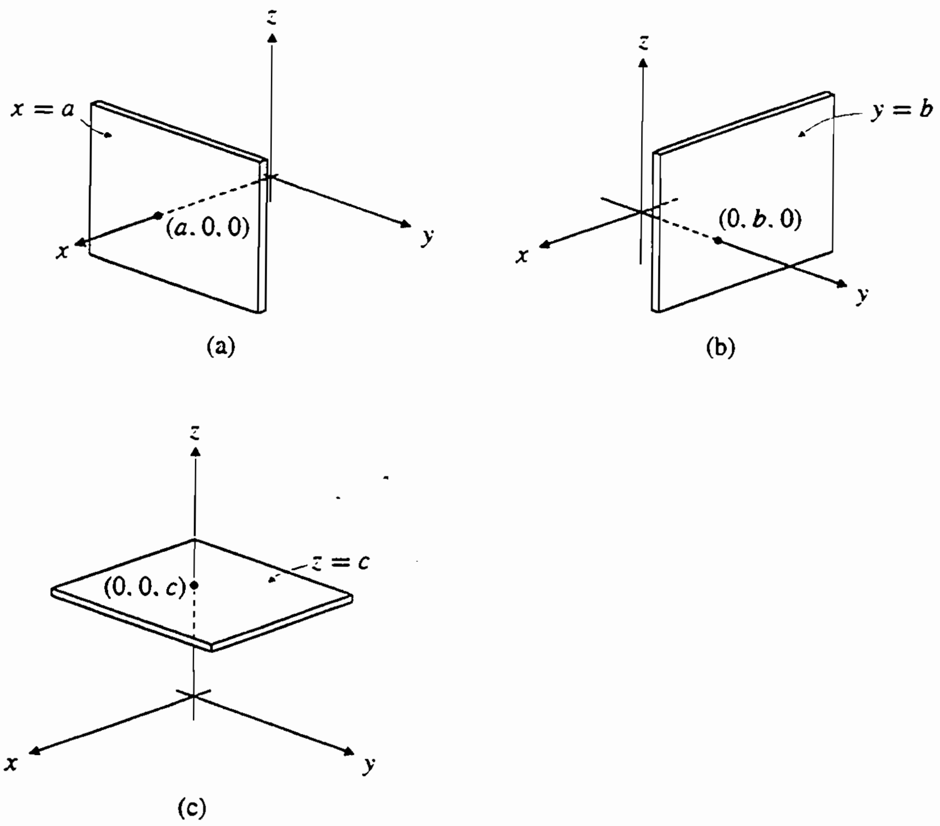


FIGURE 15.3

ment on y and z) lie in the plane indicated in Fig. 15.3(a); Figs. 15.3(b) and (c) show pieces of the two others.

Some more interesting examples of equations in three variables x, y, and z that represent surfaces in space are the following:

$$\text{(d)} \quad px + qy + rz = m, \qquad \text{(e)} \quad x^2 + y^2 + z^2 = 4$$

Equation (d) can be given an economic interpretation. Suppose a person spends an amount m on the purchase of three commodities, whose prices are respectively p, q, and r per unit. If the person buys x units of the first, y units of the second, and z units of the third commodity, then the total cost is $px + qy + rz$. Hence, (d) is the individual's *budget equation*: Only triples (x, y, z) that satisfy (d) can be bought if expenditure is m. As explained in Section 12.5, Equation (d) represents a *plane* in space, the **budget plane**. Because in most cases one also has $x \geq 0$, $y \geq 0$, and $z \geq 0$, the interesting part of the plane described by (d) is the triangle with vertices at $P = (m/p, 0, 0)$, $Q = (0, m/q, 0)$, and $R = (0, 0, m/r)$, as shown in Fig. 15.4.

Consider Equation (e) next. According to the discussion in Section 12.4 (see [12.16]), the expression $x^2 + y^2 + z^2 = (x - 0)^2 + (y - 0)^2 + (z - 0)^2$ is the square of the distance from the origin $(0, 0, 0)$ to the point (x, y, z). So the graph of (e) consists of those points (x, y, z) whose distance from the origin is 2. Thus, it

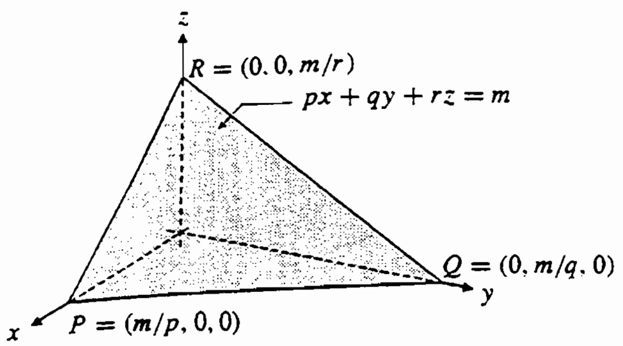


FIGURE 15.4

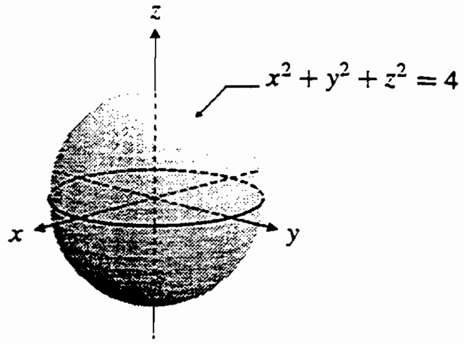


FIGURE 15.5

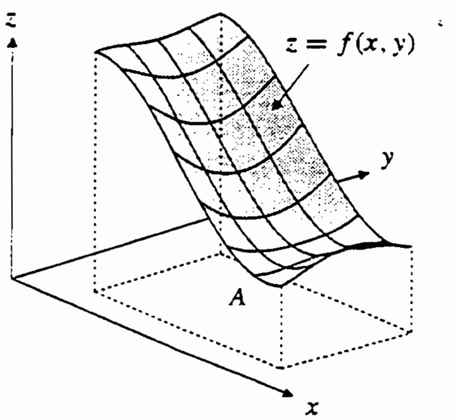
represents a sphere centered at (0, 0, 0) and with radius 2, as shown in Fig. 15.5. If (e) were replaced by the inequality $x^2 + y^2 + z^2 \leq 4$, it would represent a solid ball.

The Graph of a Function of Two Variables

Suppose that $z = f(x, y)$ represents a function of two variables defined in a set A in the xy-plane. By the graph of the function f, we understand the graph of the equation $z - f(x, y) = 0$. If f is a sufficiently "nice" function, the graph of f is a smooth surface in space, like the one shown in Fig. 15.6.

This method of representing a function of two variables helps us to visualize its behavior in broad outline. However, it requires considerable artistic ability to represent in only two dimensions the graph of $z = f(x, y)$ that lies in three-dimensional space. It is certainly difficult to use the resulting drawing for quantitative measurements. (By using modern computer graphics, however, complicated functions of two variables can be drawn fairly easily.) We now describe a second method that often does better.

FIGURE 15.6 The graph of $y = f(x, y)$.



Level Curves for $z = f(x, y)$

Map makers can describe some topographical features of the earth's surface such as hills and valleys even on a plane map. To do so, they draw a set of *level curves* or contours connecting points on the map that represent places on the earth's surface with the same elevation above sea level. For instance, there may be such contours corresponding to 100 meters above sea level, others for 200, 300, and 400 meters above sea level, and so on. Where the contours are close together, there is a steep slope. Studying the contour map gives a good idea of altitude variations on the ground.

The same idea can be used to give a geometric representation of an arbitrary function $z = f(x, y)$. The graph of the function in three-dimensional space is visualized as being cut by horizontal planes parallel to the xy-plane. The resulting intersections between the planes and the graph are then projected onto the xy-plane. If the intersecting plane is $z = c$, then the projection of the intersection onto the xy-plane is called the **level curve at height c for f.** This level curve will consist of points satisfying the equation

$$f(x, y) = c$$

Figure 15.7 illustrates such a level curve.

Example 15.9

Consider the function of two variables defined by the equation

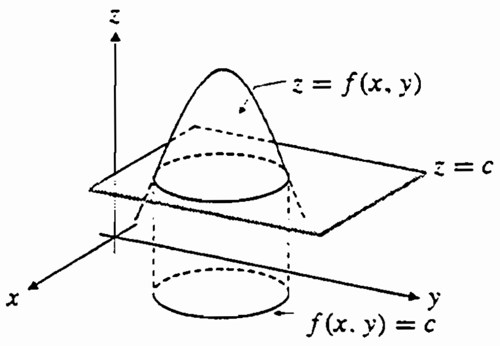
$$z = x^2 + y^2 \qquad [1]$$

What are the level curves? Draw both a set of level curves and the graph of the function.

Solution The variable z can only assume values ≥ 0. The level curves have the equation

$$x^2 + y^2 = c \qquad [2]$$

FIGURE 15.7 The graph of $z = f(x, y)$ and one of its level curves.



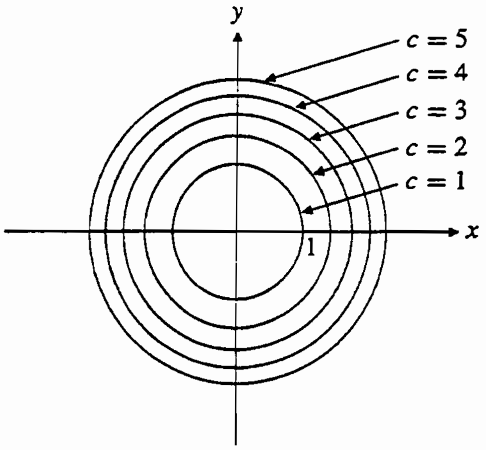


FIGURE 15.8

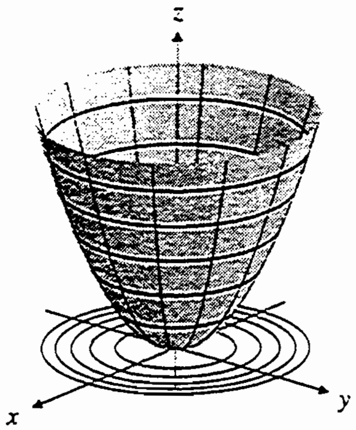


FIGURE 15.9

where $c \geq 0$. We see that these are circles in the xy-plane centered at the origin and with radius $\sqrt{c}$. See Fig. 15.8.

Concerning the graph of [1], all the level curves are circles. For $y = 0$, we have $z = x^2$. This shows that the graph of [1] cuts the xz-plane (where $y = 0$) in a parabola. Similarly, we see that for $x = 0$ one has $z = y^2$, which is the graph of a parabola in the yz-plane. It follows that the graph of [1] is obtained by rotating the parabola $z = x^2$ around the z-axis. The surface is called a **paraboloid** (of revolution), as shown in Fig. 15.9, which also shows the level curves in the xy-plane.

Example 15.10

Suppose $F(K, L)$ denotes the number of units produced by a firm when the input of capital is K and that of labor is L. A level curve for the function is a curve in the KL-plane given by

$$F(K, L) = Y_0 \qquad (Y_0 \text{ is a constant})$$

This curve is called an **isoquant** (indicating "equal quantity"). For a Cobb–Douglas function $F(K, L) = AK^aL^b$ with $a + b < 1$ and $A > 0$, Figs. 15.10 and 15.11 show a part of the graph and some of the isoquants. (Here it is convenient to view the surface from a perspective other than that used for most other figures in this section.)

Example 15.11

Show that all points (x, y) satisfying $xy = 3$ lie on a level curve for the function

$$g(x, y) = \frac{3(xy + 1)^2}{x^4y^4 - 1}$$

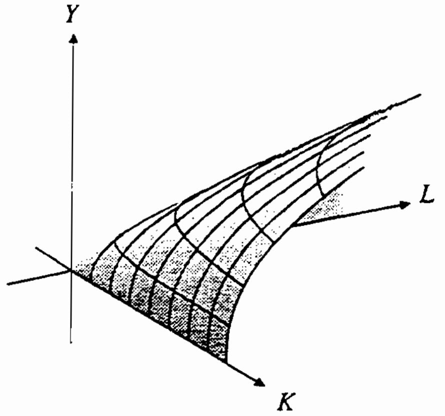


FIGURE 15.10

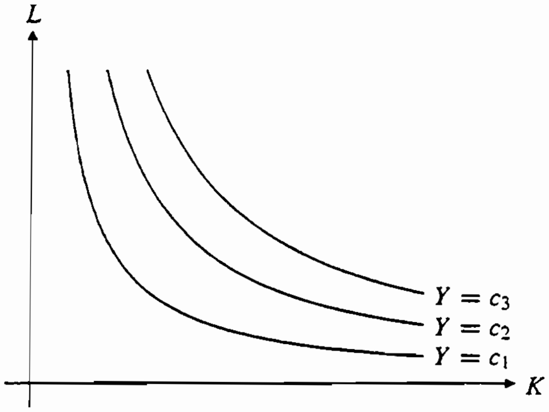


FIGURE 15.11

Solution By substituting $xy = 3$ in the expression for g, we find

$$g(x, y) = \frac{3(xy+1)^2}{(xy)^4 - 1} = \frac{3(3+1)^2}{3^4 - 1} = \frac{48}{80} = \frac{3}{5}$$

For all (x, y) where $xy = 3$, the value of $g(x, y)$ is a constant 3/5. This means that $xy = 3$ is on a level curve (at height 3/5) for g.

In fact, for any value of c other than -1 or 1, $xy = c$ is the equation of a level curve for g because $g(x, y) = 3(c+1)^2/(c^4 - 1)$ when $xy = c$.

Some Other Surfaces in Three-Dimensional Space

It is usually not at all simple to draw the graphs of equations in three variables. Yet, in recent years, a number of powerful computer programs for drawing surfaces in three-dimensional space have been developed. Two surfaces that can be drawn in this way appear in Figs. 15.12 and 15.13. (Figure 15.12 looks like a rugby football.)

FIGURE 15.12 $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1.$

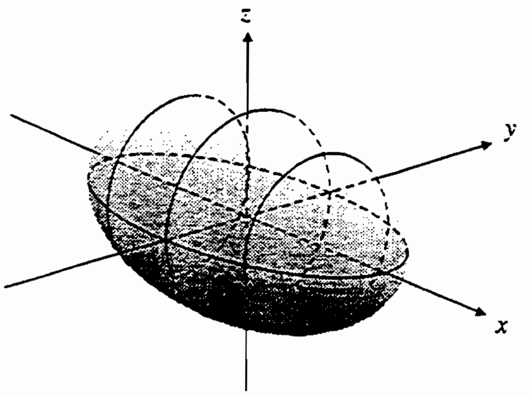
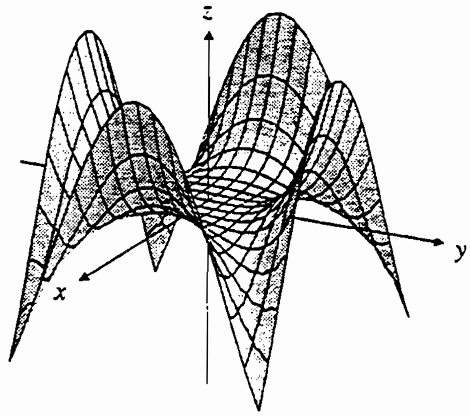


FIGURE 15.13 $z = x^4 - 3x^2y^2 + y^4.$



Functions of n Variables and the Euclidean n-Dimensional Space R^n

No concrete geometric interpretation is possible for functions of n variables in the general case when $n \geq 3$. Yet economists still use *geometric language* when dealing with functions of n variables, even though they may not think of themselves as doing geometry. It is usual to call the set of all possible n-tuples $(x_1, x_2, \ldots, x_n)$ of real numbers the **Euclidean n-dimensional space**, or **n-space**, and to denote it by R^n. For $n = 1$, 2, and 3, we have geometric interpretations of R^n as a line, a plane, and a 3-dimensional space, respectively. But for $n \geq 4$, there is no geometric interpretation.

If $z = f(x_1, x_2, \ldots, x_n)$ represents a function of n variables, we let the **graph** of f be the set of all points $\big(x_1, x_2, \ldots, x_n, f(x_1, x_2, \ldots, x_n)\big)$ in R^{n+1} for which $(x_1, x_2, \ldots, x_n)$ belongs to the domain of f. We also call the graph a **surface** (or sometimes a **hypersurface**) in R^{n+1}. For $z = z_0$ (constant), the set of points in R^n satisfying $f(x_1, x_2, \ldots, x_n) = z_0$ is called a **level surface** of f.

In production theory, it is usual to give level surfaces a different name. If $x = f(v_1, v_2, \ldots, v_n)$ is the amount produced when the input quantities of n different factors of production are respectively v_1, v_2, ..., v_n, the level surfaces where $f(v_1, v_2, \ldots, v_n) = x_0$ (constant) are called *isoquants*, as in Example 15.10.

Continuity

The concept of continuity for functions of one variable may be generalized to functions of several variables. Roughly speaking, a function of n variables is continuous if small changes in the independent variables give small changes in the function value. Just as in the one-variable case, we have the following useful rule:

> Any function of n variables that can be constructed from continuous functions by combining the operations of addition, subtraction, multiplication, division, and functional composition is continuous wherever it is defined.

If a function of one variable is continuous, it will also be continuous when considered as a function of several variables. For example, $f(x, y, z) = x^2$ is a continuous function of x, y, and z. (Small changes in x, y, and z give at most small changes in x^2.)

Example 15.12

Where are the functions given by the following formulas continuous?

(a) $f(x, y, z) = x^2y + 8x^2y^5z - xy + 8z$

(b) $g(x, y) = \dfrac{xy - 3}{x^2 + y^2 - 4}$

Solution

(a) As the sum of products of powers, f is defined and continuous for all x, y, and z.

(b) The function g is defined and continuous for all (x, y) except those that lie on the circle $x^2 + y^2 = 4$. There the denominator is zero, and so $g(x, y)$ is not defined.

Problems

1. Draw the graphs of the following functions in three-dimensional space, and draw a set of level curves for each of them:

   a. $z = 3 - x - y$ **b.** $z = \sqrt{3 - x^2 - y^2}$

2. Show that $x^2 + y^2 = 6$ is a level curve of $f(x, y) = \sqrt{x^2 + y^2} - x^2 - y^2 + 2$, and that all the level curves of f must be circles centered at the origin.

3. Show that $x^2 - y^2 = c$, for all values of the constant c lies on a level curve for $f(x, y) = e^{x^2}e^{-y^2} + x^4 - 2x^2y^2 + y^4$.

4. Let $f(x)$ represent a function of one variable. If we let $g(x, y) = f(x)$, then we have defined a function of two variables, but y is not present in its formula. Explain how the graph of g is obtained from the graph of f. Illustrate with $f(x) = -x^3$.

5. Explain why two level curves of $z = f(x, y)$ corresponding to different function values of z cannot intersect.

15.3 Partial Derivatives with Two Variables

When we study a function $y = f(x)$ of one variable, the derivative $f'(x)$ measures the function's rate of change as x changes. For functions of two variables, such as $z = f(x, y)$, we also want to examine how quickly the value of the function changes with respect to changes in the values of the independent variables. For example, if $f(x, y)$ is a firm's profit when it uses quantities x and y of two different inputs, we want to know whether and by how much profits increase as x and y are varied.

Consider the function

$$z = x^3 + 2y^2 \tag{1}$$

Suppose, first, that y is held constant. Then $2y^2$ is constant, and the rate of change of z with respect to x is given by

$$\frac{dz}{dx} = 3x^2$$

On the other hand, we can keep x fixed in [1] and examine how z varies as y varies. This involves taking the derivative of z with respect to y while keeping x constant. The result is

$$\frac{dz}{dy} = 4y$$

Of course, there are other variations we could study. For example, x and y could vary simultaneously. But in this section, we restrict our attention to variations in *either x or y*.

When we consider functions of two or more variables, we shall write $\partial z/\partial x$ instead of dz/dx for the derivative of z with respect to x. In the same way, we write $\partial z/\partial y$ instead of dz/dy. Hence, we have

$$z = x^3 + 2y^2 \Longrightarrow \frac{\partial z}{\partial x} = 3x^2 \qquad \text{and} \qquad \frac{\partial z}{\partial y} = 4y$$

In general, we introduce the following definitions:

> Suppose $z = f(x, y)$. Let $\partial z/\partial x$, called the **partial derivative of z or f with respect to x**, be the derivative of $f(x, y)$ with respect to x when y is held constant. Also, let $\partial z/\partial y$, called the **partial derivative of z or f with respect to y**, be the derivative of $f(x, y)$ with respect to y when x is held constant. [15.7]

When $z = f(x, y)$, we also denote the derivative $\partial z/\partial x$ by $\partial f/\partial x$. In the same way, $\partial z/\partial y = \partial f/\partial y$. Note that $\partial f/\partial x$ is the rate of change of $f(x, y)$ with respect to x, when y is constant, and correspondingly for $\partial f/\partial y$.

It is usually easy to find the partial derivatives of a function $z = f(x, y)$. To compute $\partial f/\partial x$, just think of y as a constant and differentiate $f(x, y)$ with respect to x as if f were a function only of x. All the rules for finding derivatives of functions of one variable can be used when we want to compute $\partial f/\partial x$. The same is true for $\partial f/\partial y$. Let us look at some further examples.

Example 15.13

Compute the partial derivatives of the following:

(a) $f(x, y) = x^3y + x^2y^2 + x + y^2$

(b) $f(x, y) = xy/(x^2 + y^2)$

Solution

(a) We find

$$\frac{\partial f}{\partial x} = 3x^2y + 2xy^2 + 1 \qquad \text{(holding } y \text{ constant)}$$
$$\frac{\partial f}{\partial y} = x^3 + 2x^2y + 2y \qquad \text{(holding } x \text{ constant)}$$

(b) For this function, the quotient gives

$$\frac{\partial f}{\partial x} = \frac{y(x^2+y^2) - 2xxy}{(x^2+y^2)^2} = \frac{y^3 - x^2y}{(x^2+y^2)^2}, \qquad \frac{\partial f}{\partial y} = \frac{x^3 - y^2x}{(x^2+y^2)^2}$$

Observe that the function in (b) is symmetric in x and y, in the sense that the function value is unchanged if we interchange x and y. By interchanging x and y in the formula for $\partial f/\partial x$, therefore, we will find the correct formula for $\partial f/\partial y$. (Compute $\partial f/\partial y$ in the usual way and check that the foregoing answer is correct.)

Other forms of notation are often used to indicate the partial derivatives of $z = f(x, y)$. Some of the most common are

$$\frac{\partial f}{\partial x} = \frac{\partial z}{\partial x} = z'_x = f'_x(x, y) = f'_1(x, y) = \frac{\partial f(x, y)}{\partial x}$$
$$\frac{\partial f}{\partial y} = \frac{\partial z}{\partial y} = z'_y = f'_y(x, y) = f'_2(x, y) = \frac{\partial f(x, y)}{\partial y}$$

Among these, $f'_1(x, y)$ and $f'_2(x, y)$ are the most satisfactory. Here the numerical subscripts refer to positions of the argument in the function. Thus, f'_1 indicates the partial derivative w.r.t. the first variable, and f'_2 w.r.t. the second variable. We are also reminded that the partial derivatives themselves are functions of x and y. Finally, $f'_1(a, b)$ and $f'_2(a, b)$ are suitable designations of the values of the partial derivatives at point (a, b) instead of at (x, y). For example, for the function in Example 15.13(a),

$$f(x, y) = x^3y + x^2y^2 + x + y^2 \Longrightarrow f'_1(x, y) = 3x^2y + 2xy^2 + 1$$

Hence, $f'_1(0, 0) = 1$ and $f'_1(-1, 2) = 3(-1)^2 2 + 2(-1)2^2 + 1 = -1$.

The notations $f'_x(x, y)$ and $f'_y(x, y)$ are often used, but especially in connection with composite functions, these notations are sometimes too ambiguous. For instance, what is the meaning of $f'_x(x^2y, x - y)$?

Higher-Order Partial Derivatives

If $z = f(x, y)$, then $\partial f/\partial x$ and $\partial f/\partial y$ are called **first-order partial derivatives**. These partial derivatives are themselves functions of two variables. From $\partial f/\partial x$, we can generate two new functions by taking the partial derivatives with respect to x and y. In the same way, we can take the partial derivatives of $\partial f/\partial y$ with respect to x and y. The four functions we obtain in this way are called **second-order partial derivatives** of $f(x, y)$. They are expressed as

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}, \qquad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}, \qquad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$$

For brevity, we sometimes refer to the first- and second-order "partials," suppressing the word "derivatives."

Example 15.14

For the function in Example 15.13(a), we obtain

$$\frac{\partial^2 f}{\partial x^2} = 6xy + 2y^2, \qquad \frac{\partial^2 f}{\partial y \partial x} = 3x^2 + 4xy = \frac{\partial^2 f}{\partial x \partial y}, \qquad \frac{\partial^2 f}{\partial y^2} = 2x^2 + 2$$

Several other kinds of notation are also frequently used for the second-order partial derivatives. For example, $\partial^2 f/\partial x^2$ is also denoted by $f''_{11}(x, y)$ or $f''_{xx}(x, y)$. In the same way, $\partial^2 f/\partial y \partial x$ can also be written as $f''_{12}(x, y)$ or $f''_{xy}(x, y)$. Note that $f''_{12}(x, y)$ means that we differentiate $f(x, y)$ first with respect to the first argument x and then with respect to the second argument y. To find $f''_{21}(x, y)$, we must differentiate in the reverse order. In Example 15.14, these two "mixed" second-order partial derivatives (or "cross-partials") are equal. For most functions $z = f(x, y)$ used in practical applications, it will actually be the case that

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} \tag{15.8}$$

Sufficient conditions for the equality in [15.8] are given in Theorem 15.1 of Section 15.5.

It is very important to note the exact meaning of the different symbols that have been introduced. If we consider [15.8], for example, it would be a serious mistake to believe that the two expressions are equal because $\partial x \partial y$ is the same as $\partial y \partial x$. Here the expression on the left-hand side is in fact the derivative of $\partial f/\partial y$ with respect to x, and the right-hand side is the derivative of $\partial f/\partial x$ with respect to y. It is a remarkable fact, and not a triviality, that the two are usually equal. As another example, we observe that $\partial^2 z/\partial x^2$ is quite different from $(\partial z/\partial x)^2$. For example, if $z = x^2 + y^2$, then $\partial z/\partial x = 2x$. Therefore, $\partial^2 z/\partial x^2 = 2$, whereas $(\partial z/\partial x)^2 = 4x^2$.

Analogously, we define partial derivatives of the third, fourth, and higher orders. For example, we obtain $\partial^4 z/\partial x \partial y^3 = z''''_{yyyx}$ when we first differentiate z three times with respect to y and then differentiate the result once more with respect to x.

Here is an additional example.

Example 15.15

If $f(x, y) = x^3 e^{y^2}$, find the first- and second-order partial derivatives at $(x, y) = (1, 0)$.

Solution To find $f_1'(x, y)$, we differentiate $x^3 e^{y^2}$ with respect to x while treating y as a constant. When y is a constant, so is e^{y^2}. Hence,

$$f_1'(x, y) = 3x^2 e^{y^2} \qquad \text{and so} \qquad f_1'(1, 0) = 3 \cdot 1^2 e^{0^2} = 3$$

To find $f_2'(x, y)$, we differentiate $f(x, y)$ with respect to y while treating x as a constant:

$$f_2'(x, y) = x^3 2y e^{y^2} = 2x^3 y e^{y^2} \qquad \text{and so} \qquad f_2'(1, 0) = 0$$

To find the second-order partial $f''_{11}(x, y)$, we must differentiate $f_1'(x, y)$ with respect to x once more, while treating y as a constant:

$$f''_{11}(x, y) = 6x e^{y^2} \qquad \text{and so} \qquad f''_{11}(1, 0) = 6 \cdot 1 e^{0^2} = 6$$

To find $f''_{22}(x, y)$, we must differentiate $f_2'(x, y) = 2x^3 y e^{y^2}$ with respect to y once more, while treating x as a constant. Because $y e^{y^2}$ is a product of two functions, each involving y, we use the product rule to obtain

$$f''_{22}(x, y) = (2x^3)(1 \cdot e^{y^2} + y 2y e^{y^2}) = 2x^3 e^{y^2} + 4x^3 y^2 e^{y^2}$$

Evaluating this at $(1, 0)$ gives $f''_{22}(1, 0) = 2$. Moreover,

$$f''_{12}(x, y) = \frac{\partial}{\partial y}\left[f_1'(x, y)\right] = \frac{\partial}{\partial y}(3x^2 e^{y^2}) = 3x^2 2y e^{y^2} = 6x^2 y e^{y^2}$$

and

$$f''_{21}(x, y) = \frac{\partial}{\partial x}\left[f_2'(x, y)\right] = \frac{\partial}{\partial x}(2x^3 y e^{y^2}) = 6x^2 y e^{y^2}$$

Hence, $f''_{12}(1, 0) = f''_{21}(1, 0) = 0$.

Approximations to Partial Derivatives

Recall how, when x is a single variable, we can often get a good approximation to $f'(x)$ by computing $f(x+1) - f(x)$ (see Example 4.5 in Section 4.3). Because $f'_x(x, y)$ is simply the derivative of $f(x, y)$ with respect to x when y is held constant, we obtain the corresponding approximation

$$f'_x(x, y) \approx f(x+1, y) - f(x, y)$$

In words:

> The partial derivative $f'_x(x, y)$ is approximately equal to the change in $f(x, y)$ that results from increasing x by one unit while holding y constant. [15.9]

> The partial derivative $f'_y(x, y)$ is approximately equal to the change in $f(x, y)$ that results from increasing y by one unit while holding x constant. [15.10]

The number $f'_x(x, y)$ measures the rate of change of f with respect to x. If $f'_x(x, y) > 0$, then a small increase in x will lead to an increase in $f(x, y)$. When the approximation in [15.9] is permissible, we can say that $f'_x(x, y) > 0$ means that a unit increase in x will lead an increase in $f(x, y)$. Similarly, $f'_x(x, y) < 0$ means that a unit increase in x will lead to a decrease in $f(x, y)$.
Note: The approximations in [15.9] and [15.10] must be used with caution. Roughly speaking, they will not be too inaccurate provided that the partial derivatives do not vary too much over the actual intervals.

Example 15.16

In Example 15.2, we studied the function $x = Ap^{-1.5}r^{2.08}$. Compute the partial derivatives of x with respect to p and r, and discuss their signs.

Solution We find

$$\frac{\partial x}{\partial p} = -1.5Ap^{-2.5}r^{2.08}, \qquad \frac{\partial x}{\partial r} = 2.08Ap^{-1.5}r^{1.08}$$

Because A, p, and r are positive, $\partial x/\partial p < 0$ and $\partial x/\partial r > 0$. These signs accord with the final remarks in Example 15.2.

Problems

1. Find $\partial z/\partial x$ and $\partial z/\partial y$ for the following:

   a. $z = x^2 + 3y^2$ **b.** $z = xy$ **c.** $z = 5x^4y^2 - 2xy^5$ **d.** $z = e^{x+y}$

e. $z = e^{xy}$ f. $z = e^x/y$ g. $z = \ln(x+y)$ h. $z = \ln(xy)$

2. Find $f_1'(x, y)$, $f_2'(x, y)$, and $f_{12}''(x, y)$ for the following:

a. $f(x, y) = x^7 - y^7$ b. $f(x, y) = x^5 \ln y$ c. $f(x, y) = (x^2 - 2y^2)^5$

3. Find all first- and second-order partial derivatives of the following:

a. $z = 3x + 4y$ b. $z = x^3y^2$ c. $z = x^5 - 3x^2y + y^6$

d. $z = x/y$ e. $z = (x-y)/(x+y)$ f. $z = \sqrt{x^2+y^2}$

4. Let $F(S, E) = 2.26\, S^{0.44} E^{0.48}$ (see Example 15.3).

a. Compute $F_S'(S, E)$ and $F_E'(S, E)$.

b. Show that $SF_S' + EF_E' = kF$ for a suitable constant k.

5. Prove that if $z = (ax + by)^2$, then $xz_x' + yz_y' = 2z$.

6. Find all the first- and second-order partial derivatives of the following:

a. $z = x^2 + e^{2y}$ b. $z = y \ln x$ c. $z = xy^2 - e^{xy}$

7. Let $f(x, y) = x \ln y - y^2 2^{xy}$. Find all the first- and second-order partial derivatives at $(x, y) = (1, 1)$.

8. Let $z = \dfrac{1}{2}\ln(x^2 + y^2)$. Show that $\partial^2 z/\partial x^2 + \partial^2 z/\partial y^2 = 0$.

Harder Problems

9. Compute $\partial^{p+q} z/\partial y^q \partial x^p$ at $(0, 0)$ for the following:

a. $z = e^x \ln(1+y)$ b. $z = e^{x+y}(xy + y - 1)$

10. Prove that if $u = Ax^a y^b$, then

$$\frac{1}{u_x'}\frac{\partial}{\partial x}\left(\frac{u_{xy}''}{u_x' u_y'}\right) = \frac{1}{u_y'}\frac{\partial}{\partial y}\left(\frac{u_{xy}''}{u_x' u_y'}\right)$$

15.4 Partial Derivatives and Tangent Planes

Partial derivatives of the first order have an interesting geometric interpretation. Let $z = f(x, y)$ be a function of two variables, with graph as shown in Fig. 15.14. Let us keep the value of y fixed at y_0. The points (x, y) on the graph of f that have $y = y_0$ are those that lie on the curve K_y indicated in the figure. The partial derivative $f_x'(x_0, y_0)$ is the derivative of $z = f(x, y_0)$ with respect to x at the point $x = x_0$, and is therefore the slope of the tangent line l_y to the curve K_y at $x = x_0$. In the same way, $f_y'(x_0, y_0)$ is the slope of the tangent line l_x to the curve K_x at $y = y_0$.

This geometric interpretation of the two partial derivatives can be explained in another way. Imagine that the graph of f describes a mountain, and suppose that we are standing at point P with coordinates $(x_0, y_0, f(x_0, y_0))$ in three

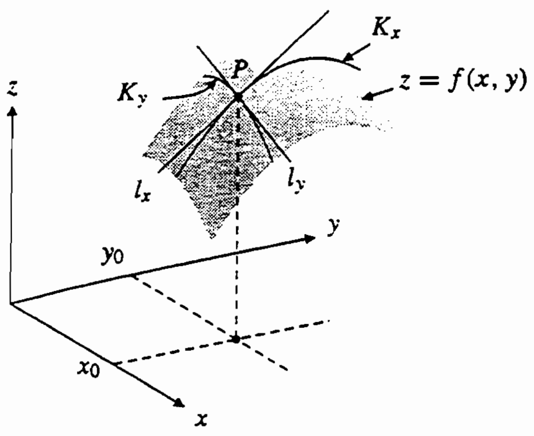


FIGURE 15.14

dimensions, where the height is $f(x_0, y_0)$ units above the xy-plane. The slope of the terrain at P depends on the direction in which we look. In particular, let us look in the direction parallel to the positive x-axis. Then $f'_x(x_0, y_0)$ is a measure of the "steepness" in this direction. In the figure, $f'_x(x_0, y_0)$ is negative, because moving from P in the direction given by the positive x-axis will take us downwards. In the same way, we see that $f'_y(x_0, y_0)$ is a measure of the "steepness" in the direction parallel to the positive y-axis. We also see that $f'_y(x_0, y_0)$ is positive, meaning that the slope is upward in this direction.

Let us now briefly consider the geometric interpretation of the "direct" second-order derivatives f''_{xx} and f''_{yy}. Consider the curve K_y on the graph of f in the figure. It seems that along this curve, $f''_{xx}(x, y_0)$ is negative, because $f'_x(x, y_0)$ decreases as x increases. In particular, $f''_{xx}(x_0, y_0) < 0$. In the same way, we see that moving along K_x makes $f'_y(x_0, y)$ decrease as y increases, so $f''_{yy}(x_0, y) < 0$ along K_x. In particular, $f''_{yy}(x_0, y_0) < 0$.

Example 15.17

Consider Fig. 15.15, showing some level curves of a function $z = f(x, y)$. On the basis of this figure, answer the following questions:

(a) What are the signs of $f'_x(x, y)$ and $f'_y(x, y)$ at P and Q?

(b) What are the solutions of the two equations: (i) $f(3, y) = 4$ and (ii) $f(x, 4) = 6$?

(c) What is the largest value that $f(x, y)$ can attain when $x = 2$, and for which y value does this maximum occur?

Solution

(a) If you stand at P, you are on the level curve $f(x, y) = 2$. If you look in the direction of the positive x-axis (along the line $y = 4$), then you will see the terrain sloping upwards, because the (nearest) level curves

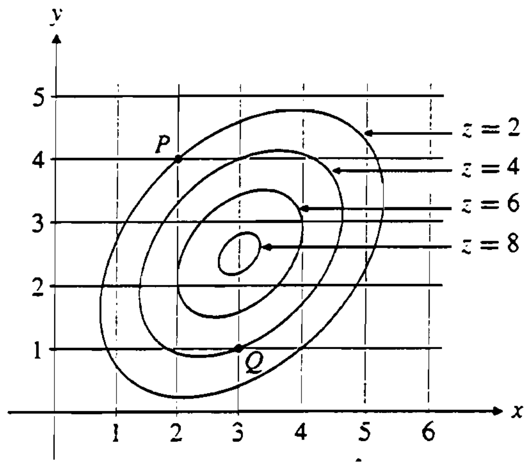


FIGURE 15.15

correspond to larger z values. Hence, $f'_x > 0$. If you stand at P and look in the direction of the positive y-axis (along $x = 2$), the terrain will slope downwards. Thus, at P, we must have $f'_y < 0$. At Q, we find similarly that $f'_x < 0$ and $f'_y > 0$.

(b) Equation (i) has the solutions $y = 1$ and $y = 4$, because the line $x = 3$ cuts the level curve $f(x, y) = 4$ at $(3, 1)$ and at $(3, 4)$. Equation (ii) has no solutions, because the line $y = 4$ does not meet the level curve $f(x, y) = 6$ at all.

(c) The highest value of c for which the level curve $f(x, y) = c$ has a point in common with the line $x = 2$ is $c = 6$. The largest value of $f(x, y)$ when $x = 2$ is therefore 6, and we see that this maximum value is attained when $y \approx 2.2$.

Tangent Planes

Look back at Fig. 15.14. The two tangent lines l_x and l_y determine a unique plane through the point $P = (x_0, y_0, f(x_0, y_0))$. This plane is called the *tangent plane* to the surface at P. From [12.23] in Section 12.5, the general equation for a plane in three-dimensional space passing through a point (x_0, y_0, z_0) is $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$. If $c = 0$, then this plane is parallel to the z-axis. If $c \neq 0$ and we define $A = -a/c$, $B = -b/c$, then solving the equation for $z - z_0$ gives

$$z - z_0 = A(x - x_0) + B(y - y_0) \tag{1}$$

So the tangent plane to the surface at P must have this form. It remains to determine A and B. Now, line l_y lies in the plane. Because the slope of the line is $f'_1(x_0, y_0)$, the points (x, y, z) that lie on l_y are characterized by the two equations $y = y_0$ and $z - z_0 = f'_1(x_0, y_0)(x - x_0)$. All these points (x, y, z) lie in the plane

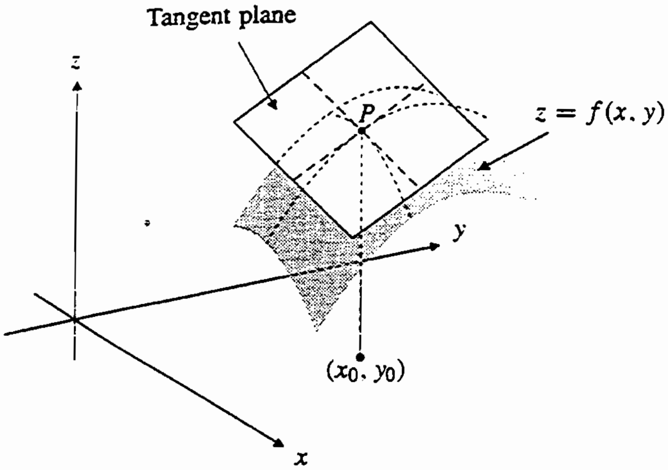


FIGURE 15.16 The graph of a function $z = f(x, y)$ and its tangent plane at P.

[1] only if $A = f_1'(x_0, y_0)$. In a similar way, we see that $B = f_2'(x_0, y_0)$. The conclusion is as follows:

> The **tangent plane** to $z = f(x, y)$ at the point (x_0, y_0, z_0), with $z_0 = f(x_0, y_0)$, has the equation
>
> $$z - z_0 = f_1'(x_0, y_0)(x - x_0) + f_2'(x_0, y_0)(y - y_0) \qquad [15.11]$$

The tangent plane is illustrated in Fig. 15.16.

Example 15.18

Find the tangent plane at $(x_0, y_0, z_0) = (1, 1, 5)$ to the graph of

$$f(x, y) = x^2 + 2xy + 2y^2$$

Solution Because $f(1, 1) = 5$, the given point lies on the graph of f. We find that

$$f_1'(x, y) = 2x + 2y, \qquad f_2'(x, y) = 2x + 4y$$

Hence, $f_1'(1, 1) = 4$ and $f_2'(1, 1) = 6$. Thus, [15.11] yields

$$z - 5 = 4(x - 1) + 6(y - 1) \qquad \text{or} \qquad z = 4x + 6y - 5$$

Problems

1. In Fig. 15.17, we have drawn some level curves for a function $z = f(x, y)$, together with the line $2x + 3y = 12$.

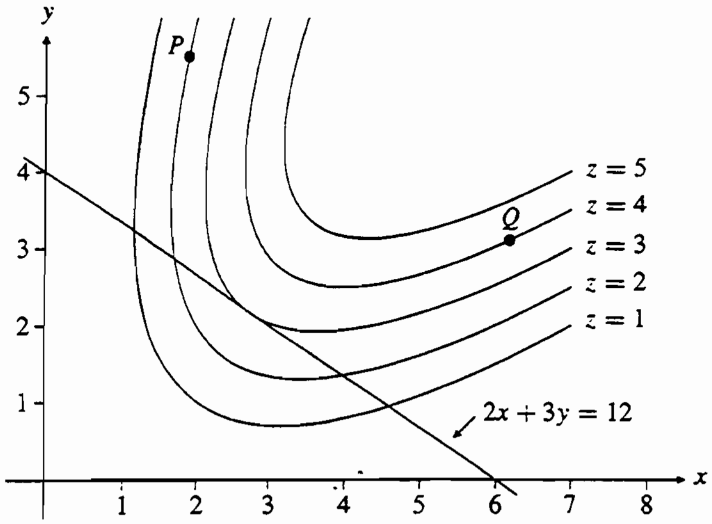


FIGURE 15.17

a. What are the signs of f'_x and f'_y at P and Q?

b. Find possible solutions of the two equations (i) $f(1, y) = 2$ and (ii) $f(x, 2) = 2$.

c. What is the largest value of $f(x, y)$ among those (x, y) that satisfy $2x + 3y = 12$?

2. Suppose $F(x, y)$ is a function about which all we know is that $F(0, 0) = 0$, $F'_1(x, y) \geq 2$ for all (x, y), and $F'_2(x, y) \leq 1$ for all (x, y). What can be said about the relative sizes of $F(0, 0)$, $F(1, 0)$, $F(2, 0)$, $F(0, 1)$, and $F(1, 1)$? Write down the inequalities that have to hold between these numbers.

3. Find the tangent planes to the following surfaces at the indicated points:

   a. $z = x^2 + y^2$ at $(1, 2, 5)$ b. $z = (y - x^2)(y - 2x^2)$ at $(1, 3, 2)$

4. Prove that all tangent planes to $z = xf(y/x)$ pass through the origin.

15.5 Partial Derivatives with Many Variables

The functions economists study usually have many variables, so we need to extend the concept of partial derivatives to such functions.

> If $z = f(x_1, x_2, \ldots, x_n)$, then $\partial f/\partial x_i$ is the derivative of $f(x_1, x_2, \ldots, x_n)$ with respect to x_i when all the other variables x_j $(j \neq i)$ are held constant. [15.12]

These n partial derivatives are of the first order. Other notation used for the first-order partials of $z = f(x_1, x_2, \ldots, x_n)$ includes

$$\frac{\partial f}{\partial x_i} = \frac{\partial z}{\partial x_i} = \partial z/\partial x_i = z'_i = f'_i(x_1, x_2, \ldots, x_n)$$

As in [15.9] and [15.10] in Section 15.3, we have the following rough approximation:

> The partial derivative $\partial z/\partial x_i$ is approximately equal to the change in $z = f(x_1, x_2, \dots, x_n)$ caused by an increase in x_i of one unit, while all the other x_j $(j \neq i)$ are held constant. [15.13]

In symbols:

$$f_i'(x_1, \dots, x_n)$$
$$\approx f(x_1, \dots, x_{i-1}, x_i + 1, x_{i+1}, \dots, x_n) - f(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_n)$$

For each of the n first-order partials of f, we have n second-order partials:

$$\frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right) = \frac{\partial^2 f}{\partial x_j \partial x_i} = z''_{ij}$$

Here both i and j may take any value 1, 2, ..., n, so there are altogether n^2 second-order partial derivatives. The $n \times n$ matrix of second-order partials

$$\begin{pmatrix} f''_{11}(\mathbf{x}) & f''_{12}(\mathbf{x}) & \dots & f''_{1n}(\mathbf{x}) \\ f''_{21}(\mathbf{x}) & f''_{22}(\mathbf{x}) & \dots & f''_{2n}(\mathbf{x}) \\ \vdots & \vdots & \ddots & \vdots \\ f''_{n1}(\mathbf{x}) & f''_{n2}(\mathbf{x}) & \dots & f''_{nn}(\mathbf{x}) \end{pmatrix} \qquad [15.14]$$

is the **Hessian** (or **Hessian matrix**) of f evaluated at $\mathbf{x} = (x_1, x_2, \dots, x_n)$. Because usually $f''_{ij}(\mathbf{x}) = f''_{ji}(\mathbf{x})$ for all i and j, the number of different partials is reduced from n^2 to at most $\frac{1}{2}n(n+1)$, and the Hessian is symmetric. (See Theorem 15.1, which follows.)

Example 15.19

Find the first-order partials with respect to A, B, and T for the function $a(A, B, T) = 122 + 3A - 25T - 75B^2 - A/B$. Also find the Hessian of $a(A, B, T)$.

Solution

$$\partial a/\partial A = 3 - 1/B, \qquad \partial a/\partial B = -150B + A/B^2, \qquad \partial a/\partial T = -25$$

and the Hessian is

$$\begin{pmatrix} \dfrac{\partial^2 a}{\partial A^2} & \dfrac{\partial^2 a}{\partial A \partial B} & \dfrac{\partial^2 a}{\partial A \partial T} \\ \dfrac{\partial^2 a}{\partial B \partial A} & \dfrac{\partial^2 a}{\partial B^2} & \dfrac{\partial^2 a}{\partial B \partial T} \\ \dfrac{\partial^2 a}{\partial T \partial A} & \dfrac{\partial^2 a}{\partial T \partial B} & \dfrac{\partial^2 a}{\partial T^2} \end{pmatrix} = \begin{pmatrix} 0 & \dfrac{1}{B^2} & 0 \\ \dfrac{1}{B^2} & -150 - 2\dfrac{A}{B^3} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Young's Theorem

We mentioned earlier that if $z = f(x_1, x_2, \ldots, x_n)$, then z''_{ij} and z''_{ji} are usually equal; this implies that the order of differentiation does not matter. The next theorem makes precise a more general result.

> **Theorem 15.1 (Young's Theorem)** Suppose that two mth-order partial derivatives of the function $f(x_1, x_2, \ldots, x_n)$ involve the same number of differentiations with respect to each of the variables, and are both continuous in an open set S. Then the two partial derivatives are necessarily equal at all points in S.

The content of this result can be explained as follows: Let $m = m_1 + \cdots + m_n$, and suppose that $f(x_1, x_2, \ldots, x_n)$ is differentiated m_1 times with respect to x_1, m_2 times with respect to x_2, ..., and m_n times with respect to x_n. Suppose that the continuity condition is satisfied for these mth-order partial derivatives. Then we end up with the same result no matter what is the order of differentiation, because each of the final partial derivatives is equal to

$$\frac{\partial^m f}{\partial x_1^{m_1} \partial x_2^{m_2} \cdots \partial x_n^{m_n}}$$

In particular, for the case when $m = 2$,

$$\frac{\partial^2 f}{\partial x_j \partial x_i} = \frac{\partial^2 f}{\partial x_i \partial x_j} \qquad (i = 1, 2, \ldots, n, \quad j = 1, 2, \ldots, n)$$

if both these partials are continuous. An example where this equality is not satisfied is presented in Problem 6. (A proof of Young's theorem is given in most advanced calculus books.)

Formal Definitions of Partial Derivatives

So far in this chapter, the functions have been given by explicit formulas and we have found the partial derivatives by using the ordinary rules for differentiation. If these rules of differentiation cannot be used, however, we must resort directly to the formal definition of partial derivatives. This corresponds closely to the definition for ordinary derivatives of functions of one variable, because partial derivatives are merely ordinary derivatives that are obtained by keeping all but one of the variables constant.

If $z = f(x_1, \dots, x_n)$, then with $g(x_i) = f(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_n)$, we have $\partial z/\partial x_i = g'(x_i)$. (Here we think of all the variables x_j other than x_i as constants.) If we use the definition of $g'(x_i)$ (see [4.3] of Section 4.2), we obtain

$$\frac{\partial z}{\partial x_i} = \lim_{h \to 0} \frac{f(x_1, \dots, x_i + h, \dots, x_n) - f(x_1, \dots, x_i, \dots, x_n)}{h} \quad [15.15]$$

(If we consider $h = 1$ as a number close to 0, we obtain the approximation in [15.13].) If the limit in [15.15] does not exist, then we say that $\partial z/\partial x_i$ *does not exist*, or that z is not differentiable with respect to x_i at the point. For instance, if a function describes the height of a pyramid, the partial derivatives will not exist at the point corresponding to the top of the pyramid.

Almost all the functions we consider will have continuous partial derivatives everywhere in their domains. If $z = f(x_1, x_2, \dots, x_n)$ has continuous partial derivatives of the first order in a domain A, we call f **continuously differentiable** in A.[3] In this case, f is called a C^1 **function** on A. If all partial derivatives up to order k exist and are continuous, f is called a C^k **function**.

Problems

1. Calculate all first-order partials of the following functions:
   a. $f(x, y, z) = x^2 + y^3 + z^4$
   b. $f(x, y, z) = 5x^2 - 3y^3 + 3z^4$
   c. $f(x, y, z) = xyz$
   d. $f(x, y, z) = x^4/yz$
   e. $f(x, y, z) = (x^2 + y^3 + z^4)^6$
   ·f. $f(x, y, z) = e^{xyz}$
2. For $F(x, y, z) = x^2e^{xz} + y^3e^{xy}$ calculate $F_1'(1, 1, 1)$, $F_2'(1, 1, 1)$, and $F_3'(1, 1, 1)$.
3. Let x and y be the populations of two cities and d the distance between them. Suppose that the number of travelers T between the cities is given by

$$T = k\frac{xy}{d^n} \qquad (k \text{ and } n \text{ are positive constants})$$

Compute $\partial T/\partial x$, $\partial T/\partial y$, and $\partial T/\partial d$, and discuss their signs.

[3]This seems appropriate, even though it is not quite standard mathematical terminology.

4. Find all the first- and second-order partials of the function $w(x, y, z) = 3xyz + x^2y - xz^3$.

5. Find all the first-order partial derivatives of the following:

a. $E(p, q) = ap^2e^{bq}$　　b. $R(p_1, p_2) = \alpha p_1^{\beta} + \gamma e^{p_1 p_2}$

c. $x(v_1, \dots, v_n) = \sum_{i=1}^{n} a_i v_i$

Harder Problems

6. Define the function $f(x, y) = xy(x^2 - y^2)/(x^2 + y^2)$ when $(x, y) \neq (0, 0)$, and $f(0, 0) = 0$. Show that Young's theorem does not apply at $(0, 0)$ by finding $f_1'(0, y)$ and $f_2'(x, 0)$, then showing that $f_{12}''(0, 0) = 1$ and that $f_{21}''(0, 0) = -1$. Show that Young's theorem is not contradicted because both f_{12}'' and f_{21}'' are discontinuous at $(0, 0)$.

7. Find all the first-order partial derivatives of $f(u, v, w) = u^{v^w}$.

15.6 Partial Derivatives in Economics

This section considers a number of economic examples of partial derivatives.

Example 15.20

Consider an agricultural production function $Y = F(K, L, T)$, where Y is the number of units produced, K capital invested, L labor input, and T the area of agricultural land that is used. Then $\partial Y/\partial K = F_K'$ is called the **marginal product of capital**. It is the rate of change of output Y with respect to K when L and T are held fixed. Similarly, $\partial Y/\partial L = F_L'$ and $\partial Y/\partial T = F_T'$ are the **marginal products of labor and of land**, respectively. For example, if K is the value of capital equipment measured in dollars, and $\partial Y/\partial K = 5$, then increasing capital input by 1 dollar would increase output by approximately 5 units.

Suppose, in particular, that F is the Cobb–Douglas function

$$F(K, L, T) = AK^aL^bT^c \qquad (A,\ a,\ b,\ \text{and } c \text{ are positive constants}) \qquad [1]$$

Find the marginal products, and the second-order partials. Discuss their signs.

Solution The marginal products are

$$\begin{aligned} F_K' &= AaK^{a-1}L^bT^c \\ F_L' &= AbK^aL^{b-1}T^c \\ F_T' &= AcK^aL^bT^{c-1} \end{aligned} \qquad [2]$$

Assuming K, L, and T are all positive, the marginal products are positive. Thus, an increase in capital, labor, or land will increase the number of units produced.

The mixed second-order partials (or cross-partials) are

$$\begin{aligned} F''_{KL} &= AabK^{a-1}L^{b-1}T^c \\ F''_{KT} &= AacK^{a-1}L^bT^{c-1} \\ F''_{LT} &= AbcK^aL^{b-1}T^{c-1} \end{aligned} \qquad [3]$$

Check for yourself that F''_{LK}, F''_{TK}, and F''_{TL} give, respectively, the same results as in [3]. Note that these partials are positive. We call each pair of factors *complementary*, because more of one increases the marginal product of the other.

The direct second-order partials are

$$\begin{aligned} F''_{KK} &= Aa(a-1)K^{a-2}L^bT^c \\ F''_{LL} &= Ab(b-1)K^aL^{b-2}T^c \\ F''_{TT} &= Ac(c-1)K^aL^bT^{c-2} \end{aligned}$$

For instance, F''_{KK} is the partial derivative of the marginal product of capital (F'_K) with respect to K. If $a < 1$, then $F''_{KK} < 0$, and there is a diminishing marginal product of capital—that is, a small increase in the capital invested will lead to a decrease in the marginal product of capital. We can interpret this to mean that although small increases in capital cause output to rise ($F'_K > 0$), this rise occurs at a decreasing rate ($F''_{KK} < 0$). Similarly for labor (if $b < 1$), and for land (if $c < 1$).

Example 15.21

Let x be an index of the total amount of goods produced and consumed in a society, and let z be a measure of the level of pollution. If $u(x, z)$ measures the total well-being of the society (not a very easy function to estimate!), what signs do you expect $u'_x(x, z)$ and $u'_z(x, z)$ to have? Can you guess what economists usually assume about the sign of $u''_{xz}(x, z)$?

Solution It is reasonable to expect that well-being increases as the amount of goods increases, but decreases as the level of pollution increases. Hence, we will usually have $u'_x(x, z) > 0$ and $u'_z(x, z) < 0$. According to [15.13] of Section 15.5, $u''_{xz} = (\partial/\partial z)(u'_x)$ is approximately equal to the change in u'_x when the level of pollution increases by one unit. Here $u'_x \approx$ the increase in welfare obtained by a unit increase in x. It is often assumed that $u''_{xz} < 0$. This implies that the increase in welfare obtained by an extra unit of x will decrease when the level of pollution increases. (An analogy: When I sit in a smoke-filled room, my increase in satisfaction from getting an extra piece of cake will decrease if the concentration of smoke increases too much.)

Example 15.22

The following modified version of the Cobb–Douglas function has been used in some economic studies:

$$F(K, L) = AK^a L^b e^{cK/L} \qquad (A,\ a,\ b, \text{ and } c \text{ are positive constants})$$

Compute the marginal products F'_K and F'_L and discuss their signs.

Solution Differentiating with respect to K while keeping L constant, AL^b is also constant, so

$$F'_K = AL^b \frac{\partial}{\partial K}\left(K^a e^{cK/L}\right)$$

We must now use the product rule for differentiation. According to the chain rule, the derivative of $e^{cK/L}$ with respect to K is $(c/L)e^{cK/L}$, so

$$\begin{aligned} F'_K &= AL^b\left[aK^{a-1}e^{cK/L} + K^a(c/L)e^{cK/L}\right] \\ &= \left(\frac{a}{K} + \frac{c}{L}\right) F(K, L) \end{aligned}$$

In the same way,

$$\begin{aligned} F'_L &= AK^a\left[bL^{b-1}e^{cK/L} + L^b(-cK/L^2)e^{cK/L}\right] \\ &= \left(\frac{b}{L} - \frac{cK}{L^2}\right) F(K, L) \end{aligned}$$

If K and L are positive, then F'_K is always positive, but F'_L is positive only if $b > cK/L$. (If $b < cK/L$, then $F'_L < 0$, so an increase in labor leads to a reduction of output. The function is, therefore, most suitable as a production function in a domain where $b > cK/L$.)

Example 15.23

For the general Cobb–Douglas function F in logarithmic form,

$$\ln F = \ln A + a_1 \ln x_1 + a_2 \ln x_2 + \cdots + a_n \ln x_n \qquad [*]$$

(see [15.6] in Section 15.1), show that

$$\sum_{i=1}^{n} x_i \frac{\partial F}{\partial x_i} = (a_1 + a_2 + \cdots + a_n)F$$

Solution Differentiating each side of [*] partially with respect to x_i by means of the chain rule gives

$$\frac{1}{F}\frac{\partial F}{\partial x_i} = a_i \frac{1}{x_i} \qquad \text{or} \qquad x_i \frac{\partial F}{\partial x_i} = a_i F$$

for $i = 1, 2, \dots, n$. So

$$\sum_{i=1}^{n} x_i \frac{\partial F}{\partial x_i} = \sum_{i=1}^{n} a_i F = \left(\sum_{i=1}^{n} a_i\right) F = (a_1 + a_2 + \cdots + a_n)F$$

Example 15.24

In Example 15.8 of Section 15.1 the function $U(x_1, x_2, \dots, x_n)$ is a measure of the satisfaction or "utility" that an individual obtains by consuming the respective quantities $x_1, x_2, \dots, x_n$ of n different goods. The partial derivative $\partial U/\partial x_i$ is called the **marginal utility** of the ith good. Usually, all the n marginal utilities are positive, because we expect utility to increase as the individual obtains more of a commodity.

For the function $U = a_1 \ln(x_1 - c_1) + a_2 \ln(x_2 - c_2) + \cdots + a_n \ln(x_n - c_n)$ specified in that example, we find

$$\frac{\partial U}{\partial x_1} = \frac{a_1}{x_1 - c_1}, \quad \frac{\partial U}{\partial x_2} = \frac{a_2}{x_2 - c_2}, \quad \dots, \quad \frac{\partial U}{\partial x_n} = \frac{a_n}{x_n - c_n}$$

If the parameters $a_1, \dots, a_n$ are all positive and $x_1 > c_1, \dots, x_n > c_n$, then we see that all the marginal utilities are positive.

Problems

1. The demand for money M in the United States for the period 1929–1952 has been estimated as

$$M = 0.14Y + 76.03(r - 2)^{-0.84} \qquad (r > 2)$$

where Y is the annual national income, and r is the interest rate (in percent per year). Compute $\partial M/\partial Y$ and $\partial M/\partial r$ and discuss their signs.

2. If a and b are constants, compute the expression $KY'_K + LY'_L$ for the following:

   a. $Y = AK^a + BL^a$ **b.** $Y = AK^a L^b$ **c.** $Y = \dfrac{K^2 L^2}{aL^3 + bK^3}$

3. Let $F(K, L, M) = AK^a L^b M^c$. Show then that $KF'_K + LF'_L + MF'_M = (a + b + c)F$.

4. Let $D(p, q)$ and $E(p, q)$ be the demands for two commodities when the prices per unit are p and q, respectively. Suppose the commodities are *substitutes* in consumption, such as butter and margarine. What are the normal signs of the partial derivatives of D and E with respect to p and q?

5. Compute $\partial U/\partial x_i$ when $U(x_1, x_2, \ldots, x_n) = 100 - e^{-x_1} - e^{-x_2} - \cdots - e^{-x_n}$.

Harder Problems

6. Compute the expression $KY'_K + LY'_L$ if $Y = Ae^{\lambda t}\left[\delta K^{-\rho} + (1-\delta)L^{-\rho}\right]^{-m/\rho}$.

15.7 Linear Models with Quadratic Objectives

In this section, we consider some simple optimization models that lead to the problem of maximizing or minimizing a quadratic objective function in two variables.

Example 15.25 (Discriminating Monopolist)
Consider a firm that sells a product in two isolated geographical areas. If it wants to, it can then charge different prices in the two different areas because what is sold in one area cannot easily be resold in the other. As an example, it seems that express mail or courier services find it possible to charge much higher prices in Europe than they can in North America. Suppose that such a firm also has some monopoly power to influence the different prices it faces in the two separate markets by adjusting the quantity it sells in each. Economists generally use the term "discriminating monopolist" to describe a firm having this power.

Faced with two such isolated markets, the discriminating monopolist has two independent demand curves. Suppose that, in inverse form, these are

$$P_1 = a_1 - b_1 Q_1, \qquad P_2 = a_2 - b_2 Q_2 \qquad [1]$$

for market areas 1 and 2, respectively. Suppose, too, that the total cost function is

$$C(Q) = \alpha(Q_1 + Q_2)$$

with total cost proportional to total production.[4]

As a function of Q_1 and Q_2, total profits are

$$\begin{aligned}\pi(Q_1, Q_2) &= P_1 Q_1 + P_2 Q_2 - C(Q_1 + Q_2)\\ &= (a_1 - b_1 Q_1)Q_1 + (a_2 - b_2 Q_2)Q_2 - \alpha(Q_1 + Q_2)\\ &= (a_1 - \alpha)Q_1 + (a_2 - \alpha)Q_2 - b_1 Q_1^2 - b_2 Q_2^2\end{aligned}$$

[4]It is true that this cost function neglects transport costs, but the point to be made is that, even though supplies to the two areas are perfect substitutes in production, the monopolist will generally charge different prices, if allowed.

We want to find the values of Q_1 and Q_2 that maximize profits. To solve this problem as in Section 3.2 by completing the square is quite simple; we just treat Q_1 and Q_2 as separate variables. Indeed,

$$\pi = -b_1\left[Q_1 - \frac{(a_1-\alpha)}{2b_1}\right]^2 - b_2\left[Q_2 - \frac{(a_2-\alpha)}{2b_2}\right]^2 + \frac{(a_1-\alpha)^2}{4b_1} + \frac{(a_2-\alpha)^2}{4b_2} \tag{2}$$

So the solution involves the optimal quantities

$$Q_1^* = (a_1-\alpha)/2b_1, \qquad Q_2^* = (a_2-\alpha)/2b_2$$

The corresponding prices can be found by inserting these values in [1] to get

$$P_1^* = a_1 - b_1Q_1^* = \tfrac{1}{2}(a_1+\alpha), \quad P_2^* = a_2 - b_2Q_2^* = \tfrac{1}{2}(a_2+\alpha)$$

From [2], maximum profits must be

$$\pi^* = \frac{(a_1-\alpha)^2}{4b_1} + \frac{(a_2-\alpha)^2}{4b_2}$$

This solution is valid as long as $a_1 \geq \alpha$ and $a_2 \geq \alpha$. In this case, P_1^* and P_2^* are both no less than α. This implies that there is no "cross subsidy" with the price in one market less than cost, and the losses in that market being subsidized by profits in the other market. Nor is there any "dumping," with price less than cost in one of the two markets. It is notable that the optimal prices are independent of b_1 and b_2. More important, note that the prices are *not* the same in the two markets, except in the special case when $a_1 = a_2$. Indeed, $P_1^* > P_2^*$ iff $a_1 > a_2$. This says that the price is higher in the market where consumers are willing to pay a higher price for each unit when the quantity is close to zero.

The foregoing analysis was simple because of the "separability" of the quadratic function $\pi(Q_1, Q_2)$, which took the form of the sum of a quadratic function $\pi_1(Q_1) = (a_1 - \alpha - b_1Q_1)Q_1$ of Q_1 and a quadratic function $\pi_2(Q_2) = (a_2-\alpha-b_2Q_2)Q_2$ of Q_2, without any term in Q_1Q_2. If we allowed the discriminating monopolist to have a quadratic cost function $C(Q) = \alpha Q + \beta Q^2$, where $Q = Q_1 + Q_2$ is total output, then the profit function $\pi(Q_1, Q_2)$ could still be maximized by completing squares. However, the analysis would become much more complicated, so we leave it out.

Example 15.26 (Discriminating Monopsonist)

A monopolist is a firm facing a downward sloping demand curve. A *discriminating monopolist* such as in Example 15.25 faces separate downward-sloping demand curves in two or more isolated markets. A *monopsonist*, on the other hand, is a firm facing an upward-sloping supply curve for one or more of its factors of production, and a *discriminating monopsonist* faces two or more upward-sloping supply curves for different kinds of the same input—for example, workers of different race or gender. Of course, discrimination by race or gender is illegal in many countries. The following analysis, however, suggests one possible reason why discrimination has had to be outlawed, and why firms might wish to discriminate if they are allowed to.

Indeed, consider a firm using quantities L_1 and L_2 of two types of labor as its only input in order to produce output Q according to the simple production function

$$Q = L_1 + L_2$$

Thus, both output and labor supply are measured so that each unit of labor produces one unit of output. Note especially how the two types of labor are essentially indistinguishable, because each unit of each type makes an equal contribution to the firm's output. Suppose, however, that there are two segmented labor markets, with different inverse supply functions specifying the wage that must be paid to attract a given labor supply. Specifically, suppose that

$$w_1 = \alpha_1 + \beta_1 L_1, \qquad w_2 = \alpha_2 + \beta_2 L_2$$

Assume that the firm is competitive in its output market, taking price P as fixed. Then the firm's profits are

$$\begin{aligned}
\pi(L_1, L_2) &= PQ - w_1 L_1 - w_2 L_2 \\
&= P(L_1 + L_2) - (\alpha_1 + \beta_1 L_1)L_1 - (\alpha_2 + \beta_2 L_2)L_2 \\
&= (P - \alpha_1)L_1 - \beta_1 L_1^2 + (P - \alpha_2)L_2 - \beta_2 L_2^2 \\
&= -\beta_1\left(L_1 - \frac{P-\alpha_1}{2\beta_1}\right)^2 - \beta_2\left(L_2 - \frac{P-\alpha_2}{2\beta_2}\right)^2 \\
&\quad + \frac{(P-\alpha_1)^2}{4\beta_1} r + \frac{(P-\alpha_2)^2}{4\beta_2}
\end{aligned}$$

It follows that the optimal labor demands are

$$L_1^* = \frac{P-\alpha_1}{2\beta_1}, \qquad L_2^* = \frac{P-\alpha_2}{2\beta_2}$$

These yield the maximum profit

$$\pi^* = \frac{(P-\alpha_1)^2}{4\beta_1} + \frac{(P-\alpha_2)^2}{4\beta_2}$$

The corresponding wages are

$$w_1^* = \alpha_1 + \beta_1 L_1^* = \tfrac{1}{2}(P + \alpha_1), \qquad w_2^* = \alpha_2 + \beta_2 L_2^* = \tfrac{1}{2}(P + \alpha_2)$$

Hence, $w_1^* = w_2^*$ only if $\alpha_1 = \alpha_2$. Generally, the wage is higher for the type of labor that demands a higher wage for very low levels of labor supply—perhaps this is the type of labor with better job prospects elsewhere.

Example 15.27 (Econometrics: Linear Regression)
Most applied economics is concerned with analyzing data in order to try to discern some pattern that helps in understanding the past, and possibly in predicting the future. For example, price and quantity data for a particular commodity such as natural gas may be used in order to try to estimate a demand curve that can be used to predict how demand will respond to future price changes. The most common technique for doing this is *linear regression*.

Suppose it is thought that variable y—say, the quantity demanded—depends upon variable x—say, price or income. Suppose that we have observations (x_t, y_t) of both variables at times $t = 1, 2, \ldots, T$. Then the technique of linear regression seeks to fit a linear function

$$y = \alpha + \beta x$$

to the data, as indicated in Fig. 15.18. Of course, an exact fit is possible only if there exist numbers α and β for which

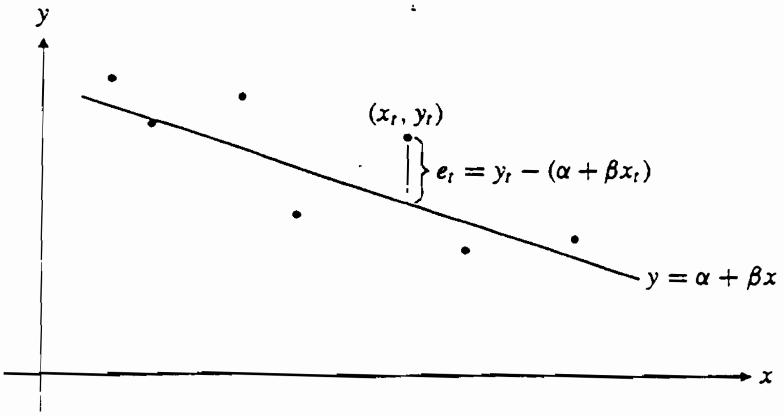
$$y_t = \alpha + \beta x_t \qquad (t = 1, 2, \ldots, T)$$

This is rarely possible. Generally, one has instead

$$y_t = \alpha + \beta x_t + e_t \qquad (t = 1, 2, \ldots, T)$$

where e_t is an *error* or *disturbance* term.

FIGURE 15.18



Obviously, one hopes that the errors will be small, on average. So the parameters α and β are chosen somehow to make the errors as "small as possible." One idea would be to make the sum $\sum_{t=1}^{T}(y_t - \alpha - \beta x_t)$ equal to zero. However, in this case, large positive discrepancies would cancel large negative discrepancies. Indeed, the sum of errors could be zero even though the line is very far from giving a perfect or even a good fit. We must somehow prevent large positive errors from canceling large negative errors. Usually, this is done by minimizing the "loss" function

$$L(\alpha, \beta) = \frac{1}{T}\sum_{t=1}^{T} e_t^2 = \frac{1}{T}\sum_{t=1}^{T}(y_t - \alpha - \beta x_t)^2 \tag{1}$$

that is the average of the squares of the errors. Expanding the square gives[5]

$$L(\alpha, \beta) = T^{-1}\sum_t (y_t^2 + \alpha^2 + \beta^2 x_t^2 - 2\alpha y_t - 2\beta x_t y_t + 2\alpha\beta x_t)$$

This is a quadratic function of α and β. We shall show how to complete the squares of this function, and so how to derive the *ordinary least-squares* estimates of α and β. Before doing so, however, it helps to introduce some standard notation. Write

$$\mu_x = \frac{x_1 + \cdots + x_T}{T} = T^{-1}\sum_t x_t, \qquad \mu_y = \frac{y_1 + \cdots + y_T}{T} = T^{-1}\sum_t y_t \tag{2}$$

for the *statistical means* of x_t and y_t, respectively. And write

$$\begin{aligned} \sigma_{xx} &= T^{-1}\sum_t (x_t - \mu_x)^2 \\ \sigma_{yy} &= T^{-1}\sum_t (y_t - \mu_y)^2 \\ \sigma_{xy} &= T^{-1}\sum_t (x_t - \mu_x)(y_t - \mu_y) \end{aligned} \tag{3}$$

for the *statistical variances* of x_t and y_t and for the *covariance*, respectively. Also note how the foregoing definition of σ_{xx} implies that

$$\begin{aligned} \sigma_{xx} &= T^{-1}\sum_t (x_t^2 - 2\mu_x x_t + \mu_x^2) = T^{-1}\sum_t x_t^2 - 2\mu_x T^{-1}\sum_t x_t + \mu_x^2 \\ &= T^{-1}\sum_t x_t^2 - 2\mu_x^2 + \mu_x^2 = T^{-1}\sum_t x_t^2 - \mu_x^2 \end{aligned}$$

[5]From now on. we often use $\sum_t$ to denote $\sum_{t=1}^{T}$.

Similarly,

$$\sigma_{yy} = T^{-1}\sum_t y_t^2 - \mu_y^2, \qquad \sigma_{xy} = T^{-1}\sum_t x_t y_t - \mu_x \mu_y$$

(You should check the last as an exercise.) Then the expression for $L(\alpha, \beta)$ becomes

$$\begin{aligned} L(\alpha, \beta) &= (\sigma_{yy} + \mu_y^2) + \alpha^2 + \beta^2(\sigma_{xx} + \mu_x^2) - 2\alpha\mu_y - 2\beta(\sigma_{xy} + \mu_x\mu_y) + 2\alpha\beta\mu_x \\ &= \alpha^2 + \mu_y^2 + \beta^2\mu_x^2 - 2\alpha\mu_y - 2\beta\mu_x\mu_y + 2\alpha\beta\mu_x + \beta^2\sigma_{xx} - 2\beta\sigma_{xy} + \sigma_{yy} \end{aligned}$$

Completing the squares then gives

$$L(\alpha, \beta) = (\mu_y - \alpha - \beta\mu_x)^2 + \sigma_{xx}\left(\beta - \frac{\sigma_{xy}}{\sigma_{xx}}\right)^2 + \sigma_{yy} - \frac{\sigma_{xy}^2}{\sigma_{xx}}$$

From this, it follows that the "ordinary least-squares" (or OLS) estimates that minimize $L(\alpha, \beta)$ with respect to α and β are given by

$$\hat{\beta} = \sigma_{xy}/\sigma_{xx}, \qquad \hat{\alpha} = \mu_y - \hat{\beta}\mu_x = \mu_y - (\sigma_{xy}/\sigma_{xx})\mu_x \tag{4}$$

Note in particular that $\hat{\alpha}$ is chosen to make the estimated straight line

$$y = \hat{\alpha} + \hat{\beta}x$$

go through the mean (μ_x, μ_y) of the observed pairs (x_t, y_t), $t = 1, \dots, T$.

Problems

1. Suppose a monopolist is practicing price discrimination in the sale of a product by charging different prices in two separate markets. Suppose the demand curves are

$$P_1 = 100 - Q_1, \qquad P_2 = 80 - Q_2$$

and suppose that the cost function is $C = 6(Q_1 + Q_2)$. How much should be sold in the two markets to maximize profits? What are the prices charged? How much profit is lost if price discrimination is made illegal?

2. Calculate the loss of profit if the discriminating monopolist of Example 15.25 is not allowed to discriminate.

3. Calculate the loss of profit if the discriminating monopsonist of Example 15.26 is not allowed to discriminate.

4. With reference to Example 15.27, find an expression for $\hat{L}$, the minimum value of $L(\alpha, \beta)$.

15.8 Quadratic Forms in Two Variables

Sections 3.1 and 3.2 presented some examples where quadratic functions in one variable could be optimized by completing the square. Examples 15.25 to 15.27 illustrate how completing squares can also work for quadratic functions of several variables. It must be admitted, however, that calculus techniques would save quite a bit of algebra in many cases. Nevertheless, even the calculus techniques that are presented in Chapter 17, especially the second-order conditions, at some stage involve examining properties of particular quadratic functions called "quadratic forms."

A **quadratic form** of two variables is a function

$$f(x, y) = ax^2 + 2bxy + cy^2 \tag{15.16}$$

where a, b, and c are three real constants. Using matrix notation, we can write (see Problem 4)

$$f(x, y) = (x, y)\begin{pmatrix} a & b \\ b & c \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

The second-order partials of f are $f''_{11} = 2a$, $f''_{12} = f''_{21} = 2b$, and $f''_{22} = 2c$, so according to [15.14] of Section 15.5, the Hessian of f is

$$2\begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

The quadratic form $f(x, y) = ax^2+2bxy+cy^2$ is said to be **positive definite** if $f(x, y) > 0$ for all $(x, y) \neq (0, 0)$, and **positive semidefinite** if $f(x, y) \geq 0$ for all (x, y).

Next, $f(x, y)$ is **negative definite** if $f(x, y) < 0$ for all $(x, y) \neq (0, 0)$, and **negative semidefinite** if $f(x, y) \leq 0$ for all (x, y).

Finally, $f(x, y)$ is **indefinite** if there are two different pairs (x^-, y^-) and (x^+, y^+) such that $f(x^-, y^-) < 0$ and $f(x^+, y^+) > 0$.

Example 15.28

Discuss the definiteness properties of the five quadratic forms:

(a) $x^2 + y^2$ **(b)** $(x + y)^2$ **(c)** $-x^2 - y^2$ **(d)** $-(x + y)^2$ **(e)** $x^2 - y^2$

Solution

(a) $x^2 + y^2 > 0$ for all $(x, y) \neq (0, 0)$, so $x^2 + y^2$ is positive definite.

(b) $(x+y)^2 \geq 0$ for all (x, y), but $(x+y)^2 = 0$ when $(x, y) = (1, -1)$, for instance. So $(x + y)^2$ is positive semidefinite, but not positive definite.

(c) and **(d)** are simply (a) and (b) with the signs reversed; the quadratic forms are respectively negative definite and negative semidefinite.

(e) $x^2 - y^2 > 0$ if $(x, y) = (1, 0)$, and $x^2 - y^2 < 0$ if $(x, y) = (0, 1)$. Hence, the quadratic form is indefinite.

Note that $f(0, 0) = 0$ whatever the constants a, b, and c may be, so the preceding definitions of positive and negative definiteness have to exclude the point $(0, 0)$.

These definitions evidently imply that (a) a positive definite or semidefinite quadratic form has a minimum at $(0, 0)$, (b) a negative definite or semidefinite quadratic form has a maximum at $(0, 0)$, and (c) an *indefinite* quadratic form has no maximum or minimum anywhere. When the form is definite (positive or negative), the minimum or maximum is *strict*.

The definiteness of a quadratic form depends entirely on the values of the coefficients a, b, and c. In fact, we shall prove the following very important results:

The **quadratic form** $f(x, y) = ax^2 + 2bxy + cy^2$ is

positive definite	$\iff$	$a > 0,\ c > 0,$ and $\begin{vmatrix} a & b \\ b & c \end{vmatrix} > 0$	[15.17]
positive semidefinite	$\iff$	$a \geq 0,\ c \geq 0,$ and $\begin{vmatrix} a & b \\ b & c \end{vmatrix} \geq 0$	[15.18]
negative definite	$\iff$	$a < 0,\ c < 0,$ and $\begin{vmatrix} a & b \\ b & c \end{vmatrix} > 0$	[15.19]
negative semidefinite	$\iff$	$a \leq 0,\ c \leq 0,$ and $\begin{vmatrix} a & b \\ b & c \end{vmatrix} \geq 0$	[15.20]
indefinite	$\iff$	$\begin{vmatrix} a & b \\ b & c \end{vmatrix} < 0$	[15.21]

(Recall that

$$\begin{vmatrix} a & b \\ b & c \end{vmatrix} = ac - b^2$$

by the definition of a 2×2 determinant (see [13.4] in Section 13.1).)

Proof We prove [15.18] first. Suppose that $f(x, y)$ is positive semidefinite. Then, in particular, $f(1, 0) = a \geq 0$ and $f(0, 1) = c \geq 0$. If $a = 0$, then $f(x, 1) = 2bx + c$, which can only be ≥ 0 for all x provided $b = 0$. (If $b > 0$, choosing x as a large negative number makes $f(x, 1)$ negative. If $b < 0$, choosing x as a large positive number makes $f(x, 1)$ negative.) Thus, $ac - b^2 = 0$. If $a > 0$, then $f(-b, a) = ab^2 - 2ab^2 + ca^2 = a(ac - b^2)$, which must be nonnegative, so $ac - b^2 \geq 0$.

To prove the reverse implication in [15.18], suppose that $a \geq 0$, $c \geq 0$, and $ac - b^2 \geq 0$. If $a = 0$, then $ac - b^2 \geq 0$ implies $b = 0$, and then $f(x, y) = cy^2 \geq 0$ for all (x, y). If $a > 0$, then we can write

$$f(x, y) = a\left(x + \frac{b}{a}y\right)^2 + \left(c - \frac{b^2}{a}\right)y^2 \qquad [*]$$

Because $c - b^2/a \geq 0$ and $a > 0$, we see that $f(x, y) \geq 0$ for all (x, y).

The equivalence in [15.20] is proved in the same way as [15.18] after reversing signs.

To prove [15.17], suppose $f(x, y)$ is positive definite. Then $f(1, 0) = a > 0$ and $f(0, 1) = c > 0$. But then [*] yields $f(-b/a, 1) = c - b^2/a = (ac - b^2)/a > 0$, so $ac - b^2 > 0$. To prove the reverse implication in [15.17], suppose $a > 0$ and $ac - b^2 > 0$. By [*], $f(x, y) \geq 0$ for all (x, y). If $f(x, y) = 0$, then $x + by/a = 0$ and $y = 0$, so $x = y = 0$. Hence, $f(x, y)$ is positive definite. The equivalence in [15.19] is proved in the same way—just reverse signs.

Finally, we prove [15.21]. Suppose $f(x, y)$ is indefinite. Because neither the inequalities in [15.18] nor those in [15.20] are satisfied, either a and c have opposite signs or $ac - b^2 < 0$. But if a and c do have opposite signs, then $ac < 0 \leq b^2$ anyway, so $ac - b^2 < 0$ in all cases.

To prove the reverse implication in [15.21], suppose $ac - b^2 < 0$. If $a \neq 0$, then $f(1, 0) = a$ and $f(-b, a) = a(ac - b^2)$ have opposite signs, so $f(x, y)$ is indefinite. If $a = 0$ and $c = 0$, then $f(1, 1) = 2b$ and $f(-1, 1) = -2b$. Because $ac - b^2 < 0$ implies $b^2 > 0$ in this case, one has $b \neq 0$ and so $f(x, y)$ is indefinite. If $a = 0$ and $c \neq 0$, then $f(0, 1) = c$ and $f(c, -b) = -b^2c$ have different signs, so $f(x, y)$ is indefinite.

General Quadratic Functions in Two Variables

Adding any linear function $px + qy + r$ of x and y to the terms in [15.16] gives

$$f(x, y) = ax^2 + 2bxy + cy^2 + px + qy + r \qquad [15.22]$$

This is the general quadratic function of x and y. It can be expressed as

$$f(x, y) = a(x + \xi)^2 + 2b(x + \xi)(y + \eta) + c(y + \eta)^2 + d \qquad [15.23]$$

provided that we arrange to have

$$2a\xi + 2b\eta = p, \qquad 2b\xi + 2c\eta = q, \qquad a\xi^2 + 2b\xi\eta + c\eta^2 + d = r$$

so that the coefficients of x and y match, as well as the constant term. This involves choosing

$$\xi = \frac{cp - bq}{2(ac - b^2)}, \quad \eta = \frac{aq - bp}{2(ac - b^2)}, \quad d = r - (a\xi^2 + 2b\xi\eta + c\eta^2) \qquad [15.24]$$

which is possible provided that $ac \neq b^2$. If $ac = b^2 \neq 0$, then

$$f(x, y) = a\left(x + \sqrt{\frac{c}{a}}y\right)^2 + px + qy + r$$

and it is easy to study the quadratic function directly. A similar transformation works if $ac = b^2 = 0$ and either a or c is $\neq 0$. If $a = b = c = 0$, the function is not even quadratic.

Thus, the only interesting cases arise when, after changing variables if necessary by replacing x with $x + \xi$ and y with $y + \eta$, the function f can be written in the form

$$f(x, y) = ax^2 + 2bxy + cy^2 + d$$

Of course, the constant d does not change the essential behavior of $f(x, y)$. Thus, in all interesting cases, the general quadratic function [15.22] is reduced to the quadratic form [15.16] that was studied in detail earlier.

Quadratic Forms with Linear Constraints

Consider the quadratic form $Q = ax^2 + 2bxy + cy^2$ and assume that the variables are subject to the linear constraint $px + qy = 0$, where $q \neq 0$. Solving the constraint for y, we have $y = -px/q$, and substituting this value for y into the expression for Q yields

$$Q = ax^2 + 2bx\left(-\frac{px}{q}\right) + c\left(-\frac{px}{q}\right)^2 = \frac{1}{q^2}(aq^2 - 2bpq + cp^2)x^2 \qquad [*]$$

We say that $Q(x, y)$ is **positive (negative) definite subject to the constraint** $px + qy = 0$ provided Q is positive (negative) for all $(x, y) \neq (0, 0)$ satisfying the constraint $px + qy = 0$. By expanding the determinant, it is easy to verify that

$$aq^2 - 2bpq + cp^2 = -\begin{vmatrix} 0 & p & q \\ p & a & b \\ q & b & c \end{vmatrix} \qquad [**]$$

Combining this result with [*] gives

$$\left.\begin{array}{l} Q = ax^2 + 2bxy + cy^2 \text{ is positive definite} \\ \text{subject to the constraint } px + qy = 0 \end{array}\right\} \iff \begin{vmatrix} 0 & p & q \\ p & a & b \\ q & b & c \end{vmatrix} < 0 \qquad [15.25]$$

Problems

1. Use [15.17] to [15.21] to determine the definiteness of the following quadratic forms:

   a. $4x^2 + 8xy + 5y^2$ **b.** $-x^2 + xy - 3y^2$ **c.** $x^2 - 6xy + 9y^2$

   d. $4x^2 - y^2$ **e.** $\frac{1}{2}x^2 - xy + \frac{1}{4}y^2$ **f.** $6xy - 9y^2 - x^2$

2. Show that the following quadratic functions can be expressed in the form [15.23] by using [15.24]:
   a. $f(x, y) = 2x^2 - 4xy + y^2 - 3x + 4y$
   b. $f(x, y) = -x^2 - xy + y^2 - x - y + 5$
3. Examine the definiteness of the following quadratic forms subject to the given linear constraint:
   a. $x^2 - 2xy + y^2$ subject to $x + y = 0$.
   b. $2x^2 - 4xy + y^2$ subject to $3x + 4y = 0$.
   c. $-x^2 + xy - y^2$ subject to $5x - 2y = 0$.
4. Verify the matrix equation that was displayed following Equation [15.16].

15.9 Quadratic Forms in Many Variables

We often encounter quadratic forms in more than two variables, such as that in three variables with $Q(x_1, x_2, x_3) = 2x_1^2 + 4x_1x_2 - x_1x_3 + x_2^2 + 5x_2^2 - x_3^2$. The sum of the exponents of the variables in each term is 2.

A general **quadratic form** in n variables is a function $Q = Q(x_1, \dots, x_n)$ given by the double sum

$$\begin{aligned} Q &= \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_ix_j \\ &= a_{11}x_1^2 + a_{12}x_1x_2 + \cdots + a_{ij}x_ix_j + \cdots + a_{nn}x_n^2 \end{aligned} \qquad [15.26]$$

Suppose we put

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \qquad \mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$$

Then it follows from the definition of matrix multiplication that

$$Q(x_1, \dots, x_n) = Q(\mathbf{x}) = \mathbf{x}'\mathbf{A}\mathbf{x} \qquad [15.27]$$

Of course, $x_ix_j = x_jx_i$, so we can write $a_{ij}x_ix_j + a_{ji}x_jx_i = (a_{ij} + a_{ji})x_ix_j$. If we replace a_{ij} and a_{ji} by $\frac{1}{2}(a_{ij} + a_{ji})$, then a_{ij} and a_{ji} become equal without changing $Q(x_1, \dots, x_n)$. Thus, we can assume in [15.26] that

$$a_{ij} = a_{ji} \qquad \text{(for all } i \text{ and } j\text{)} \qquad [15.28]$$

which means that matrix $\mathbf{A}$ is symmetric.

Example 15.29

Write

$$Q(x_1, x_2, x_3) = 5x_1^2 + x_1x_2 - 3x_1x_3 + 3x_2x_1 + x_2^2 - 2x_2x_3 + 5x_3x_1 + 2x_3x_2 + x_3^2$$

in the form [15.27], both with **A** not symmetric and **A** symmetric.

Solution

$$Q(x_1, x_2, x_3) = (x_1, x_2, x_3)\begin{pmatrix} 5 & 1 & -3 \\ 3 & 1 & -2 \\ 5 & 2 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$= (x_1, x_2, x_3)\begin{pmatrix} 5 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

The Sign of a Quadratic Form

We are particularly interested in conditions ensuring that Q is always positive or always negative, thus generalizing some of the results from the previous section.

In general, a symmetric $n \times n$ matrix **A** and its associated quadratic form Q are both said to be **positive definite** if

$$Q(x_1, \ldots, x_n) = \mathbf{x}'\mathbf{A}\mathbf{x} > 0 \qquad \text{for all } (x_1, \ldots, x_n) \neq (0, \ldots, 0) \qquad [15.29]$$

They are said to be **negative definite** if

$$Q(x_1, \ldots, x_n) = \mathbf{x}'\mathbf{A}\mathbf{x} < 0 \qquad \text{for all } (x_1, \ldots, x_n) \neq (0, \ldots, 0) \qquad [15.30]$$

If we replace $>$ in [15.29] with ≥ 0, then **A** is **positive semidefinite**, and **A** is **negative semidefinite** if we replace $<$ in [15.30] with $\leq$. Finally, **A** is **indefinite** if it is neither positive semidefinite nor negative semidefinite. (In this case, there must exist a vector $\mathbf{x}_0$ and a vector $\mathbf{y}_0$ such that $\mathbf{x}_0'\mathbf{A}\mathbf{x}_0 < 0$ and $\mathbf{y}_0'\mathbf{A}\mathbf{y}_0 > 0$.) Note that $Q(0, \ldots, 0) = 0$ whatever the constants a_{ij} may be, so the foregoing definitions of positive and negative definiteness have to exclude the point $(0, \ldots, 0)$.

Example 15.30

Let $\mathbf{D} = \text{diag}\,(\lambda_1, \ldots, \lambda_n)$ be an $n \times n$ diagonal matrix. When is the matrix **D**: (a) negative definite, (b) positive semidefinite, and (c) indefinite?

Solution **D** is a symmetric matrix whose associated quadratic form is

$$Q = (x_1, x_2, \ldots, x_n)\begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$$= (x_1, x_2, \ldots, x_n)\begin{pmatrix} \lambda_1 x_1 \\ \lambda_2 x_2 \\ \vdots \\ \lambda_n x_n \end{pmatrix} = \sum_{i=1}^{n} \lambda_i x_i^2$$

(a) This quadratic form is obviously negative definite if $\lambda_i < 0$ for $i = 1, 2, \ldots, n$. Conversely, if $\lambda_i \geq 0$ for any i, then $Q \geq 0$ when $\mathbf{x}$ is the unit vector $\mathbf{e}_i$ whose ith component is 1, but all other components are zero. So if $\mathbf{D}$ is negative definite, then $\lambda_i < 0$ for $i = 1, 2, \ldots, n$.

(b) $\mathbf{D}$ is evidently positive semidefinite iff $\lambda_i \geq 0$ for $i = 1, 2, \ldots, n$.

(c) $\mathbf{D}$ is evidently indefinite iff there is at least one positive diagonal element as well as at least one negative diagonal element.

By Example 14.13 of Section 14.4, the diagonal elements of a diagonal matrix are its eigenvalues. So Example 15.30 shows that the definiteness properties of a quadratic form depends upon those eigenvalues. The same is true for the general 2×2 symmetric matrix

$$\mathbf{A} = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

By Example 14.14, the eigenvalues λ_1 and λ_2 of $\mathbf{A}$ are real, with $\lambda_1 + \lambda_2 = a + c$ and $\lambda_1\lambda_2 = \det(\mathbf{A})$. By [15.18], $\mathbf{A}$ is positive semidefinite iff $a \geq 0$, $c \geq 0$, and $\det(\mathbf{A}) \geq 0$. But because $\det(\mathbf{A}) \geq 0$ ensures that a and c cannot have opposite signs, $\mathbf{A}$ is positive semidefinite iff $\lambda_1 + \lambda_2 = a + c \geq 0$ and $\det(\mathbf{A}) = \lambda_1\lambda_2 \geq 0$. It follows that $\mathbf{A}$ is positive semidefinite iff λ_1 and λ_2 are both nonnegative. The cases of negative semidefinite, of positive or negative definite, and of indefinite matrices, are entirely similar. Indeed, the sign of any quadratic form in n variables is determined by the signs of the eigenvalues of the associated matrix, because of the following:

Theorem 15.2 Suppose $\mathbf{A}$ is a symmetric matrix. Then:

(a) $\mathbf{A}$ is positive definite $\iff$ all eigenvalues of $\mathbf{A}$ are positive.

(b) $\mathbf{A}$ is positive semidefinite $\iff$ all eigenvalues of $\mathbf{A}$ are ≥ 0.

(c) $\mathbf{A}$ is negative definite $\iff$ all eigenvalues of $\mathbf{A}$ are negative.

(d) $\mathbf{A}$ is negative semidefinite $\iff$ all eigenvalues of $\mathbf{A}$ are ≤ 0.

(e) $\mathbf{A}$ is indefinite $\iff$ $\mathbf{A}$ has at least two eigenvalues with opposite signs.

Proof Let λ be any eigenvalue of $\mathbf{A}$. Then there is a corresponding eigenvector $\mathbf{x}_\lambda \neq \mathbf{0}$ such that $\mathbf{A}\mathbf{x}_\lambda = \lambda\mathbf{x}_\lambda$. So $Q(\mathbf{x}_\lambda) = \mathbf{x}_\lambda'\mathbf{A}\mathbf{x}_\lambda = \mathbf{x}_\lambda'\lambda\mathbf{x}_\lambda = \lambda\mathbf{x}_\lambda'\mathbf{x}_\lambda$, which has the same sign as λ. Now, if $\mathbf{A}$ is positive definite, then $Q(\mathbf{x}_\lambda) = \mathbf{x}_\lambda'\mathbf{A}\mathbf{x}_\lambda = \lambda\mathbf{x}_\lambda'\mathbf{x}_\lambda > 0$ for all eigenvectors $\mathbf{x}_\lambda \neq \mathbf{0}$ and so for all eigenvalues

λ. Thus, **A** can have only positive eigenvalues. This also holds for negative definite, for positive or negative semidefinite, and for indefinite matrices.

Conversely, by Theorem 14.8 in Section 14.6, there exists an orthogonal matrix **U** (with $\mathbf{U}^{-1} = \mathbf{U}'$) such that

$$\mathbf{U}'\mathbf{A}\mathbf{U} = \text{diag}\,(\lambda_1, \dots, \lambda_n) = \mathbf{D}$$

where $\lambda_1, \dots, \lambda_n$ are the eigenvalues of **A**. Hence,

$$\mathbf{A} = (\mathbf{U}')^{-1}(\mathbf{U}'\mathbf{A}\mathbf{U})\mathbf{U}^{-1} = (\mathbf{U}')^{-1}\mathbf{D}\mathbf{U}^{-1} = \mathbf{U}\mathbf{D}\mathbf{U}'$$

But now, for any **x** in R^n, it must be true that

$$\mathbf{x}'\mathbf{A}\mathbf{x} = \mathbf{x}'\mathbf{U}\mathbf{D}\mathbf{U}'\mathbf{x} = (\mathbf{U}'\mathbf{x})'\mathbf{D}(\mathbf{U}'\mathbf{x}) = \mathbf{y}'\mathbf{D}\mathbf{y} = \sum_{k=1}^{n} \lambda_k y_k^2$$

where $\mathbf{y} = \mathbf{U}'\mathbf{x}$. Moreover, if $\mathbf{x} \neq \mathbf{0}$, then $\mathbf{y} \neq \mathbf{0}$ because $\mathbf{x} = \mathbf{U}\mathbf{y}$. So now, in case (a), when all eigenvalues of **A** are positive, then $\mathbf{y}'\mathbf{D}\mathbf{y} > 0$ for all $\mathbf{y} \neq \mathbf{0}$, and so $\mathbf{x}'\mathbf{A}\mathbf{x} > 0$ for all $\mathbf{x} \neq \mathbf{0}$, implying that **A** is positive definite. The proofs for the cases (b) to (e) are entirely similar.

Example 15.31

Check the sign of the quadratic form in Example 15.29.

Solution The characteristic equation of the corresponding symmetric matrix is

$$\begin{vmatrix} 5-\lambda & 2 & 1 \\ 2 & 1-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{vmatrix} = 0$$

Thus, $(5-\lambda)(1-\lambda)^2 - 4(1-\lambda) - (1-\lambda) = 0$, which reduces to $\lambda(1-\lambda)(\lambda-6) = 0$. So the eigenvalues are 0, 1, and 6. From (b) in Theorem 15.2, the quadratic form is positive semidefinite.

In order to apply Theorem 15.2, we have to compute the eigenvalues of the associated matrix. The next theorem makes it possible to decide the definiteness of a matrix **A** by checking the signs of certain minors of **A**.

Let $\mathbf{A} = (a_{ij})$ be any $n \times n$ matrix. The **leading principal minors** of **A** are the n determinants:

$$D_k = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1k} \\ a_{21} & a_{22} & \dots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & \dots & a_{kk} \end{vmatrix} \qquad (k = 1, \dots, n) \qquad [15.31]$$

Note that D_k is obtained from $|\mathbf{A}|$ by crossing out the last $n-k$ columns and the corresponding last $n-k$ rows. Thus, for $k = 1, 2, 3, \dots, n$, the leading principal minors are, respectively,

$$a_{11}, \quad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \quad \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}, \ldots, \quad \begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{vmatrix} \tag{15.32}$$

One can prove the following result:[6]

Theorem 15.3 Let $\mathbf{A} = (a_{ij})_{n\times n}$ be a symmetric matrix, with leading principal minors D_k $(k = 1, 2, \ldots, n)$ defined by [15.31]. Then:

(a) A is positive definite $\iff$ $D_k > 0$ for $k = 1, 2, \ldots, n$.

(b) A is negative definite $\iff$ $(-1)^k D_k > 0$ for $k = 1, 2, \ldots, n$.

Though the proof of Theorem 15.3 is too advanced for this book, it can easily be illustrated for the case when A is a diagonal matrix with $\mathbf{A} = \text{diag}(\lambda_1, \ldots, \lambda_n)$. For then Example 14.13 of Section 14.4 and Theorem 15.2 imply that **A** is positive definite iff $\lambda_i > 0$ for $i = 1, 2, \ldots, n$. However, the leading principal minors of A are $\lambda_1, \lambda_1\lambda_2, \lambda_1\lambda_2\lambda_3, \ldots, \lambda_1\lambda_2\lambda_3 \ldots \lambda_n$, which are all positive iff $\lambda_i > 0$ for $i = 1, 2, \ldots, n$. So case (a) of Theorem 15.3 is verified. On the other hand, **A** is negative definite iff $\lambda_i < 0$ for $i = 1, 2, \ldots, n$. However, the leading principal minors alternate in sign iff $\lambda_i < 0$ for $i = 1, 2, \ldots, n$, which is case (b) of Theorem 15.3.

Also, when **A** is a 2×2 matrix, conditions [15.17] and [15.19] of Section 15.8 are the appropriate versions of Theorem 15.3. This is because

$$\begin{vmatrix} a & b \\ b & c \end{vmatrix} > 0$$

implies that $ac > b^2 \geq 0$, so $ac > 0$, implying that a and c must have the same sign.

Example 15.32

Prove that the following matrix is negative definite:

$$\mathbf{A} = \begin{pmatrix} -3 & 2 & 0 \\ 2 & -3 & 0 \\ 0 & 0 & -5 \end{pmatrix}$$

Solution In this case,

$$-3 < 0, \qquad \begin{vmatrix} -3 & 2 \\ 2 & -3 \end{vmatrix} = 5 > 0, \qquad \begin{vmatrix} -3 & 2 & 0 \\ 2 & -3 & 0 \\ 0 & 0 & -5 \end{vmatrix} = -25 < 0$$

[6] See, for example, Hadley (1973).

By Theorem 15.3 (b), we see that **A** is negative definite. As an exercise, you should also apply the eigenvalue test, Theorem 15.2.

The Semidefinite Case

It is tempting to conjecture that a matrix will be positive semidefinite iff all the strict inequalities in Theorem 15.3(a) are replaced by weak inequalities. This is wrong. The quadratic form $Q(x_1, x_2) = 0x_1^2 + 0 \cdot x_1x_2 - x_2^2 = -x_2^2$ is negative semidefinite, not positive semidefinite. Yet the leading principal minors of the associated matrix

$$\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$$

are both ≥ 0 (in fact, both $= 0$).

In order to check semidefiniteness by calculating minors, one has to consider the signs of *all* the principal minors of **A**, not only the leading principal minors. An arbitrary principal minor of order $(n-r) \times (n-r)$ in **A** is obtained by crossing out any r rows and the corresponding r columns—not necessarily the last r rows and columns. *One can prove that a quadratic form* $\mathbf{x}'\mathbf{A}\mathbf{x}$ *is positive semidefinite iff all the principal minors in* **A** *are* ≥ 0. For the 2×2 case, [15.18] of Section 15.8 confirms this result. Also, *one can prove that* $\mathbf{x}'\mathbf{A}\mathbf{x}$ *is negative semidefinite iff all the principal minors of order* k *in* **A** *have the same sign as* $(-1)^k$. For the 2×2 case, [15.20] confirms this.

Another case in which it is easy to confirm these two results occurs when **A** is diagonal. For then **A** is positive semidefinite iff all its diagonal elements are nonnegative; the principal minors of **A**, which are products of its diagonal elements, will also be nonnegative iff all its diagonal elements are nonnegative. There is an obvious corresponding argument when **A** is negative semidefinite.

Problems

1. Write the quadratic form [15.26] in full when $n = 3$.
2. Write the following quadratic forms in the matrix form [15.27] with **A** symmetric:

   a. $x^2 + 2xy + y^2$ **b.** $3x_1^2 - 2x_1x_2 + 3x_1x_3 + x_2^2 - 4x_2x_3 + 3x_3^2$
3. Use Theorem 15.3 to classify the following quadratic forms in the three variables x_1, x_2, and x_3:

   a. $x_1^2 + 2x_2^2 + 8x_3^2$ **b.** $x_2^2 + 8x_3^2$ **c.** $-3x_1^2 + 2x_1x_2 - x_2^2 + 4x_2x_3 - 8x_3^2$
4. Suppose **A** is positive semidefinite and symmetric. Prove that **A** is positive definite if $|\mathbf{A}| \neq 0$.

A

Elementary Algebra

Is it right I ask; is it even prudence;
to bore thyself and bore the students?
—Mephistopheles to Faust (from Goethe's "Faust")

This appendix is for students who need to review elementary algebra. To save time, you should quickly glance through the sections, and do some of the problems. (Answers to all the problems in this appendix are given in the back of the book.) If you have difficulties with any of these problems, read the preceding theory carefully and then redo the problems. If you have considerable difficulties with this appendix, turn to a more elementary book on algebra.

A.1 Powers

You probably recall that instead of the product $3 \cdot 3 \cdot 3 \cdot 3$, we often write 3^4, that $\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}$ can be written as $\left(\frac{1}{2}\right)^5$, and that $(-10)^3 = (-10)(-10)(-10) = -1000$. If a is any number and n is any natural number, then a^n is defined by

$$a^n = \underbrace{a \cdot a \cdots a}_{n \text{ times}} \qquad (a \text{ occurs as a factor } n \text{ times}) \qquad [\text{A.1}]$$

In fact, a^n is called the *nth power of a*; a is the *base*, and n is the *exponent*. We have, for example, $a^1 = a$, $a^2 = a \cdot a$, $x^4 = x \cdot x \cdot x \cdot x$, and

$$\left(\frac{p}{q}\right)^5 = \frac{p}{q} \cdot \frac{p}{q} \cdot \frac{p}{q} \cdot \frac{p}{q} \cdot \frac{p}{q}$$

where $a = p/q$, and $n = 5$. A further example:

$$(r+1)^3 = (r+1)\cdot(r+1)\cdot(r+1) \qquad (\text{where } a = r+1, \text{ and } n = 3)$$

The product $(r+1)\cdot(r+1)\cdot(r+1)$ can be expanded further. See Example A.5 in Section A.3. We usually drop the multiplication sign if this is unlikely to create misunderstanding. For example, we write abc instead of $a \cdot b \cdot c$.

We define further

$$a^0 = 1 \qquad \text{for } a \neq 0 \qquad [\text{A.2}]$$

Thus, $5^0 = 1$, $(-16.2)^0 = 1$, and $(x \cdot y)^0 = 1$ (if $x \cdot y \neq 0$). But if $a = 0$, we do not assign a numerical value to a^0; the expression 0^0 is *undefined.*

We also need to define powers with negative exponents. What do we mean by 3^{-2}? It turns out that the sensible definition is to set 3^{-2} equal to $1/3^2 = 1/9$. In general, we define

$$a^{-n} = \frac{1}{a^n} \qquad [\text{A.3}]$$

whenever n is a natural number and $a \neq 0$. For example,

$$a^{-1} = \frac{1}{a}, \qquad 8^{-3} = \frac{1}{8^3} = \frac{1}{512}, \qquad (x^2+5)^{-16} = \frac{1}{(x^2+5)^{16}}$$

Note: Students often make serious mistakes by misplacing parentheses or by interpreting them incorrectly. The following examples highlight some common sources of confusion over the use of parentheses.

1. There is an important difference between $(-10)^2 = (-10)(-10) = 100$, and $-10^2 = -(10 \cdot 10) = -100$. The square of minus 10 is not equal to minus the square of 10.
2. Note that $(2x)^{-1} = 1/2x$, whereas $2x^{-1} = 2 \cdot (1/x) = 2/x$.
3. As we shall see in what follows, $1000 \cdot (1.08)^5$ is the amount you will have in your account after 5 years if you invest \$1000 at 8% interest per year. Using a calculator, you quickly find that you will have approximately \$1469.33. One student put $1000 \cdot (1.08)^5 = (1000 \cdot 1.08)^5 = (1080)^5$, which is a horrible mistake because it is 10^{12} (or a trillion) times the right answer.
4. The area of a square with sides of length x is x^2. What is the area if the sides are doubled? *Solution*: The area expands to $(2x)^2 = (2x)(2x) = 4x^2$, so it grows four times. If $(2x)^2$ is incorrectly replaced by $2x^2$, the result would only be a doubling of the area. (Use a drawing to prove that the latter is wrong.)
5. The volume of a ball with radius r is $\frac{4}{3}\pi r^3$. What is the volume if the radius is doubled? *Solution*: The new volume is $\frac{4}{3}\pi(2r)^3 = \frac{4}{3}\pi(2r)(2r)(2r) =$

$\frac{4}{3}\pi 8r^3 = 8(\frac{4}{3}\pi r^3)$, so the volume is eight times as large. (If we make the mistake of "simplifying" $(2r)^3$ to $2r^3$, the result would imply only a doubling of the volume; this should be contrary to common sense.)

Properties of Exponents

The following properties of exponents are very important and must be memorized.

General Properties of Exponents

$$a^n \cdot a^m = a^{n+m} \quad \text{(a)}$$

$$a^n / a^m = a^{n-m} \quad \text{(b)}$$

$$(a^n)^m = a^{n \cdot m} \quad \text{(c)} \qquad \text{[A.4]}$$

$$(a \cdot b)^n = a^n \cdot b^n \quad \text{(d)}$$

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n} \quad \text{(e)}$$

Here $a^n \cdot a^m = a^{n+m}$ and $(a^n)^m = a^{n \cdot m}$ are the fundamental rules, because all the others follow from these two and from the definitions of powers. Here are some examples indicating why the rules in [A.4] are valid:

$$a^3 \cdot a^2 = (a \cdot a \cdot a) \cdot (a \cdot a) = a \cdot a \cdot a \cdot a \cdot a = a^5 = a^{3+2}$$

$$a^3 \div a^2 = \frac{a^3}{a^2} = \frac{a \cdot a \cdot a}{a \cdot a} = a = a^{3-2}$$

$$(a^2)^3 = (a^2)(a^2)(a^2) = a^{2+2+2} = a^6 = a^{2 \cdot 3}$$

$$(a \cdot b)^3 = (a \cdot b)(a \cdot b)(a \cdot b) = a \cdot a \cdot a \cdot b \cdot b \cdot b = a^3 b^3$$

$$\left(\frac{a}{b}\right)^4 = \left(\frac{a}{b}\right) \cdot \left(\frac{a}{b}\right) \cdot \left(\frac{a}{b}\right) \cdot \left(\frac{a}{b}\right) = \frac{a^4}{b^4}$$

Property [A.4](a) says that exponents with the same base are multiplied by adding the exponents. Try to formulate the other properties in words as well. Study the examples carefully.

The properties in [A.4] hold also if m and/or n are negative integers. For example,

$$a^{-3} \cdot a^5 = a^{-3+5} = a^2, \qquad (x \cdot y)^{-2} = x^{-2} \cdot y^{-2}$$

Also, using the rules for fractions (see Section A.5), we get

$$\left(\frac{a}{b}\right)^{-n} = \frac{a^{-n}}{b^{-n}} = \frac{1/a^n}{1/b^n} = \frac{(1/a^n) \cdot a^n \cdot b^n}{(1/b^n) \cdot a^n \cdot b^n} = \frac{b^n}{a^n} \qquad \text{[A.5]}$$

This result can be applied whenever a fraction is raised to a negative power. For example,

$$\left(\frac{5}{4}\right)^{-3} = \frac{4^3}{5^3} = \frac{64}{125}$$

Example A.1

If $ab^2 = 2$, compute the following:

$$\text{(a) } a^2b^4 \qquad \text{(b) } a^{-4}b^{-8} \qquad \text{(c) } a^3b^6 + a^{-1}b^{-2}$$

Solution

(a) $a^2b^4 = (ab^2)^2 = 2^2 = 4$

(b) $a^{-4}b^{-8} = (ab^2)^{-4} = 2^{-4} = 1/2^4 = 1/16$

(c) $a^3b^6 + a^{-1}b^{-2} = (ab^2)^3 + (ab^2)^{-1} = 2^3 + 2^{-1} = 8 + 1/2 = 17/2$

Note: An important motivation for introducing definitions [A.2] and [A.3] is that we would like the properties in [A.4] to be valid for all exponents. For example, consider the consequences of requiring [A.4](a) to be valid for $a^5 \cdot a^0$. We obtain $a^{5+0} = a^5$, so that $a^5 \cdot a^0 = a^5$, and hence we must choose $a^0 = 1$. If [A.4](a) is to be valid for $m = -n$, we must have $a^n \cdot a^{-n} = a^{n+(-n)} = a^0 = 1$. Because $a^n \cdot (1/a^n) = 1$, we *must* define a^{-n} by [A.3].

Compound Interest

Powers are used in practically every branch of applied mathematics, including economics. To illustrate their use, consider how they are needed to calculate compound interest.

Suppose you deposit \$1000 in a bank at 8% interest per year.[1] After one year you will have earned \$1000 · 0.08 = \$80 in interest, so the amount in your bank account at the end of the year will be \$1080. This can be rewritten as

$$1000 + \frac{1000 \cdot 8}{100} = 1000\left(1 + \frac{8}{100}\right) = 1000 \cdot 1.08$$

If this new amount of \$1000 · 1.08 is left in the bank for another year at an interest

[1] Remember that 1% means one in a hundred, or 0.01. To calculate, say, 23% of \$4000, we write

$$\frac{4000 \cdot 23}{100} = 920 \qquad \text{or} \qquad 4000 \cdot 0.23 = 920$$

rate of 8%, after a second year, the amount will have grown to a total of

$$1000 \cdot 1.08 + \frac{(1000 \cdot 1.08) \cdot 8}{100} = 1000 \cdot 1.08 \left(1 + \frac{8}{100}\right) = 1000 \cdot (1.08)^2$$

Each year the amount will increase by the factor 1.08, and we see that at the end of t years, it will have grown to $\$1000 \cdot (1.08)^t$. If the original amount is $\$K$ and the interest rate is $p\%$ per year, by the end of the first year, the amount will be $K + K \cdot p/100 = K(1 + p/100)$ dollars. The growth factor per year is thus $1 + p/100$. In general, after t (whole) years, the original investment of $\$K$ will have grown to an amount

$$K\left(1 + \frac{p}{100}\right)^t$$

when the interest rate is $p\%$ per year (and interest is added to the capital every year—that is, compound interest).

If you see an expression like $(1.08)^t$, you should immediately be able to recognize it as the amount to which \$1 has grown after t years when the interest rate is 8% per year. What would be the interpretation of $(1.08)^0$? You deposit \$1 at 8% per year, and leave the amount for 0 years. Then you still have only \$1, because there has been no time to accumulate any interest, so that $(1.08)^0$ *must* equal 1.

Are Negative Exponents Useful?

How much money should you have deposited in the bank 5 years ago in order to have \$1000 today, given that the interest rate has been 8% per year over this period? If we call this amount x, the requirement is that $x \cdot (1.08)^5$ must equal \$1000, or that

$$x \cdot (1.08)^5 = 1000$$

The solution for x is

$$x = \frac{1000}{(1.08)^5} = 1000 \cdot (1.08)^{-5}$$

(which is approximately \$681). It turns out that $\$(1.08)^{-5}$ is what you should have deposited 5 years ago in order to have \$1 today, given the constant interest rate of 8%.

In general, $\$P\,(1 + p/100)^{-t}$ is what you should have deposited t years ago in order to have $\$P$ today if the interest rate has been $p\%$ every year.

Problems

1. Compute the following:

a. 6^3 **b.** $\left(\frac{2}{3}\right)^2$ **c.** $(-1)^5$ **d.** $(0.3)^2$

e. $(4.5 - 2.5)^4$ **f.** $2^2 \cdot 2^4$ **g.** $2^2 \cdot 3^2 \cdot 4^2$ **h.** $(2^2 \cdot 3^2)^3$

2. Express the following as powers:

a. $15 \cdot 15 \cdot 15$ b. $\left(-\frac{1}{3}\right)\left(-\frac{1}{3}\right)\left(-\frac{1}{3}\right)$ c. $\frac{1}{10}$

d. 0.0000001 e. $t\,t\,t\,t\,t\,t$ f. $(a-b)(a-b)(a-b)$

g. $a\,a\,b\,b\,b\,b$ h. $(-a)(-a)(-a)$

3. Simplify:

a. $a^4 \cdot a^2$ b. $(a^4)^2$ c. $x^6 \div x^3$ d. $\dfrac{b^2}{b^5}$

e. $(x^2y^3)^3$ f. $\dfrac{x^n \cdot x}{x^{n-1}}$ g. $\dfrac{z^2 \div z^5}{z^3 \cdot z^{-4}}$ h. $\dfrac{3^3 \cdot 3^{-2}}{3^2 \cdot 3^5}$

4. Compute the following:

a. $2^0 \cdot 2^1 \cdot 2^2 \cdot 2^3$ b. $\left(\dfrac{4}{3}\right)^3$ c. $\dfrac{4^2 \cdot 6^2}{3^3 \cdot 2^3}$

d. x^5x^4 e. $y^5y^4y^3$ f. $(2xy)^3$

g. $\dfrac{10^2 \cdot 10^{-4} \cdot 10^3}{10^0 \cdot 10^{-2} \cdot 10^5}$ h. $\dfrac{(k^2)^3k^4}{(k^3)^2}$ i. $\dfrac{(x+1)^3(x+1)^{-2}}{(x+1)^2(x+1)^{-3}}$

5. Which of the following expressions are defined and what are their values?

a. $\dfrac{0}{26}$ b. $\dfrac{2-x}{0}$ c. $0 \cdot 0$ d. 0^{25}

e. $(0+2)^0$ f. 0^{-2} g. $\dfrac{(10)^0}{(0+1)^0}$ h. $\dfrac{(0+1)^0}{(0+2)^0}$

6. Find the solution x of the following equations:

a. $5^2 \cdot 5^x = 5^7$ b. $10^x = 1$ c. $10^x \div 10^5 = 10^{-2}$

d. $(25)^2 = 5^x$ e. $2^{10} - 2^2 \cdot 2^x = 0$ f. $(x+3)^2 = x^2 + 3^2$

7. Which of the following equalities are correct?

a. $3^5 = 5^3$ b. $(5^2)^3 = 5^{2^3}$ c. $(3^3)^4 = (3^4)^3$

d. $0^3 \cdot 4^0 = 0$ e. $(0^{-2})(-2)^0 = 1$ f. $(5+7)^2 = 5^2 + 7^2$

g. $\dfrac{2x+4}{2} = x + 4$ h. $2(x-y) = x \cdot 2 - y \cdot 2$ i. $-x + y = y - x$

8. Which of the following equalities are true and which are false? Justify your answer. (Note: a and b are positive, m and n are integers.)

a. $a^0 = 0$ b. $(a+b)^{-n} = 1/(a+b)^n$ c. $a^m \cdot a^m = a^{2m}$

d. $a^m \cdot b^m = (ab)^{2m}$ e. $(a+b)^m = a^m + b^m$ f. $a^n \cdot b^m = (ab)^{n+m}$

9. Complete the following:

a. $xy = 3 \Longrightarrow x^3y^3 = \cdots$

b. $ab = -2 \Longrightarrow (ab)^4 = \cdots$

c. $a^2 = 4 \Longrightarrow (a^{20})^0 = \cdots$

d. n integer $\Longrightarrow (-1)^{2n} = \cdots$

e. $x^{-1}y^{-1} = 3 \Longrightarrow x^3y^3 = \cdots$

f. $x^7 = 2 \Longrightarrow (x^{-3})^6(x^2)^2 = \cdots$

g. $\left(\frac{xy}{z}\right)^{-2} = 3 \Longrightarrow \left(\frac{z}{xy}\right)^6 = \cdots$

h. $a^{-1}b^{-1}c^{-1} = 1/4 \Longrightarrow (abc)^4 = \cdots$

10. Compute the following:

a. $(2x)^4$ **b.** $(2^{-1} - 4^{-1})^{-1}$ **c.** $\dfrac{24x^3y^2z^3}{4x^2yz^2}$

d. $\left[-(-ab^3)^{-3}(a^6b^6)^2\right]^3$ **e.** $\dfrac{a^5 \cdot a^3 \cdot a^{-2}}{a^{-3} \cdot a^6}$ **f.** $\left[\left(\dfrac{x}{2}\right)^3 \cdot \dfrac{8}{x^{-2}}\right]^{-3}$

g. $\dfrac{a^{2n+3}}{a^{2n-1}}$ **h.** $\dfrac{5^{pq+p}5^{2p}}{5^{q+3p}5^{pq}}$

11. Compute the following:

a. 13% of 150 **b.** 6% of 2400 **c.** 5.5% of 200

12. A box containing 5 balls costs \$8.50. If the balls are bought individually, they cost \$2.00 each. How much cheaper is it, in percentage terms, to buy the box as opposed to buying 5 individual balls?

13. Give economic interpretations for each of the following expressions and then use a calculator to find the approximate values:

a. $50 \cdot (1.11)^8$ **b.** $10\,000 \cdot (1.12)^{20}$ **c.** $5000 \cdot (1.07)^{-10}$

14. Compute 2^{10}. Is 2^{10} bigger than 10^3? Explain on the basis of your answer why 2^{30} is bigger than 10^9. Check by using a calculator.

A.2 Square Roots

So far the power a^x has been defined for integer exponents—that is, when $x = 0$, ± 1, ± 2, ± 3, ... If $a \geq 0$ and $x = 1/2$, then we define $a^x = a^{1/2}$ as equal to $\sqrt{a}$, the **square root** of a. Thus, $a^{1/2} = \sqrt{a}$ is defined as the nonnegative number that when multiplied by itself gives a. This definition makes sense because $a^{1/2} \cdot a^{1/2} = a^{1/2+1/2} = a^1 = a$. Note that a real number multiplied by itself must always be ≥ 0, whether that number is positive, negative, or zero. Therefore, we do not define square roots of negative numbers. For example, $(16)^{1/2} = \sqrt{16} = 4$ because $4 \cdot 4 = 16$, and $(1/9)^{1/2} = \sqrt{1/9} = 1/3$ because $(1/3)(1/3) = 1/9$, while $(-25)^{1/2} = \sqrt{-25}$ is not defined. Usually, the square root of a natural number is an irrational number. For example, $\sqrt{2} \approx 1.414$, $\sqrt{3} \approx 1.732$ are irrational numbers.

Properties [A.4](d) and (e) are also valid for square roots. For instance,

$$\sqrt{16 \cdot 25} = \sqrt{16} \cdot \sqrt{25} = 4 \cdot 5 = 20, \qquad \sqrt{\frac{9}{4}} = \frac{\sqrt{9}}{\sqrt{4}} = \frac{3}{2}$$

These can be formulated alternatively as

$$(16 \cdot 25)^{1/2} = (16)^{1/2} \cdot (25)^{1/2} = 4 \cdot 5 = 20, \qquad \left(\frac{9}{4}\right)^{1/2} = \frac{9^{1/2}}{4^{1/2}} = \frac{3}{2}$$

In general, if a and b are nonnegative numbers with $b \neq 0$, then

$$\text{(a)}\ \sqrt{a \cdot b} = \sqrt{a}\sqrt{b} \qquad \text{(b)}\ \sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}} \qquad \text{[A.6]}$$

Note that the formulas in [A.6] are not valid if a or b or both are negative. For example, $\sqrt{(-1)(-1)} = \sqrt{1} = 1$, whereas $\sqrt{-1} \cdot \sqrt{-1}$ is not defined.

By using a calculator, we find that $\sqrt{2} \div \sqrt{3} \approx 0.816$. Without a calculator, the division $\sqrt{2} \div \sqrt{3} \approx 1.414 \div 1.732$ would be tedious. It becomes easier if we rationalize the denominator—that is, if we multiply both numerator and denominator by the same term in order to remove expressions with roots in the denominator. Thus,

$$\frac{\sqrt{2}}{\sqrt{3}} = \frac{\sqrt{2} \cdot \sqrt{3}}{\sqrt{3} \cdot \sqrt{3}} = \frac{\sqrt{2 \cdot 3}}{3} = \frac{\sqrt{6}}{3} \approx \frac{2.448}{3} = 0.816$$

Example A.2

Rationalize the denominators: (a) $\dfrac{5}{\sqrt{5}}$ (b) $\dfrac{(a+1)\sqrt{a}}{\sqrt{a+1}}$.

Solution

$$\text{(a)}\quad \frac{5}{\sqrt{5}} = \frac{5 \cdot \sqrt{5}}{\sqrt{5} \cdot \sqrt{5}} = \frac{5 \cdot \sqrt{5}}{5} = \sqrt{5}$$

$$\text{(b)}\quad \frac{(a+1) \cdot \sqrt{a}}{\sqrt{a+1}} = \frac{(a+1) \cdot \sqrt{a} \cdot \sqrt{a+1}}{\sqrt{a+1} \cdot \sqrt{a+1}} = \frac{(a+1) \cdot \sqrt{a} \cdot \sqrt{a+1}}{a+1}$$

$$= \sqrt{a} \cdot \sqrt{a+1} = \sqrt{a(a+1)}$$

Note: One of the most common errors committed in elementary algebra is to replace $\sqrt{a+b}$ by $\sqrt{a} + \sqrt{b}$. For example, $\sqrt{9+16} = \sqrt{25} = 5$, whereas $\sqrt{9} + \sqrt{16} = 3 + 4 = 7$. Thus, we have

$$\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$$

The following statistics illustrate just how frequently this error occurs. During an examination for a basic course in mathematics for economists, 43 out of 190 students simplified $\sqrt{1/16 + 1/25}$ incorrectly and claimed that it was equal to $1/4 + 1/5 = 9/20$. (The correct answer is $\sqrt{41/400} = \sqrt{41}/20$.)

Problems

1. Compute the following:

 a. $\sqrt{9}$ b. $\sqrt{1600}$ c. $(100)^{1/2}$ d. $\sqrt{9+16}$

 e. $(36)^{-1/2}$ f. $(0.49)^{1/2}$ g. $\sqrt{0.01}$ h. $\sqrt{\frac{1}{25}}$

2. Solve for x:

 a. $\sqrt{x}=9$ b. $\sqrt{x}\cdot\sqrt{4}=4$ c. $\sqrt{x+2}=25$

 d. $\sqrt{3}\cdot\sqrt{5}=\sqrt{x}$ e. $2^{2-x}=8$ f. $2^x-2^{x-1}=4$

3. Rationalize the denominator and simplify:

 a. $\dfrac{6}{\sqrt{7}}$ b. $\dfrac{\sqrt{32}}{\sqrt{2}}$ c. $\dfrac{\sqrt{3}}{4\sqrt{2}}$ d. $\dfrac{\sqrt{54}-\sqrt{24}}{\sqrt{6}}$

 e. $\dfrac{2}{\sqrt{3}\sqrt{8}}$ f. $\dfrac{4}{\sqrt{2y}}$ g. $\dfrac{x}{\sqrt{2x}}$ h. $\dfrac{x(\sqrt{x}+1)}{\sqrt{x}}$

4. Decide whether each "?" should be replaced by $=$ or $\neq$. Justify your answer. (*Note*: a and b are positive.)

 a. $\sqrt{25\cdot 16}$? $\sqrt{25}\cdot\sqrt{16}$ b. $\sqrt{25+16}$? $\sqrt{25}+\sqrt{16}$

 c. $(a+b)^{1/2}$? $a^{1/2}+b^{1/2}$ d. $(a+b)^{-1/2}$? $(\sqrt{a+b})^{-1}$

A.3 Rules of Algebra

You are probably already familiar with the most common rules of algebra. Nevertheless, it may be useful at this stage to recall those that are most important. If a, b, and c are arbitrary real numbers, then:

(a)	$a+b=b+a$	(g)	$1\cdot a=a$	
(b)	$(a+b)+c=a+(b+c)$	(h)	$aa^{-1}=1$ for $a\neq 0$	
(c)	$a+0=a$	(i)	$(-a)b=a(-b)=-ab$	[A.7]
(d)	$a+(-a)=0$	(j)	$(-a)(-b)=ab$	
(e)	$ab=ba$	(k)	$a(b+c)=ab+ac$	
(f)	$(ab)c=a(bc)$	(l)	$(a+b)c=ac+bc$	

These rules are used in the following examples:

$5+x^2=x^2+5$

$x\frac{1}{3}=\frac{1}{3}x$

$(-3)5=3(-5)=-(3\cdot 5)=-15$

$3x(y+2z)=3xy+6xz$

$(a+2b)+3b=a+(2b+3b)=a+5b$

$(xy)y^{-1}=x(yy^{-1})=x$

$(-6)(-20)=120$

$(t^2+2t)4t^3=t^24t^3+2t4t^3=4t^5+8t^4$

Rules [A.7](k) and (l) can be combined with the others in several ways to give:

$$\text{(m)} \quad a(b-c) = a[b+(-c)] = ab + a(-c) = ab - ac$$

$$\text{(n)} \quad x(a+b-c+d) = xa + xb - xc + xd \qquad \text{[A.8]}$$

$$(a+b)(c+d) = ac + ad + bc + bd$$

A geometric argument for [A.8] (0) requires considering areas in Fig. A.1.

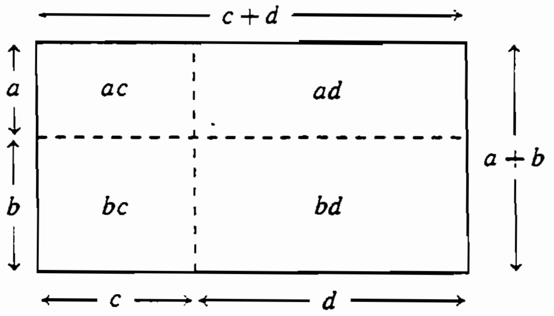


FIGURE A.1

We often encounter parentheses with a minus sign in front. Because $(-1)x = -x$, using (n) gives

$$-(a+b-c+d) = -a - b + c - d \qquad \text{[A.9]}$$

In words: *When removing a pair of parentheses with a minus in front, change the signs of all the terms within the parentheses—do not leave any out.*

Some Important Identities

Three special cases of equation [A.8] are so important that you should memorize them.

$$(a+b)^2 = a^2 + 2ab + b^2 \qquad \text{[A.10]}$$

$$(a-b)^2 = a^2 - 2ab + b^2 \qquad \text{[A.11]}$$

$$(a+b)(a-b) = a^2 - b^2 \qquad \text{[A.12]}$$

Formula [A.12] is often called the *difference-of-squares formula.* (Proof of [A.10]: $(a+b)^2$ means $(a+b)(a+b)$, which according to [A.8] is equal to $aa + ab + ba + bb = a^2 + 2ab + b^2$. Prove [A.11] and [A.12] yourself.)

Example A.3

Use [A.10] to [A.12] to expand each of the following:

(a) $(2x+3y)^2$ (b) $\left(1 - \frac{1}{2}z\right)^2$ (c) $\left(\sqrt{3}+\sqrt{6}\right)\left(\sqrt{3}-\sqrt{6}\right)$

Solution

(a) $(2x+3y)^2 = (2x)^2 + 2\cdot 2x\cdot 3y + (3y)^2 = 4x^2 + 12xy + 9y^2$

(b) $\left(1-\frac{1}{2}z\right)^2 = 1^2 - 2\cdot 1\cdot \frac{1}{2}z + \left(\frac{1}{2}z\right)^2 = 1 - z + \frac{1}{4}z^2$

(c) $\left(\sqrt{3}+\sqrt{6}\right)\left(\sqrt{3}-\sqrt{6}\right) = \left(\sqrt{3}\right)^2 - \left(\sqrt{6}\right)^2 = 3-6 = -3$

Example A.4

Expand $\left(\sqrt{x}+1-\sqrt{x+1}\right)^2$.

Solution Set $a = \sqrt{x}+1$ and $b = \sqrt{x+1}$. Then using [A.11] and [A.10] gives

$$\begin{aligned}\left[(\sqrt{x}+1)-\sqrt{x+1}\right]^2 &= (a-b)^2 = a^2 - 2ab + b^2\\ &= (\sqrt{x}+1)^2 - 2(\sqrt{x}+1)\sqrt{x+1} + (\sqrt{x+1})^2\\ &= x + 2\sqrt{x} + 1 - 2\sqrt{x}\sqrt{x+1} - 2\sqrt{x+1} + x + 1\\ &= 2\left(x+1+\sqrt{x}-\sqrt{x}\sqrt{x+1}-\sqrt{x+1}\right)\end{aligned}$$

Alternatively, set $\left(\sqrt{x}+1-\sqrt{x+1}\right)^2 = (a+b)^2$, where $a=\sqrt{x}$ and $b = 1-\sqrt{x+1}$; then use [A.10] and [A.12]. Do you get the same solution?

In [A.8], we multiplied two factors, $(a+b)$ and $(c+d)$. How do we compute such products when there are several factors? Consider the following:

$$\begin{aligned}(a+b)(c+d)(e+f) &= \left[(a+b)(c+d)\right](e+f)\\ &= \left(ac+ad+bc+bd\right)(e+f)\\ &= (ac+ad+bc+bd)e + (ac+ad+bc+bd)f\\ &= ace+ade+bce+bde+acf+adf+bcf+bdf\end{aligned}$$

Alternatively, write $(a+b)(c+d)(e+f) = (a+b)\left[(c+d)(e+f)\right]$, then expand and show that you get the same answer.

Example A.5

Compute $(r+1)^3$.

Solution

$$(r+1)^3 = \left[(r+1)(r+1)\right](r+1) = (r^2+2r+1)(r+1) = r^3+3r^2+3r+1$$

Let us illustrate this last result with an example. A ball with radius r meters has a volume of $\frac{4}{3}\pi r^3$. By how much does the volume expand if the radius increases by 1 meter? The solution is

$$\frac{4}{3}\pi(r+1)^3 - \frac{4}{3}\pi r^3 = \frac{4}{3}\pi(r^3 + 3r^2 + 3r + 1) - \frac{4}{3}\pi r^3 = \frac{4}{3}\pi(3r^2 + 3r + 1)$$

Algebraic Expressions

Expressions such as $3xy - 5x^2y^3 + 2xy + 6y^3x^2 - 3x + 5yx$ are called *algebraic expressions*. We call $3xy$, $-5x^2y^3$, $2xy$, $6y^3x^2$, $-3x$, and $5yx$ the *terms* in the expression that is formed by adding all the terms together. The numbers 3, -5, 2, 6, -3, and 5 are the *numerical coefficients* of the terms. Two terms where only the numerical coefficients are different, such as $-5x^2y^3$ and $6y^3x^2$, are called *terms of the same type*. In order to simplify expressions, we usually collect terms of the same type. Then within each term, we usually put numerical coefficients first and place the letters in alphabetical order. Thus,

$$3xy - 5x^2y^3 + 2xy + 6y^3x^2 - 3x + 5yx = x^2y^3 + 10xy - 3x$$

Example A.6

Expand and simplify:

(a) $(2pq - 3p^2)(p + 2q) - (q^2 - 2pq)(2p - q)$
(b) $(xy - 3y^2)(x^2y - x^3 + 3xy^2)$

Solution

(a) $(2pq - 3p^2)(p + 2q) - (q^2 - 2pq)(2p - q)$

$$= 2pqp + 2pq2q - 3p^3 - 6p^2q - (q^2 2p - q^3 - 4pqp + 2pq^2)$$
$$= 2p^2q + 4pq^2 - 3p^3 - 6p^2q - 2pq^2 + q^3 + 4p^2q - 2pq^2$$
$$= -3p^3 + q^3$$

(b) $(xy - 3y^2)(x^2y - x^3 + 3xy^2)$

$$= xyx^2y - xyx^3 + xy3xy^2 - 3y^2x^2y + 3y^2x^3 - 3y^2 3xy^2$$
$$= x^3y^2 - x^4y + 3x^2y^3 - 3x^2y^3 + 3x^3y^2 - 9xy^4$$
$$= -x^4y + 4x^3y^2 - 9xy^4$$

Problems

1. Simplify the following:

a. $-3 + (-4) - (-8)$ **b.** $(-3)(2 - 4)$ **c.** $(-3)(-12)\left(-\frac{1}{2}\right)$
d. $-3[4 - (-2)]$ **e.** $-3(-x - 4)$ **f.** $(5x - 3y)9$

g. $2x\left(\frac{3}{2x}\right)$ **h.** $0 \cdot (1 - x)$ **i.** $-7x\frac{2}{14x}$

In Problems 2–6, expand and collect terms.

2. **a.** $5a^2 - 3b - (-a^2 - b) - 3(a^2 + b)$
b. $-x(2x - y) + y(1 - x) + 3(x + y)$
c. $12t^2 - 3t + 16 - 2(6t^2 - 2t + 8)$
d. $r^3 - 3r^2s + 3rs^2 + s^3 - (-s^3 - r^3 + 3r^2s)$

3. **a.** $-3(n^2 - 2n + 3)$ **b.** $x^2(1 + x^3)$
c. $(4n - 3)(n - 2)$ **d.** $6a^2b(5ab - 3ab^2)$
e. $(a^2b - ab^2)(a + b)$ **f.** $(x - y)(x - 2y)(x - 3y)$

4. **a.** $a(a - 1)$ **b.** $(x - 3)(x + 7)$ **c.** $-\sqrt{3}\left(\sqrt{3} - \sqrt{6}\right)$
d. $\left(1 - \sqrt{2}\right)^2$ **e.** $(x - 1)^3$ **f.** $(1 - b^2)(1 + b^2)$
g. $(1 + x + x^2 + x^3)(1 - x)$ **h.** $(1 + x)^4$

5. **a.** $3(x - y) + (3y - x)$ **b.** $(a - 2b)^2$
c. $\left(\frac{1}{2}x - \frac{1}{3}y\right)\left(\frac{1}{2}x + \frac{1}{3}y\right)$ **d.** $2x^2y - 3x - (2 + 3x^2y)$
e. $(x + a)(x + b)$ **f.** $(x - 2y)^3$

6. **a.** $(2t - 1)(t^2 - 2t + 1)$ **b.** $(a + 1)^2 + (a - 1)^2 - 2(a + 1)(a - 1)$
c. $(x + y + z)^2$ **d.** $(x + y + z)^2 - (x - y - z)^2$

7. Use [A.10] to [A.12] to expand each of the following:

a. $(3x + 2y)^2$ **b.** $\left(\sqrt{3} + \sqrt{2}\right)^2$
c. $(-3u + 8v)^2$ **d.** $(u - 5v)(u + 5v)$

8. Compute $(1000)^2/[(252)^2 - (248)^2]$ without using a calculator.

9. Expand and collect terms:

a. $(x^2 - y^2)^2$ **b.** $\frac{1}{\left(\sqrt{5} + \sqrt{3}\right)\left(\sqrt{5} - \sqrt{3}\right)}$ **c.** $(a - b + 1)^2$
d. $(\sqrt{a} - \sqrt{b})^2$ **e.** $\left[(\sqrt{2} + 1)(\sqrt{2} - 1)\right]^{100}$ **f.** $(n - 1)^4$

10. Expand and collect terms:

a. $(ax + b)(cx + d)$ **b.** $(2 - t^2)(2 + t^2)$
c. $(a + b + c)^2$ **d.** $(a^5 - b^5)(a^5 + b^5)$
e. $(\sqrt{3} + \sqrt{5} + \sqrt{7})(\sqrt{3} + \sqrt{5} - \sqrt{7})$ **f.** $(u - v)^2(u + v)^2$

11. Use the diagrams in Fig. A.2 to give a geometric interpretation of [A.10] and [A.11].

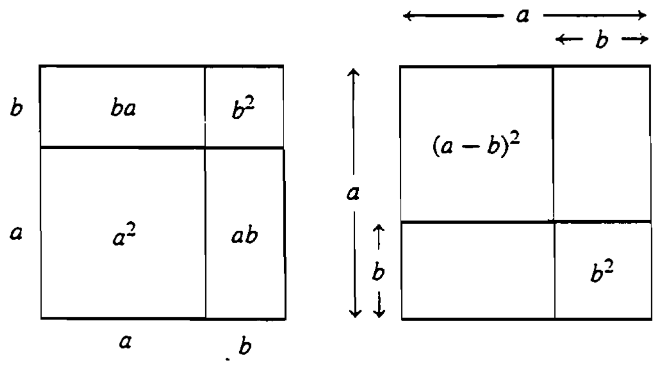


FIGURE A.2

Harder Problems

12. Show that

$$(a-b)(a+b) = a^2 - b^2$$

$$(a-b)(a^2+ab+b^2) = a^3 - b^3$$

$$(a-b)(a^3+a^2b+ab^2+b^3) = a^4 - b^4$$

$$(a-b)(a^4+a^3b+a^2b^2+ab^3+b^4) = a^5 - b^5$$

Guess the result of the division $(a^{10} - b^{10})/(a-b)$, and try to verify your guess.

A.4 Factors

When we write $49 = 7 \cdot 7$, $125 = 5 \cdot 5 \cdot 5$, or $672 = 2 \cdot 3 \cdot 4 \cdot 4 \cdot 7$, we say that we have *factored* these numbers. Algebraic expressions can often be factored in a similar way. For example,

$$6x^2y = 2 \cdot 3 \cdot x \cdot x \cdot y \qquad \text{and} \qquad 5x^2y^3 - 15xy^2 = 5 \cdot x \cdot y \cdot y(xy-3)$$

Example A.7

Factor each of the following:

(a) $5x^2 + 15x$ (b) $-18b^2 + 9ab$

(c) $K(1+r) + K(1+r)r$ (d) $\delta L^{-3} + (1-\delta)L^{-2}$

Solution

(a) $5x^2 + 15x = 5x(x + 3)$

(b) $-18b^2 + 9ab = 9ab - 18b^2 = 3 \cdot 3b(a - 2b)$

(c) $K(1 + r) + K(1 + r)r = K(1 + r)(1 + r) = K(1 + r)^2$

(d) $\delta L^{-3} + (1 - \delta)L^{-2} = L^{-3}[\delta + (1 - \delta)L]$

Formulas [A.10] to [A.12] can often be used "in reverse" for factorization. They sometimes enable us to factor expressions that otherwise appear to have no factors.

Example A.8
Factor each of the following:

(a) $16a^2 - 1$ (b) $x^2y^2 - 25z^2$ (c) $4u^2 + 8u + 4$ (d) $x^2 - x + \frac{1}{4}$

Solution

(a) $16a^2 - 1 = (4a + 1)(4a - 1)$ (apply [A.12])

(b) $x^2y^2 - 25z^2 = (xy + 5z)(xy - 5z)$ (apply [A.12])

(c) $4u^2 + 8u + 4 = (2u + 2)^2 = 4(u + 1)^2$ (apply [A.10])

(d) $x^2 - x + \frac{1}{4} = (x - \frac{1}{2})^2$ (apply [A.11])

Usually, it is easy to verify that an algebraic expression has been factored correctly by simply multiplying the factors. For example, we check that

$$x^2 - (a + b)x + ab = (x - a)(x - b) \qquad [A.13]$$

by expanding $(x - a)(x - b)$. Formula [A.13] itself is important because it can often be used to factor quadratic expressions.

Example A.9
Factor (if possible) the following:

(a) $x^2 - 8x + 15$ (b) $x^2 + 5x + 6$ (c) $x^2 + 2x + 2$

Solution

(a) Comparing this with [A.13], we see that making $x^2 - 8x + 15$ equal to $(x - a)(x - b)$ requires that $a + b = 8$ and $ab = 15$. We need to find two numbers, a and b, whose sum is 8 and whose product is 15. Such numbers are $a = 3$ and $b = 5$ (or $a = 5$ and $b = 3$), so the factorization is

$$x^2 - 8x + 15 = (x - 3)(x - 5)$$

(b) In a similar way, we find that $x^2 + 5x + 6 = (x+2)(x+3)$.

(c) In this case, we need two numbers, a and b, whose sum is -2 and whose product is 2. Because there is no pair of real numbers with these properties, factorization is not possible.

Section A.8 will investigate factorization of expressions like $ax^2 + bx + c$ more systematically.

We conclude this section with some examples where a suitable grouping of terms is the crucial point.

Example A.10

Try to factor the following:

(a) $x^2 + 2xy^2 + xy + 2y^3$ (b) $a^3 - 4b^3 - 4ab^2 + a^2b$

Solution

(a) This is quite simple:

$$\begin{aligned} x^2 + 2xy^2 + xy + 2y^3 &= (x^2 + 2xy^2) + (xy + 2y^3) \\ &= (x + 2y^2)x + (x + 2y^2)y \\ &= (x + 2y^2)(x + y) \end{aligned}$$

(b) This demands some careful rearrangement of terms:

$$\begin{aligned} a^3 - 4b^3 - 4ab^2 + a^2b &= a^3 - 4ab^2 + a^2b - 4b^3 \\ &= a(a^2 - 4b^2) + b(a^2 - 4b^2) \\ &= (a + b)(a^2 - 4b^2) \\ &= (a + b)(a + 2b)(a - 2b) \end{aligned}$$

Note: If we write $15+25 = 3\cdot5+5\cdot5$, then we have factored 15 and 25, but *not* the sum 15+25. Correspondingly, $9x^2-25y^2 = 3\cdot3\cdot x\cdot x-5\cdot5\cdot y\cdot y$ is *not* a factorization of $9x^2 - 25y^2$. (The correct factorization is $9x^2 - 25y^2 = (3x - 5y)(3x + 5y)$.)

Problems

In Problems 1 to 3, factor the given expressions.

1. **a.** $28a^2b^3$ **b.** $4x + 8y - 24z$ **c.** $2x^2 - 6xy$
   d. $4a^2b^3 + 6a^3b^2$ **e.** $7x^2 - 49xy$ **f.** $5xy^2 - 45x^3y^2$
   g. $16 - b^2$ **h.** $3x^2 - 12$

2. a. $x^2 - 4x + 4$ b. $4t^2s - 8ts^2$
   c. $16a^2 + 16ab + 4b^2$ d. $5x^4 - 10x^2y^2$
3. a. $K^3 - K^2L$ b. $KL^3 + KL$ c. $L^2 - K^2$
   d. $K^2 - 2KL + L^2$ e. $K^3L - 4K^2L^2 + 4KL^3$ f. $K^{-\rho} - K^{-2\rho}$
4. Use the method in Example A.9 to factor the following:
   a. $x^2 + 5x + 6$ b. $x^2 + 2x - 15$ c. $p^2 - 3p + 2$
   d. $2q^2 + 16q - 66$ e. $3x^2 - 12x - 15$ f. $-p^2 - p + 20$
5. Factor (see Example A.10) the following:
   a. $5(x + 2y) + a(x + 2y)$ b. $(a + b)c - d(a + b)$
   c. $ax + ay + 2x + 2y$ d. $2x^2 - 5yz + 10xz - xy$
   e. $p^2 - q^2 + p - q$ f. $u^3 + v^3 - u^2v - v^2u$
6. Factor the following:
   a. $p + prt$ b. $\pi r^2 + 2\pi rh$ c. $\frac{1}{4}mnr - \frac{1}{4}mn$
   d. $PR^2 - PQR$ e. $(1 + r)^2 + (1 + r)^3$ f. $\frac{1}{2}gh^4 - \frac{1}{8}g^3h^2$

Harder Problems

7. Factor the following:
   a. $a^{2n} - 25$ b. $(a + b)^2 - c^2$ c. $(a + b + c)^2 - (a + b)^2$
   d. $x^4 - y^4$ e. $81 - c^4$ f. $x^6 - y^6$
   (Hint for (d): $x^4 - y^4 = (x^2 + y^2)(x^2 - y^2)$ and so on.)

A.5 Fractions

Recall that

$$a \div b = \frac{a}{b} \quad \begin{array}{l} \leftarrow \text{numerator} \\ \leftarrow \text{denominator} \end{array}$$

For example, $5 \div 8 = \frac{5}{8}$. For typographical reasons, we often write 5/8 instead of $\frac{5}{8}$. Of course, $5 \div 8 = 0.625$, and $5/8 = 0.625$. In this case, we have written the fraction as a decimal number. Fraction 5/8 is called a *proper fraction* because 5 is less than 8. Fraction 19/8 is an *improper fraction* because the numerator is larger than (or equal to) the denominator. An improper fraction can be written as a *mixed number:*

$$\tfrac{19}{8} = 2 + \tfrac{3}{8} = 2\tfrac{3}{8}$$

Note: $2\frac{3}{8}$ means 2 *plus* 3/8. On the other hand, $2 \cdot \frac{3}{8} = \frac{2 \cdot 3}{8} = \frac{3}{4}$ (by the rules reviewed in what follows). Note, however, that $2\frac{x}{8}$ means $2 \cdot \frac{x}{8}$; the notation $\frac{2x}{8}$

or $2x/8$ is obviously preferable in this case. Indeed, $\frac{19}{8}$ is probably better than $2\frac{3}{8}$ because it also helps avoid ambiguity.

Note that $0 \div 5 = 0$. In general, $0 \div b = 0$ whatever the number b, except if $b = 0$. On the other hand, $b \div 0$ is not defined for any number b.

$$\frac{0}{b} = 0 \quad (b \neq 0), \qquad \frac{b}{0} \text{ is not defined}$$

Reducing and Extending Fractions

You should know that

$$\frac{a \cdot c}{b \cdot c} = \frac{a}{b} \quad (b \neq 0 \text{ and } c \neq 0) \qquad [A.14]$$

In general, we reduce fractions by factoring the numerator and the denominator and canceling *common factors* (that is, dividing both the numerator and denominator by the same nonzero quantity). Thus:

(a) $\frac{189}{135} = \frac{\cancel{3} \cdot \cancel{3} \cdot \cancel{3} \cdot 7}{\cancel{3} \cdot \cancel{3} \cdot \cancel{3} \cdot 5} = \frac{7}{5}$

(b) $\frac{5x^2yz^3}{25xy^2z} = \frac{\cancel{5} \cdot \cancel{x} \cdot x \cdot \cancel{y} \cdot \cancel{z} \cdot z \cdot z}{\cancel{5} \cdot 5 \cdot \cancel{x} \cdot \cancel{y} \cdot y \cdot \cancel{z}} = \frac{xz^2}{5y}$

(c) $\frac{x^2 + xy}{x^2 - y^2} = \frac{x(x+y)}{(x-y)(x+y)} = \frac{x}{x-y}$

(d) $\frac{4 - 4a + a^2}{a^2 - 4} = \frac{(a-2)(a-2)}{(a-2)(a+2)} = \frac{a-2}{a+2}$

When we use property [A.14] in reverse, we are *expanding* the fraction:

$$\frac{5}{8} = \frac{5 \cdot 125}{8 \cdot 125} = \frac{625}{1000} = 0.625$$

$$\frac{1}{\sqrt{5} + \sqrt{3}} = \frac{\sqrt{5} - \sqrt{3}}{(\sqrt{5} + \sqrt{3})(\sqrt{5} - \sqrt{3})} = \frac{\sqrt{5} - \sqrt{3}}{5 - 3} = \frac{1}{2}(\sqrt{5} - \sqrt{3})$$

Note the trick we used in the last example to make the denominator rational.

When we simplify fractions, only common factors can be removed. Two frequently occurring errors are illustrated in the following examples:

$$\textbf{Wrong!} \rightarrow \quad \frac{2\cancel{x} + 3y}{\cancel{x}y} = \frac{2 + 3\cancel{y}}{\cancel{y}} = \frac{2+3}{1} = 5$$

and

$$\textbf{Wrong!} \rightarrow \quad \frac{x-1}{x^2-1} = \frac{x-1}{(x-1)(x+1)} = \frac{0}{x+1} = 0$$

In the first case, we cannot simplify the expression because the numerator and the denominator do not have common factors. In the second case, the correct way of reducing the fraction gives the answer $1/(x+1)$.

Sign Rules

The following sign rules are important:

$$\frac{-a}{-b} = \frac{(-a)\cdot(-1)}{(-b)\cdot(-1)} = \frac{a}{b} \quad \text{and} \quad -\frac{a}{b} = (-1)\frac{a}{b} = \frac{(-1)a}{b} = \frac{-a}{b} \qquad [A.15]$$

These equalities are derived from [A.14], the equation $-x = (-1)x$, and property [A.19], which follows.

Addition of Fractions

Here are three basic rules for adding fractions:

$$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c} \qquad [A.16]$$

$$\frac{a}{b} + \frac{c}{d} = \frac{a\cdot d + b\cdot c}{b\cdot d} \qquad [A.17]$$

$$a + \frac{c}{d} = \frac{a\cdot d + c}{d} \qquad [A.18]$$

Because $a/1 = a$, [A.18] follows from [A.17] by letting $b = 1$. Formula [A.17] follows from [A.14] and [A.16]:

$$\frac{a}{b} + \frac{c}{d} = \frac{a\cdot d}{b\cdot d} + \frac{c\cdot b}{d\cdot b} = \frac{a\cdot d + b\cdot c}{b\cdot d}$$

Example A.11

(a) $\frac{5}{3} + \frac{13}{3} = \frac{18}{3} = 6$

(b) $\frac{3}{5} + \frac{1}{6} = \frac{3\cdot 6 + 5\cdot 1}{5\cdot 6} = \frac{23}{30}$

(c) $\frac{a-1}{6a} + \frac{a}{6a} + \frac{1}{6a} = \frac{a-1+a+1}{6a} = \frac{2a}{6a} = \frac{2\cdot a}{2\cdot 3\cdot a} = \frac{1}{3}$

Combining [A.16], [A.17], and [A.18] with [A.15] gives

(d) $\frac{a}{c} + \frac{b}{c} - \frac{d}{c} = \frac{a+b-d}{c}$

(e) $\frac{a}{b} - \frac{c}{d} + \frac{e}{f} = \frac{adf}{bdf} - \frac{cbf}{bdf} + \frac{ebd}{bdf} = \frac{adf - cbf + ebd}{bdf}$

If the numbers b, d, and f have common factors, the computation carried out in (e) involves unnecessarily large numbers. We can simplify the process by first finding the least common denominator (LCD) of the fractions. To do so, factor each denominator completely; the LCD is the product of all the distinct factors that appear in any denominator, each raised to the highest power to which it gets raised in any denominator. The use of the LCD is demonstrated in the following example.

Example A.12

Simplify the following:

$$\text{(a)}\ \frac{1}{2}-\frac{1}{3}+\frac{1}{6} \qquad \text{(b)}\ \frac{2+a}{a^2b}+\frac{1-b}{ab^2}-\frac{2b}{a^2b^2}$$

$$\text{(c)}\ \frac{x-y}{x+y}-\frac{x}{x-y}+\frac{3xy}{x^2-y^2}$$

Solution

(a) The LCD is 6 and so

$$\frac{1}{2}-\frac{1}{3}+\frac{1}{6}=\frac{1\cdot 3}{2\cdot 3}-\frac{1\cdot 2}{2\cdot 3}+\frac{1}{2\cdot 3}=\frac{3-2+1}{6}=\frac{2}{6}=\frac{1}{3}$$

(b) The LCD is a^2b^2 and so

$$\frac{2+a}{a^2b}+\frac{1-b}{ab^2}-\frac{2b}{a^2b^2}=\frac{(2+a)b}{a^2b^2}+\frac{(1-b)a}{a^2b^2}-\frac{2b}{a^2b^2}$$

$$=\frac{2b+ab+a-ba-2b}{a^2b^2}=\frac{a}{a^2b^2}=\frac{1}{ab^2}$$

(c) The LCD is $(x+y)(x-y)$ and so

$$\frac{x-y}{x+y}-\frac{x}{x-y}+\frac{3xy}{x^2-y^2}=\frac{(x-y)(x-y)}{(x-y)(x+y)}-\frac{x(x+y)}{(x-y)(x+y)}+\frac{3xy}{(x-y)(x+y)}$$

$$=\frac{x^2-2xy+y^2-x^2-xy+3xy}{(x-y)(x+y)}=\frac{y^2}{x^2-y^2}$$

An Important Note

What do we mean by $1-\frac{5-3}{2}$? It means that from the number 1, we subtract the number $\frac{5-3}{2}=\frac{2}{2}=1$. Therefore, $1-\frac{5-3}{2}=0$. Alternatively,

$$1-\frac{5-3}{2}=\frac{2}{2}-\frac{(5-3)}{2}=\frac{2-(5-3)}{2}=\frac{2-5+3}{2}=\frac{0}{2}=0$$

In the same way,

$$\frac{2+b}{ab^2}-\frac{a-2}{a^2b}$$

means that we subtract

$$\frac{a-2}{a^2b} \quad \text{from} \quad \frac{2+b}{ab^2}$$

$$\frac{2+b}{ab^2} - \frac{a-2}{a^2b} = \frac{(2+b)a}{a^2b^2} - \frac{(a-2)b}{a^2b^2} = \frac{(2+b)a-(a-2)b}{a^2b^2} = \frac{2(a+b)}{a^2b^2}$$

It is a good idea first to enclose in parentheses the numerators of the fractions that are being subtracted.

Example A.13
Simplify the expression

$$\frac{x-1}{x+1} - \frac{1-x}{x-1} - \frac{-1+4x}{2(x+1)}$$

Solution

$$\frac{x-1}{x+1} - \frac{1-x}{x-1} - \frac{-1+4x}{2(x+1)} = \frac{(x-1)}{x+1} - \frac{(1-x)}{x-1} - \frac{(-1+4x)}{2(x+1)}$$

$$= \frac{2(x-1)^2 - 2(1-x)(x+1) - (-1+4x)(x-1)}{2(x+1)(x-1)}$$

$$= \frac{2(x^2-2x+1) - 2(1-x^2) - (4x^2-5x+1)}{2(x+1)(x-1)}$$

$$= \frac{(x-1)}{2(x+1)(x-1)} = \frac{1}{2(x+1)}$$

Multiplication and Division of Fractions

Here are three basic rules for multiplication and division of fractions:

$$a \cdot \frac{b}{c} = \frac{a \cdot b}{c} \qquad \text{[A.19]}$$

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d} \qquad \text{[A.20]}$$

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{a \cdot d}{b \cdot c} \qquad \text{[A.21]}$$

Example of [A.19]:

$$3 \cdot \frac{b}{c} = \frac{b}{c} + \frac{b}{c} + \frac{b}{c} = \frac{b+b+b}{c} = \frac{3b}{c}$$

We prove [A.21] by writing $(a/b) \div (c/d)$ as a ratio of fractions:

$$\frac{a}{b} \div \frac{c}{d} = \frac{\frac{a}{b}}{\frac{c}{d}} = \frac{b \cdot d \cdot \frac{a}{b}}{b \cdot d \cdot \frac{c}{d}} = \frac{\frac{\cancel{b} \cdot d \cdot a}{\cancel{b}}}{\frac{b \cdot \cancel{d} \cdot c}{\cancel{d}}} = \frac{d \cdot a}{b \cdot c} = \frac{a \cdot d}{b \cdot c} = \frac{a}{b} \cdot \frac{d}{c}$$

Example A.14

Simplify the following:

(a) $6 \cdot \frac{5}{13}$ (b) $\frac{4}{7} \cdot \frac{5}{8}$ (c) $\frac{3}{8} \div \frac{6}{14}$ (d) $2xy^2 \cdot \frac{x+2y}{3xy^3}$

(e) $\frac{1-36x^2}{18x+9y} \div \frac{(2-12x)(1+6x)}{16x^2-4y^2}$ (f) $\frac{\frac{a}{2}-\frac{a}{5}}{\frac{a}{5}-\frac{a}{6}}$

Solution

(a) $6 \cdot \frac{5}{13} = \frac{6 \cdot 5}{13} = \frac{30}{13} = 2\frac{4}{13}$

(b) $\frac{4}{7} \cdot \frac{5}{8} = \frac{4 \cdot 5}{7 \cdot 8} = \frac{\cancel{4} \cdot 5}{7 \cdot 2 \cdot \cancel{4}} = \frac{5}{14}$

(c) $\frac{3}{8} \div \frac{6}{14} = \frac{3}{8} \cdot \frac{14}{6} = \frac{\cancel{3} \cdot \cancel{2} \cdot 7}{\cancel{2} \cdot 2 \cdot 2 \cdot 2 \cdot \cancel{3}} = \frac{7}{8}$

(d) $2xy^2 \cdot \frac{x+2y}{3xy^3} = \frac{2\cancel{x}\cancel{y^2}(x+2y)}{3\cancel{x}y^{\cancel{3}}} = \frac{2(x+2y)}{3y}$

(e) $\frac{1-36x^2}{18x+9y} \div \frac{(2-12x)(1+6x)}{16x^2-4y^2}$

$$= \frac{(1-36x^2)}{(18x+9y)} \cdot \frac{(16x^2-4y^2)}{(2-12x)(1+6x)}$$

$$= \frac{(1-36x^2) \cdot 4(4x^2-y^2)}{9(2x+y) \cdot 2 \cdot (1-6x)(1+6x)}$$

$$= \frac{(1-6x)(1+6x) \cdot 4(2x-y)(2x+y)}{9(2x+y) \cdot 2 \cdot (1-6x)(1+6x)} = \frac{2(2x-y)}{9}$$

(f) $$\frac{\frac{a}{2}-\frac{a}{5}}{\frac{a}{5}-\frac{a}{6}} = \frac{\frac{5a}{10}-\frac{2a}{10}}{\frac{6a}{30}-\frac{5a}{30}} = \frac{\frac{3a}{10}}{\frac{a}{30}} = \frac{\frac{3a}{10} \cdot 30}{\frac{a}{30} \cdot 30} = \frac{9a}{a} = 9$$

When we deal with fractions of fractions, we should be sure to emphasize which is the fraction line of the dominant fraction. For example,

$$\frac{a}{\frac{b}{c}} \text{ means } a \div \left(\frac{b}{c}\right) = \frac{ac}{b} \qquad \text{whereas} \qquad \frac{\frac{a}{b}}{c} \text{ means } \frac{a}{b} \div c = \frac{a}{bc}$$

However, the first should be written as $\frac{a}{b/c}$, and the second as $\frac{a/b}{c}$.

Problems

In Problems 1 to 3, simplify the various expressions.

1. **a.** $\frac{3}{7}+\frac{4}{7}-\frac{5}{7}$ **b.** $\frac{3}{4}+\frac{4}{3}-1$ **c.** $\frac{3}{12}-\frac{1}{24}$ **d.** $\frac{1}{5}-\frac{2}{25}-\frac{3}{75}$

e. $3\frac{3}{5}-1\frac{4}{5}$ **f.** $\frac{3}{5}\cdot\frac{5}{6}$ **g.** $\left(\frac{3}{5}\div\frac{2}{15}\right)\cdot\frac{1}{9}$ **h.** $\frac{\frac{2}{3}+\frac{1}{4}}{\frac{3}{4}+\frac{3}{2}}$

2. **a.** $\frac{x}{10}-\frac{3x}{10}+\frac{17x}{10}$ **b.** $\frac{9a}{10}-\frac{a}{2}+\frac{a}{5}$ **c.** $\frac{b+2}{10}-\frac{3b}{15}+\frac{b}{10}$

d. $\frac{x+2}{3}+\frac{1-3x}{4}$ **e.** $\frac{3}{2b}-\frac{5}{3b}$ **f.** $\frac{3a-2}{3a}-\frac{2b-1}{2b}+\frac{4b+3a}{6ab}$

3. **a.** $\frac{1}{x-2}-\frac{1}{x+2}$ **b.** $\frac{6x+25}{4x+2}-\frac{6x^2+x-2}{4x^2-1}$

c. $\frac{18b^2}{a^2-9b^2}-\frac{a}{a+3b}+2$ **d.** $\frac{1}{8ab}-\frac{1}{8b(a+2)}+\frac{1}{b(a^2-4)}$

e. $\frac{2t-t^2}{t+2}\cdot\left(\frac{5t}{t-2}-\frac{2t}{t-2}\right)$ **f.** $2-\frac{a\left(1-\frac{1}{2a}\right)}{0.25}$

4. If $x = 3/7$ and $y = 1/14$, find the simplest forms of these fractions:

a. $x+y$ **b.** $\frac{x}{y}$ **c.** $\frac{x-y}{x+y}$ **d.** $\frac{13(2x-3y)}{2x+1}$

5. Reduce the following expressions by making the denominators rational:

a. $\frac{1}{\sqrt{7}+\sqrt{5}}$ **b.** $\frac{\sqrt{5}-\sqrt{3}}{\sqrt{5}+\sqrt{3}}$ **c.** $\frac{x}{\sqrt{3}-2}$

d. $\frac{x\sqrt{y}-y\sqrt{x}}{x\sqrt{y}\div y\sqrt{x}}$ **e.** $\frac{h}{\sqrt{x+h}-\sqrt{x}}$ **f.** $\frac{1-\sqrt{x+1}}{1+\sqrt{x+1}}$

6. Simplify the following:

a. $\frac{2}{x}+\frac{1}{x\div 1}-3$ **b.** $\frac{t}{2t+1}-\frac{t}{2t-1}$ **c.** $\frac{3x}{x+2}-\frac{4x}{2-x}-\frac{2x-1}{x^2-4}$

d. $\dfrac{\dfrac{1}{x}+\dfrac{1}{y}}{\dfrac{1}{xy}}$ e. $\dfrac{\dfrac{1}{(x+h)^2}-\dfrac{1}{x^2}}{h}$ f. $\dfrac{\dfrac{10x^2}{x^2-1}}{\dfrac{5x}{x+1}}$

7. Prove that $x^2+2xy-3y^2=(x+3y)(x-y)$, and then simplify:

$$\frac{x-y}{x^2+2xy-3y^2}-\frac{2}{x-y}-\frac{7}{x+3y}$$

8. Simplify the following expressions:

a. $n-\dfrac{n}{1-\dfrac{1}{n}}$ **b.** $\dfrac{\dfrac{1}{x-1}+\dfrac{1}{x^2-1}}{x-\dfrac{2}{x+1}}$

9. Simplify the following expressions:

a. $\left(\dfrac{1}{4}-\dfrac{1}{5}\right)^{-2}$ **b.** $\dfrac{1}{1+x^{p-q}}+\dfrac{1}{1+x^{q-p}}$ **c.** $\dfrac{a^{-2}-b^{-2}}{a^{-1}-b^{-1}}$

10. Reduce the following fractions:

a. $\dfrac{25a^3b^2}{125ab}$ **b.** $\dfrac{x^2-y^2}{x+y}$ **c.** $\dfrac{4a^2-12ab+9b^2}{4a^2-9b^2}$ **d.** $\dfrac{4x-x^3}{4-4x+x^2}$

A.6 Simple Equations and How to Solve Them

Some equations can be solved easily. Consider, for example, the equation $3x+10=28$. To solve this, we ask: What number must be added to 10 to get 28? Answer: 18. Hence, $3x=18$. Because 3 times x is 18, x must be 6. There are no other solutions.

In more complicated cases, we need a more systematic procedure for solving equations. Two equations that have exactly the same solutions are called *equivalent*. The main principles used in solving equations are summarized as follows:

> We get equivalent equations if on both sides of the equality sign we do the following:
>
> **(a)** add the same number
> **(b)** subtract the same number
> **(c)** multiply by the same number $\neq 0$
> **(d)** divide by the same number $\neq 0$

[A.22]

In order to solve simple equations, we can use [A.22] as follows. First, apply rules (a) and (b) to isolate all terms containing the unknown on the same side of the equality sign. Next, combine all the terms containing the unknown. Finally, use rules (c) and (d) to find the unknown.

The principle is illustrated by the following examples.

Example A.15

Solve the equation $3x + 10 = x + 4$.

Solution By using the rules in [A.22] systematically, we obtain:

$$3x + 10 = x + 4$$
$$3x + 10 - 10 = x + 4 - 10$$
$$3x = x - 6$$
$$3x - x = x - x - 6$$
$$2x = -6$$
$$\frac{2x}{2} = \frac{-6}{2}$$
$$x = -3$$

With experience, it is possible to shorten the number of steps involved:

$$3x + 10 = x + 4 \iff 3x - x = 4 - 10 \iff 2x = -6 \iff x = -3$$

Here the *equivalence arrow* $\iff$ means: "has the same solutions as."

When faced with more complicated equations involving parentheses and fractions, we usually begin by multiplying out the parentheses, and then we multiply both sides of the equation by the common denominator for all the fractions. We illustrate the procedure in the following example.

Example A.16

Solve the equation $6p - \frac{1}{2}(2p - 3) = 3(1 - p) - \frac{7}{6}(p + 2)$.

Solution

$$6p - p + \tfrac{3}{2} = 3 - 3p - \tfrac{7}{6}p - \tfrac{7}{3}$$
$$36p - 6p + 9 = 18 - 18p - 7p - 14$$
$$55p = -5$$
$$p = -\frac{5}{55} = -\frac{1}{11}$$

The next two examples show that at times great care is needed to find the right solutions.

Example A.17

Solve the equation

$$\frac{x+2}{x-2} - \frac{8}{x^2-2x} = \frac{2}{x}$$

Solution The equation is equivalent to

$$\frac{x+2}{x-2} - \frac{8}{x(x-2)} = \frac{2}{x}$$

We see that $x = 2$ and $x = 0$ both make the equation absurd, because at least one of the denominators becomes 0. If $x \neq 0$ and $x \neq 2$, we can multiply both sides of the equation by the common denominator $x(x-2)$ to obtain

$$\frac{x+2}{x-2} \cdot x(x-2) - \frac{8}{x(x-2)} \cdot x(x-2) = \frac{2}{x} \cdot x(x-2)$$

$$(x+2)x - 8 = 2(x-2)$$

$$x^2 + 2x - 8 = 2x - 4$$

$$x^2 = 4$$

Because $2^2 = 4$ and $(-2)^2 = (-2)(-2) = 4$, both $x = 2$ and $x = -2$ satisfy the last equation. But because $x = 2$ makes the original equation absurd, *only* $x = -2$ *is a solution.*

Example A.18

Solve the equation

$$\frac{z}{z-5} + \frac{1}{3} = \frac{-5}{5-z}$$

Solution We see that z cannot be 5. In order to remember this, we often write

$$\frac{z}{z-5} + \frac{1}{3} = \frac{-5}{5-z} \qquad (z \neq 5)$$

We continue by multiplying both sides by $3(z-5)$. This gives

$$3z + z - 5 = 15$$

which has the unique solution $z = 5$. Because we had to assume $z \neq 5$, we must conclude that no solution exists for this equation.

It is often the case that in order to solve a problem, especially in economics, you must first formulate an appropriate algebraic equation.

Example A.19

A firm manufactures a commodity that costs \$20 per unit to produce. In addition, the firm has fixed costs of \$2000. Each unit is sold for \$75. How many units must be sold if the firm is to have a profit of \$14,500?

Solution If the number of units produced and sold is denoted by Q, then the revenue of the firm is $75Q$ and the total cost of production is $20Q+2000$. Because profit is the difference between total revenue and total cost, it can be written as

$$75Q - (20Q + 2000)$$

Because we want the profit to be 14,500, the equation

$$75Q - (20Q + 2000) = 14,500$$

must be satisfied. It is easy to find the solution $Q = 16,500/55 = 300$ units.

Problems

In Problems 1 to 3, solve the equations.

1. **a.** $5x - 10 = 15$
   b. $2x - (5 + x) = 16 - (3x + 9)$
   c. $-5(3x - 2) = 16(1 - x)$
   d. $4x + 2(x - 4) - 3 = 2(3x - 5) - 1$
   e. $\frac{2}{3}x = -8$
   f. $(8x - 7)5 - 3(6x - 4) + 5x^2 = (x + 1)(5x + 2)$
   g. $x^2 + 10x + 25 = 0$
   h. $(3x - 1)^2 + (4x + 1)^2 = (5x - 1)(5x + 1) + 1$
2. **a.** $3x + 2 = 11$ **b.** $-3x = 21$
   c. $3x = \frac{1}{4}x - 7$ **d.** $\frac{x-3}{4} + 2 = 3x$
   e. $\frac{1}{2x+1} = \frac{1}{x+2}$ **f.** $\sqrt{2x + 14} = 16$
3. **a.** $\frac{x-3}{x+3} = \frac{x-4}{x+4}$ **b.** $\frac{3}{x-3} - \frac{2}{x+3} = \frac{x^2}{x^2-9} - 1$
   c. $\frac{6x}{5} - \frac{5}{x} = \frac{2x-3}{3} + \frac{8x}{15}$
4. Solve the following problems by first formulating an equation:
   a. The sum of twice a number and 5 is equal to the difference between the number and 3. Find the number.
   b. The sum of three successive natural numbers is 10 more than twice the smallest of them.

c. Ann receives double pay for every hour she works over and above 38 hours per week. Last week, she worked 48 hours and earned a total of \$812. What is Ann's regular hourly wage?

d. John has invested \$15,000 at an annual interest rate of 10%. How much additional money should he invest at the interest rate of 12% if he wants the total interest income earned by the end of the year to equal \$2100?

e. When Mr. Barne passed away, his estate was divided in the following manner: 2/3 of the estate was left to his wife, 1/4 to his children, and the remainder, \$1000, was donated to a charitable organization. How big was Mr. Barne's estate?

5. Solve the following equations:

a. $\dfrac{3y-1}{4} - \dfrac{1-y}{3} + 2 = 3y$

b. $\dfrac{4}{x} + \dfrac{3}{x+2} = \dfrac{2x+2}{x^2+2x} + \dfrac{7}{2x+4}$

c. $\dfrac{2 - \dfrac{z}{1-z}}{1+z} = \dfrac{6}{2z+1}$

d. $\dfrac{1}{2}\left(\dfrac{p}{2} - \dfrac{3}{4}\right) - \dfrac{1}{4}\left(1 - \dfrac{p}{3}\right) - \dfrac{1}{3}(-p+1) = -\dfrac{1}{3}$

6. A swimming pooling can be filled by any one of three different hosepipes in 20, 30, and 60 minutes, respectively. How long will it take to fill the pool if all three hosepipes are used at the same time?

A.7 Inequalities

The real numbers consist of the positive numbers, 0, and the negative numbers. If a is a positive number, we write $a > 0$ (or $0 < a$), and we say that a is greater than zero. A fundamental property of the set of positive numbers is

$$a > 0 \text{ and } b > 0 \text{ imply } a + b > 0 \text{ and } a \cdot b > 0 \qquad \text{[A.23]}$$

If the number c is negative, we write $c < 0$ (or $0 > c$).

In general, we say that *the number a is greater than the number b*, and we write $a > b$ (or $b < a$), if $a - b$ is positive:

$$a > b \quad \text{means that} \quad a - b > 0 \qquad \text{[A.24]}$$

Thus, $4.11 > 3.12$ because $4.11 - 3.12 = 0.99 > 0$, and $-3 > -5$ because $-3 - (-5) = 2 > 0$. On the number line (see Fig. A.3), $a > b$ means that a lies to right of b.

When $a > b$, we often say that a *is strictly greater than* b in order to emphasize that $a = b$ is ruled out. If $a > b$ or $a = b$, then we write $a \geq b$ (or $b \leq a$) and say that a is *greater than or equal to* b.

$$a \geq b \quad \text{means that} \quad a - b \geq 0 \tag{A.25}$$

For example, $4 \geq 4$ and $4 \geq 2$. Note in particular that it *is* correct to write $4 \geq 2$, because $4 - 2$ *is* positive or 0.

One can prove a number of important properties of $>$ and $\geq$. For example,

$$a > b \text{ and } c \text{ arbitrary implies } a + c > b + c \tag{A.26}$$

The argument is simple: For all a, b, and c, $(a+c)-(b+c) = a+c-b-c = a-b$. Hence, if $a - b > 0$, then $a + c - (b + c) > 0$. So [A.26] follows from [A.24]. On the number line in Fig. A.3, [A.26] is:

FIGURE A.3

At the risk of being trivial, here is another interpretation of [A.26]. If one day the temperature in Paris is higher than that in London, and the temperature at both places then increases (or decreases) by the same number of degrees, then the ensuing Paris temperature is still higher than that in London.

In order to find when a given inequality is satisfied, property [A.26] is basic.

Example A.20

Find the values of x satisfying $2x - 5 > x - 3$.

Solution Adding 5 to both sides of the inequality yields

$$2x - 5 + 5 > x - 3 + 5 \quad \text{or} \quad 2x > x + 2$$

Adding $(-x)$ to both sides yields $2x - x > x - x + 2$, so the solution is $x > 2$.

Further Properties of Inequalities

To deal with more complicated inequalities involves using the following further properties:

$$a > b \quad \text{and} \quad b > c \implies a > c \tag{A.27}$$

$$a > b \quad \text{and} \quad c > 0 \implies ac > bc \tag{A.28}$$

$$a > b \quad \text{and} \quad c < 0 \implies ac < bc \tag{A.29}$$

$$a > b \quad \text{and} \quad c > d \implies a + c > b + d \tag{A.30}$$

The corresponding properties are valid when each $>$ is replaced by $\geq$. These four properties all follow easily from [A.23]. For example, [A.29] is proved in this way. Suppose $a > b$ and $c < 0$. Then $a - b > 0$ and $-c > 0$, so according to [A.23], $(a-b)(-c) > 0$. Hence, $-ac + bc > 0$ and, therefore, $ac < bc$.

According to [A.28], if an inequality is multiplied by a positive number, the direction of the inequality is preserved. On the other hand, according to [A.29], *if we multiply an inequality by a negative number, the direction of the inequality is reversed.* It is important that you understand these rules, and realize that they correspond to everyday experience. For instance, [A.28] can be interpreted this way: given two rectangles with the same base, the one with the larger height has the larger area.

Sign Diagrams

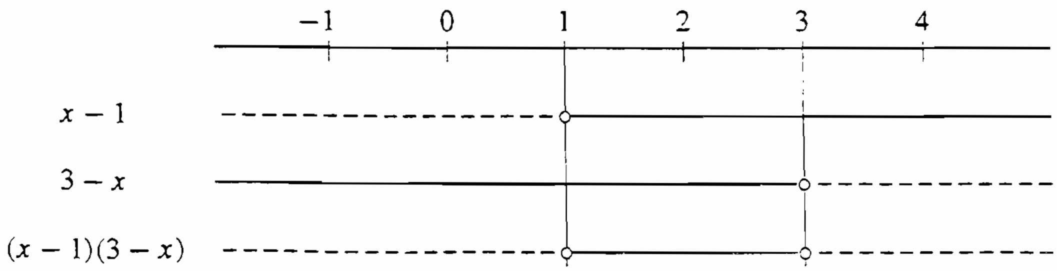
We often need to find the values of a variable for which a given inequality is satisfied.

Example A.21

Check whether the inequality $(x-1)(3-x) > 0$ is satisfied for $x = -3$, $x = 2$, and $x = 5$. Then find all the values x that satisfy the same inequality.

Solution For $x = -3$, we have $(x-1)(3-x) = (-4)\cdot 6 = -24 < 0$; for $x = 2$, we have $(x-1)(3-x) = 1 \cdot 1 = 1 > 0$; and for $x = 5$, we have $(x-1)(3-x) = 4\cdot(-2) = -8 < 0$. Hence, the inequality is satisfied for $x = 2$, but not for $x = -3$ or $x = 5$.

To find the entire solution set, we use a sign diagram. The sign variation for each factor in the product is determined. For example, the factor $x-1$ is negative when $x < 1$, is 0 when $x = 1$, and is positive when $x > 1$. This sign variation is represented in the diagram below. The dashed line to the left of the vertical line $x = 1$ indicates that $x - 1 < 0$ if $x < 1$; the small circle indicates that $x - 1 = 0$ when $x = 1$; and the solid line to the right of $x = 1$ indicates that $x - 1 > 0$ if $x > 1$. In a similar way, we represent the sign variation for $3 - x$. The sign variation of the product is obtained as follows. When $x < 1$, then $x - 1$ is negative and $3 - x$ is positive, so the product is negative. When $1 < x < 3$, both factors are positive, so the product is positive. When $x > 3$, then $x - 1$ is positive and $3 - x$ is negative, so the product is negative. Conclusion: The solution set consists of those x's that are greater than 1, but less than 3. So $(x-1)(3-x) > 0$ if and only if $1 < x < 3$.



Example A.22

Find the solution set of

$$\frac{2p-3}{p-1} > 3-p$$

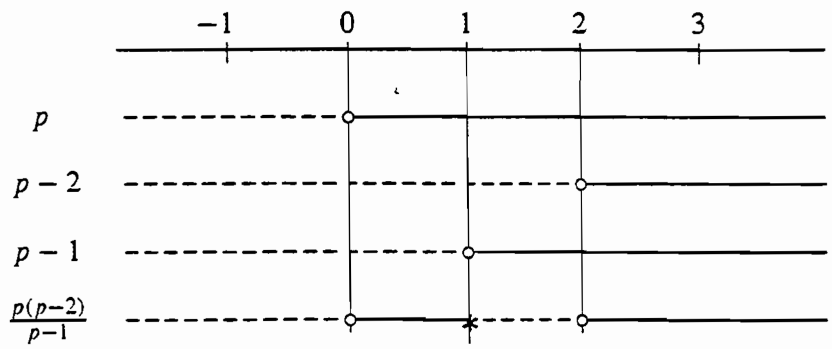
Solution It is tempting to begin by multiplying each side of the inequality by $p-1$. However, then we must distinguish between the two cases, $p-1>0$ and $p-1<0$. If we multiply through by $p-1$ when $p-1<0$, we have to reverse the inequality sign. There is an alternative method that can be used, which makes it unnecessary to distinguish between two different cases. We begin by adding $p-3$ to both sides. This yields

$$\frac{2p-3}{p-1} + p - 3 > 0$$

Making $p-1$ the common denominator gives

$$\frac{2p-3+(p-3)(p-1)}{p-1} > 0 \qquad \text{or} \qquad \frac{p(p-2)}{p-1} > 0$$

because $2p-3+(p-3)(p-1) = 2p-3+p^2-4p+3 = p^2-2p = p(p-2)$. To find the solution set of this inequality, we again use a sign diagram. On the basis of the sign variations for p, $p-2$, and $p-1$, the sign variation for $p(p-2)/(p-1)$ is determined. For example, if $0<p<1$, then p is positive and $(p-2)$ is negative, so $p(p-2)$ is negative. But $p-1$ is also negative on this interval, so $p(p-2)/(p-1)$ is positive. Arguing this way for all the relevant intervals leads to the following conclusion: The fraction $p(p-2)/(p-1)$ is positive if and only if $0<p<1$ or $p>2$. (The original inequality has no meaning when $p=1$. This is symbolized by a cross in the diagram.) So the original inequality is satisfied if and only if $0<p<1$ or $p>2$.

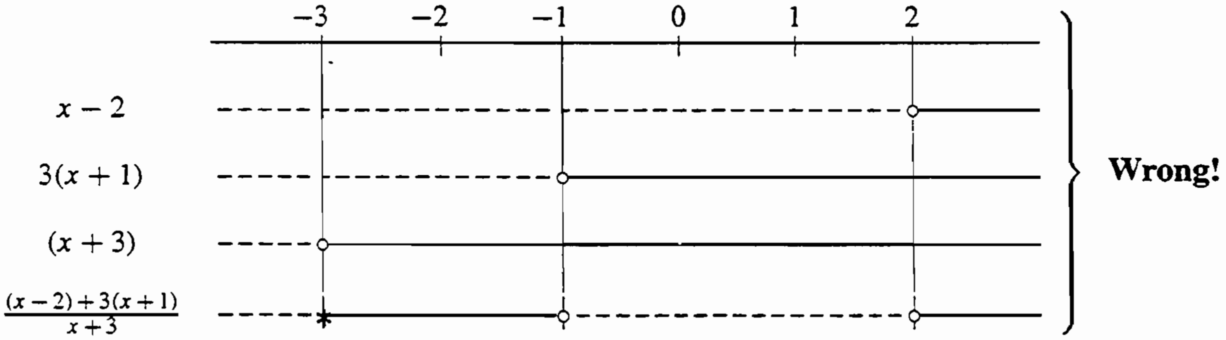


Warning 1: The most common error committed in solving inequalities is precisely the one that was indicated in Example A.22: If we multiply by $p-1$, the inequality is *only* preserved if $p-1$ is positive, that is, if $p>1$.

Warning 2: It is vital that you really understand the method of sign diagrams. A standard error is illustrated by the following example. Find the solution set for

$$\frac{(x-2)+3(x+1)}{x+3} \leq 0$$

"Solution": We construct the sign diagram:



According to this diagram, the inequality should be satisfied for $x < -3$ and for $-1 \leq x \leq 2$. However, for $x = -4$ (< -3), the fraction reduces to 15, which is positive. What went wrong? Suppose $x < -3$. Then $x - 2 < 0$ and $3(x + 1) < 0$ and, therefore, the numerator $(x - 2) + 3(x + 1)$ is negative. Because the denominator $x + 3$ is also negative for $x < -3$, the fraction is positive. The sign variation for the fraction in the diagram above is, therefore, completely wrong. We obtain a correct solution to the given problem by first collecting terms in the numerator so that the inequality becomes $(4x + 1)/(x + 3) \leq 0$. A sign diagram for this inequality reveals the correct answer, which is $-3 < x \leq -1/4$.

Double Inequalities

Two inequalities that are valid simultaneously are often written as a double inequality. If, for example, $a \leq z$ and moreover $z < b$, it is natural to write $a \leq z < b$. (On the other hand, if $a \leq z$ and $z > b$, but we do not know which is the larger of a and b, then we cannot write $a \leq b < z$ or $b \leq a \leq z$, and we do *not* write $a \leq z > b$.)

Example A.23

One day, the lowest temperature in a certain city is 50°F, and the highest is 77°F. What is the corresponding temperature variation in degrees Celsius? (If F denotes degrees Fahrenheit and C denotes degrees Celsius, then $F = \frac{9}{5}C + 32$.)

Solution

$$50 \le F \le 77$$
$$50 \le \tfrac{9}{5}C + 32 \le 77$$
$$50 - 32 \le \tfrac{9}{5}C \le 77 - 32$$
$$18 \le \tfrac{9}{5}C \le 45$$
$$90 \le 9C \le 225$$
$$10 \le C \le 25$$

The temperature varied between 10 and 25°C.

Problems

1. Decide which of the following inequalities are true:

a. $-6.15 > -7.16$ **b.** $6 \ge 6$ **c.** $(-5)^2 \le 0$ **d.** $-\frac{1}{2}\pi < -\frac{1}{3}\pi$

e. $\dfrac{4}{5} > \dfrac{6}{7}$ **f.** $2^3 < 3^2$ **g.** $2^{-3} < 3^{-2}$ **h.** $\dfrac{1}{2} - \dfrac{2}{3} < \dfrac{1}{4} - \dfrac{1}{3}$

Find the solution sets for the inequalities in Problems 2 to 5.

2. **a.** $3x + 5 < x - 13$ **b.** $3x - (x - 1) \ge x - (1 - x)$

c. $\dfrac{2x - 4}{3} \le 7$ **d.** $\frac{1}{3}(1 - x) \ge 2(x - 3)$

e. $\dfrac{t}{24} - (t + 1) + \dfrac{3t}{8} < \dfrac{5}{12}(t + 1) - \dfrac{7}{6}$ **f.** $\dfrac{x + 2}{x + 4} \le 3$

3. **a.** $\dfrac{x + 2}{x - 1} < 0$ **b.** $\dfrac{2x + 1}{x - 3} > 1$ **c.** $5a^2 \le 125$

d. $2 < \dfrac{3x + 1}{2x + 4}$ **e.** $\dfrac{120}{n} + 1.1 \le 1.85$ **f.** $g^2 - 2g \le 0$

g. $\dfrac{1}{p - 2} + \dfrac{3}{p^2 - 4p + 4} \ge 0$ **h.** $\dfrac{-n - 2}{n + 4} > 2$ **i.** $x^4 < x^2$

4. **a.** $(x - 1)(x + 4) > 0$ **b.** $(x - 1)^2(x + 4) > 0$

c. $(x - 1)(x - 2)(x - 3) \le 0$ **d.** $(5x - 1)^{10}(x - 1) < 0$

e. $(5x - 1)^{11}(x - 1) < 0$ **f.** $\dfrac{3x - 1}{x} > x + 3$

g. $\dfrac{x - 3}{x + 3} < 2x - 1$ **h.** $x^2 + 4x - 5 \le 0$

i. $-\frac{1}{3}x^3 - x^2 + 6x \le 0$

5. **a.** $1 \le \frac{1}{3}(2x-1) + \frac{8}{3}(1-x) < 16$ **b.** $-5 < \dfrac{1}{x} < 0$ **c.** $\dfrac{\dfrac{1}{x} - 1}{\dfrac{1}{x} + 1} \ge 1$

6. Decide whether the following inequalities are valid for all x and y:

   a. $x + 1 > x$ **b.** $x^2 > x$ **c.** $x + x > x$ **d.** $x^2 + y^2 \ge 2xy$

7. In 1993, there was a fixed charge of approximately \$120 per year for having a telephone in Zimbabwe, and an additional \$0.167 per call unit used.
   a. What was the total cost for 1 year in which x call units are used?
   b. What were the smallest and largest numbers of call units one could use if the annual telephone bill were to be between \$170.10 and \$186.80?

8. **a.** The temperature for storing potatoes should be between 4 and 6°C. What are the corresponding temperatures in degrees Fahrenheit?
   b. The freshness of a bottle of milk is guaranteed for 7 days if it is kept at between 36 and 40°F. Find the corresponding temperature variation in degrees Celsius.

A.8 Quadratic Equations

This section reviews the method for solving quadratic equations. An example of such an equation is $12x^2 - 16x - 3 = 0$. We could, of course, try to find the values of x that satisfy the equation by trial and error. However, then it is not easy to find the only two solutions, which are $x = 3/2$ and $x = -1/6$.

Note: The methods for solving equations we have used so far are based on [A.22] in Section A.6. Many students try to use similar ideas to solve quadratic equations like $12x^2 - 16x - 3 = 0$. One attempt is this: $12x^2 - 16x - 3 = 0$, so $12x^2 - 16x = 3$, and $2x(6x - 8) = 3$. Thus, the product of $2x$ and $6x - 8$ must be 3. But there are infinitely many pairs of numbers whose product is 3, so this is of very little help in finding x.

Some others first try to divide each term by x. Then $12x^2 - 16x - 3 = 0$ yields $12x - 16 = 3/x$. Because the unknown x now occurs on both sides of the equation, we are stuck. Evidently, we need a completely new idea in order to find the solution.

The general quadratic equation has the form

$$ax^2 + bx + c = 0 \qquad (a \neq 0) \tag{A.31}$$

where a, b, and c are given constants, and x is the unknown. Some simple examples:

(a) $x^2 - 4 = 0$ $\quad (a = 1,\ b = 0, \text{ and } c = -4)$

(b) $5x^2 - 8x = 0$ $\quad (a = 5,\ b = -8, \text{ and } c = 0)$

(c) $x^2 + 3 = 0$ $\quad (a = 1,\ b = 0, \text{ and } c = 3)$

In each case, we are interested in finding the solutions (if there are any).

(a) The equation yields $x^2 = 4$, and, hence, $x = \pm\sqrt{4} = \pm 2$, which means that x is either 2 or -2. (Alternatively: $x^2 - 4 = (x+2)(x-2) = 0$, so $x = 2$ or $x = -2$.)

(b) Here x is a common factor on the left-hand side, so $x(5x - 8) = 0$. But the product of two numbers is 0 if and only if at least one of the factors is 0. Hence, there are two possibilities: either $x = 0$ or $5x - 8 = 0$, so $x = 0$ or $x = 8/5$.

(c) Because x^2 is always ≥ 0, the equation $x^2 + 3 = 0$ has no solution.

Now we turn to two examples of equations that are more difficult to solve.

Example A.24

$$x^2 + 8x - 9 = 0 \qquad [1]$$

It is natural to begin by moving 9 to the right-hand side:

$$x^2 + 8x = 9 \qquad [2]$$

However, because x occurs in two terms, it is not obvious how to proceed. A method called *completing the square*, one of the oldest tricks in mathematics, turns out to work. To see how, recall from [A.10] that

$$(x + a)^2 = x^2 + 2ax + a^2 \qquad [*]$$

where $x^2 + 2ax + a^2$ is called a *complete square*. Now look at the expression $x^2 + 8x$ on the left-hand side of [2]. What must be added to this expression to make it a complete square? Comparing the left-hand side of [2] with the right-hand side of [*], we see that we should have $2a = 8$ and, hence, $a = 4$. Thus, $a^2 = 4^2$, and by adding 4^2 to the left-hand side of [2], we complete the square of $x^2 + 8x$ to get

$$x^2 + 8x + 4^2 = (x + 4)^2$$

Let us now add 4^2 to both sides of Equation [2]. We then obtain an equation that has precisely the same solutions as [2] and where, moreover, the left-hand side is a complete square:

$$x^2 + 8x + 4^2 = 9 + 4^2$$

Thus, [2] is equivalent to

$$(x + 4)^2 = 25 \qquad [3]$$

Now, the equation $z^2 = 25$ has two solutions, $z = \sqrt{25} = 5$ and $z =$

$-\sqrt{25} = -5$. Thus, from [3], either $x + 4 = 5$ or $x + 4 = -5$. The solutions to Equation [1] are, therefore, $x = 1$ and $x = -9$.

Equation [3] can be written as

$$(x + 4)^2 - 5^2 = 0 \tag{4}$$

Then, using the difference of squares formula [A.12] yields

$$(x + 4 - 5)(x + 4 + 5) = 0 \qquad \text{or} \qquad (x - 1)(x + 9) = 0$$

So we have the following *factorization* of the left-hand side of [1]:

$$x^2 + 8x - 9 = (x - 1)(x + 9)$$

Example A.25

Solve

$$12x^2 - 16x - 3 = 0$$

and factor the left-hand side.

Solution The given equation is equivalent to

$$12\left(x^2 - \tfrac{4}{3}x - \tfrac{1}{4}\right) = 0 \tag{1}$$

This equation clearly has the same solutions as

$$x^2 - \tfrac{4}{3}x = \tfrac{1}{4} \tag{2}$$

Now complete the square for $x^2 - \frac{4}{3}x = x^2 + \left(-\frac{4}{3}\right)x$. One-half of the coefficient of x is $-\frac{2}{3}$, and, therefore, we add the square of $-\frac{2}{3}$ to each side of [2], thus obtaining

$$x^2 - \tfrac{4}{3}x + \left(-\tfrac{2}{3}\right)^2 = \tfrac{1}{4} + \left(-\tfrac{2}{3}\right)^2 = \tfrac{1}{4} + \tfrac{4}{9}$$

$$\left(x - \tfrac{2}{3}\right)^2 = \tfrac{25}{36} \tag{3}$$

Hence, $x - \frac{2}{3} = \sqrt{\frac{25}{36}} = \frac{5}{6}$ or $x - \frac{2}{3} = -\sqrt{\frac{25}{36}} = -\frac{5}{6}$. The two solutions are, therefore, $x = \frac{2}{3} + \frac{5}{6} = \frac{9}{6} = \frac{3}{2}$ and $x = \frac{2}{3} - \frac{5}{6} = -\frac{1}{6}$.

As demonstrated in Example A.24, we see that [3] can be written as

$$\left(x - \tfrac{2}{3} - \tfrac{5}{6}\right)\left(x - \tfrac{2}{3} + \tfrac{5}{6}\right) = 0 \qquad \text{or} \qquad \left(x - \tfrac{3}{2}\right)\left(x + \tfrac{1}{6}\right) = 0$$

Hence, we obtain the factorization

$$12\left(x^2 - \tfrac{4}{3}x - \tfrac{1}{4}\right) = 12\left(x - \tfrac{3}{2}\right)\left(x + \tfrac{1}{6}\right) \tag{4}$$

Check that this is correct by expanding the right-hand side.

The General Case

We will now apply the method of completing the square to the general quadratic equation [A.31]. We begin by taking the nonzero coefficient of x^2 outside the parentheses so that [A.31] becomes

$$a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) = 0 \qquad \text{[A.32]}$$

Because $a \neq 0$ this equation has the same solutions as

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

One-half of the coefficient of x is $b/2a$. Adding the square of this number to each side of the equation yields

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$$

or

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2} \qquad \text{[A.33]}$$

Note that $a^2 > 0$ and, if $b^2 - 4ac < 0$, then the right-hand side of [A.33] is negative. Because $(x + b/2a)^2$ is nonnegative for all choices of x, we conclude that if $b^2 - 4ac < 0$, then equation [A.33] has no solutions. On the other hand, if $b^2 - 4ac \geq 0$, then [A.33] yields two possibilities:

$$x + \frac{b}{2a} = \frac{\sqrt{b^2 - 4ac}}{2a} \quad \text{or} \quad x + \frac{b}{2a} = -\frac{\sqrt{b^2 - 4ac}}{2a}$$

Then the values of x are easily found. In conclusion:

For $b^2 - 4ac \geq 0$, $a \neq 0$,

$$ax^2 + bx + c = 0 \iff x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad \text{[A.34]}$$

It is a good idea to spend 10 minutes to memorize this formula thoroughly. Once you have done so, you can immediately write the solutions of any quadratic equation. Only if $b^2 - 4ac \geq 0$ are the solutions real numbers. If we use the formula when $b^2 - 4ac < 0$, the square root of a negative number appears and no real solution exists.

Example A.26

Use [A.34] to find the solutions of

$$2x^2 - 4x - 7 = 0$$

Solution Write the equation as $2x^2 + (-4)x + (-7) = 0$. Because $a = 2$, $b = -4$, and $c = -7$, formula [A.34] yields

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4 \cdot 2 \cdot (-7)}}{2 \cdot 2} = \frac{4 \pm \sqrt{16 + 56}}{4}$$

$$= \frac{4 \pm \sqrt{72}}{4} = \frac{4 \pm 6\sqrt{2}}{4} = 1 \pm \frac{3}{2}\sqrt{2}$$

The solutions are, therefore, $x = 1 + \frac{3}{2}\sqrt{2}$ and $x = 1 - \frac{3}{2}\sqrt{2}$.

Suppose $b^2 - 4ac \geq 0$. By using the square of the difference formula as we did in Examples A.24 and A.25, it follows that [A.33] is equivalent to

$$\left(x + \frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}\right)\left(x + \frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}\right) = 0 \qquad \text{[A.35]}$$

Denoting the two solutions in [A.34] by x_1 and x_2, Equation [A.35] can be written as $(x - x_1)(x - x_2) = 0$. Therefore, $x^2 + (b/a)x + c/a = (x - x_1)(x - x_2)$. Hence:

> Provided that $b^2 - 4ac \geq 0$ and $a \neq 0$, we have
>
> $$ax^2 + bx + c = a(x - x_1)(x - x_2) \qquad \text{[A.36]}$$
>
> where
>
> $$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

This is a very important result, because it shows how to factor a general quadratic function. If $b^2 - 4ac < 0$, there is no factorization of $ax^2 + bx + c$.

Expanding the right-hand side of the identity $x^2 + (b/a)x + c/a = (x - x_1)(x - x_2)$ yields $x^2 + (b/a)x + c/a = x^2 - (x_1 + x_2)x + x_1x_2$. Equating like powers of x gives $x_1 + x_2 = -b/a$ and $x_1x_2 = c/a$. Thus:

> If x_1 and x_2 are the roots of $ax^2 + bx + c = 0$, then
>
> $$x_1 + x_2 = -b/a \quad \text{and} \quad x_1x_2 = c/a \qquad \text{[A.37]}$$

Example A.27

Factor (if possible) the following second-degree polynomials:

$$\text{(a)}\ \tfrac{1}{3}x^2 + \tfrac{2}{3}x - \tfrac{14}{3} \qquad \text{(b)}\ -2x^2 + 40x - 600$$

Solution

(a) For $\frac{1}{3}x^2 + \frac{2}{3}x - \frac{14}{3} = 0$, we have $a = \frac{1}{3}$, $b = \frac{2}{3}$, and $c = -\frac{14}{3}$, so [A.34] gives

$$x_{1,2} = \frac{-\frac{2}{3} \pm \sqrt{(\frac{2}{3})^2 - 4 \cdot (\frac{1}{3}) \cdot (-\frac{14}{3})}}{2 \cdot (\frac{1}{3})} = \frac{-\frac{2}{3} \pm \sqrt{\frac{4}{9} + \frac{56}{9}}}{\frac{2}{3}}$$

$$= \frac{-\frac{2}{3} \pm \sqrt{60}/3}{\frac{2}{3}} = \frac{-2 \pm \sqrt{60}}{2} = \frac{-2 \pm 2\sqrt{15}}{2} = -1 \pm \sqrt{15}$$

The solutions are, therefore, $x_1 = -1 + \sqrt{15}$ and $x_2 = -1 - \sqrt{15}$, so [A.36] yields

$$\tfrac{1}{3}x^2 + \tfrac{2}{3}x - \tfrac{14}{3} = \tfrac{1}{3}\left[x - (-1 + \sqrt{15})\right]\left[x - (-1 - \sqrt{15})\right]$$

(b) For $-2x^2 + 40x - 600 = 0$, $a = -2$, $b = 40$, and $c = -600$, so $b^2 - 4ac = 1600 - 4800 = -3200$. Therefore, no factorization like that in [A.36] exists in this case.

Note: The general formula for the solution of a second-degree equation is very useful. However, if b or c is 0, then it is unnecessary to use the formula.

1. If $ax^2 + bx = 0$ (the quadratic equation lacks the constant term), then factorization yields $x(ax + b) = 0$, which gives the solutions $x = 0$ and $x = -b/a$ directly.
2. If $ax^2 + c = 0$ (the equation lacks the term involving x), then $a(x^2 + c/a) = 0$ and there are two possibilities. If $c/a > 0$, then the equation $x^2 + c/a = 0$ has no solutions. If $c/a < 0$, then the solutions are $x = \pm\sqrt{-c/a}$.

Problems

1. Solve the following quadratic equations (if they have solutions):

   a. $15x - x^2 = 0$ **b.** $p^2 - 16 = 0$ **c.** $(q - 3)(q + 4) = 0$

   d. $2x^2 + 9 = 0$ **e.** $x(x + 1) = 2x(x - 1)$ **f.** $x^2 - 4x + 4 = 0$

2. Solve the following quadratic equations by using the method of completing the square, and factor (if possible) the left-hand side:

   a. $x^2 - 5x + 6 = 0$ **b.** $y^2 - y - 12 = 0$

c. $2x^2 + 60x + 800 = 0$ d. $-\frac{1}{4}x^2 + \frac{1}{2}x + \frac{1}{2} = 0$

e. $m(m - 5) - 3 = 0$ f. $0.1p^2 + p - 2.4 = 0$

3. Solve the following quadratic equations:

a. $r^2 + 11r - 26 = 0$ b. $3p^2 + 45p = 48$

c. $20\,000 = 300K - K^2$ d. $r^2 + (\sqrt{3} - \sqrt{2})r = \sqrt{6}$

e. $0.03x^2 - 0.009x = 0.012$ f. $\dfrac{1}{24} = p^2 - \dfrac{1}{12}p$

4. Solve the following equations by using formula [A.34]:

a. $x^2 - 3x + 2 = 0$ b. $5t^2 - t = 3$ c. $6x = 4x^2 - 1$

d. $9x^2 + 42x + 44 = 0$ e. $30\,000 = x(x + 200)$ f. $3x^2 = 5x - 1$

5. a. Find the rectangle whose circumference is 40 cm and area is 75 cm^2.

b. Find two successive natural numbers whose sum of squares is 13.

c. In a right-angled triangle, the hypotenuse is 34 cm. One of the short sides is 14 cm longer than the other. Find the lengths of the two short sides.

d. A motorist drove 80 km. In order to save 16 minutes, he had to drive 10 km/h faster than usual. What was his usual driving speed?

6. Solve the following equations:

a. $x^3 - 4x = 0$ b. $x^4 - 5x^2 + 4 = 0$ c. $z^{-2} - 2z^{-1} - 15 = 0$

7. Prove formula [A.34] using the following approach. Multiply Equation [A.31] by $4a$; after rearranging, this yields $4a^2x^2 + 4abx = -4ac$. Now add b^2 to both sides. Notice that the left-hand side is then a complete square.

A.9 Two Equations with Two Unknowns

This section reviews some methods for solving equations with two unknowns.

Example A.28

Find the values of x and y that satisfy the two equations

$$\begin{aligned} 2x + 3y &= 18 \\ 3x - 4y &= -7 \end{aligned} \quad [1]$$

We need to find the values of x and y that satisfy *both* equations. Suppose we start by trying $x = 0$ in the first equation; this implies $y = 6$. Given $x = 0$ and $y = 6$, $2x + 3y = 18$ and $3x - 4y = -24$. Thus, the first equation in [1] *is* satisfied, but not the second. Hence, $x = 0$ and $y = 6$ is *not* a solution to [1]. Only if we are very lucky will we find the solution to [1] by such trial and error.

Fortunately, there exist more systematic methods for solving [1].

Method 1: First, solve one of the equations for one of the variables in terms of the other and then substitute the result into the other equation. This leaves only one equation in one unknown, which is easily solved.

Applying this method to [1], we solve the first equation for y in terms of x: $2x + 3y = 18$ implies that $3y = 18 - 2x$ and, hence, $y = 6 - \frac{2}{3}x$. Substituting this expression for y into the second equation in [1] gives

$$\begin{aligned} 3x - 4\left(6 - \tfrac{2}{3}x\right) &= -7 \\ 3x - 24 + \tfrac{8}{3}x &= -7 \\ 9x - 72 + 8x &= -21 \\ 17x &= 51 \end{aligned}$$

Hence, $x = 3$. Then we find y by using $y = 6 - \frac{2}{3}x$ once again, thus implying that $y = 6 - \frac{2}{3} \cdot 3 = 4$. The solution of [1] is, hence, $x = 3$ and $y = 4$. (Such a solution should always be checked by direct substitution.)

Method 2: This method is based on eliminating one of the variables by adding or subtracting a multiple of one equation from the other. For system [1], suppose we want to eliminate y; a similar method could be used to eliminate x instead. If we multiply the first equation in [1] by 4 and the second by 3, then the coefficients of the y terms in both equations will be the same except for the sign. If we then add the transformed equations together, we obtain

$$\begin{array}{rcr} 8x + 12y &=& 72 \\ \underline{9x - 12y} &\underline{=}& \underline{-21} \\ 17x \qquad &=& 51 \end{array} \qquad [2]$$

Hence, $x = 3$. To find the value for y, substitute 3 for x in either of the original equations and solve for y. This gives $y = 4$, which agrees with the earlier result.

Some prefer to find both x and y by using the following setup:

$$\begin{array}{rcr|c|c|} 2x + 3y &=& 18 & 4 & 3 \\ 3x - 4y &=& -7 & 3 & -2 \end{array} \qquad [3]$$

The first column on the right-hand side of [3] suggests that we multiply the first equation by 4 and the second by 3. This leads to the arrangement in [2]. The second column on the right-hand side of [3] suggests that we multiply the first equation by 3 and the second by -2. Doing this yields

$$\begin{aligned} 6x + 9y &= 54 \\ -6x + 8y &= 14 \\ \hline 17y &= 68 \\ y &= 4 \end{aligned}$$

We end this section by using the elimination method to solve a general linear system of equations with two equations and two unknowns:

$$\begin{aligned} ax + by &= c \\ dx + ey &= f \end{aligned} \qquad [A.38]$$

Here a, b, c, d, e, and f are arbitrary given numbers, whereas x and y are the unknowns. If we let $a = 2$, $b = 3$, $c = 18$, $d = 3$, $e = -4$, and $f = -7$, then [A.38] reduces to system [1]. Using the elimination method for the general case, we obtain

$$\begin{array}{l|c|c|} ax + by = c & e & d \\ dx + ey = f & -b & -a \end{array}$$

$$\begin{aligned} aex + bey &= ce \\ -bdx - bey &= -bf \\ \hline (ae - bd)x \quad &= ce - bf \end{aligned} \qquad \begin{aligned} adx \quad + bdy &= cd \\ -adx \quad - aey &= -af \\ \hline (bd - ae)y &= cd - af \end{aligned}$$

which gives

$$x = \frac{ce - bf}{ae - bd} \qquad y = \frac{cd - af}{bd - ae} = \frac{af - cd}{ae - bd} \qquad [A.39]$$

We have found expressions for both x and y.

The formulas in [A.39] break down if the denominator $ae - bd$ in both fractions is equal to 0. This case requires special attention—see Section 14.3.

Problems

1. Solve each of the following systems of two simultaneous equations:

(a) $x - y = 5$, $x + y = 11$ (b) $4x - 3y = 1$, $2x + 9y = 4$ (c) $3x + 4y = 2.1$, $5x - 6y = 7.3$

2. Solve each of the following systems of two simultaneous equations:

(a) $2K + L = 11.35$, $K + 4L = 25.8$ (b) $230p + 450q = 1810$, $100p + 150q = 650$

$$\begin{aligned}(c)\quad 0.01r + 0.21s &= 0.042\\ -0.25r + 0.55s &= -0.47\end{aligned}$$

3. **a.** Find two numbers whose sum is 52 and whose difference is 26.
   b. Five tables and 20 chairs cost \$1800, whereas 2 tables and 3 chairs cost \$420. What is the price of each table and each chair?
   c. A firm produces a good in two qualities, A and B. The estimate for the coming year's sales of A is 50% higher than the estimate of the sales of B. The profit per unit sold of the two qualities is \$300 for A and \$200 for B. If the goal is a profit of \$13,000 over the next year, how much of each of the two qualities must be produced?
   d. A person has saved a total of \$10,000 in two accounts. The interest rates are 5 and 7.2% per year, respectively. If the person earns \$676 interest in 1 year, what was the balance in each of the two accounts?

B

Sums, Products, and Induction

—Mathematicians are like lovers ... consent to the most innocent principle —the mathematician draws from it a conclusion that you also must accept, and from this conclusion another ...
—Fontenelle (1657–1757)

B.1 Summation Notation

Suppose we are interested in the population of a country that is divided into six regions. Let N_i denote the population in region i. Then

$$N_1 + N_2 + N_3 + N_4 + N_5 + N_6$$

is the total population. It is often convenient to have an abbreviated notation for such lengthy sums. The capital Greek letter sigma Σ is used as a **summation symbol**, and the previous sum is written as

$$\sum_{i=1}^{6} N_i$$

This expression means the "sum from $i = 1$ to $i = 6$ of N_i." Suppose, in general, that there are n regions. Then

$$N_1 + N_2 + \cdots + N_n \quad [1]$$

is one possible notation for the total population. Here $\cdots$ indicates that the obvious pattern continues. In summation notation, we write

$$\sum_{i=1}^{n} N_i$$

This notation tells us to form the sum of all the terms that result when we substitute successive integers for i, starting with $i = 1$ and ending with $i = n$. The symbol i is called the **index of summation**. It is a "dummy variable" that can be replaced by any other letter (provided that the letter has not already been used for something else). That is, both $\sum_{j=1}^{n} N_j$ and $\sum_{i=1}^{n} N_i$ represent the same sum [1].

The upper and lower limits of summation can both vary. For example,

$$\sum_{i=30}^{35} N_i = N_{30} + N_{31} + N_{32} + N_{33} + N_{34} + N_{35}$$

is the total population in the six regions numbered from 30 to 35.

More generally, if p and q are integers with $q \geq p$, then

$$\sum_{i=p}^{q} a_i = a_p + a_{p+1} + \cdots + a_q$$

denotes the sum of the terms that result when we substitute successive integers for i, starting with $i = p$ and ending with $i = q$. If the upper and lower limits of summation are the same, then the "sum" reduces to one term. For example,

$$\sum_{i=1}^{1} N_i = N_1, \qquad \sum_{i=3}^{3} \frac{1}{i^2} = \frac{1}{3^2} = \frac{1}{9}$$

And if the upper limit is less than the lower limit, then there are no terms at all, so the "sum" reduces to zero.

Example B.1

Compute the following sums:

$$\text{(a)} \sum_{i=1}^{5} i^2 \qquad \text{(b)} \sum_{k=3}^{6} (5k - 3) \qquad \text{(c)} \sum_{j=0}^{2} \frac{1}{(j+1)(j+3)}$$

Solution

$$\text{(a)} \sum_{i=1}^{5} i^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 1 + 4 + 9 + 16 + 25 = 55$$

(b) $\sum_{k=3}^{6}(5k-3) = (5\cdot 3-3)+(5\cdot 4-3)+(5\cdot 5-3)+(5\cdot 6-3) = 78$

(c) $\sum_{j=0}^{2}\frac{1}{(j+1)(j+3)} = \frac{1}{1\cdot 3}+\frac{1}{2\cdot 4}+\frac{1}{3\cdot 5} = \frac{40+15+8}{120} = \frac{63}{120}$
$= \frac{21}{40}$

Sums and summation notation occur frequently in books and papers on economics. Often, there are several variables or parameters in addition to the summation index. It is important to be able to read such sums. In each case, the summation symbol tells you that there is a sum of terms. The sum results from substituting successive integers for the summation index, starting with the lower limit and ending with the upper limit.

Example B.2

Expand the following sums:

(a) $\sum_{i=1}^{n} p_t^{(i)}q^{(i)}$ (b) $\sum_{j=-2}^{1} x^{5-j}y^j$ (c) $\sum_{i=1}^{N}(x_{ij}-\bar{x}_j)^2$

Solution

(a) $\sum_{i=1}^{n} p_t^{(i)}q^{(i)} = p_t^{(1)}q^{(1)} + p_t^{(2)}q^{(2)} + \cdots + p_t^{(n)}q^{(n)}$

(b) $\sum_{j=-2}^{1} x^{5-j}y^j = x^{5-(-2)}y^{-2} + x^{5-(-1)}y^{-1} + x^{5-0}y^0 + x^{5-1}y^1$

$$= x^7y^{-2} + x^6y^{-1} + x^5 + x^4y$$

(c) $\sum_{i=1}^{N}(x_{ij}-\bar{x}_j)^2 = (x_{1j}-\bar{x}_j)^2 + (x_{2j}-\bar{x}_j)^2 + \cdots + (x_{Nj}-\bar{x}_j)^2$

Note that t is *not* an index of summation in (a), and j is *not* an index of summation in (c).

Example B.3

Write the following sums using summation notation:

(a) $1+3+3^2+3^3+\cdots+3^{81}$

(b) $a_i^6 + a_i^5b_j + a_i^4b_j^2 + a_i^3b_j^3 + a_i^2b_j^4 + a_ib_j^5 + b_j^6$

Solution

(a) This is easy if we note that the first two terms of the sum can be written as $3^0 = 1$ and $3^1 = 3$. The general term is 3^i, and we have

$$1 + 3 + 3^2 + 3^3 + \cdots + 3^{81} = \sum_{i=0}^{81} 3^i$$

(b) This is more difficult. Note, however, that the indices i and j never change. Also the exponent for a_i decreases step by step from 6 to 0, whereas that for b_j increases from 0 to 6. The general term has the form $a_i^{6-k} b_j^k$, where k varies from 0 to 6. Thus,

$$a_i^6 + a_i^5 b_j + a_i^4 b_j^2 + a_i^3 b_j^3 + a_i^2 b_j^4 + a_i b_j^5 + b_j^6 = \sum_{k=0}^{6} a_i^{6-k} b_j^k$$

Example B.4

To measure variations in the cost of living, a number of different *price indices* have been suggested.

Consider a "basket" of n commodities. For $i = 1, \ldots, n$, define

$q^{(i)}$ = number of units of good i in the basket

$p_0^{(i)}$ = price per unit of good i in year 0

$p_t^{(i)}$ = price per unit of good i in year t

Then

$$\sum_{i=1}^{n} p_0^{(i)} q^{(i)} = p_0^{(1)} q^{(1)} + p_0^{(2)} q^{(2)} + \cdots + p_0^{(n)} q^{(n)} \qquad [1]$$

is the cost of the basket in year 0, and

$$\sum_{i=1}^{n} p_t^{(i)} q^{(i)} = p_t^{(1)} q^{(1)} + p_t^{(2)} q^{(2)} + \cdots + p_t^{(n)} q^{(n)} \qquad [2]$$

is the cost of the basket in year t. A price index for year t, with year 0 as the base year, is defined as

$$\frac{\sum_{i=1}^{n} p_t^{(i)} q^{(i)}}{\sum_{i=1}^{n} p_0^{(i)} q^{(i)}} \cdot 100 \qquad [B.1]$$

If the cost of the basket is 1032 in year 0 and the price of the same basket in year t is 1548, then the price index is $(1548/1032) \cdot 100 = 150$.

In case the quantities $q^{(i)}$ are levels of consumption in the base year 0, this index is called the **Laspeyres price index**. But if the quantities $q^{(i)}$ are levels of consumption in the year t, this index is called the **Paasche price index**.

Problems

1. Evaluate the following:

a. $\sum_{i=1}^{10} i$ **b.** $\sum_{k=2}^{6}(5 \cdot 3^{k-2} - k)$ **c.** $\sum_{m=0}^{5}(2m+1)$

d. $\sum_{l=0}^{2} 2^{2^l}$ **e.** $\sum_{i=1}^{10} 2$ **f.** $\sum_{j=1}^{4} \frac{j+1}{j}$

2. Expand the following sums:

a. $\sum_{k=-2}^{2} 2\sqrt{k+2}$ **b.** $\sum_{i=0}^{3}(x+2i)^2$

c. $\sum_{k=1}^{n} a_{ki}b^{k+1}$ **d.** $\sum_{j=0}^{m} f(x_j)\,\Delta x_j$

3. Write these sums by using summation notation:

a. $4 + 8 + 12 + 16 + \cdots + 4n$
b. $1^3 + 2^3 + 3^3 + 4^3 + \cdots + n^3$
c. $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots + (-1)^n \frac{1}{2n+1}$
d. $a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$
e. $3x + 9x^2 + 27x^3 + 81x^4 + 243x^5 + 729x^6$
f. $a_i^3 b_{i+3} + a_i^4 b_{i+4} + \cdots + a_i^p b_{i+p}$
g. $a_i^3 b_{i+3} + a_{i+1}^4 b_{i+4} + \cdots + a_{i+p}^{p+3} b_{i+p+3}$
h. $81{,}297 + 81{,}495 + 81{,}693 + 81{,}891$

4. Compute the price index [B.1] if $n = 3$, $p_0^{(1)} = 1$, $p_0^{(2)} = 2$, $p_0^{(3)} = 3$, $p_t^{(1)} = 2$, $p_t^{(2)} = 3$, $p_t^{(3)} = 4$, $q^{(1)} = 3$, $q^{(2)} = 5$, and $q^{(3)} = 7$.

5. **a.** Expand $\sum_{i=1}^{5}(x_i - \bar{x})$, and prove that it is equal to $\sum_{i=1}^{5} x_i - 5\bar{x}$.
b. Prove in general that

$$\sum_{i=1}^{n}(x_i - \bar{x}) = \sum_{i=1}^{n} x_i - n\bar{x}$$

6. Consider a country divided into 100 regions. For a certain year, let c_{ij} be the number of persons who move from region i to region j. If, say, $i = 25$

and $j = 10$, then we write $c_{25,10}$ for c_{ij}. Explain the meaning of the sums:

a. $\sum_{j=1}^{100} c_{ij}$ **b.** $\sum_{i=1}^{100} c_{ij}$

7. Decide which of the following equalities are generally valid:

a. $\sum_{k=1}^{n} ck^2 = c\sum_{k=1}^{n} k^2$ **b.** $\left(\sum_{i=1}^{n} a_i\right)^2 = \sum_{i=1}^{n} a_i^2$

c. $\sum_{j=1}^{n} b_j + \sum_{j=n+1}^{N} b_j = \sum_{j=1}^{N} b_j$ **d.** $\sum_{k=3}^{7} 5^{k-2} = \sum_{k=0}^{4} 5^{k+1}$

e. $\sum_{i=0}^{n-1} a_{i,j}^2 = \sum_{k=1}^{n} a_{k-1,j}^2$ **f.** $\sum_{k=1}^{n} \frac{a_k}{k} = \frac{1}{k}\sum_{k=1}^{n} a_k$

B.2 Rules for Sums

The following algebraic properties of the sigma notation are helpful when manipulating sums:

$$\sum_{i=1}^{n}(a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i \qquad \textbf{(additivity property)} \qquad [B.2]$$

$$\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i \qquad \textbf{(homogeneity property)} \qquad [B.3]$$

The proofs of these properties are straightforward. For example, [B.3] is proved by noting that

$$\sum_{i=1}^{n} ca_i = ca_1 + ca_2 + \cdots + ca_n = c(a_1 + a_2 + \cdots + a_n) = c\sum_{i=1}^{n} a_i$$

Property [B.3] states that a constant factor can be moved across the summation sign. If $a_i = 1$ for all i in [B.3], then

$$\sum_{i=1}^{n} c = nc \qquad [B.4]$$

which just states that a constant c summed n times is equal to n times c.

Properties [B.2] to [B.4] are also valid if the lower index of summation is an integer other than 1. For example,

$$\sum_{k=3}^{6} 7 = 7 + 7 + 7 + 7 = 28$$

because the number 7 is summed 4 times.

Rules [B.2] to [B.4] can be applied in combination to give formulas like

$$\sum_{i=1}^{n}(a_i + b_i - c_i + d) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i - \sum_{i=1}^{n} c_i + nd$$

Example B.5

Evaluate the sum

$$\sum_{m=2}^{n} \frac{1}{(m-1)m} = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{(n-1)n}$$

by using the identity

$$\frac{1}{(m-1)m} = \frac{1}{m-1} - \frac{1}{m}$$

Solution

$$\begin{aligned}
&\sum_{m=2}^{n} \frac{1}{m(m-1)} \\
&\quad = \sum_{m=2}^{n}\left(\frac{1}{m-1} - \frac{1}{m}\right) = \sum_{m=2}^{n} \frac{1}{m-1} - \sum_{m=2}^{n} \frac{1}{m} \\
&\quad = \left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1}\right) - \left(\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} + \frac{1}{n}\right) \\
&\quad = 1 - \frac{1}{n}
\end{aligned}$$

To derive the last equality, note that all the terms cancel pairwise, except the first term within the first parentheses and the last term within the last parentheses. This is a commonly used and powerful trick for calculating certain kinds of sums.

Example B.6

The **arithmetic mean** $\bar{x}$ of n numbers $x_1, x_2, \ldots, x_n$ is the sum of all the numbers divided by the number of terms, n:

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

Prove that

$$\sum_{i=1}^{n}(x_i - \bar{x}) = 0 \qquad \text{and} \qquad \sum_{i=1}^{n}(x_i - \bar{x})^2 = \sum_{i=1}^{n} x_i^2 - n\bar{x}^2$$

Solution The difference $x_i - \bar{x}$ is the deviation between x_i and the mean. We prove first that the sum of these deviations is 0, using the foregoing definition of $\bar{x}$:

$$\sum_{i=1}^{n}(x_i - \bar{x}) = \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} \bar{x} = \sum_{i=1}^{n} x_i - n\bar{x} = n\bar{x} - n\bar{x} = 0$$

Furthermore,

$$\sum_{i=1}^{n}(x_i - \bar{x})^2 = \sum_{i=1}^{n}(x_i^2 - 2\bar{x}x_i + \bar{x}^2) = \sum_{i=1}^{n} x_i^2 - 2\bar{x}\sum_{i=1}^{n} x_i + \sum_{i=1}^{n} \bar{x}^2$$

$$= \sum_{i=1}^{n} x_i^2 - 2\bar{x}n\bar{x} + n\bar{x}^2 = \sum_{i=1}^{n} x_i^2 - n\bar{x}^2$$

Note: We have considered some useful algebraic properties of sums. A frequent error is a failure to observe that, in general,

$$\sum_{i=1}^{n} x_i^2 \neq \left(\sum_{i=1}^{n} x_i\right)^2$$

It is important to note that the sum of the squares is not generally equal to the square of the sum. For example, $\sum_{i=1}^{2} x_i^2 = x_1^2 + x_2^2$ whereas $(\sum_{i=1}^{2} x_i)^2 = (x_i + x_2)^2 = x_1^2 + 2x_1x_2 + x_2^2$, so the two are equal iff $x_1x_2 = 0$—that is, x_1 or x_2 (or both) must be zero. More generally,

$$\sum_{i=1}^{n} x_i y_i \neq \left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right)$$

so the sum of the cross products is not equal to the products of the individual sums.

Useful Formulas

If you asked a group of 10–12-year-old students to sum all the numbers from 1 to 100, would you expect to have a correct answer within 1 hour? According to reliable sources, Carl F. Gauss solved a similar problem in his tenth year. His teacher asked his students to sum $81,297 + 81,495 + 81,693 + \cdots + 100,899$. There are 100 terms and the difference between successive terms is constant and equal to 198. Obviously, the teacher chose this sum knowing that a trick could yield the answer quickly. Thus, the laboriously derived answers of the students could easily be checked. But Gauss, who later became one of the world's leading mathematicians, gave the right answer, which is 9,109,800, in only a few minutes.

Applied to the easier problem of finding the sum $1 + 2 + \cdots + n$, Gauss' argument was probably as follows: First, write the sum x in two ways

$$x = 1 + 2 + \cdots + (n-1) + n$$
$$x = n + (n-1) + \cdots + 2 + 1$$

Summing vertically gives

$$\begin{aligned} 2x &= (1+n) + [2 + (n-1)] + \cdots + [(n-1) + 2] + (n+1) \\ &= (1+n) + (1+n) + \cdots + (1+n) + (1+n) \\ &= n(1+n) \end{aligned}$$

Thus, we have the result:

$$\sum_{i=1}^{n} i = 1 + 2 + \cdots + n = \tfrac{1}{2}n(n+1) \qquad \text{[B.5]}$$

The following two summation formulas are sometimes useful:

$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \tfrac{1}{6}n(n+1)(2n+1) \qquad \text{[B.6]}$$

$$\sum_{i=1}^{n} i^3 = 1^3 + 2^3 + 3^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2 \qquad \text{[B.7]}$$

Check to see if these formulas are true for $n = 1$, 2, and 3. One way of proving that they are valid generally is to use mathematical induction, as discussed in Section B.5.

Newton's Binomial Formula

We all know that $(a+b)^1 = a+b$ and $(a+b)^2 = a^2+2ab+b^2$. Using the latter equality and writing $(a+b)^3 = (a+b)^2(a+b)$ and $(a+b)^4 = (a+b)^2(a+b)^2$, we find that

$$
\begin{aligned}
(a+b)^1 &= a+b \\
(a+b)^2 &= a^2+2ab+b^2 \\
(a+b)^3 &= a^3+3a^2b+3ab^2+b^3 \\
(a+b)^4 &= a^4+4a^3b+6a^2b^2+4ab^3+b^4
\end{aligned}
$$

What is the corresponding formula for $(a+b)^m$, where m is an arbitrary positive integer? The answer is given by the Newton binomial formula:

$$
\begin{aligned}
(a+b)^m = a^m + \binom{m}{1}a^{m-1}b + \binom{m}{2}a^{m-2}b^2 + \cdots \\
+ \binom{m}{m-1}ab^{m-1} + \binom{m}{m}b^m
\end{aligned}
\tag{B.8}
$$

Here the binomial coefficient

$$\binom{m}{k} = \frac{m(m-1)\cdots(m-k+1)}{k!}$$

as explained in Section 7.4. Formula [B.8] is proved in Section 7.4. In general, $\binom{m}{1} = m$ and $\binom{m}{m} = 1$. For $m = 5$, we have

$$\binom{5}{2} = \frac{5\cdot 4}{1\cdot 2} = 10, \qquad \binom{5}{3} = \frac{5\cdot 4\cdot 3}{1\cdot 2\cdot 3} = 10, \qquad \binom{5}{4} = \frac{5\cdot 4\cdot 3\cdot 2}{1\cdot 2\cdot 3\cdot 4} = 5$$

So [B.8] yields

$$(a+b)^5 = a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5$$

If we study the coefficients in the expansions for the successive powers of $(a+b)$, we have the following pattern, called **Pascal's triangle** (though it was actually known in China by around 1100, long before Blaise Pascal was born):

									1									
								1		1								
							1		2		1							
						1		3		3		1						
					1		4		6		4		1					
				1		5		10		10		5		1				
			1		6		15		20		15		6		1			
		1		7		21		35		35		21		7		1		
	1		8		28		56		70		56		28		8		1	
1		9		36		84		126		126		84		36		9		1

This table can be continued indefinitely. The numbers in this triangle are the binomial coefficients. For instance, the numbers in row 6 (when the first row is numbered 0) are

$$\binom{6}{0} \quad \binom{6}{1} \quad \binom{6}{2} \quad \binom{6}{3} \quad \binom{6}{4} \quad \binom{6}{5} \quad \binom{6}{6}$$

Note, first, that the numbers are symmetric about the middle line. This symmetry can be expressed as

$$\binom{m}{k} = \binom{m}{m-k} \qquad \text{[B.9]}$$

For example, $\binom{6}{2} = 15 = \binom{6}{4}$. Second, the sum of two adjacent numbers in any row is equal to the number between the two in the row below. For instance, the sum of 21 and 35 in the seventh row is equal to 56 in the eighth row. In terms of symbols,

$$\binom{m}{k} + \binom{m}{k+1} = \binom{m+1}{k+1} \qquad \text{[B.10]}$$

In Problem 4, you are asked to prove [B.9] and [B.10].

Problems

1. Use the results in [B.5] and [B.6] to find $\sum_{k=1}^{n}(k^2 + 3k + 2)$.
2. Use Newton's binomial formula [B.8] to find $(a + b)^6$.

3. a. Prove that $\sum_{k=1}^{8}(a_{k+1} - a_k) = a_9 - a_1$, and, more generally, that $\sum_{k=1}^{n}(a_{k+1} - a_k) = a_{n+1} - a_1$.
   b. Use the result in (a) to compute the following:
   (i) $\sum_{k=1}^{50}\left(\frac{1}{k} - \frac{1}{k+1}\right)$ (ii) $\sum_{k=1}^{12}(3^{k+1} - 3^k)$
   (iii) $\sum_{k=1}^{n}(ar^{k+1} - ar^k)$
4. a. Verify that
$$\binom{8}{3} = \binom{8}{8-3} \quad \text{and that} \quad \binom{8}{3} + \binom{8}{3+1} = \binom{8+1}{3+1}$$
   b. Verify [B.9] and [B.10] by using the definition of $\binom{m}{k}$.
5. Find the sum
$$\sum_{k=0}^{n-1} \frac{n}{x}\left(\frac{kx}{n}\right)^2$$
6. Prove the summation formula for an arithmetic series,
$$\sum_{i=0}^{n-1}(a + id) = na + \frac{n(n-1)d}{2}$$
by using the idea in the proof of [B.5]. Then verify the summation result of Gauss mentioned earlier.

B.3 Double Sums

Often one has to combine several summation signs. Consider, for example, the following rectangular array of numbers:

$$\begin{matrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{matrix} \qquad [B.11]$$

A typical number here is of the form a_{ij}, where $1 \le i \le m$ and $1 \le j \le n$. (For example, a_{ij} may indicate the total revenue of a firm from its sales in region i in month j.) There are $n \cdot m$ numbers in all. Let us find the sum of all the numbers in the array by first finding the sum of the numbers in each of the m rows and then adding all these row sums. The m row sums can be written in the form $\sum_{j=1}^{n} a_{1j}$,

$\sum_{j=1}^{n} a_{2j}, \ldots, \sum_{j=1}^{n} a_{mj}$. (In our example, these row sums are the total revenues in each region summed over all the n months.) The sum of these m sums is equal to $\sum_{j=1}^{n} a_{1j} + \sum_{j=1}^{n} a_{2j} + \cdots + \sum_{j=1}^{n} a_{mj}$, which can be written as $\sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij}\right)$.

If we add the numbers in each of the n columns first and then take the sum of these columns, we get instead

$$\sum_{i=1}^{m} a_{i1} + \sum_{i=1}^{m} a_{i2} + \cdots + \sum_{i=1}^{m} a_{in} = \sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ij} \right)$$

(How do you interpret this sum in our economic example?) In both these cases, we have calculated the sum of all the numbers in the array, so that we must have

$$\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} = \sum_{j=1}^{n} \sum_{i=1}^{m} a_{ij} \qquad \text{[B.12]}$$

where, according to usual practice, we have deleted the parentheses. Formula [B.12] says that *in a (finite) double sum, the order of summation is immaterial*. Here it is important to note that the summation limits for i and j are independent of each other. (See Problem 2 for a case in which the summation limits are not independent.)

Example B.7

Compute

$$\sum_{i=1}^{3} \sum_{j=1}^{4} (i + 2j)$$

Solution

$$\sum_{i=1}^{3} \sum_{j=1}^{4} (i + 2j) = \sum_{i=1}^{3} \left[(i + 2) + (i + 4) + (i + 6) + (i + 8)\right]$$

$$= \sum_{i=1}^{3} (4i + 20) = 24 + 28 + 32 = 84$$

(Perform the summation by first summing over i, and then over j, and show that the result is the same.)

Example B.8

Consider the $m \cdot n$ numbers a_{ij} in [B.11]. Denote the arithmetic mean of them all by $\bar{a}$, and the mean of all the numbers in the jth column by $\bar{a}_j$, so that

$$\bar{a} = \frac{1}{mn} \sum_{r=1}^{m} \sum_{s=1}^{n} a_{rs}, \qquad \bar{a}_j = \frac{1}{m} \sum_{r=1}^{m} a_{rj}$$

Prove that

$$\sum_{r=1}^{m}\sum_{s=1}^{m}(a_{rj}-\bar{a})(a_{sj}-\bar{a}) = m^2(\bar{a}_j-\bar{a})^2 \qquad [*]$$

Solution Because $a_{rj}-\bar{a}$ is independent of the summation index s, we get

$$\sum_{s=1}^{m}(a_{rj}-\bar{a})(a_{sj}-\bar{a}) = (a_{rj}-\bar{a})\sum_{s=1}^{m}(a_{sj}-\bar{a})$$

for each r, and so

$$\sum_{r=1}^{m}\sum_{s=1}^{m}(a_{rj}-\bar{a})(a_{sj}-\bar{a}) = \left[\sum_{r=1}^{m}(a_{rj}-\bar{a})\right]\left[\sum_{s=1}^{m}(a_{sj}-\bar{a})\right] \qquad [**]$$

Using [B.2] to [B.4] and the previous expression $\bar{a}_j$, we have

$$\sum_{r=1}^{m}(a_{rj}-\bar{a}) = \sum_{r=1}^{m}a_{rj} - \sum_{r=1}^{m}\bar{a} = m\bar{a}_j - m\bar{a} = m(\bar{a}_j-\bar{a}) \qquad [***]$$

Moreover, replacing r with s as the index of summation in [***] gives

$$\sum_{s=1}^{m}(a_{sj}-\bar{a}) = m(\bar{a}_j-\bar{a})$$

Substituting these values into [**] then confirms [*].

Problems

1. Expand the following:

   a. $\displaystyle\sum_{i=1}^{3}\sum_{j=1}^{4} i\cdot 3^j$ **b.** $\displaystyle\sum_{s=0}^{2}\sum_{r=2}^{4}\left(\frac{rs}{r+s}\right)^2$ **c.** $\displaystyle\sum_{i=1}^{m}\sum_{j=1}^{n} i\cdot k^j \quad (k\neq 1)$

2. Prove that the sum of all the numbers in the triangular table

$$\begin{array}{ccccc} a_{11} & & & & \\ a_{21} & a_{22} & & & \\ a_{31} & a_{32} & a_{33} & & \\ \vdots & \vdots & \vdots & \ddots & \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mm} \end{array}$$

can be written as

$$\sum_{i=1}^{m}\left(\sum_{j=1}^{i} a_{ij}\right) \quad \text{or as} \quad \sum_{j=1}^{m}\left(\sum_{i=j}^{m} a_{ij}\right)$$

3. Consider a group of individuals each having a certain number of units of m different goods. Let a_{ij} denote the number of units of good i owned by person j ($i = 1, \ldots, m$, $j = 1, \ldots, n$). Explain in words the meaning of the following sums:

a. $\sum_{j=1}^{n} a_{ij}$ **b.** $\sum_{i=1}^{m} a_{ij}$ **c.** $\sum_{j=1}^{n}\sum_{i=1}^{m} a_{ij}$

B.4 Products

There is a special notation for products, analogous to the $\sum$ notation for sums. If $a_1, a_2, \ldots, a_n$ are numbers, then we write $\prod_{i=1}^{n} a_i$ for the product of the numbers $a_1, a_2, \ldots a_n$. That is,

$$\prod_{i=1}^{n} a_i = a_1 a_2 \cdots a_n \tag{B.13}$$

The values of all the a_i as i ranges from 1 to n are multiplied together. More generally, if m and n are integers with $m \leq n$, then we write

$$\prod_{i=m}^{n} a_i = a_m a_{m+1} \cdots a_n \tag{B.14}$$

This product consists of $n - m + 1$ factors. For instance, if $n = 6$ and $m = 3$, then $\prod_{i=3}^{6} a_i = a_3 a_4 a_5 a_6$ is a product of $6 - 3 + 1 = 4$ factors.

Example B.9

Expand the following products:

$$\text{(a)} \prod_{i=3}^{6} (2i - 1) \qquad \text{(b)} \prod_{k=1}^{n} \left(1 + \frac{1}{k}\right)^{k}$$

Solution

$$\text{(a)} \prod_{i=3}^{6} (2i - 1) = 5 \cdot 7 \cdot 9 \cdot 11 = 3465$$

(b) $\prod_{k=1}^{n}\left(1+\frac{1}{k}\right)^{k} = \left(1+\frac{1}{1}\right)^{1}\left(1+\frac{1}{2}\right)^{2}\cdots\left(1+\frac{1}{n}\right)^{n}$

If all the a_i's in [B.13] and [B.14] are equal to the same number a, then

$$\prod_{i=1}^{n} a = a^n, \qquad \prod_{i=m}^{n} a = a^{n-m+1} \qquad \text{[B.15]}$$

In Section 20.3, we consider sums of products. Let us consider a typical example.

Example B.10

Evaluate the expression

$$\sum_{k=1}^{3}\left(\prod_{s=k+1}^{5} x_s\right) y^k$$

Solution Each product in parentheses obviously depends on k. If we let $D_k = \prod_{s=k+1}^{5} x_s$, the sum becomes $\sum_{k=1}^{3} D_k y^k = D_1 y^1 + D_2 y^2 + D_3 y^3$, so

$$\sum_{k=1}^{3}\left(\prod_{s=k+1}^{5} x_s\right) y^k = \left(\prod_{s=2}^{5} x_s\right) y^1 + \left(\prod_{s=3}^{5} x_s\right) y^2 + \left(\prod_{s=4}^{5} x_s\right) y^3$$
$$= (x_2x_3x_4x_5)y + (x_3x_4x_5)y^2 + (x_4x_5)y^3$$

Problems

1. Evaluate the following:

a. $\prod_{s=1}^{6} 2^{-s}$ b. $\prod_{k=3}^{6} k^3$ c. $\prod_{j=-2}^{1} \frac{j}{j+3}$ d. $\frac{\prod_{s=3}^{5}(1+r_s)}{\prod_{s=1}^{5}(1+r_s)}$

2. Expand the following expressions:

a. $\prod_{k=1}^{n} \frac{2k}{2k-1}\frac{2k}{2k+1}$ b. $\prod_{i=1}^{n} \frac{a_i}{b_i}$ c. $\sum_{i=1}^{n}\left(\prod_{s=i+1}^{n} a_s\right) b_i$

3. Which of the following equalities are true?

a. $\prod_{i=1}^{n} ka_i = k\prod_{i=1}^{n} a_i$ b. $\prod_{i=1}^{n} y_i^3 = \left(\prod_{i=1}^{n} y_i\right)^3$

c. $\prod_{i=1}^{n} x_i y_i = \left(\prod_{i=1}^{n} x_i\right)\left(\prod_{i=1}^{n} y_i\right)$ d. $\prod_{i=1}^{n}\left(\prod_{j=1}^{i} a_{ij}\right) = \prod_{j=1}^{n}\left(\prod_{i=j}^{n} a_{ij}\right)$

B.5 Induction

Proof by induction is an important technique for verifying formulas involving natural numbers. For instance, consider the sum of the first n odd numbers. We observe that

$$1 = 1 \quad = 1^2$$
$$1 + 3 = 4 \quad = 2^2$$
$$1 + 3 + 5 = 9 \quad = 3^2$$
$$1 + 3 + 5 + 7 = 16 = 4^2$$
$$1 + 3 + 5 + 7 + 9 = 25 = 5^2$$

This suggests a general pattern, with the sum of the first n odd numbers equal to n^2:

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2 \qquad [*]$$

To prove that this is generally valid, we can proceed as follows. Suppose that the formula in [*] is correct for a certain natural number $n = k$, so that

$$1 + 3 + 5 + \cdots + (2k - 1) = k^2$$

By adding the next odd number $2k + 1$ to each side, we get

$$1 + 3 + 5 + \cdots + (2k - 1) + (2k + 1) = k^2 + (2k + 1) = (k + 1)^2$$

But this is the formula [*] with $n = k + 1$. Hence, we have proved that if the sum of the first k odd numbers is k^2, then the sum of the first $k + 1$ odd numbers equals $(k + 1)^2$. This implication, together with the fact that [*] *is* valid for $n = 1$, implies that [*] is generally valid. For we have just shown that if [*] is true for $n = 1$, then it is true for $n = 2$; that if it is true for $n = 2$, then it is true for $n = 3$; ...; that if it is true for $n = k$, then it is true for $n = k + 1$; and so on.

A proof of this type is called a *proof by induction*. We show that the formula is valid for $n = 1$ and moreover that *if* the formula is valid for $n = k$, then it is also valid for $n = k + 1$. It follows by induction that the formula is valid for all natural numbers n. Our next example requires you to know about differentiation.

Example B.11

Prove by induction that, for all positive integers n,

$$f(x) = x^n \Longrightarrow f'(x) = nx^{n-1} \qquad [1]$$

Solution Formula [1] is correct for $n = 1$, because $f(x) = x \Rightarrow f'(x) = 1 = 1 \cdot x^{1-1}$. *Suppose* that [1] is valid for $n = k$. Then

$$f(x) = x^k \Longrightarrow f'(x) = kx^{k-1} \tag{2}$$

We have to prove that [1] is valid also for $n = k + 1$. To this end, we must differentiate $f(x) = x^{k+1}$. We cannot use [1] for $n = k + 1$, but because we have assumed the correctness of [2], we write $f(x) = x^{k+1} = x^k \cdot x$. Using the product rule for derivatives together with [2], we get

$$f(x) = x^k \cdot x \Longrightarrow f'(x) = kx^{k-1} \cdot x + x^k \cdot 1 = (k+1)x^k$$

But this result is precisely what we get by replacing the number n in [1] by $k + 1$. Using induction, we have proved [1] for all positive integers.

On the basis of these examples, the general structure of an induction proof can be explained as follows: We want to prove that a mathematical formula $A(n)$ that depends on n is valid for all natural numbers n. In the two previous examples, the respective statements $A(n)$ were

$$A(n): 1 + 3 + 5 + \cdots + (2n - 1) = n^2$$

$$A(n): f(x) = x^n \Longrightarrow f'(x) = nx^{n-1}$$

The steps required in each proof are as follows: First, verify that $A(1)$ is valid, which means that the formula is correct for $n = 1$. Then prove that for each natural number k, if $A(k)$ is true, it follows that $A(k + 1)$ must be true. Here $A(k)$ is called *the induction hypothesis*, and the step from $A(k)$ to $A(k + 1)$ is called *the induction step* in the proof. When the induction step is proved for an arbitrary natural number k, then, by induction, statement $A(n)$ is true for all n. The general principle can now be formulated:

The Principle of Mathematical Induction

Suppose that $A(n)$ is a statement for all natural numbers n and that

(a) $A(1)$ is true [B.16]

(b) if the induction hypothesis $A(k)$ is true, then $A(k + 1)$ is true for each natural number k

Then $A(n)$ is true for all natural numbers n.

The principle of induction seems intuitively evident. If the truth of $A(k)$ for each k implies the truth of $A(k+1)$, then because $A(1)$ is true, $A(2)$ must be true, which, in turn, means that $A(3)$ is true, and so on. (An analogy: Consider a ladder with an infinite number of steps. Suppose you can climb the first step and suppose, moreover, that after each step, you can always climb the next. Then you are able to climb up to any step.)

The principle of mathematical induction can easily be generalized to the case in which we have a statement $A(n)$ for each natural number greater than or equal to an arbitrary natural number n_0. If we can prove that $A(n_0)$ is valid, and moreover we can prove that if $A(k)$ is true, then $A(k+1)$ is true for each $k \geq n_0$, then $A(n)$ is true for all $n \geq n_0$.

Problems

1. Prove statement [B.5] in Sec. B.2 by induction, namely:

$$1 + 2 + 3 + \cdots + n = \tfrac{1}{2}n(n+1) \qquad [*]$$

2. Prove formulas [B.6] and [B.7] in Section B.2 by induction.
3. Prove the following by induction:
   a. $\dfrac{1}{1 \cdot 2} + \dfrac{1}{2 \cdot 3} + \dfrac{1}{3 \cdot 4} + \cdots + \dfrac{1}{n(n+1)} = \dfrac{n}{n+1}$
   b. $3 + 3^2 + 3^3 + 3^4 + \cdots + 3^n = \frac{1}{2}(3^{n+1} - 3)$
4. $1^3 + 2^3 + 3^3 = 36$ is divisible by 9. Prove by induction that the sum $n^3 + (n+1)^3 + (n+2)^3$ of three consecutive cubes is always divisible by 9.
5. Prove by induction that for $k \neq 1$,

$$a + ak + \cdots + ak^{n-1} = a\frac{1-k^n}{1-k}$$

   (See also (6.10) in Section 6.5.)
6. Let n be a positive integer and consider the expression $s_n = n^2 - n + 41$. Verify that s_n is a prime number (and so has no factor except 1 and itself) for $n = 1, 2, 3, 4$, and 5. With some efforts, one can prove that s_n is a prime number for $n = 6, 7, \ldots,$ and 40 as well. Is s_n a prime for all n? (This problem was first suggested by the Swiss mathematician L. Euler.)

Answers to Odd-Numbered Problems

Chapter 1

1.3

1. (a) $p_1x_1 + p_2x_2 + p_3x_3$ (b) $F + bx$ (c) $(F + cx)/x = F/x + c$
(d) After the $p\%$ raise, his salary is $L + pL/100 = L(1 + p/100)$. A $q\%$ raise of this new salary gives the final answer: $L(1 + p/100)(1 + q/100)$.
(e) $V = x(18 - 2x)(18 - 2x) = 4x(9 - x)^2$. (See Figs. 1 and 2.)

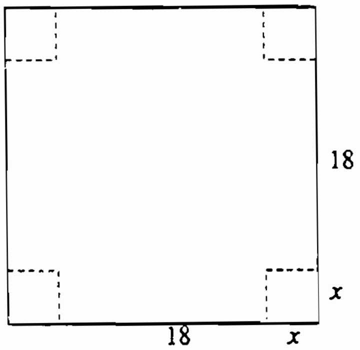


FIGURE 1

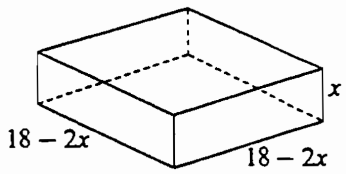


FIGURE 2

3. (a) $y = \frac{3}{5}(x + 2)$ (b) $x = \dfrac{b + d}{a - c}$ (when $a \neq c$)
(c) Square each side: $A^2K^2L = Y_0^2$. Solving for L yields $L = Y_0^2/A^2K^2$.
(d) $y = \dfrac{m}{q} - \dfrac{p}{q}x$ (when $q \neq 0$)
(e) $\dfrac{1}{1+r} - a = c\left(\dfrac{1}{1+r} + b\right)$. Multiplying by $1 + r$ and solving for r yields $r = \dfrac{(1 - a) - c(1 + b)}{a + bc}$ (f) $Y = \dfrac{I_p + G - ak + b}{1 - a + at}$

5. $2\pi(r+1) - 2\pi r = 2\pi$, where r is the radius of the earth (as an approximate sphere).

7. Let each side have length s. Then the area K is the sum of the areas of the triangles APB, APC, and BPC in Fig. 3, so $\frac{1}{2}sh_1 + \frac{1}{2}sh_2 + \frac{1}{2}sh_3 = K$. Thus, $h_1 + h_2 + h_3 = 2K/s$, which is independent of where P is placed.

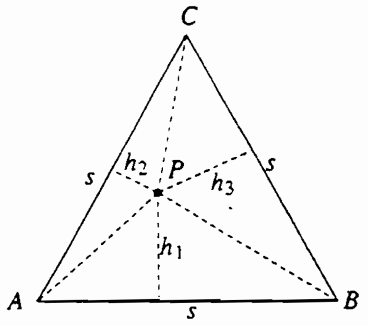


FIGURE 3

1.4

1. (a) Rational (note that this is only an approximation to the irrational number π).
   (b) $\sqrt{\frac{9}{2} - \frac{1}{2}} = \sqrt{4} = 2$, a natural number. (c) $3-2 = 1$, a natural number.
   (d) $3\pi - 1/4$ is irrational: If $3\pi - 1/4$ were rational, then there would exist integers p and q such that $3\pi - 1/4 = p/q$. Hence, $\pi = (4p+q)/12q$, which would imply that π is rational, a contradiction.

3. (a) $x \neq 4$ (b) $x \neq 0$ and $x \neq -2$ (c) $x \neq -5$ and $x \neq 1$. (The quadratic equation $x^2+4x-5 = 0$ has the solutions $x = -5$ and $x = 1$.) (d) $x \neq -2$. (The quadratic equation $x^2+4x+4 = (x+2)^2 = 0$ has the solution $x = -2$.)

5. $F/x + c < q$, which yields $x > F/(q-c)$. For $F = 100{,}000$, $c = 120$, and $q = 160$: $100{,}000/x + 120 < 160$, that is, $100{,}000/x < 40$, and so $x > 100{,}000/40 = 2500$.

7. (a) $|5-3(-1)| = |5+3| = 8$, $|5-3\cdot 2| = 1$, $|5-3\cdot 4| = 7$ (b) $x = 5/3$
   (c) $|5-3x| = 5-3x$ for $x \leq 5/3$, $|5-3x| = 3x-5$ for $x > 5/3$

9. (a) $4.999 < x < 5.001$ (b) $|x-5| < 0.001$

1.5

1. (a) $2x - 4 = 2 \Longrightarrow x = 3$ (b) $x = 3 \Longrightarrow 2x - 4 = 2$ (c) $x = 1 \Longrightarrow x^2 - 2x + 1 = 0$ (d) $x^2 > 4 \Longleftrightarrow x > 2$ or $x < -2$

3. (a) $x \geq 0$ is necessary, but not sufficient. (b) $x \geq 50$ is sufficient, but not necessary. (c) $x \geq 4$ is necessary and sufficient.

5. (a) $x = 3$ (b) $x = 0$ or $x = -5$ (c) $x = -3$ or $x = 3$

7. (a) Iff. (Note: $\sqrt{4}$ means 2, not ± 2.) (b) If (c) Only if (d) Iff (e) Only if

9. (a) $x < 0$ or $y < 0$ (b) $x < a$ for at least one x. (c) x and/or y is less than 5. (Is it easier if the statement were: neither Ron nor Rita is less than 5 years old?) (d) There exists an $\varepsilon > 0$ such that B is not satisfied for any $\delta > 0$. (e) Someone may not like cats. (f) Someone never loves anyone.

1.6

1. (b), (d), and (e) all express the same condition. (a) and (c) are different.

3. We should show why the fact that p^2 has 2 as a factor implies that p has 2 as a factor. Assume on the contrary that p does not have 2 as a factor. Then $p = 2m + 1$, for some natural number m. But then $p^2 = (2m+1)^2 = 4m^2 + 4m + 1 = 2(2m^2 + 2m) + 1$, which is odd and therefore does not have 2 as a factor.

1.7

1. (a) $5 \in C$, $D \subset C$, and $B = C$ are true. The three others are false. (b) $A \cap B = \{2\}$, $A \cup B = \{2, 3, 4, 5, 6\}$, $A \setminus B = \{3, 4\}$, $B \setminus A = \{5, 6\}$, $(A \cup B) \setminus (A \cap B) = \{3, 4, 5, 6\}$, $A \cup B \cup C \cup D = \{2, 3, 4, 5, 6\}$, $A \cap B \cap C = \{2\}$, and $A \cap B \cap C \cap D = \emptyset$

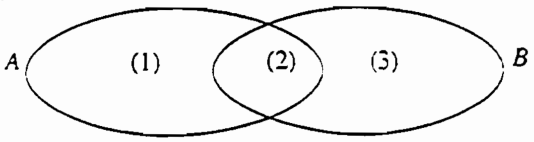
3. (a) $B \subset M$ (b) $F \cap B \cap C \neq \emptyset$ (c) $F \setminus (T \cup C) \subset B$

5. (a) to (e) follow immediately from the definitions and also from obvious Venn diagrams. Both sets in (f) are seen to consist of the areas (1), (2), (3), (4), and (7) in Fig. 1.8 in Section 1.7.

7. The $2^3 = 8$ subsets of $\{a, b, c\}$ are the set itself, the empty set, $\{a\}$, $\{b\}$, $\{c\}$, $\{a, b\}$, $\{a, c\}$, and $\{b, c\}$. The $2^4 = 16$ subsets of $\{a, b, c, d\}$ are the eight preceding sets together with $\{d\}$, $\{a, d\}$, $\{b, d\}$, $\{c, d\}$, $\{a, b, d\}$, $\{a, c, d\}$, $\{b, c, d\}$, and $\{a, b, c, d\}$.

9. (a) Look at Fig. 4. $n(A \cup B)$ is the sum of the numbers of elements in (1), (2), and (3) respectively—that is, $n(A \setminus B)$, $n(A \cap B)$, and $n(B \setminus A)$. $n(A) + n(B)$ is the number of elements in (1) and (2) together, plus the number of elements in (2) and (3) together. Thus, the elements in (2) are counted twice. Hence, you must subtract the number of elements in (2) (that is $n(A \cap B)$) to have equality. (b) Look again at Fig. 4. $n(A \setminus B)$ is the number of elements in (1), and $n(A) - n(A \cap B)$ is the number of elements in (1) and (2) together, minus the number of elements in (2). Hence, it is the number of elements in (1).

FIGURE 4



11. (b) is not generally valid. For example, in Problem 1, $(A \cap C) \triangle B = \{2\} \triangle \{2, 5, 6\} = \{5, 6\}$, whereas $(A \triangle B) \cap (C \triangle B) = \{3, 4, 5, 6\} \cap \emptyset = \emptyset$. (Alternatively, use Fig. 1.8 in Section 1.7.)

Chapter 2

2.2

1. (a) $f(0) = 0^2 + 1 = 1$, $f(-1) = (-1)^2 + 1 = 2$, $f(1/2) = 5/4$, and $f(\sqrt{2}) = 3$. (b) (i) For all x. (ii) When $(x+1)^2 + 1 = (x^2+1) + 2$, so $x^2 + 2x + 2 = x^2 + 3$, or $2x + 2 = 3$, or $x = 1/2$. (iii) $x = \pm\sqrt{1/2}$.

3. (a) $f(0) = 0$, $f(a) = a^2$, $f(-a) = a^2 - 4a^2 = -3a^2$, and $f(2a) = 0$ (b) $3f(a) + f(-2a) = 3a^2 + [a^2 - (-2a - a)^2] = 3a^2 + a^2 - 9a^2 = -5a^2$

5. (a) $C(0) = 1000$, $C(100) = 41{,}000$, and $C(101) - C(100) = 501$ (b) $C(x+1) - C(x) = 2x + 301 =$ incremental cost of producing one unit more than x.

7. (a) $D(8) = 4$, $D(10) = 3.4$, and $D(10.22) = 3.334$ (b) $P = 10.9$

9. (a) $f(tx) = 100(tx)^2 = 100t^2x^2 = t^2 100x^2 = t^2 f(x)$ (b) $P(tx) = (tx)^{1/2} = t^{1/2}x^{1/2} = t^{1/2}P(x)$

11. (a) $f(a+b) = A(a+b) = Aa + Ab = f(a) + f(b)$ (b) $f(a+b) = 10^{a+b} = 10^a \cdot 10^b = f(a) \cdot f(b)$

13. (a) $x \le 5$ (b) $x \ne 0$ and $x \ne 1$ (c) $-3 < x \le 1$ or $x > 2$ (d) $x > 1$

15. (a) $D_g = [-2, \infty)$, $R_g = (-\infty, 1]$. (The largest value of $g(x)$ is 1 for $x = -2$. As x increases from -2 to ∞, $g(x)$ decreases from 1 to $-\infty$.)

17.
$$f\left(\frac{ax+b}{cx-a}\right) = \frac{a\left(\dfrac{ax+b}{cx-a}\right) + b}{c\left(\dfrac{ax+b}{cx-a}\right) - a}$$
$$= \frac{a(ax+b) + b(cx-a)}{c(ax+b) - a(cx-a)} = \frac{a^2x + bcx}{a^2 + bc} = x$$

2.3

1. See Fig. 5.

3. See Figs. 6 to 8.

5. (a) $\sqrt{2}$ (b) $\sqrt{29}$ (c) $\frac{1}{2}\sqrt{205}$ (d) $\sqrt{x^2+9}$ (e) $2|a|$ (f) $\sqrt{8}$

7. (a) 5.362 (b) $\sqrt{(2\pi)^2 + (2\pi - 1)^2} = \sqrt{8\pi^2 - 4\pi + 1} \approx 8.209$

9. (a) $(x+5)^2 + (y-3)^2 = 4$ has center at $(-5, 3)$ and radius 2. (b) $(x+3)^2 + (y-4)^2 = 12$ has center at $(-3, 4)$ and radius $\sqrt{12}$.

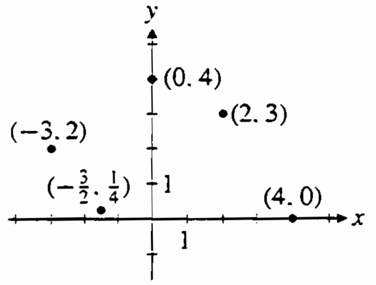


FIGURE 5

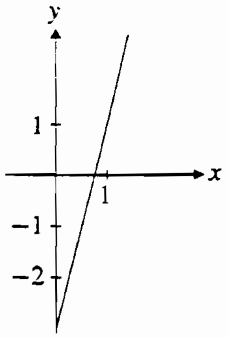


FIGURE 6

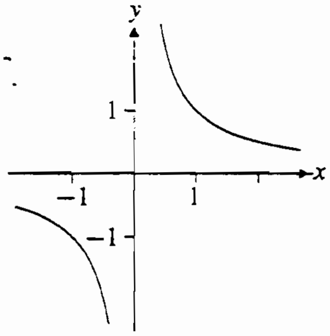


FIGURE 7

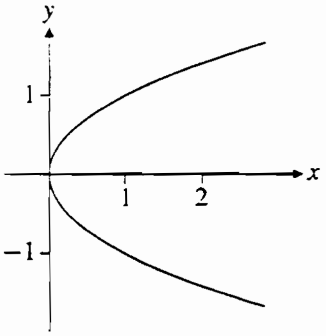


FIGURE 8

11. The condition is that $\sqrt{(x+2)^2+y^2} = 2\sqrt{(x-4)^2+y^2}$, or equivalently $(x-6)^2 + y^2 = 4^2$.

13. (a) The two expressions give total price per unit (product price plus shipping costs) for the product delivered at (x, y) from A and B respectively. (b) The condition is that $p + 10\sqrt{x^2+y^2} = p + 5\sqrt{(x-60)^2+y^2}$, which reduces to $(x+20)^2 + y^2 = 40^2$.

15. $x^2 + y^2 + Ax + By + C = 0 \iff x^2 + Ax + y^2 + By + C = 0 \iff x^2 + Ax + (\frac{1}{2}A)^2 + y^2 + By + (\frac{1}{2}B)^2 = \frac{1}{4}(A^2 + B^2 - 4C) \iff (x + \frac{1}{2}A)^2 + (y + \frac{1}{2}B)^2 = \frac{1}{4}(A^2 + B^2 - 4C)$. The last is the equation of a circle centered at $(-\frac{1}{2}A, -\frac{1}{2}B)$ with radius $\frac{1}{2}\sqrt{A^2 + B^2 - 4C}$. If $A^2 + B^2 = 4C$, the graph consists only of the point $(-\frac{1}{2}A, -\frac{1}{2}B)$. For $A^2 + B^2 < 4C$, the solution set is empty.

2.4

1. (a) All x (b) $x = 0$ (c) All x (d) $x = 0$ (For $x > 0$, the equation $y^4 = x$ has *two* solutions.) (e) $x = \pm 1$ (f) All $x \neq 3$ (g) All x (h) All x

3. Suppose c is positive. Then $f(x) + c$ is obtained by raising the graph of $f(x)$ by c units. $f(x + c)$ is obtained by shifting the graph of $f(x)$ by c units to the left. $-f(x)$ is obtained by reflecting the graph of $f(x)$ in the x-axis. $f(-x)$ is obtained by reflecting the graph of $f(x)$ in the y-axis.

2.5

1. (a) Slope = $(8-3)/(5-2) = 5/3$ (b) $-2/3$ (c) $51/5$

3. L_1 is $y = x + 2$, with slope 1; L_2 is $y = -\frac{3}{5}x + 3$, with slope $-3/5$; L_3 is $y = 1$, with slope 0; L_4 is $y = 3x - 14$, with slope 3; L_5 is $y = \frac{1}{9}x + 2$, with slope 1/9.

5. (a), (b), and (d) are all linear; (c) is not.

7. (a) L_1 is $(y-3) = 2(x-1)$ or $y = 2x+1$ (b) L_2 is $y-2 = \frac{3-2}{3-(-2)}[x-(-2)]$ or $y = x/5 + 12/5$ (c) L_3 is $y = -x/2$ (d) L_4 is $x/a + y/b = 1$, or $y = -bx/a + b$. The graphs are shown in Figs. 9 and 10.

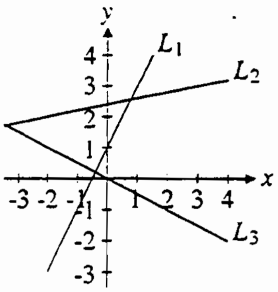


FIGURE 9

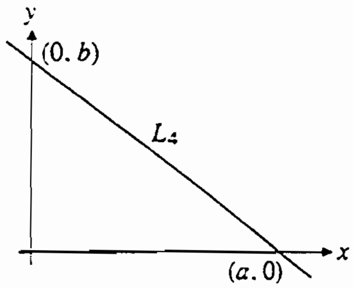


FIGURE 10

9. The point–point formula gives $y - 200 = \dfrac{275-200}{150-100}(x-100)$ or $y = \frac{3}{2}x + 50$.

11. (a) April 1960 corresponds to $t = 9/4$, when $N(9/4) = -17{,}400 \cdot (9/4) + 151{,}000 = 111{,}850$. (b) $-17{,}400t + 151{,}000 = 0$ implies $t = 8.68$, which corresponds roughly to September 1966.

13. For (a), shown in Fig. 11, the solution is $x = 3$, $y = -2$. For (b), shown in Fig. 12, the solution is $x = 2$, $y = 0$. For (c), shown in Fig. 13, there are no solutions, because the two lines are parallel.

15. See Fig. 14. $C = 0.8824Y - 1.3941$. The slope is an estimate of the marginal propensity to consume.

17. See Fig. 15. Each arrow shows the side of the line on which the relevant inequality is satisfied. The shaded triangle is the required set.

FIGURE 11

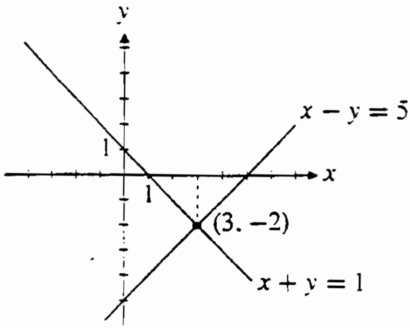
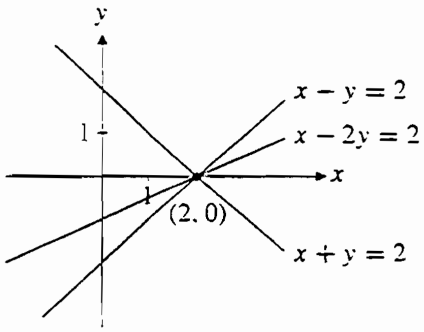


FIGURE 12



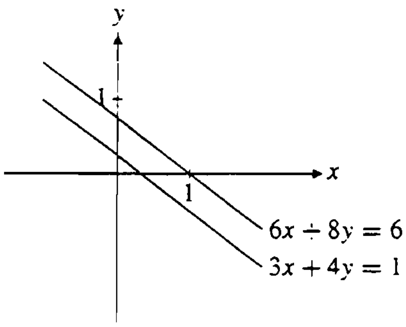


FIGURE 13

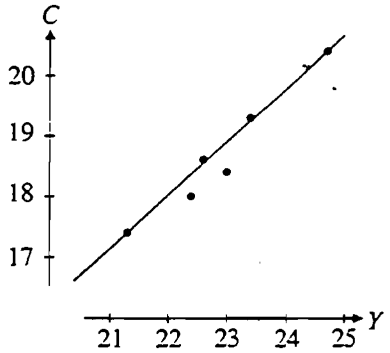


FIGURE 14

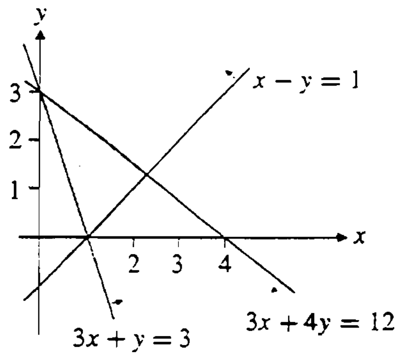


FIGURE 15

Chapter 3

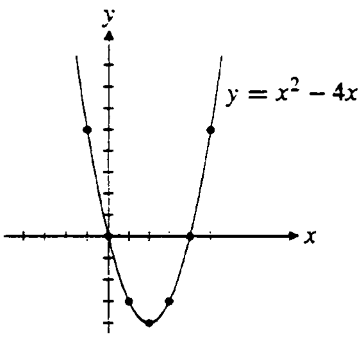
3.1

1. (a)

x	−1	0	1	2	3	4	5
$f(x) = x^2 - 4x$	5	0	−3	−4	−3	0	5

(b) See Fig. 16. (c) $f(x) = (x-2)^2 - 4$. Minimum at $(2, -4)$. (d) $x = 0$ and $x = 4$.

FIGURE 16



3. (a) $x^2 + 4x = (x+2)^2 - 4$. Minimum -4 for $x = -2$. (b) $x^2 + 6x + 18 = (x+3)^2 + 9$. Minimum 9 for $x = -3$. (c) $-3x^2 + 30x - 30 = -3(x-5)^2 + 45$. Maximum 45 for $x = 5$. (d) $9x^2 - 6x - 44 = 9(x - 1/3)^2 - 45$. Minimum -45 for $x = 1/3$. (e) $-x^2 - 200x + 30{,}000 = -(x + 100)^2 + 40{,}000$. Maximum 40,000 for $x = -100$. (f) $x^2 + 100x - 20{,}000 = (x+50)^2 - 22{,}500$. Minimum $-22{,}500$ for $x = -50$.

5. (a) $x = 2p$ and $x = p$ (b) $x = p$ and $x = q$ (c) $x = -\frac{1}{2}p \pm \sqrt{\frac{1}{4}p^2 - q}$

7. $x = 250$ maximizes A. ($A = -x^2 + 500x = -(x - 250)^2 + 62{,}500$.)

9. $x = 4(r-1)/(r^2+1)$. (Use [3.4].)

11. Using [3.3], we see that $f(x) = a(x - \bar{x})^2 - A$, with $\bar{x} = -b/2a$, and $A = (b^2 - 4ac)/4a$. Now, $f(\bar{x} - t) = a(\bar{x} - t - \bar{x})^2 - A = a(-t)^2 - A = at^2 - A$, and $f(\bar{x} + t) = a(\bar{x} + t - \bar{x})^2 - A = at^2 - A$. So $f(\bar{x} - t) = f(\bar{x} + t)$ for all t, and the graph is symmetric about the line $x = \bar{x}$.

3.2

1. (a) $\pi(Q) = -\frac{1}{2}Q^2 + (\alpha_1 - \alpha_2 - \gamma)Q$ (b) Using [3.4], $Q^* = \alpha_1 - \alpha_2 - \gamma$ maximizes profits if $\alpha_1 - \alpha_2 - \gamma > 0$. If $\alpha_1 - \alpha_2 - \gamma \le 0$, then $Q^* = 0$. (c) $\pi(Q) = -\frac{1}{2}Q^2 + (\alpha_1 - \alpha_2 - \gamma - t)Q$ and $Q^* = \alpha_1 - \alpha_2 - \gamma - t$ if $\alpha_1 - \alpha_2 - \gamma - t > 0$. (d) Tax revenue $= tQ^* = t(\alpha_1 - \alpha_2 - \gamma - t)$. Then $t = \frac{1}{2}(\alpha_1 - \alpha_2 - \gamma)$ maximizes tax revenues.

3.3

1. (a) 1 and -2 (b) 1, 5, and -5 (c) -1

3. (a) $x + 4$ (b) $x^2 + x + 1$ (c) $-3x^2 - 12x$

5. (a) $x^3 - x - 1$ is not 0 for $x = 1$, so the division leaves a remainder.
(b) $2x^3 - x - 1$ is 0 for $x = 1$, so the division leaves no remainder.
(c) $x^3 - ax^2 + bx - ab$ is 0 for $x = a$, so the division leaves no remainder.
(d) $x^{2n} - 1$ is 0 for $x = -1$, so the division leaves no remainder.

7. (a) $\frac{1}{2}(x+1)(x-3)$ (b) $-2(x-1)(x-2)(x+3)$ (c) $\frac{1}{2}(x-2)^2(x+3)$

3.4

1. (a) 2 (b) 1/3 (c) 5 (d) 1/8

3. (a) 4/3 (b) 2.5 (c) 1/5

5. (a) $a^{1/5}$ (b) $a^{163/60}$ (c) $9a^7/2$ (d) $a^{1/4}$

7. (b) and (c) are valid; the others are not generally valid.

9. The surface area is $4.84 \cdot (100)^{2/3}\ \text{m}^2 \approx 104.27\ \text{m}^2$. You need $104 \cdot 27/5$ or slightly less than 21 liters.

11. (a) $a - b$ (b) $(a + x)/2bx^{3/2}$

3.5

1. The doubling time t^* is determined by the equation $(1.0072)^{t^*} = 2$. Using a calculator, we find $t^* \approx 96.6$.
3. (a) The amount of savings after t years: $100\,(1 + 12/100)^t = 100 \cdot (1.12)^t$
(b)

x	1	2	5	10	20	30	50
$100 \cdot (1.12)^x$	112	125.44	176.23	310.58	964.63	2995.99	28,900.21

5. The graphs are drawn in Fig. 17.

x	−3	−2	−1	0	1	2	3
2^x	1/8	1/4	1/2	1	2	4	8
2^{-x}	8	4	2	1	1/2	1/4	1/8

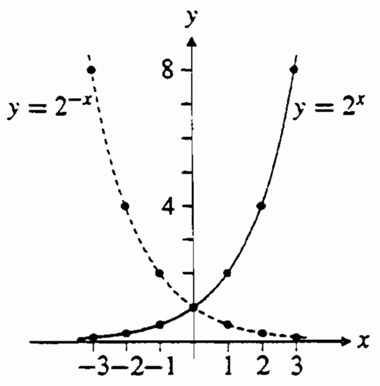


FIGURE 17

7. If $5.1 \cdot (1.035)^t = 3.91 \cdot 10^{11}$, then $(1.035)^t = 3.91 \cdot 10^{11}/5.1 \approx 0.7667 \cdot 10^{11}$. We find (using a calculator) that $t \approx 728$, so the year is $728 + 1969 = 2697$. This is when each Zimbabwean would have only 1 m^2 of land.
9. $(1 + p/100)^{15} = 2$, so $p \approx 4.7\%$
11. (b) and (d) do not define exponential functions. (In (f): $y = (1/2)^x$.)
13. Solve the equation $y = Ab^x$ for A and b by using the two indicated points on each graph. This gives $y = 2 \cdot 2^x$, $y = 2 \cdot 3^x$, and $y = 4(1/2)^x$.

3.6

1. Only (c) does not define a function. (Rectangles with different areas can have the same perimeter.)
3. The function cannot be one-to-one because at least two persons must have the same blood group.

Chapter 4

4.2

1. $f(5+h)-f(5)=4(5+h)^2-4\cdot 5^2=4(25+10h+h^2)-100=40h+4h^2$. So $[f(5+h)-f(5)]/h=40+4h\to 40$ as $t\to 0$. Hence, $f'(5)=40$. This accords with [4.8] when $a=4$ and $b=c=0$.

3. $f'(a)<0,\quad f'(b)=0,\quad f'(c)>0,\quad f'(d)<0$

5. (a) $f'(0)=3$ (b) $f'(1)=2$ (c) $f'(3)=-1/3$ (d) $f'(0)=-2$ (e) $f'(-1)=0$ (f) $f'(1)=4$

7. (a) $dD(P)/dP=-b$ (b) $C'(x)=2qx$

9. (a) $f(x+h)-f(x)=a(x+h)^3+b(x+h)^2+c(x+h)+d-(ax^3+bx^2+cx+d)$. Here $(x+h)^3=x^3+3x^2h+3xh^2+h^3$ and $(x+h)^2=x^2+2xh+h^2$. Easy algebra then gives $[f(x+h)-f(x)]/h=3ax^2+2bx+c+3ahx+ah^2+bh$, which evidently tends to $3ax^2+2bx+c$ as $h\to 0$. (b) If $a=1$ and $b=c=d=0$, then the result in Example 4.3 follows. So does that in Problem 6 when $a=0$.

4.3

1. $C'(100)=203$ and $C'(x)=2x+3$.

3. (a) $S'(Y)=b$ (b) $S'(Y)=4Y+10$

5. (a) $R'(x)=a$, $C'(x)=2a_1x+b_1$, and $\pi'(x)=a-2a_1x-b_1$. The marginal profit is 0 when $x=(p-b_1)/2a_1$. (b) $R'(x)=2ax$, $C'(x)=a_1$, and $\pi'(x)=2ax-a_1$. The marginal profit is 0 when $x=a_1/2a$. (b) With $R(x)=ax-bx^3$, $R'(x)=a-2bx$, $C'(x)=a_1$, and $\pi'(x)=a-2bx-a_1$. The marginal profit is 0 when $x=(a-a_1)/2b$.

4.4

1. (a) 3 (b) $-1/2$ (c) $(13)^3=2197$ (d) 40 (e) 1 (f) $-3/4$

3. The $\lim_{t\to a}h(t)$ is equal to 0 for $a=-1$, it is 1 for $a=0$, and it is 3 for $a=2$. For $a=3$ and 4, the limit does not exist.

5. (a) For $h\neq 2$, $\dfrac{\frac{1}{3}-\frac{2}{3h}}{h-2}=\dfrac{\left(\frac{1}{3}-\frac{2}{3h}\right)3h}{(h-2)3h}=\dfrac{h-2}{(h-2)3h}=\dfrac{1}{3h}$, which tends to 1/6 as $h\to 2$. (b) $-\infty$ (limit does not exist) (c) 2 (*Hint:* $t-3$ is a common factor for $32t-96$ and t^2-2t-3.) (d) $\sqrt{3}/6$ (*Hint:* Multiply numerator and denominator by $\sqrt{h+3}+\sqrt{3}$.) (e) $-2/3$ (*Hint:* $t+2$ is a common factor.) (f) 1/4 (*Hint:* $4-x=(2+\sqrt{x})(2-\sqrt{x})$.)

7. (a) 0.6931 (b) 1.0986 (c) 2.7183

4.5

1. (a) 0 (b) $4x^3$ (c) $90x^9$ (d) 0 (Remember that π is a constant!)

3. (a) $6x^5$ (b) $33x^{10}$ (c) $50x^{49}$ (d) $28x^{-8}$ (e) x^{11} (f) $4x^{-3}$
(g) $(-3/2)x^{-3/2}$ (h) $3x^{-5/2}$

5. Let $x - a = h$. Then $x = a + h$, and the result follows from [4.3], the ordinary definition of the derivative. For $f(x) = x^2$, the equation gives $f'(a) = 2a$.

7. (a) Let $f(x) = x^2$. Then $f'(x) = 2x$, and the limit is equal to $f'(5) = 2 \cdot 5 = 10$. (b) Let $f(x) = x^5$. Then $f'(x) = 5x^4$, and the limit is equal to $f'(1) = 5 \cdot 1^4 = 5$. (c) Let $f(x) = 5x^2 + 10$. Then $f'(x) = 10x$, and this is the value of the limit.

4.6

1. (a) 1 (b) $1 + 2x$ (c) $15x^4 + 8x^3$ (d) $32x^3 + x^{-1/2}$ (e) $1/2 - 3x + 15x^2$
(f) $-21x^6$

3. (a) $-6x^{-7}$ (b) $(3/2)x^{1/2} - (1/2)x^{-3/2}$ (c) $(-3/2)x^{-5/2}$ (d) $-2/(x-1)^2$
(e) $(-4x - 5)/x^6$ (f) $34/(2x+8)^2$ (g) $-33x^{-12}$
(h) $(-3x^2 + 2x + 4)/(x^2 + x + 1)^2$

5. $R'(P) = D(P) + PD'(P)$

7. (a) $y = -3x+4$ (b) $y = x - 1$ (c) $y = (17x - 19)/4$ (d) $y = -(x-3)/9$

9. (a) $\dfrac{2(AD - BC)p}{(Cp^2 + D)^2}$ (b) $-6y^{-7} - 16y^{-9}$ (c) $-2f'(x)/\left[1 + f(x)\right]^2$

11. If $f(x) = 1/x^n$, then by the rule for differentiating a quotient, $f'(x) = (0 \cdot x^n - 1 \cdot nx^{n-1})/(x^n)^2 = -nx^{-n-1}$, which is the power rule.

4.7

1. (a) $y'' = 20x^3 - 36x^2$ (b) $y'' = (-1/4)x^{-3/2}$ (c) $y'' = (-2)(x+1)^{-3}$

3. (a) $y'' = 18x$ (b) $Y''' = 36$ (c) $d^3z/dt^3 = -2$ (d) $f^{(4)}(1) = 84.000$.

5. $y'' = f''(x)g(x) + 2f'(x)g'(x) + f(x)g''(x)$,
$y''' = f'''(x)g(x) + 3f''(x)g'(x) + 3f'(x)g''(x) + f(x)g'''(x)$

7. $f(x) = x^{16/3}$ has this property. Successive differentiations eventually yield $f^{(5)}(x) = (16/3)(13/3)(10/3)(7/3)(4/3)x^{1/3}$ for all x, but only for $x \neq 0$ is $f^{(6)}(x) = (16/3)(13/3)(10/3)(7/3)(4/3)(1/3)x^{-2/3}$.

Chapter 5

5.1

1. (a) $f(x) = 9x^4 + 6x^2 + 1$, so $f'(x) = 36x^3 + 12x$.
(b) Using [5.1]: $f'(x) = 2(3x^2 + 1)^{2-1}6x = 36x^3 + 12x$.

3. (a) $\frac{1}{2}(1+x)^{-1/2}$ (b) $\frac{3}{2}x^2(x^3+1)^{-1/2}$ (c) $-\frac{3}{2}(x-1)^{-2}\left(\frac{2x+1}{x-1}\right)^{-1/2}$
(d) $-66x(1-x^2)^{32}$ (e) $3x^2\sqrt{1-x}-\dfrac{x^3}{2\sqrt{1-x}}$
(f) $\frac{1}{3}(1+x)^{-2/3}(1-x)^{1/5}-\frac{1}{5}(1+x)^{1/3}(1-x)^{-4/5}$

5. (a) $1+f'(x)$ (b) $2f(x)f'(x)-1$ (c) $4\big[f(x)\big]^3 f'(x)$
(d) $2xf(x)+x^2f'(x)+3\big[f(x)\big]^2f'(x)$ (e) $f(x)+xf'(x)$ (f) $\dfrac{f'(x)}{2\sqrt{f(x)}}$
(g) $[2xf(x)-x^2f'(x)]/[f(x)]^2$ (h) $[2xf(x)f'(x)-3(f(x))^2]/x^4$
(i) $\frac{1}{3}\{f(x)+\big[f(x)\big]^3+x\}^{-2/3}\{f'(x)+3\big[f(x)\big]^2f'(x)+1\}$

7. $dy/dv = Aapqv^{p-1}(av^p+b)^{q-1}$

9. $f'(x)=m(x-a)^{m-1}\cdot(x-b)^n+n(x-a)^m\cdot(x-b)^{n-1}$. Factoring this expression yields $f'(x)=[m(x-b)+n(x-a)](x-a)^{m-1}\cdot(x-b)^{n-1}=0$ at $x_0=(na+mb)/(n+m)$. Clearly, $a<(na+mb)/(n+m)<b$ iff $a(n+m)<na+mb<b(n+m)$. Here $a(n+m)<na+mb$ iff $(a-b)m<0$, which is true because $a<b$ and m is positive. Moreover, $na+mb<b(n+m)$ iff $0<n(b-a)$, which is also true.

11. Use the product rule and [5.1]. $m=n=1$ yields the product rule for derivatives. $m=-n=1$ yields the quotient rule.

5.2

1. (a) $dy/du=20u^3$ and $du/dx=2x$ so $dy/dx=(dy/du)\cdot(du/dx)=40u^3x=40x(1+x^2)^3$
(b) $dy/dx=(1-6u^5)(du/dx)=\big[1-6(1/x+1)^5\big]\big(-1/x^2\big)$

3. (a) $y'=-5(x^2+x+1)^{-6}(2x+1)$ (b) $y'=\frac{1}{2}\big[x+(x+x^{1/2})^{1/2}\big]^{-1/2}\big[1+\frac{1}{2}(x+x^{1/2})^{-1/2}(1+\frac{1}{2}x^{-1/2})\big]$ (c) $y'=ax^{a-1}(px+q)^b+x^abp(px+q)^{b-1}$

5. $dY/dt=F'\big[h(t)\big]\cdot h'(t)$

7. $h'(x)=f'(x^2)2x$

9. (a) $\dot{x}/x=2\dot{a}/a+\dot{b}/b$ (b) $\dot{x}/x=5\dot{a}/a-\dot{b}/b$
(c) $\dot{x}/x=(\alpha+\beta)(\alpha a^{\alpha-1}\dot{a}+\beta b^{\beta-1}\dot{b})/(a^\alpha+b^\beta)$ (d) $\dot{x}/x=\alpha\dot{a}/a+\beta\dot{b}/b$

11. (a) $h(x)=f\big(g(x)\big)$, where $g(x)=1+x+x^2$ and $f(u)=u^{1/2}$, so $h'(x)=(1/2)(1+x+x^2)^{-1/2}(1+2x)$. (b) $h(x)=f(g(x))$, where $g(x)=x^{100}+28$ and $f(u)=u^{-1}$, so $h'(x)=-(x^{100}+28)^{-2}100x^{99}$.

13. (a) $y'=5(x^4)^4\cdot4x^3=20x^{19}$ (b) $y'=3(1-x)^2(-1)=-3+6x-3x^2$

15. $dR/dt=(dR/dS)(dS/dK)(dK/dt)=\alpha S^{\alpha-1}\beta\gamma K^{\gamma-1}Apt^{p-1}$

5.3

1. (a) Differentiation w.r.t. x yields $1\cdot y+xy'=0$, so $y'=-y/x$. Because $y=1/x$, this gives $y'=-1/x^2$. (b) $y'=(1+3y)/(1-3x)$. Because

$y = (x-2)/(1-3x)$, this gives $y' = -5/(1-3x)^2$. (c) $y' = 5x^4/6y^5$. Because $y = x^{5/6}$, this gives $y' = (5/6)x^{-1/6}$.

3. $\dfrac{dv}{du} = \dfrac{2u+v}{3v^2-u}$. Hence, $dv/du = 0$ when $v = -2u$. Inserting this value into the original equation implies that $dv/du = 0$ at $(u, v) = (1/8, -1/4)$.

5. $dQ/dP = -19/P^{3/2}$

7. $Y = f(Y) + I + \bar{A} - g(Y)$. Differentiating w.r.t. I using the chain rule yields $dY/dI = f'(Y)(dY/dI) + 1 - g'(Y)(dY/dI)$. Solving for dY/dI gives $dY/dI = 1/\left[1 - f'(Y) + g'(Y)\right]$. Imports should increase when income increases, so $g'(Y) > 0$. We find that $dY/dI > 0$.

9. Differentiation w.r.t. x yields $g'\big(f(x)\big)f'(x) = 1$, so $f'(x) = 1/g'\big(f(x)\big)$ (provided that $g'\big(f(x)\big) \neq 0$).

5.4

1. If $f(x) = \sqrt{1+x}$, then $f'(x) = 1/2\sqrt{1+x}$, so $f(0) = 1$ and $f'(0) = 1/2$. Hence, [5.5] gives $\sqrt{1+x} \approx 1 + \frac{1}{2}(x-0) = 1 + \frac{1}{2}x$. See Fig. 18.

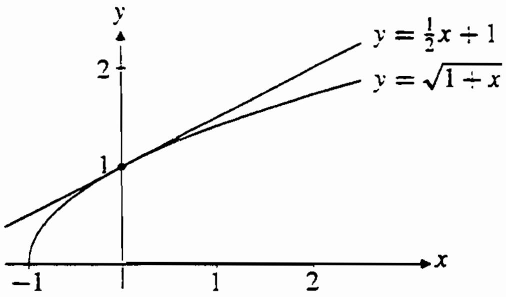


FIGURE 18

3. (a) $1/(1+x) \approx 1 - x$ (b) $(1+x)^5 \approx 1 + 5x$ (c) $(1-x)^{1/4} \approx 1 - x/4$

5. (a) $\sqrt[3]{1.1} = (1 + 1/10)^{1/3} \approx 1 + (1/3)(1/10) \approx 1.033$ (b) $\sqrt[5]{33} = 2(1 + 1/32)^{1/5} \approx 2(1 + 1/160) = 2.0125$ (c) $\sqrt[3]{9} = 2(1 + 1/8)^{1/3} \approx 2(1 + 1/24) \approx 2.083$ (d) $(1.02)^{25} = (1 + 1/50)^{25} \approx 1 + 1/2 = 1.5$ (e) $\sqrt{37} = (36+1)^{1/2} = 6(1 + 1/36)^{1/2} \approx 6(1 + 1/72) \approx 6.083$ (f) $(26.95)^{1/3} = (27 - 5/100)^{1/3} = 3(1 - 0.05/27)^{1/3} \approx 3 - 0.05/27 \approx 2.998$

7. $V(r) = (4/3)\pi r^3$. The linear approximation is: $V(2 + 0.03) - V(2) \approx 0.03 \cdot V'(2) = 0.48\pi$. Actual increase: $V(2.03) - V(2) = 0.487236\pi$.

5.5

1. (a) $(1+x)^5 \approx 1 + 5x + 10x^2$ (b) $AK^\alpha \approx A + \alpha A(K-1) + \frac{1}{2}\alpha(\alpha-1)A(K-1)^2$ (c) $(1 + \frac{3}{2}\varepsilon + \frac{1}{2}\varepsilon^2)^{1/2} \approx 1 + \frac{3}{4}\varepsilon - \frac{1}{32}\varepsilon^2$ (d) $(1-x)^{-1} \approx 1 + x + x^2$

3. Implicit differentiation yields: [*] $3x^2y + x^3y' + 1 = \frac{1}{2}y^{-1/2}y'$. Inserting $x = 0$ and $y = 1$ gives $1 = (\frac{1}{2})1^{-1/2}y'$, so $y' = 2$. Differentiating [*] once

more w.r.t. x yields $6xy + 3x^2y' + 3x^2y' + x^3y'' = -\frac{1}{4}y^{-3/2}(y')^2 + \frac{1}{2}y^{-1/2}y''$. Inserting $x = 0$, $y = 1$, and $y' = 2$ gives $y'' = 2$. Hence, $y(x) \approx 1 + 2x + x^2$.

5. Use [5.10] with $f(x) = (1+x)^n$ and $x = p/100$. Then $f'(x) = n(1+x)^{n-1}$ and $f''(x) = n(n-1)(1+x)^{n-2}$. The approximation follows.

5.6

1. (a) -3 (b) 100 (c) 1/2 (d) $-3/2$

3. (a) An increase in prices by 10% leads to a decrease in traffic by approximately 4%. (b) One reason could be that for long-distance travel, more people fly when rail fares go up.

5. $\text{El}_x f(x)^p = \dfrac{x}{f(x)^p} p(f(x))^{p-1} f'(x) = p\dfrac{x}{f(x)} f'(x) = p\,\text{El}_x f(x)$

7. (a) $\text{El}_x A = (x/A)(dA/dx) = 0$

(b) $\text{El}_x(fg) = \dfrac{x}{fg}(fg)' = \dfrac{x}{fg}(f'g + fg') = \dfrac{xf'}{f} + \dfrac{xg'}{g} = \text{El}_x f + \text{El}_x g$

(c) $\text{El}_x \dfrac{f}{g} = \dfrac{xg}{f}\left(\dfrac{f}{g}\right)' = \dfrac{xg}{f}\left(\dfrac{gf' - fg'}{g^2}\right) = \dfrac{xf'}{f} - \dfrac{xg'}{g} = \text{El}_x f - \text{El}_x g$

(d) $\text{El}_x(f+g) = \dfrac{x(f'+g')}{f+g} = \dfrac{f\dfrac{xf'}{f} + g\dfrac{xg'}{g}}{f+g} = \dfrac{f\,\text{El}_x f + g\,\text{El}_x g}{f+g}$

(e) Is like (d), but with $+g$ replaced by $-g$, and $+g'$ by $-g'$.

(f) $z = f(g(u))$, $u = g(x) \Rightarrow \text{El}_x z = \dfrac{x}{z}\dfrac{dz}{dx} = \dfrac{x}{u}\dfrac{u}{z}\dfrac{dz}{du}\dfrac{du}{dx} = \text{El}_u f(u)\,\text{El}_x u$

9. (a) $5 + \text{El}_x f(x)$ (b) $\frac{3}{2}\text{El}_x f(x)$ (c) $\dfrac{x + \frac{1}{2}\sqrt{f(x)}\,\text{El}_x f(x)}{x + \sqrt{f(x)}}$ (d) $-\text{El}_x f(x)$

Chapter 6

6.1

1. (a) -4 (b) 0 (c) 2 (d) $-\infty$ (e) ∞ (f) $-\infty$

3. (a) (i) $-\infty$ (ii) ∞ (iii) 0 (iv) A (b) $\lim_{x\to 0} f(x) = B$

5. (a) $y = x - 1$ ($x = -1$ is a vertical asymptote). (b) $y = 2x - 3$ (c) $y = 3x + 5$ ($x = 1$ is a vertical asymptote). (d) $y = 5x$ ($x = 1$ is a vertical asymptote).

6.2

1. (b) and (d) are continuous; the others are not necessarily so. (As for (c), consider Problem 6.)

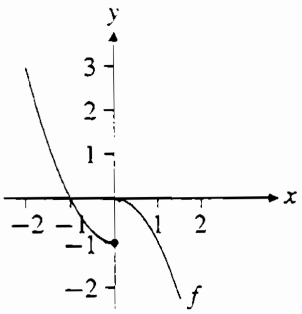


FIGURE 19

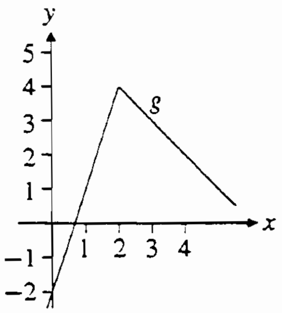


FIGURE 20

3. f is discontinuous at $x = 0$. g is continuous at $x = 2$. The graphs of f and g are shown in Figs. 19 and 20.
5. $a = 5$, so that $\lim_{x\to 1^+} f(x) = 4 = \lim_{x\to 1^-} f(x) = a - 1$.
7. No. Let $f(x) = g(x) = 1$ for $x < a$, let $f(x) = -1$ and $g(x) = 3$ for $x \geq a$. Then f and g are both discontinuous at $x = a$, but $f(x) + g(x) = 2$ for all x, and therefore $f + g$ is continuous for all x. (Draw a figure!) Let $h(x) = -f(x)$ for all x. Then h is also discontinuous at $x = a$, whereas $f(x)h(x) = -1$ for all x, and so $f \cdot h$ is continuous for all x.

6.3

1. $f'(0^+) = 1$ and $f'(0^-) = 0$. See Fig. 21.
3. If $x > 0$, then $f'(x) = \frac{1}{3}x^{-2/3} \to \infty$ as $x \to 0^+$. Also $f'(x) \to -\infty$ as $x \to 0^-$. Hence, the graph has a cusp at $x = 0$. See Fig. 22.

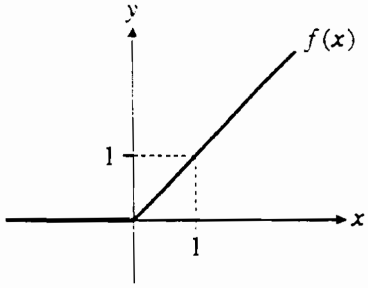


FIGURE 21

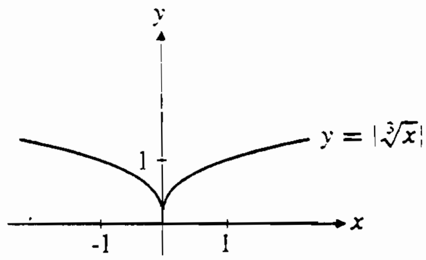


FIGURE 22

6.4

1. (a) $\alpha_n = \dfrac{(3/n) - 1}{2 - (1/n)} \to -\dfrac{1}{2}$ as $n \to \infty$ (b) $\beta_n = \dfrac{1 + (2/n) - (1/n^2)}{3 - (2/n^2)} \to \dfrac{1}{3}$ as $n \to \infty$ (c) $3 \cdot (-1/2) + 4 \cdot (1/3) = -1/6$ (d) $(-1/2) \cdot (1/3) = -1/6$ (e) $(-1/2)/(1/3) = -3/2$ (f) $\sqrt{(1/3) - (-1/2)} = \sqrt{5/6} = \sqrt{30}/6$

Chapter 7

7.1

1. (a) Let $f(x) = x^7 - 5x^5 + x^3 - 1$. Then f is continuous, $f(-1) = 2$, and $f(1) = -4$, so according to Theorem 7.2, the equation $f(x) = 0$ has a solution in $(-1, 1)$. Parts (b) and (c) are done in the same way.

3. Your height is a continuous function of time. You were once less than 1 meter tall and (unless you are unduly precocious) you are probably now above 1 meter tall. The intermediate value theorem (and common sense) give the conclusion.

5. (a) See Fig. 23. (b) Define $g(x) = f(x) - x$. Then $g(a) \geq 0$ and $g(b) \leq 0$. If either $g(a)$ or $g(b)$ is 0, we have a fixed point for f. If $g(a) > 0$ and $g(b) < 0$, then $g(x) = 0$ for some x^* in (a, b) by the intermediate-value theorem. This x^* is a fixed point for f.

7.2

1. f is continuous on $[0, 5]$, so the extreme-value theorem applies.

3. f has a maximum at $x = 1$ and a minimum at all $x > 1$. (Draw your own graph.) Yet the function is discontinuous at $x = 1$, and its domain of definition is neither closed nor bounded.

7.3

1. (a) $\xi = 3/2$ (b) $\xi = \sqrt{2}/2$ (c) $\xi = \sqrt{12}$ (d) $\xi = \sqrt{3}$

3. $\xi = \pm 1/\sqrt{27}$. The conditions of the mean-value theorem are not satisfied, because f is not differentiable at $x = 0$. See Fig. 24.

7.4

1. $(1 + x)^{-1} = 1 - x + x^2 - (1 + c)^{-4}x^3$

3. $(1+1/8)^{1/3} = 1+1/24-1/576+R_3(1/8)$, where $0 < R_3(1/8) < 5/(81 \cdot 8^3)$. Thus, $\sqrt[3]{9} = 2(1 + 1/8)^{1/3} \approx 2.080$, with three correct decimals.

FIGURE 23

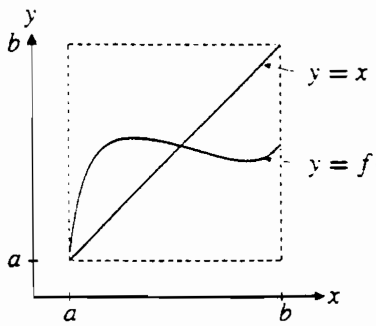
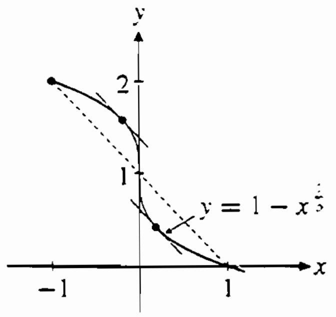


FIGURE 24



5. The idea is precisely the same as in Problem 4. The expressions get so big that we do not reproduce them here.

7.5

1. (a) $\lim_{x\to 1}(x-1)/(x^2-1) = \lim_{x\to 1} 1/2x = 1/2$ (b) $\lim_{x\to a} 2x/1 = 2a$ (c) $\lim_{x\to -2}(3x^2+6x)/(3x^2+10x+8) = \lim_{x\to -2}(6x+6)/(6x+10) = 3$
3. The second fraction is not "0/0". The correct limit is 5/2.
5. Does not exist if $b \neq d$. If $b = d$, the limit is "0/0" and by l'Hôpital's rule, it is $\lim_{x\to 0}[\frac{1}{2}a(ax+b)^{-1/2} - \frac{1}{2}c(cx+d)^{-1/2}]/1 = a/2\sqrt{b} - c/2\sqrt{d} = (a-c)/2\sqrt{b}$.
7. $\sqrt[n]{x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n} - x =$ $x\left(\sqrt[n]{1 + a_1/x + \cdots + a_n/x^n} - 1\right) = \dfrac{\sqrt[n]{1 + a_1/x + \cdots + a_n/x^n} - 1}{1/x}$. Now use l'Hôpital's rule.

7.6

1. $p = 64/3 - 10D/3$
3. (a) $x = -y/3$ (b) $x = 1/y$ (c) $x = y^{1/3}$
5. (a) $g(x) = (x^3+1)^{1/3}$ (b) $g(x) = (2x+1)/(x-1)$ (c) $g(x) = \left[1-(x-2)^5\right]^{1/3}$
7. (a) $f'(x) = 4x - 4x^3 > 0$ for $x \in (0,1)$, so $R_f = [0,1]$. (b) $g(x) = \sqrt{1-\sqrt{1-x}}$
9. (a) $f(x) = x/2$ and $g(x) = 2x$ are inverse functions. (b) $f(x) = 3x-2$ and $g(x) = \frac{1}{3}(x+2)$ are inverse functions. (c) $C = \frac{5}{9}(F-32)$ and $F = \frac{9}{5}C+32$ are inverse functions.
11. f^{-1} determines how much it costs to buy a specified number of kilograms of meat.
13. $f'(x) = 4x^2(3-x^2)/3\sqrt{4-x^2} > 0$ for $x \in (0,\sqrt{3})$, so f has an inverse on $[0,\sqrt{3}]$. $g'(\sqrt{3}/3) = 1/f'(1) = 3\sqrt{3}/8$.

Chapter 8

8.1

1. (a) $y' = -3e^{-3x}$ (b) $y' = 6x^2e^{x^3}$ (c) $y' = (-1/x^2)e^{1/x}$ (d) $y' = 5(4x-3)e^{2x^2-3x+1}$
3. (a) $f'(x) = e^x + xe^x = e^x(1+x)$, $f''(x) = e^x(2+x)$. f is increasing for $x > -1$. (b) See Fig. 25.

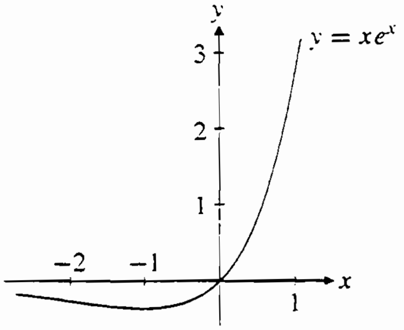


FIGURE 25

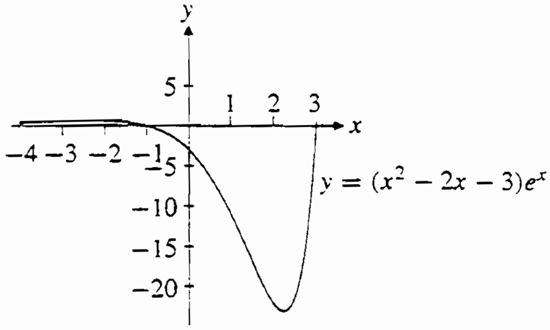


FIGURE 26

5. See Fig. 26.

7. $y = xe^x$ has the derivative $y' = e^x + xe^x = (x+1)e^x$, so the formula *is* correct for $n = 1$. *Assume* that the kth derivative is $y^{(k)} = (x+k)e^x$. Then by the product rule, $y^{(k+1)} = (d/dx)(x+k)e^x = e^x + (x+k)e^x = [x+(k+1)]e^x$. Thus, the given formula is valid also for $n = k+1$. By induction, it is valid generally.

8.2

1. (a) $\ln 9 = \ln 3^2 = 2\ln 3$ (b) $\frac{1}{2}\ln 3$ (c) $\ln\sqrt[5]{3^2} = \ln 3^{2/5} = \frac{2}{5}\ln 3$
(d) $\ln(1/81) = \ln 3^{-4} = -4\ln 3$

3. (a) $3^x 4^{x+2} = 8$ when $3^x 4^x 4^2 = 8$ or $12^x = 1/2$. Hence, $x = -\ln 2/\ln 12$.
(b) $\ln x^3 + \ln x^4 = 6$, or $\ln x^7 = 6$, so $7\ln x = 6$ and then $x = e^{6/7}$.
(c) $4^x(1-4^{-1}) = 3^x(3-1)$, so $(4/3)^x = 8/3$, implying that $x = \ln(8/3)/\ln(4/3)$. (Also, $(4/3)^{x-1} = 2$, so an alternative correct answer is $x = 1 + \ln 2/\ln(4/3)$.)

5. We show how to prove (c) and (e). For (c), when $x > 0$: $\ln(e^3x^2) = \ln e^3 + \ln x^2 = 3\ln e + 2\ln x = 3 + 2\ln x$. For (e): Note that $p_i\ln(1/p_i) = p_i(\ln 1 - \ln p_i) = -p_i\ln p_i$ when $p_i > 0$.

7. (a) Wrong. (Let $A = e$.) (b) and (c) are right.

9. (a) $x > -1$ (b) $1/3 < x < 1$ (c) $x \neq 0$ (d) $x > 1$ or $x < -1$ (e) $x > 1$
(f) $x \neq e^e$ and $x > 1$

11. (a) $\dfrac{1}{x\ln x}$ (b) $\dfrac{-x}{1-x^2}$ (c) $e^x\left(\ln x + \dfrac{1}{x}\right)$ (d) $e^{x^3}\left(3x^2\ln x^2 + \dfrac{2}{x}\right)$
(e) $\dfrac{e^x}{e^x+1}$ (f) $\dfrac{2x+3}{x^2+3x-1}$

13. (a) $\ln f(x) = \frac{1}{3}[\ln(x+1) - \ln(x-1)]$, so $\dfrac{f'(x)}{f(x)} = \dfrac{1}{3}\left(\dfrac{1}{x+1} - \dfrac{1}{x-1}\right) = \dfrac{-2}{3(x^2-1)}$ (b) $\ln f(x) = x\ln x$, so $f'(x)/f(x) = \ln x + 1$ (c) $\ln f(x) = \frac{1}{2}\ln(x-2) + \ln(x^2+1) + \ln(x^4+6)$, so $\dfrac{f'(x)}{f(x)} = \dfrac{1}{2x-4} + \dfrac{2x}{x^2+1} + \dfrac{4x^3}{x^4+6}$

15. (a) $f'(x) = e^{x-1} - 1$ is < 0 for $x < 1$, and > 0 for $x > 1$. But $f(1) = 0$. So $f(x) > 0$ for all $x \neq 0$. See Fig. 27. (b) $f(-1) = e^{-2} + 1 > 1$, $f(1) = 0$, and $f(3) = e^2 - 3 > 1$. By the intermediate-value theorem, there have to be solutions of $f(x) = 1$ in each of the intervals $(-1, 1)$ and $(1, 3)$. Because $f'(x) < 0$ for $x > 1$ and $f'(x) > 0$ for $x > 1$, there is only one solution in each of the two intervals, and these are the only two solutions. (c) g is defined for all $x \neq 1$, $x \neq x_1$, and $x \neq x_2$, where x_1 and x_2 are the two solutions in (b). $g(x) \to 0$ as $x \to \pm\infty$. (d) See Fig. 28.

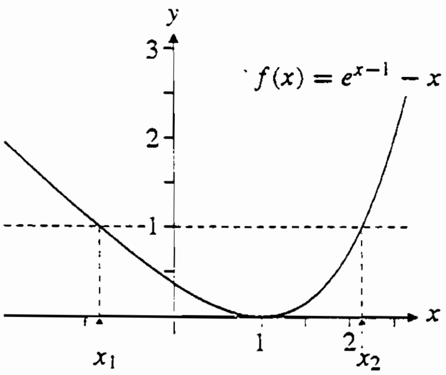


FIGURE 27

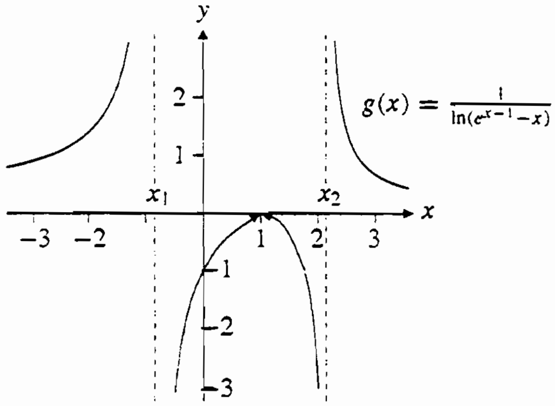


FIGURE 28

17. (a) $F(x) = e^{-e^{-x}}$ (b) $f(x) = e^{-x}e^{-e^{-x}} = \exp(-x)\exp[-\exp(-x)]$ (c) $f'(x) = f(x)(e^{-x} - 1)$

19. (a) $xy'/y = ax$ (b) $3 + 2x$ (c) $1 + \dfrac{x}{(x+1)\ln(x+1)}$ (d) $\dfrac{x^{-\delta}}{x^{-\delta} + 1}$

21. $\ln y = v \ln u$, so $y'/y = v' \ln u + vu'/u$.

23. We must solve $x = \frac{1}{2}(e^y - e^{-y})$ for y. Multiply the equation by e^y to get $\frac{1}{2}e^{2y} - \frac{1}{2} = xe^y$ or $e^{2y} - 2xe^y - 1 = 0$. Letting $e^y = z$ yields $z^2 - 2xz - 1 = 0$, with solution $z = x \pm \sqrt{x^2 + 1}$. The minus sign makes z negative, so $z = e^y = x + \sqrt{x^2 + 1}$. This gives $y = \ln(x + \sqrt{x^2 + 1})$ as the inverse function.

8.3

1. (a) $\log_5 25 = \log_5 5^2 = 2\log_5 5 = 2$ (b) 3/2, because $125 = 5^3$. (c) -2 (d) -6

3. (a) $y' = 5 \cdot 3^x \ln 3$ (b) $y' = 2^x \ln 2 \ln x + 2^x/x$ (c) $y' = \log_2 x + 1/\ln 2$ (d) $y' = x/[(1 + x^2)\ln 2]$

5. (a) $0 < x \leq 1/e$ (b) $x^2 - x - 1 \geq 1$, so $x \leq -1$ or $x \geq 2$ (c) One must have $x > 3$. Then $\ln x + \ln(x - 3) = \ln x(x - 3) \leq \ln 4$, and so $x(x - 3) \leq 4$. Thus, $3 < x \leq 4$.

7. Applying l'Hôpital's rule, $\lim_{\lambda \to 0^+}(x^\lambda - y^\lambda)/\lambda = \lim_{\lambda \to 0^+}(x^\lambda \ln x - y^\lambda \ln y)/1 = \ln x - \ln y$.

9. Inserting $e^x \approx 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3$ and $e^{-x} \approx 1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3$ in the given equation yields, after rearranging, $x(x^2 - 1) \approx 0$. So $x = 0$ (exact), $x \approx -1$, and $x \approx 1$.

8.4

1. (a) $\dot{x}/x = 5/(5t + 10)$ (b) $\dot{x}/x = 1/(t + 1)\ln(t + 1)$ (c) $\dot{x}/x = 1$ (d) $\dot{x}/x = \ln 2$ (e) $\dot{x}/x = 2t$ (f) $\dot{x}/x = (e^t - e^{-t})/(e^t + e^{-t})$. So (c) and (d) have a constant relative rate of increase, which accords with [8.22].

3. (a) $P(25) \approx 6595$ (millions) (b) Doubling time: $t = (\ln 2)/0.02 \approx 34.7$ years.

5. (a) 710 (b) A little more than 21 days. (c) Yes, after about 35 days, 999 will have or have had influenza, and $N(t) \to 1000$.

7. (a) $f'(x) = k - A\alpha e^{-\alpha x} = 0$ when $x_0 = (1/\alpha)\ln(A\alpha/k)$ (b) $x_0 > 0$ iff $A\alpha > k$. Then $f'(x) < 0$ if $x < x_0$ and $f'(x) > 0$ if $x > x_0$. (c) x_0 increases as p_0 increases, and as V increases; x_0 decreases as δ increases, and as k increases.

9. $\ln m = -0.02 + 0.19 \ln N$. When $N = 480{,}000$, then $m \approx 11.77$.

11. $\ln z = \ln 694{,}500 - 0.3 \ln p$, and $p = (694{,}500/z)^{10/3}$.

13. (a) See Fig. 29.

T	36.3	35.0	33.9	32.4	24.7	24.2
$\ln n$	5.04	4.89	4.70	4.54	3.64	3.58

(b) $f(T) = 1.99e^{0.12T}$

(c) The fall in temperature that halves the pulse rate is $(\ln 2)/0.12 \approx 5.8$ degrees.

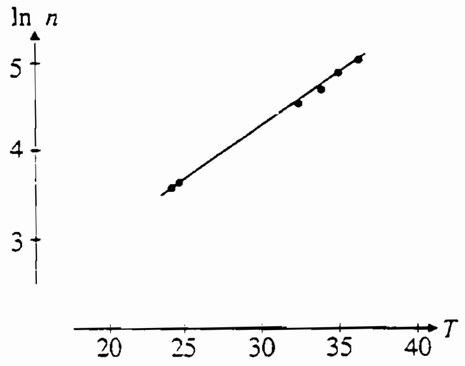


FIGURE 29

15. $t_0 = 1972 + 8000 \ln 0.886$, which gives the approximate date 1004.

8.5

1. (a) (i) $\$1000(1.05)^{10} \approx \$1629$ (ii) $\$1000(1.05)^{50} \approx \$11{,}467$
(b) (i) $\$1000(1+0.05/12)^{120} \approx \$1647$
(ii) $\$1000(1+0.05/12)^{600} \approx \$12{,}119$
(c) (i) $\$1000e^{0.05\cdot 10} \approx \$1649$ (ii) $\$1000e^{0.05\cdot 50} \approx \$12{,}182$.

3. $h'(u) = u/(1+u)^2 > 0$ for $u > 0$, so $h(u) > 0$ for $u > 0$, implying that $g'(x)/g(x) = h(r/x) > 0$ for all $x > 0$. So $g(x)$ is strictly increasing for $x > 0$. Because $g(x) \to e^r$ as $x \to \infty$, it follows that $g(x) < e^r$ for all $x > 0$. Continuous compounding of interest is best for the lender.

Chapter 9

9.2

1. $y' = 1.06 - 0.08x$. So y has a maximum at $x = 1.06/0.08 = 13.25$.

3. (a) $V'(x) = 12(x-3)(x-9)$. So V has maximum 432 at $x = 3$. (b) The box has maximum volume when the square cut out from each corner has a length of 3 cm. (c) Logarithmic differentiation yields $V'(x)/V(x) = 3(3-x)/x(9-x)$.

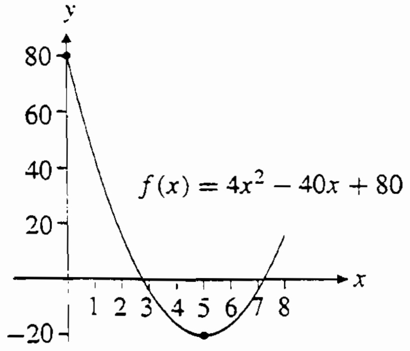
5. (a) Maximum 2 at $x = 0$. No minimum. (b) Maximum 3 at $x = 2$. No minimum. (c) Minimum -3 at $x = -2$. No maximum. (d) Minimum -1 at $x = 0$. No maximum. (e) Maximum 2 at $x = 1$. No minimum. (f) Maximum 1 at $x = 0$. Minimum 1/2 at $x = \pm 1$.

7. $\overline{T}'(Y) = a(bY+c)^{p-1}(pbY - bY - c)/Y^2$, which is 0 for $Y^* = c/b(p-1)$. This must be the minimum point because $\overline{T}'(Y)$ is negative for $Y < Y^*$ and positive for $Y > Y^*$.

9.3

1. Maximum 80 at $x = 0$. Minimum -20 at $x = 5$. See Fig. 30.

FIGURE 30



3. Choose both numbers equal to 8. (If $x + y = 16$, then $xy = x(16 - x) = -x^2 + 16x$, with $x \in (0, 16)$, and this function of x has its maximum at $x = 8$.)

5. (a) $Q^* = (p - \beta)/2\gamma$ (b) (i) $Q = 450$ (ii) $Q = 500$ (iii) $Q = 0$

7. $y' = e^{-x}(2x - x^2)$ is positive in $(0, 2)$ and negative in $(2, 4)$, so y has a maximum $4e^{-2} \approx 0.52$ at $x = 2$.

9. $A(Q) = C(Q)/Q = aQ^2 + bQ + c + d/Q$. Then $A'(Q) = 2aQ + b - d/Q^2$, so $A'(Q) \to -\infty$ as $Q \to 0^+$, and $A'(Q) \to \infty$ as $Q \to \infty$. Moreover, $A''(Q) = 2a + 2d/Q^3$, which is positive for all $Q > 0$. Hence, $A'(Q)$ is strictly increasing from $-\infty$ to ∞ in $(0, \infty)$. Thus, there is a unique point Q^* at which $A'(Q^*) = 0$, and Q^* minimizes $A(Q)$. For $b = 0$, $A(Q)$ has a minimum at $Q^* = (d/2a)^{1/3}$.

9.4

1. $f'(x) = 3x^2 - 12 = 0$ at $x = \pm 2$, and $f''(x) = 6x$. So $x = 2$ is a local minimum point, whereas $x = -2$ is a local maximum point.

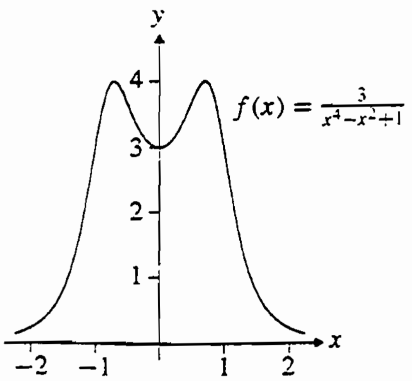
3. (a) $D_f = [-6, 0) \cup (0, \infty)$; $f(-2) = f(-6) = 0$; moreover $f(x) > 0$ in $(-6, -2) \cup (0, \infty)$. (b) Local maximum $\frac{1}{2}\sqrt{2}$ at $x = -4$. Local minimum $(8/3)\sqrt{3}$ at $x = 6$ and local minimum 0 at $x = -6$ (where $f'(x)$ is undefined). (c) $f(x) \to -\infty$ as $x \to 0^-$, $f(x) \to \infty$ as $x \to 0^+$, $f(x) \to \infty$ as $x \to \infty$, and $f'(x) \to 0$ as $x \to \infty$. f attains neither a maximum nor a minimum.

5. (a) $f(t) = \dfrac{P(t)e^{-rt}}{1 - e^{-rt}} = \dfrac{P(t)}{e^{rt} - 1}$ (b) Here $f'(t) = \dfrac{P'(t)(e^{rt} - 1) - P(t)re^{rt}}{(e^{rt} - 1)^2}$, and $t^* > 0$ can only maximize $f(t)$ if $f'(t^*) = 0$, that is, if $P'(t^*)(e^{rt^*} - 1) = rP(t^*)e^{rt^*}$, which implies that $P'(t^*) = rP(t^*)/(1 - e^{-rt^*})$.

7. a and d are local minimum points, whereas c is a local maximum point for f.

9. (a) $f'(x) = \dfrac{-12x^3 + 6x}{(x^4 - x^2 + 1)^2}$. $x = 0$ is a local minimum point; $x = \pm\sqrt{2}/2$ are global maximum points. (b) See Fig. 31.

FIGURE 31



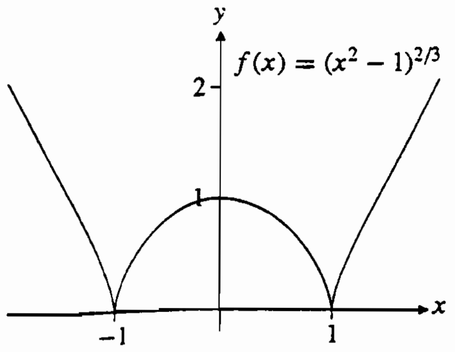


FIGURE 32

11. (a) $f'(x) = 4x/3(x^2-1)^{1/3}$ and $f''(x) = 4(x-\sqrt{3})(x+\sqrt{3})/9(x^2-1)^{4/3}$. (b) $x = -1$ and $x = 1$ are local (and global) minimum points. $x = 0$ is a local maximum point. The graph is shown in Fig. 32.

9.5

1. $f'(x) = -(2x/3) + 8$ and $f''(x) = -2/3 < 0$ for all x, so f is concave.

3. (a) $R = p\sqrt{x}$, $C = wx + F$, and $\pi(x) = p\sqrt{x} - wx - F$. (b) $\pi'(x) = p(1/2\sqrt{x}) - w = 0$, or $p(1/2\sqrt{x}) = w$. (Marginal cost = price per unit of output.) (c) $\pi''(x) = -\frac{1}{4}px^{-3/2} < 0$ for all $x > 0$, so profit is maximized. (d) No maximum exists, because $\pi(x) \to \infty$ as $x \to \infty$.

5. (a) $f''(x) = 2x(x^2-3)(1+x^2)^{-3}$, so f is convex in $[-\sqrt{3}, 0]$ and in $[\sqrt{3}, \infty)$. Inflection points are at $x = -\sqrt{3}$, 0, and $\sqrt{3}$. (b) $g''(x) = 4(1+x)^{-3} > 0$ when $x > -1$, so g is (strictly) convex in $(-1, \infty)$. No inflection points. (c) $h''(x) = (2+x)e^x$, so h is convex in $[-2, \infty)$ and $x = -2$ is an inflection point.

7. $C''(x) = 6aQ + 2b$, so $C(Q)$ is concave in $[0, -b/3a]$ and convex in $[-b/3a, \infty)$. $Q = -b/3a$ is an inflection point.

9. (a) $f'(v) = \frac{1}{3}(v-1)^{-2/3}$ is positive in $[0, 1)$ and in $(1, \infty)$. Because $f(v)$ is continuous at $v = 1$, it is (strictly) increasing in $[0, \infty)$. Moreover, $f''(v) = -\frac{2}{9}(v-1)^{-5/3}$, so $f''(v) > 0$ in $[0, 1)$ and $f''(v) < 0$ in $(1, \infty)$. See Fig. 33. (b) $f'(v_m) = p$, so $v_m = 1 + (3p)^{-3/2}$ (c) See Fig. 33. (d) $\pi(v) = 0$ when $v - 1 = (pv-1)^3$, or $p^3v^3 - 3p^2v^2 + (3p-1)v = 0$. This equation always has $v = 0$ as a root. Any other root must satisfy the quadratic equation $p^3v^2 - 3p^2v + 3p - 1 = 0$. For $0 < p < 1/3$, the only positive root is $v = \left(3+\sqrt{4/p-3}\right)/2p$. For $1/3 < p < 4/3$, there are two positive roots, which are $v = \left(3 \pm \sqrt{4/p-3}\right)/2p$; for $p = 1/3$, $v = 9$; for $p = 4/3$, $v = 9/8$; for $p > 4/3$, the only root is $v = 0$. (e) The solutions are: For $0 < p < 4/3$, v_m is given by (b). For $p = 4/3$, $v = 0$ and $v = 9/8$. For $p > 4/3$, $v = 0$.

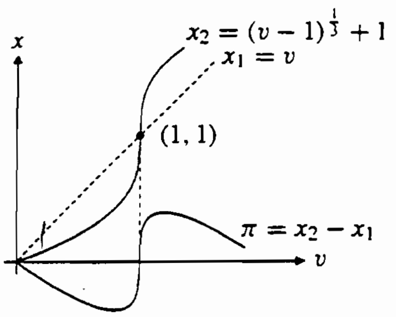


FIGURE 33

9.6

1. (a) Straightforward algebra. (b) $D \geq 0$, so f is concave. (c) $D > 0$ if $\lambda \in (0, 1)$ and $a \neq b$, so f is strictly concave. (d) $f''(x) = -2 < 0$, so according to [9.17], f is strictly concave.
3. (a) Convex, as a sum of convex functions. (b) Concave, as a sum of concave functions. (c) Concave, as a sum of concave functions. (d) Convex, as a sum of convex functions. (See the note immediately preceding Jensen's inequality.)
5. Apply [9.14] to the set shaded in Fig. 34, which is the set of points below the graph of $h(x)$.

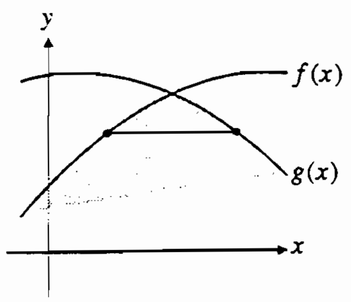


FIGURE 34

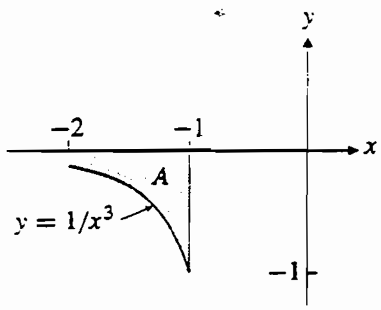


FIGURE 35

Chapter 10

10.1

1. $F(x) = \frac{1}{4}x^4$ and $A = F(1) - F(0) = \frac{1}{4}$.
3. See Fig. 35. $F(x) = -\frac{1}{2}x^{-2}$ so $F(-1) = -\frac{1}{2}(-1)^{-2} = -\frac{1}{2}$ and $F(-2) = -\frac{1}{2}(-2)^{-2} = -\frac{1}{8}$. Because $f(x)$ is negative in $[-2, -1]$, so the area $A = -[F(-1) - F(-2)] = -[-\frac{1}{2} - (-\frac{1}{8})] = \frac{1}{2} - \frac{1}{8} = \frac{3}{8}$.

10.2

1. (a) $\frac{1}{14}x^{14} + C$ (b) $\frac{2}{5}x^2\sqrt{x} + C$ (c) $2\sqrt{x} + C$ (d) $\frac{8}{15}x^{15/8} + C$, because $\sqrt{x\sqrt{x\sqrt{x}}} = x^{7/8}$.

3. (a) $\frac{2}{5}y^2\sqrt{y} - \frac{8}{3}y\sqrt{y} + 8\sqrt{y} + C$ (b) $\frac{1}{3}x^3 - \frac{1}{2}x^2 + x - \ln|x+1| + C$, because $x^3/(x+1) = x^2 - x + 1 - 1/(x+1)$ (c) $\frac{1}{32}(1+x^2)^{16} + C$

5. Differentiate the right-hand side to get $x\sqrt{ax+b}$.

7. $c(x) = \frac{3}{2}x^2 + 4x + 40$

9. (a) $f'(x) = 2x + 1$ and $f(x) = x^2 + x + 2$ (b) $f'(x) = -1/x + \frac{1}{4}x^4 + 2x - 1$ and $f(x) = -\ln x + (1/20)x^5 + x^2 - x - 1/20$

10.3

1. (a) $\int_0^1 x\,dx = \Big|_0^1 \frac{1}{2}x^2 = (\frac{1}{2})1^2 - (\frac{1}{2})0^2 = 1/2$ (b) $\Big|_1^2 (x^2 + \frac{1}{3}x^3) = 16/3$ (c) $\Big|_{-2}^3 (\frac{1}{6}x^3 - \frac{1}{12}x^4) = 5/12$

3. (a) $x = 1000\sqrt{3}$ maximizes profits. See Fig. 36.
(b) $I = \frac{1}{2000}\Big|_{1000}^{3000} (4000x - \frac{1}{2}x^2 - 3{,}000{,}000\ln x) = 2000 - 1500\ln 3 \approx 352$

5. (a) t^2 (b) $-e^{-t^2}$ (c) $2e^{-t^2}$ (d) $\ln t(1 - t^{-1/2}/4)$ (e) $\frac{1}{3}t^{4/3} - \frac{1}{6}t^{1/6}$ (f) $2/\sqrt{t^4+1}$

7. $A = \int_0^3 (\sqrt{3x} - x^2 + 2x)\,dx = 6$. See Fig. 37.

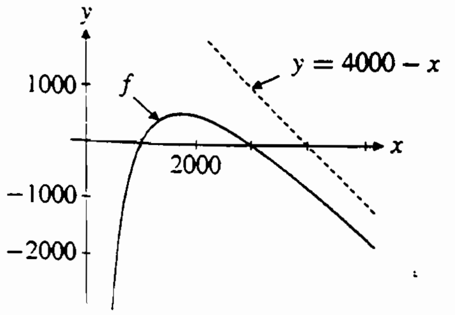


FIGURE 36

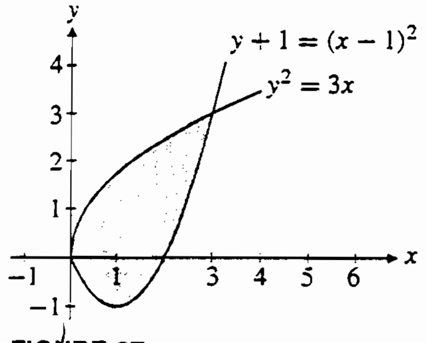


FIGURE 37

9. $W(T) = K(1 - e^{-\rho T})/\rho T$. Here $W(T) \to 0$ as $T \to \infty$, and using l'Hôpital's rule, $W(T) \to K$ as $T \to 0^+$. For $T > 0$, we find $W'(T) = Ke^{-\rho T}(1 + \rho T - e^{\rho T})/\rho T^2 < 0$ because $e^{\rho T} > 1 + \rho T$ (see Problem 14 in Section 8.2). We conclude that $W(T)$ is strictly decreasing and that $W(T) \in (0, K)$.

10.4

1. $x(t) = x_0 - \int_0^t \bar{u}e^{-as}\,ds = x_0 - \bar{u}(1-e^{-at})/a$. We see that $x(t) \to x_0 - \bar{u}/a$ as $t \to \infty$. If $x_0 \geq \bar{u}/a$, the reservoir will never be empty.
3. (a) $K(5) - K(0) = \int_0^5 (3t^2 + 2t + 5)\,dt = 175$
(b) $K(T) - K_0 = (T^3 - t_0^3) + (T^2 - t_0^2) + 5(T - t_0)$
5. (a) $\int_0^T ae^{-rt}\,dt = (a/r)(1 - e^{-rT})$ (b) a/r

Chapter 11

11.1

1. (a) Use [11.1] with $f(x) = x$ and $g'(x) = e^{-x}$. Then $g(x) = -e^{-x}$ and $\int xe^{-x}\,dx = x(-e^{-x}) - \int 1\cdot(-e^{-x})\,dx = -xe^{-x} - e^{-x} + C$. (b) Use [11.1] with $f(x) = 3x$ and $g'(x) = e^{4x}$ to get $\frac{3}{4}xe^{4x} - \frac{3}{16}e^{4x} + C$. (c) Use [11.1] with $f(x) = 1 + x^2$ and $g'(x) = e^{-x}$, as well as the answer to (a), to get $-x^2e^{-x} - 2xe^{-x} - 3e^{-x} + C$. (d) Use [11.1] with $f(x) = \ln x$ and $g'(x) = x$ to get $\frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + C$.
3. The general formula follows from [11.1]. $\int \ln x\,dx = x\ln x - x + C$.
5. Let $f(x) = \ln x$ and $g'(x) = x^\rho$ and use [11.1].
7. Use [11.2] with $f(u) = u^2$ and $g'(u) = (\bar{u} - u)^{\gamma-1}$. This gives $g(u) = -(1/\gamma)(\bar{u} - u)^\gamma$, so $T^* = (2k/\gamma)\int_0^{\bar{u}} u(\bar{u} - u)^\gamma\,du$. Then integrate by parts once more to get

$$T^* = -\frac{2k}{\gamma}\left[\Big|_0^{\bar{u}} \frac{u(\bar{u}-u)^{\gamma+1}}{\gamma+1} - \int_0^{\bar{u}} \frac{(\bar{u}-u)^{\gamma+1}}{\gamma+1}\,du\right] = \frac{2k\bar{u}^{\gamma+2}}{\gamma(\gamma+1)(\gamma+2)}$$

11.2

1. (a) $\frac{1}{9}(x^2+1)^9 + C$ (Let $f(u) = u^8$, $u = g(x) = x^2+1$.) (b) $\frac{1}{11}(x+2)^{11} + C$ (Let $u = x + 2$.) (c) $\ln|x^2 - x + 8| + C$ (Let $u = x^2 - x + 8$.)
3. (a) Substitute $u = \sqrt{1+x^2}$. Then $u^2 = 1 + x^2$, so $u\,du = x\,dx$. If $x = 0$, then $u = 1$; if $x = 1$, then $u = \sqrt{2}$. Hence, $\int_0^1 x\sqrt{1+x^2}\,dx = \int_1^{\sqrt{2}} u\,u\,du = \int_1^{\sqrt{2}} u^2\,du = \Big|_1^{\sqrt{2}} \frac{1}{3}u^3 = \frac{1}{3}(2\sqrt{2} - 1)$. (b) 1/2 (Let $u = \ln y$.) (c) $\frac{1}{2}(e^2 - e^{2/3})$ (Let $u = 2/x$.)
5. (a) 1/70 (The integrand is $-x^4(x^5 - 1)^{13}$. Let $u = x^5 - 1$.)
(b) $2\sqrt{x}\ln x - 4\sqrt{x} + C$. (Let $u = \sqrt{x}$.) (c) 8/3 (Let $u = \sqrt{1 + \sqrt{x}}$.)
7. (a) Make a common denominator on the right-hand side. (b) (i) $3\ln 2 - \ln 3$
(ii) $16\ln 2 - 7\ln 3$

9. $I = \dfrac{A(1+C)}{\beta C}\ln\left(\dfrac{1+CD}{1+CDe^{\beta t}}\right) + At$. (The suggested substitution $x = CDe^{\beta\tau}$ implies $dx = CD\beta e^{\beta\tau}\,d\tau = \beta x d\tau$. Moreover,

$$\frac{A(1-De^{\beta\tau})}{1+CDe^{\beta\tau}}\,d\tau = \frac{A(1-x/C)\,dx}{1+x\quad \beta x} = \frac{A}{\beta C}\left(\frac{C}{x} - \frac{1+C}{1+x}\right)dx$$

and so on.)

11. (a) $\ln(x+\sqrt{x^2+1}) + C$ (b) $\frac{1}{2}x\sqrt{x^2+1} + \frac{1}{2}\ln(x+\sqrt{x^2+1}) + C$

11.3

1. (a) $\int_1^b (1/x^3)\,dx = \int_1^b x^{-3}\,dx = \Big|_1^b(-\frac{1}{2}x^{-2}) = \frac{1}{2} - \frac{1}{2}b^{-2} \to \frac{1}{2}$ as $b \to \infty$. So $\int_1^b (1/x^3)\,dx = \frac{1}{2}$. (b) $\int_1^b x^{-1/2}\,dx = \Big|_1^b 2x^{1/2} \to \infty$ as $b \to \infty$, so the integral diverges. (c) 1 (d) $\int_0^a \dfrac{x\,dx}{\sqrt{a^2-x^2}} = \Big|_0^a -\sqrt{a^2-x^2} = a$

3. Using a simplified notation and the result in Example 11.7, we have:
(a) $\int_0^\infty x\lambda e^{-\lambda x}\,dx = -\Big|_0^\infty xe^{-\lambda x} + \int_0^\infty e^{-\lambda x}\,dx = 1/\lambda$
(b) $\int_0^\infty (x-\lambda^{-1})^2\lambda e^{-\lambda x}\,dx = -\Big|_0^\infty (x-\lambda^{-1})^2 e^{-\lambda x} + \int_0^\infty 2(x-\lambda^{-1})\, e^{-\lambda x}\,dx = \lambda^{-2} + 2\int_0^\infty xe^{-\lambda x}\,dx - 2\lambda^{-1}\int_0^\infty e^{-\lambda x}\,dx = \lambda^{-2} + 2\lambda^{-2} - 2\lambda^{-2} = \lambda^{-2}$
(c) $\int_0^\infty (x-\lambda^{-1})^3\lambda e^{-\lambda x}\,dx = -\Big|_0^\infty (x-\lambda^{-1})^3 e^{-\lambda x} + \int_0^\infty 3(x-\lambda^{-1})^2\, e^{-\lambda x}\,dx = -\lambda^{-3} + 3\lambda^{-1}\int_0^\infty (x-\lambda^{-1})^2\lambda e^{-\lambda x}\,dx = -\lambda^{-3} + 3\lambda^{-1}\lambda^{-2} = 2\lambda^{-3}$

5. (a) $f'(x) = 1/x^4 - 3\ln x/x^4 = 0$ at $x = e^{1/3}$. $f(x) \to -\infty$ as $x \to 0^+$ and $f(x) \to 0$ as $x \to \infty$. $f(e^{1/3}) = 1/3e > 0$. Hence, f has a maximum at $(e^{1/3}, 1/3e)$, but no minimum exists. (b) $\int_a^b x^{-3}\ln x\,dx = -\Big|_a^b \frac{1}{2}x^{-2}\ln x + \int_a^b \frac{1}{2}x^{-3}\,dx = \Big|_a^b(-\frac{1}{2}x^{-2}\ln x - \frac{1}{4}x^{-2})$. This diverges when $b = 1$ and $a \to 0$. But $\int_1^\infty x^{-3}\ln x\,dx = 1/4$.

7. If both limits exist, the integral is the sum of the following two limits: $I_1 = \lim_{\epsilon\to 0^+}\int_{-2+\epsilon}^3 (1/\sqrt{x+2})\,dx$ and $I_2 = \lim_{\epsilon\to 0^+}\int_{-2}^{3-\epsilon}(1/\sqrt{3-x})\,dx$. Here $I_1 = \lim_{\epsilon\to 0^+}\Big|_{-2+\epsilon}^3(2\sqrt{x+2}) = \lim_{\epsilon\to 0^+}(2\sqrt{5} - 2\sqrt{\epsilon}) = 2\sqrt{5}$, and $I_2 = \lim_{\epsilon\to 0^+}\Big|_{-2}^{3-\epsilon}(-2\sqrt{3-x}) = \lim_{\epsilon\to 0^+}(-2\sqrt{\epsilon} + 2\sqrt{5}) = 2\sqrt{5}$. The answer is $4\sqrt{5}$.

9. $f(x) = 1/x^2$ is not defined at $x = 0$, so f is not continuous in $[-1, 1]$.

11. $\int_1^A [k/x - k^2/(1+kx)]\,dx = \Big|_1^A [k\ln x - k\ln(1+kx)] = \Big|_1^A k\ln[x/(1+kx)] = k\ln[A/(1+kA)] - k\ln[1/(1+k)] = k\ln[1/(1/A+k)] - k\ln[1/(1+k)]$, which tends to $k\ln(1/k) - k\ln[1/(1+k)] = \ln(1+1/k)^k$ as $A \to \infty$. So $I_k = \ln(1+1/k)^k$, which tends to $\ln e = 1$ as $k \to \infty$.

13. The substitution suggested in the hint and [11.13] are useful in all three cases. For (b), you also need the result in Example 11.7. For (c), the suggested substitution leads to the integral $I = (1/\sqrt{\pi}) \int_{-\infty}^{+\infty} (2\sigma^2 z^2 + 2\sqrt{2}\sigma\mu z + \mu^2) e^{-z^2} \, dz$. Note that $\int_{-\infty}^{+\infty} e^{-z^2} \, dz = \sqrt{\pi}$ by [11.13], and $\int_{-\infty}^{+\infty} z e^{-z^2} \, dz = 0$ by Example 11.9. Finally, integration by parts gives $\int z^2 e^{-z^2} \, dz = -\frac{1}{2} z e^{-z^2} + \int \frac{1}{2} e^{-z^2} \, dz$, so $\int_{-\infty}^{+\infty} z^2 e^{-z^2} \, dz = \frac{1}{2}\sqrt{\pi}$. Thus $I = (1/\sqrt{\pi})(2\sigma^2 \cdot \frac{1}{2}\sqrt{\pi} + 0 + \mu^2\sqrt{\pi}) = \sigma^2 + \mu^2$.

11.4

1. See Fig. 38. The solid curve represents the U.S. income distribution in 1980. The dotted curve gives the distribution in the Netherlands in 1959 (almost the same as the U.S. 1980 curve). The dashed curve gives the distribution in the Netherlands in 1985.

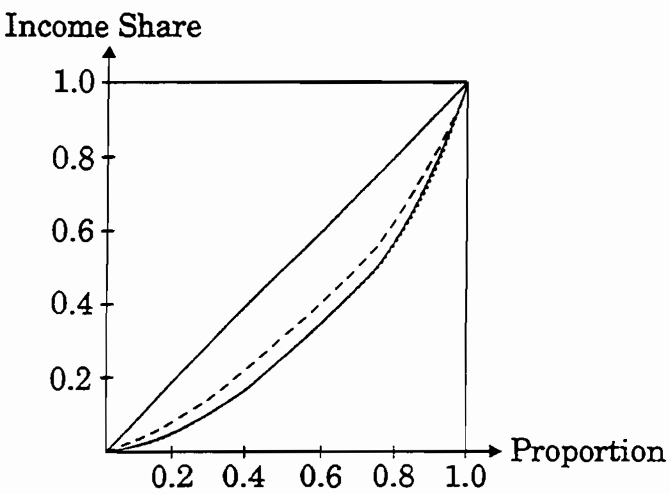


FIGURE 38

Chapter 12

12.1

1. (a) Let x and y denote total production in industries A and I, respectively. Then $x = \frac{1}{6}x + \frac{1}{4}y + 60$ and $y = \frac{1}{4}x + \frac{1}{4}y + 60$. So $\frac{5}{6}x - \frac{1}{4}y = 60$ and $-\frac{1}{4}x + \frac{3}{4}y = 60$. (b) $x = 320/3$ and $y = 1040/9$.

3. $0.8x_1 - 0.3x_2 = 120$ and $-0.4x_1 + 0.9x_2 = 90$, with solution $x_1 = 225$ and $x_2 = 200$.

12.2

1. $\mathbf{a}+\mathbf{b} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$, $\mathbf{a}-\mathbf{b} = \begin{pmatrix} -1 \\ -5 \end{pmatrix}$, $2\mathbf{a}+3\mathbf{b} = \begin{pmatrix} 13 \\ 10 \end{pmatrix}$, and $-5\mathbf{a}+2\mathbf{b} = \begin{pmatrix} -4 \\ 13 \end{pmatrix}$.

3. $a_1 = 0$, $a_2 = 1/3$, and $a_3 = 1$.

5. (a) $x_i = 0$ for all i. (b) Nothing, because $0 \cdot \mathbf{x} = \mathbf{0}$ for all $\mathbf{x}$.

7. $\mathbf{x} = 3\mathbf{a} + 4\mathbf{b}$

9. (a) (i) Possible, with $\theta = 1/2$. (ii) Impossible. (iii) Impossible.
(b) (i) Proportion of lead $\theta = 1/2$. (ii) If output can be thrown away, the proportion of lead can be $\theta = 2/3$. (iii) Impossible in any case, because $(1-\theta)4 + \theta 3 < 9$ for all $\theta \in [0, 1]$.

12.3

1. $\mathbf{a} + \mathbf{b} = (3, 3)$ and $-\frac{1}{2}\mathbf{a} = (-2.5, 0.5)$. See Fig. 39.

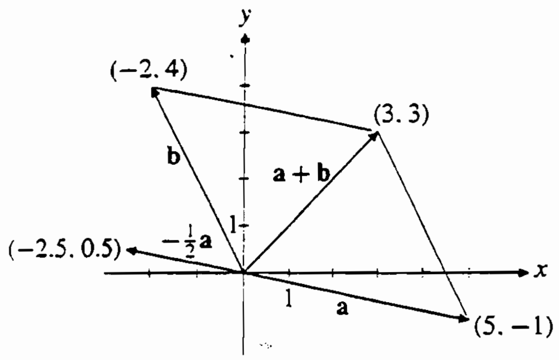


FIGURE 39

3. (a) $x_1 = 2$, $x_2 = -1$. (b) Suppose $x_1(1, 2, 1) + x_2(-3, 0, -2) = (-3, 6, 1)$. Then $x_1 - 3x_2 = -3$, $2x_1 = 6$, and $x_1 - 2x_2 = 1$. The first two equations yield $x_1 = 3$ and $x_2 = 2$; then the last equation is not satisfied.

5. (a) A straight line through $(0, 2, 3)$ parallel to the x-axis.
(b) A plane parallel to the z-axis through the line $y = x$ in the xy-plane.

12.4

1. $\mathbf{a} \cdot \mathbf{a} = 5$, $\mathbf{a} \cdot \mathbf{b} = 2$, and $\mathbf{a} \cdot (\mathbf{a} + \mathbf{b}) = 7$. We see that $\mathbf{a} \cdot \mathbf{a} + \mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot (\mathbf{a} + \mathbf{b})$.

3. The pairs of vectors in (a) and (c) are orthogonal.

5. The vectors are orthogonal iff their scalar product is 0, that is iff $x^2 - x - 8 - 2x + x = x^2 - 2x - 8 = 0$, which is the case for $x = -2$ and $x = 4$.

7. Use the rules in [12.14].

9. (a) The firm's revenue is $\mathbf{p} \cdot \mathbf{z}$. Its costs are $\mathbf{p} \cdot \mathbf{x}$. (b) Profit = revenue − costs = $\mathbf{p} \cdot \mathbf{z} - \mathbf{p} \cdot \mathbf{x} = \mathbf{p} \cdot (\mathbf{z} - \mathbf{x}) = \mathbf{p} \cdot \mathbf{y}$. If $\mathbf{p} \cdot \mathbf{y} < 0$, the firm makes a loss equal to $-\mathbf{p} \cdot \mathbf{y}$.

11. (a) Input vector = $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ (b) Output vector = $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ (c) Cost = $(1, 3)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 3$
(d) Revenue = $(1, 3)\begin{pmatrix} 2 \\ 0 \end{pmatrix} = 2$ (e) Value of net output = $(1, 3)\begin{pmatrix} 2 \\ -1 \end{pmatrix} = 2 - 3 = -1$ (f) Loss = cost − revenue = $3 - 2 = 1$.

12.5

1. (a) $x_1 = 3 + 7t$, $x_2 = -2 + 4t$, and $x_3 = 2 - t$ (b) $x_1 = 1$, $x_2 = 3 - t$, and $x_3 = 2 + t$

3. $x_1 - 3x_2 - 2x_3 = -3$. (One method: $(5, 2, 1) - (3, 4, -3) = (2, -2, 4)$ and $(2, -1, 4) - (3, 4, -3) = (-1, -5, 7)$ are two vectors in the plane. The normal (p_1, p_2, p_3) must be orthogonal to both these vectors, so $(2, -2, 4) \cdot (p_1, p_2, p_3) = 2p_1 - 2p_2 + 4p_3 = 0$ and $(-1, -5, 7) \cdot (p_1, p_2, p_3) = -p_1 - 5p_2 + 7p_3 = 0$. One solution to these two equations is $(p_1, p_2, p_3) = (1, -3, -2)$. Then using formula [12.23] with $(a_1, a_2, a_3) = (2, -1, 4)$ yields $(1, -3, -2) \cdot (x_1 - 2, x_2 + 1, x_3 - 4) = 0$, which reduces to $x_1 - 3x_2 - 2x_3 = -3$.)

12.6

1. $\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

3. $u = 3$ and $v = -2$

5. $\mathbf{A} + \mathbf{B} = \begin{pmatrix} 1 & 0 & 4 \\ 2 & 4 & 16 \end{pmatrix}$, $\mathbf{A} - \mathbf{B} = \begin{pmatrix} -1 & 2 & -6 \\ 2 & 2 & -2 \end{pmatrix}$, and $5\mathbf{A} - 3\mathbf{B} = \begin{pmatrix} -3 & 8 & -20 \\ 10 & 12 & 8 \end{pmatrix}$

12.7

1. (a) $\mathbf{AB} = \begin{pmatrix} -2 & -10 \\ -2 & 17 \end{pmatrix}$ and $\mathbf{BA} = \begin{pmatrix} 12 & 6 \\ 15 & 3 \end{pmatrix}$

(b) $\mathbf{AB} = \begin{pmatrix} 26 & 3 \\ 6 & -22 \end{pmatrix}$ and $\mathbf{BA} = \begin{pmatrix} 14 & 6 & -12 \\ 35 & 12 & 4 \\ 3 & 3 & -22 \end{pmatrix}$

(c) $\mathbf{AB} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 4 & -6 & -2 \\ 0 & -8 & 12 & 4 \\ 0 & -2 & 3 & 1 \end{pmatrix}$ and $\mathbf{BA} = (17)$, a 1×1 matrix.

(d) $\mathbf{AB}$ is not defined. $\mathbf{BA} = \begin{pmatrix} -1 & 4 \\ 3 & 4 \\ 4 & 8 \end{pmatrix}$

3. $\mathbf{B} = \begin{pmatrix} w - y & y \\ y & w \end{pmatrix}$, for arbitrary y, w. (*Hint:* Let $\mathbf{B} = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$. Then $\mathbf{AB} = \mathbf{BA}$ iff: [1] $x + 2y = x + 2z$; [2] $2x + 3y = y + 2w$; [3] $z + 2w = 2x + 3z$; and [4] $2z + 3w = 2y + 3w$. Now [1] and [4] are true iff $z = y$. So [2] and [3] are true as well iff $x = w - y$.)

12.8

1. $\mathbf{A}(\mathbf{B}+\mathbf{C}) = \mathbf{AB}+\mathbf{AC} = \begin{pmatrix} 3 & 2 & 6 & 2 \\ 7 & 4 & 14 & 6 \end{pmatrix}$

3. It is straightforward to show that $(\mathbf{AB})\mathbf{C}$ and $\mathbf{A}(\mathbf{BC})$ are both equal to the 2×2 matrix $\mathbf{D} = (d_{ij})_{2\times 2}$ with $d_{ij} = a_{i1}b_{11}c_{1j} + a_{i1}b_{12}c_{2j} + a_{i2}b_{21}c_{1j} + a_{i2}b_{22}c_{2j}$ for $i = 1, 2$ and $j = 1, 2$.

5. (a) $\begin{pmatrix} 5 & 3 & 1 \\ 2 & 0 & 9 \\ 1 & 3 & 3 \end{pmatrix}$ (b) $(1, 2, -3)$.

7. (a) Direct verification. (b) $\mathbf{AA} = (\mathbf{AB})\mathbf{A} = \mathbf{A}(\mathbf{BA}) = \mathbf{AB} = \mathbf{A}$, so $\mathbf{A}$ is idempotent. Then just interchange $\mathbf{A}$ and $\mathbf{B}$ to show that $\mathbf{B}$ is idempotent. (c) As the induction hypothesis, suppose that $\mathbf{A}^k = \mathbf{A}$, which is true for $k = 2$. Then $\mathbf{A}^{k+1} = \mathbf{A}^k\mathbf{A} = \mathbf{AA} = \mathbf{A}$, which completes the proof by induction.

12.9

1. $\mathbf{A}' = \begin{pmatrix} 3 & -1 \\ 5 & 2 \\ 8 & 6 \\ 3 & 4 \end{pmatrix}$ and $\mathbf{B}' = (0, 1, -1, 2)$

3. $\mathbf{A} = \mathbf{A}'$ and $\mathbf{B} = \mathbf{B}'$

5. No! For example: $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 3 & 3 \end{pmatrix}$.

7. (a) Direct verification. (b) $\begin{pmatrix} p & q \\ -q & p \end{pmatrix}\begin{pmatrix} p & -q \\ q & p \end{pmatrix} = \begin{pmatrix} p^2+q^2 & 0 \\ 0 & p^2+q^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ iff $p^2 + q^2 = 1$. (c) Suppose $\mathbf{P}$ and $\mathbf{Q}$ are orthogonal, that is, $\mathbf{P}'\mathbf{P} = \mathbf{I}_n$ and $\mathbf{Q}'\mathbf{Q} = \mathbf{I}_n$. Then $(\mathbf{PQ})'(\mathbf{PQ}) = (\mathbf{Q}'\mathbf{P}')(\mathbf{PQ}) = \mathbf{Q}'(\mathbf{P}'\mathbf{P})\mathbf{Q} = \mathbf{Q}'\mathbf{I}_n\mathbf{Q} = \mathbf{Q}'\mathbf{Q} = \mathbf{I}_n$, so $\mathbf{PQ}$ is orthogonal.
(d) If $\mathbf{P}$ is orthogonal and $\mathbf{c}_i$ and $\mathbf{c}_j$ are two different columns of $\mathbf{P}$, then $\mathbf{c}_i'\mathbf{c}_j$ is the element in row i and column j of $\mathbf{P}'\mathbf{P} = \mathbf{I}$, so $\mathbf{c}_i'\mathbf{c}_j = 0$. If $\mathbf{r}_i$ and $\mathbf{r}_j$ are two different rows of $\mathbf{P}$, then $\mathbf{r}_i\mathbf{r}_j'$ is the element in row i and column j of $\mathbf{PP}' = \mathbf{I}' = \mathbf{I}$, so again $\mathbf{r}_i\mathbf{r}_j' = 0$.

9. Suppose $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Then: [1] $a^2 + bc = 0$; [2] $ab + bd = 0$; [3] $ac + cd = 0$; [4] $bc + d^2 = 0$. We claim that $\operatorname{tr}(\mathbf{A}) = a+d = 0$. Subtracting [4] from [1] yields $a^2 - d^2 = 0$, or $(a-d)(a+d) = 0$. Either $a + d = 0$ and we are through, or $a = d$. But if $a = d$, then [2] implies that $ab = 0$, so [1] implies that $a^3 = -abc = 0$. Hence, $a = 0$ and $\operatorname{tr}(\mathbf{A}) = a + d = 0$ even when $a = d$.

Chapter 13

13.1

1. (a) 18 (b) 0 (c) $(a+b)^2-(a-b)^2=4ab$ (d) $3^t2^{t-1}-3^{t-1}2^t=6^{t-1}$

3. (a) $x_1 = 11/5$ and $x_2 = -7/5$ (b) $x = 4$ and $y = -1$ (c) Provided that $a^2+b^2 \neq 0$, $x = \dfrac{a+2b}{a^2+b^2}$ and $y = \dfrac{2a-b}{a^2+b^2}$.

5. If $\mathbf{A} = \mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, then $|\mathbf{A}+\mathbf{B}| = 4$, whereas $|\mathbf{A}|+|\mathbf{B}| = 2$. ($\mathbf{A}$ and $\mathbf{B}$ can be chosen almost at random.)

7. (a) $X_1 = M_2$ because nation 1's exports are nation 2's imports. Similarly, $X_2 = M_1$. (b) Substituting for X_1, X_2, M_1, M_2, C_1, and C_2 gives the two equations: [1] $Y_1(1-c_1+m_1)-m_2Y_2 = A_1$; [2] $Y_2(1-c_2+m_2)-m_1Y_1 = A_2$. Using Cramer's rule with $D = (1-c_2+m_2)(1-c_1+m_1)-m_1m_2$ yields:

$$Y_1 = \frac{1}{D}[A_2m_2 + A_1(1-c_2+m_2)], \qquad Y_2 = \frac{1}{D}[A_1m_1 + A_2(1-c_1+m_1)]$$

(c) Y_2 depends linearly on A_1. Increasing A_1 by one unit changes Y_2 by the factor m_1/D. Because c_1 is the proportion of income consumed, we can assume that $0 < c_1 < 1$. Likewise, $0 < c_2 < 1$. Because m_1 and m_2 are nonnegative, we see that $D > 0$ and that Y_2 increases when A_1 increases. Here is an economic explanation: An increase in A_1 increases nation 1's income, Y_1. This in turn increases nation 1's imports, M_1. However, nation 1's imports are nation 2's exports, so this causes nation 2's income, Y_2, to increase, and so on.

9. $(d/dt)[a(t)b'(t)-a'(t)b(t)] = a(t)b''(t)-a''(t)b(t)$

13.2

1. (a) -2 (b) -2 (c) adf (d) $e(ad-bc)$

3. (a) $x_1 = 1$, $x_2 = 2$, and $x_3 = 3$ (b) $x_1 = x_2 = x_3 = 0$ (c) $x = 1$, $y = 2$, and $z = 3$

5. (a) Successive substitutions give $Y = C + A_0 = a + b(Y-T) + A_0 = a + bY - b(d+tY) + A_0$. Thus,

$$Y = \frac{A_0 + a - bd}{1-b+bt}, \qquad C = \frac{a - bd + bA_0 - btA_0}{1-b+bt},$$

$$T = \frac{d - bd + tA_0 + ta}{1-b+bt}$$

(b) The matrix equation is

$$\begin{pmatrix} 1 & -1 & 0 \\ -b & 1 & b \\ -t & 0 & 1 \end{pmatrix} \begin{pmatrix} Y \\ C \\ T \end{pmatrix} = \begin{pmatrix} A_0 \\ a \\ d \end{pmatrix}$$

and by Cramer's rule the solution is as in (a).

13.3

1. (a) 24 (b) 1 (c) $d - a$ (d) 0
3. $-a_{15}a_{24}a_{32}a_{43}a_{51}$. (There are nine rising lines.)

13.4

1. (a) $\mathbf{AB} = \begin{pmatrix} 13 & 16 \\ 29 & 36 \end{pmatrix}$, $\mathbf{BA} = \begin{pmatrix} 15 & 22 \\ 23 & 34 \end{pmatrix}$, $\mathbf{A'B'} = \begin{pmatrix} 15 & 23 \\ 22 & 34 \end{pmatrix}$, $\mathbf{B'A'} = \begin{pmatrix} 13 & 29 \\ 16 & 36 \end{pmatrix}$

   (b) $|\mathbf{A}| = |\mathbf{A}'| = -2$ and $|\mathbf{B}| = |\mathbf{B}'| = -2$, so $|\mathbf{AB}| = 4 = |\mathbf{A}| \cdot |\mathbf{B}|$, $|\mathbf{A'B'}| = |\mathbf{A}'| \cdot |\mathbf{B}'| = 4$
3. (a) 0 (one column has only zeros). (b) 0 (rows 1 and 4 are proportional). (c) Subtract $3a$ times the second column from the first, so that the new fourth row is (0, 1, 0, 0). Interchange the first and fourth rows and then the first and second column. Explain why you can now drop the first row and the first column. The answer is $6a^4 + 29a^2 - 6a - 1$.
5. $\mathbf{X'X} = \begin{pmatrix} 4 & 3 & 2 \\ 3 & 5 & 1 \\ 2 & 1 & 2 \end{pmatrix}$ and $|\mathbf{X'X}| = 10$
7. Because $\mathbf{P'P} = \mathbf{I}_n$, it follows from [13.15] and [13.14] that $|\mathbf{P}'||\mathbf{P}| = |\mathbf{I}_n| = 1$. Because $|\mathbf{P}'| = |\mathbf{P}|$, we get $|\mathbf{P}|^2 = 1$, so $|\mathbf{P}| = \pm 1$.
9. Let $\mathbf{A} = \begin{pmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{pmatrix}$. Then compute $\mathbf{A}^2$ and recall [13.15].
11. (a) For the first equality, you can use the definition of a determinant. The second equality is quite easy if you perform the operations indicated. For the last equality, you can again use the definition of a determinant. (From the first column, you have to pick $-1, \dots$) (b) If you can do it, you deserve a break.

13.5

1. (a) 2. (Subtract row 1 from both row 2 and row 3 to get a determinant whose first column has elements 1, 0, 0. Then expand by the first column.) (b) 30 (c) 0. (Columns 2 and 4 are proportional.)

3. For $n = 4$, the Vandermonde determinant is $\begin{vmatrix} 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 1 & x_3 & x_3^2 & x_3^3 \\ 1 & x_4 & x_4^2 & x_4^3 \end{vmatrix}$

Now multiply the third, the second, and the first columns successively by $-x_1$ and add the results to the next column. This yields the determinant

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & x_2 - x_1 & x_2^2 - x_2 x_1 & x_2^3 - x_2^2 x_1 \\ 1 & x_3 - x_1 & x_3^2 - x_3 x_1 & x_3^3 - x_3^2 x_1 \\ 1 & x_4 - x_1 & x_4^2 - x_4 x_1 & x_4^3 - x_4^2 x_1 \end{vmatrix} = (x_2 - x_1)(x_3 - x_1)(x_4 - x_1) \begin{vmatrix} 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \\ 1 & x_4 & x_4^2 \end{vmatrix}$$

The last equality is obtained by noting that $x_2 - x_1$, $x_3 - x_1$, and $x_4 - x_1$ are common factors in the second, third, and fourth rows, respectively, of the preceding determinant. The last determinant is again a Vandermonde determinant, and the conclusion follows.

13.6

1. $\begin{pmatrix} 3 & 0 \\ 2 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1/3 & 0 \\ 2/3 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

3. $\mathbf{AB} = \begin{pmatrix} 1 & 0 & 0 \\ a+b & 2a+1/4+3b & 4a+3/2+2b \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{I}$ iff $a + b = 4a + 3/2 + 2b = 0$ and $2a + 1/4 + 3b = 1$. This is true iff $a = -3/4$ and $b = 3/4$.

5. $\mathbf{A}^{-1} = \mathbf{A}^2 = \frac{1}{2}\begin{pmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix}$

7. (a) $\mathbf{AA}' = \begin{pmatrix} 21 & 11 \\ 11 & 10 \end{pmatrix}$, $|\mathbf{AA}'| = 89$, and $(\mathbf{AA}')^{-1} = \dfrac{1}{89}\begin{pmatrix} 10 & -11 \\ -11 & 21 \end{pmatrix}$.
(b) No, $\mathbf{AA}'$ is always symmetric (see Note 2 and Example 12.28 in Section 12.9).

9. $\mathbf{B}^2 + \mathbf{B} = \mathbf{I}$ and $\mathbf{B}^3 - 2\mathbf{B} + \mathbf{I} = 0$. From $\mathbf{B}^2 + \mathbf{B} = \mathbf{I}$, we get $\mathbf{B}(\mathbf{B} + \mathbf{I}) = \mathbf{I}$, so $\mathbf{B}^{-1} = \mathbf{B} + \mathbf{I} = \begin{pmatrix} 1/2 & 5 \\ 1/4 & 1/2 \end{pmatrix}$.

11. Let $\mathbf{B} = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$. Then $\mathbf{A}^2 = (\mathbf{I}_m - \mathbf{B})(\mathbf{I}_m - \mathbf{B}) = \mathbf{I}_m - \mathbf{B} - \mathbf{B} + \mathbf{B}^2$. Here $\mathbf{B}^2 = (\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}')(\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}') = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}(\mathbf{X}'\mathbf{X})(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}' = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}' = \mathbf{B}$. Thus, $\mathbf{A}^2 = \mathbf{I}_m - \mathbf{B} - \mathbf{B} + \mathbf{B} = \mathbf{I}_m - \mathbf{B} = \mathbf{A}$. (b) Direct verification.

13. $\mathbf{D}^3 = \mathbf{D}^2\mathbf{D} = (2\mathbf{D} + 3\mathbf{I}_n)\mathbf{D} = 2\mathbf{D}^2 + 3\mathbf{D} = 2(2\mathbf{D} + 3\mathbf{I}_n) + 3\mathbf{D} = 7\mathbf{D} + 6\mathbf{I}_n$. $\mathbf{D}^6 = \mathbf{D}^3\mathbf{D}^3 = (7\mathbf{D} + 6\mathbf{I}_n)(7\mathbf{D} + 6\mathbf{I}_n) = 49\mathbf{D}^2 + 84\mathbf{D} + 36\mathbf{I}_n = 49(2\mathbf{D} + 3\mathbf{I}_n) + 84\mathbf{D} + 36\mathbf{I}_n = 182\mathbf{D} + 183\mathbf{I}_n$. To find $\mathbf{D}^{-1}$, note that from $\mathbf{D}^2 = 2\mathbf{D} + 3\mathbf{I}_n$, we obtain $\mathbf{D}(\mathbf{D} - 2\mathbf{I}_n) = 3\mathbf{I}_n$, and so $\mathbf{D}\frac{1}{3}(\mathbf{D} - 2\mathbf{I}_n) = \mathbf{I}_n$. Thus, $\mathbf{D}^{-1} = \frac{1}{3}\mathbf{D} - \frac{2}{3}\mathbf{I}_n$.

13.7

1. (a) $\begin{pmatrix} -5/2 & 3/2 \\ 2 & -1 \end{pmatrix}$ (b) $\dfrac{1}{9}\begin{pmatrix} 1 & 4 & 2 \\ 2 & -1 & 4 \\ 4 & -2 & -1 \end{pmatrix}$ (c) The matrix has no inverse.

3. $(\mathbf{I}-\mathbf{A})^{-1} = \dfrac{5}{62}\begin{pmatrix} 18 & 16 & 10 \\ 2 & 19 & 8 \\ 4 & 7 & 16 \end{pmatrix} \approx \begin{pmatrix} 1.45 & 1.29 & 0.81 \\ 0.16 & 1.53 & 0.65 \\ 0.32 & 0.56 & 1.29 \end{pmatrix}$

5. (a) $\begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{pmatrix}$ (c) There is no inverse.

13.8

1. (a) $x = 1$, $y = -2$, and $z = 2$ (b) $x = -3$, $y = 6$, $z = 5$, and $u = -5$
3. Show that the determinant of the coefficient matrix is equal to the expression $-(a^3 + b^3 + c^3 - 3abc)$, and then use Theorem 13.7.

Chapter 14

14.1

1. $\begin{pmatrix} 8 \\ 9 \end{pmatrix} = x\begin{pmatrix} 2 \\ 5 \end{pmatrix} + y\begin{pmatrix} -1 \\ 3 \end{pmatrix}$ requires $8 = 2x - y$ and $9 = 5x + 3y$. Solving these equations gives $x = 3$ and $y = -2$.
3. The determinant of the matrix with the three vectors as columns is equal to 3, so the vectors are linearly independent.
5. Suppose $\alpha(\mathbf{a}+\mathbf{b}) + \beta(\mathbf{b}+\mathbf{c}) + \gamma(\mathbf{a}+\mathbf{c}) = \mathbf{0}$. Then $(\alpha+\gamma)\mathbf{a} + (\alpha+\beta)\mathbf{b} + (\beta+\gamma)\mathbf{c} = \mathbf{0}$. Because **a**, **b**, and **c** are linearly independent, $\alpha+\gamma = 0$, $\alpha+\beta = 0$, and $\beta+\gamma = 0$. It follows that $\alpha = \beta = \gamma = 0$, which means that $\mathbf{a}+\mathbf{b}$, $\mathbf{b}+\mathbf{c}$, and $\mathbf{a}+\mathbf{c}$ are linearly independent. The vectors $\mathbf{a}-\mathbf{b}$, $\mathbf{b}+\mathbf{c}$, and $\mathbf{a}+\mathbf{c}$ are linearly *dependent* because $(\mathbf{a}-\mathbf{b}) + (\mathbf{b}+\mathbf{c}) - (\mathbf{a}+\mathbf{c}) = \mathbf{0}$.
7. Both these two statements follow immediately from the definitions.

14.2

1. (a) 1. (The determinant of the matrix is 0, so the rank is less than 2. Because not all entries are 0, the rank is 1.) (b) 2 (c) 2 (d) 3 (e) 2 (f) 3
3. $\mathbf{A} = \begin{pmatrix} 3 & 1 \\ 6 & 2 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 2 \\ -3 & -6 \end{pmatrix}$. Here $r(\mathbf{AB}) = 0$ and $r(\mathbf{BA}) = 1$.

14.3

1. (a) No solutions. (b) $x_1 = 1 + (2/3)b$, $x_2 = 1 + a - (5/3)b$, $x_3 = a$, and $x_4 = b$, with a, b arbitrary. Two degrees of freedom. (c) $x_1 = (-1/3)a$.

$x_2 = (5/3)a$, $x_3 = a$, and $x_4 = 1$, with a arbitrary. One degree of freedom. (d) No solutions. (e) $x_1 = x_2 = x_3 = 0$ is the only solution. There are 0 degrees of freedom. (f) $x_1 = a$, $x_2 = -a$, $x_3 = -a$, and $x_4 = a$, with a arbitrary. One degree of freedom.

3. For $a \neq 0$ and $a \neq 7$, the system has a unique solution. For $a = 0$ and $b = 9/2$, or for $a = 7$ and $b = 10/3$, the system has an infinite number of solutions. For other values of the parameters, there are no solutions.

5. (a) Unique solution for $p \neq 3$. For $p = 3$ and $q = 0$, there are infinitely many solutions (1 degree of freedom). For $p = 3$ and $q \neq 0$, there are no solutions. (b) For $p \neq 3$, only $\mathbf{z} = \mathbf{0}$ is orthogonal to the three vectors. For $p = 3$, the vector $\mathbf{z} = (-a, 0, a)$ is orthogonal to the three vectors, for all values of a. (c) Let the n vectors be $\mathbf{a}_i = (a_{i1}, \dots, a_{in})$, $i = 1, \dots, n$. If $\mathbf{b} = (b_1, \dots, b_n)$ is orthogonal to each of these n vectors, then the scalar product of $\mathbf{b}$ with each $\mathbf{a}_i$ is 0,

$$a_{i1}b_1 + \cdots + a_{in}b_n = 0 \qquad (i = 1, \dots, n)$$

Because $\mathbf{a}_1, \dots, \mathbf{a}_n$ are linearly independent, this homogeneous system of equations only has the solution $b_1 = \cdots = b_n = 0$, so $\mathbf{b} = \mathbf{0}$.

14.4

1. With r, s, and t as arbitrary real numbers, we have:

(a) Eigenvalues: $-1, -5$. Eigenvectors: $r\begin{pmatrix} 7 \\ 3 \end{pmatrix}$, $s\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

(b) No real eigenvalues.

(c) Eigenvalues: $5, -5$. Eigenvectors: $r\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $s\begin{pmatrix} -2 \\ 3 \end{pmatrix}$.

(d) Eigenvalues: 2, 3, 4. Eigenvectors: $r\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $s\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $t\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

(e) Eigenvalues: $-1, 0, 2$. Eigenvectors: $r\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$, $s\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$, $t\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$.

(f) Eigenvalues: 0, 1, 3. Eigenvectors: $r\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $s\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$, $t\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$.

3. From [14.8], we see that $\lambda = 0$ is an eigenvalue iff $|\mathbf{A}| = 0$. If $\lambda \neq 0$ is an eigenvalue for $\mathbf{A}$, then $|\mathbf{A} - \lambda\mathbf{I}| = 0$. But we have $|\mathbf{A}^{-1} - (1/\lambda)\mathbf{I}| = |\mathbf{A}^{-1}(\mathbf{I} - (1/\lambda)\mathbf{A})| = |\mathbf{A}^{-1}||1/\lambda||\lambda\mathbf{I} - \mathbf{A}| = 0$, which shows that $1/\lambda$ is an eigenvalue for $\mathbf{A}^{-1}$.

5. The eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 0.75$, and $\lambda_3 = 0.5$. Because $\mathbf{T}(\mathbf{v}) = \mathbf{v}$, it follows that $\mathbf{T}^n(\mathbf{v}) = \mathbf{v}$, for all n.

7. $|\mathbf{A}-\mathbf{I}| = \begin{vmatrix} a_{11}-1 & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22}-1 & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn}-1 \end{vmatrix}$. Add all the last $n-1$ rows to the first row. Because all the column sums are 1, all entries in the first row are 0. Hence, $|\mathbf{A}-\mathbf{I}| = 0$, so 1 is an eigenvalue for $\mathbf{A}$.

14.5

1. $\mathbf{D}^2 = \text{diag}(1/4, 1/9, 1/16)$, $\mathbf{D}^n = \text{diag}((1/2)^n, (1/3)^n, (1/4)^n)$, $\mathbf{D}^n \to 0$ as $n \to \infty$.

3. (a) The matrix has eigenvalues 2 and -1, with corresponding eigenvectors $(1, 0)$ and $(1, -3)$, respectively. So take $\mathbf{P} = \begin{pmatrix} 1 & 1 \\ 0 & -3 \end{pmatrix}$, and then $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \text{diag}(2, -1)$.

(b) $\mathbf{P} = \begin{pmatrix} 14 & 1 & 7 \\ 3 & 0 & 6 \\ -3 & 0 & 3 \end{pmatrix}$ and $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \text{diag}(3, 6, -6)$

(c) $\mathbf{P} = \begin{pmatrix} 1 & -3 & 3 \\ 0 & 5 & 2 \\ 3 & 1 & -1 \end{pmatrix}$ and $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \text{diag}(1, -4, 3)$.

14.6

1. (a) The matrix has eigenvalues 1 and 3, with corresponding eigenvectors $(1, -1)$ and $(1, 1)$, respectively. After normalizing these eigenvectors, the appropriate orthogonal matrix is $\mathbf{U} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$, and then $\mathbf{U}^{-1}\mathbf{A}\mathbf{U} = \text{diag}(1, 3)$.

(b) Matrix $\mathbf{A}$ has only two eigenvalues, which are 0 and 2, but three linearly independent eigenvectors, which are $(1, -1, 0)$, $(1, 1, 0)$, and $(0, 0, 1)$. This suggests taking $\mathbf{U} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$, and then $\mathbf{U}^{-1}\mathbf{A}\mathbf{U} = \text{diag}(0, 2, 2)$.

(c) $\mathbf{U} = \begin{pmatrix} 0 & \sqrt{2}/2 & -\sqrt{2}/2 \\ -4/5 & 3\sqrt{2}/10 & 3\sqrt{2}/10 \\ 3/5 & 2\sqrt{2}/5 & 2\sqrt{2}/5 \end{pmatrix}$, and then $\mathbf{U}^{-1}\mathbf{A}\mathbf{U} = \text{diag}(1, 6, -4)$.

3. Let $\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_n$ be the column vectors of $\mathbf{P}$. Then the diagonal elements of the matrix product $\mathbf{P}'\mathbf{P}$ are all 1, because the length of each column vector is 1. Moreover, the off-diagonal elements are 0 because the column vectors of $\mathbf{P}$ are mutually orthogonal. Hence, $\mathbf{P}'\mathbf{P}$ is the identity matrix, and thus $\mathbf{P}$ is orthogonal.

Chapter 15

15.1

1. $f(0, 1) = 0$, $f(-1, 2) = -4$, and $f(a, a) = a^3$
3. (a) $f(-1, 2) = 1$, $f(a, a) = 4a^2$, and $f(a+h, b) - f(a, b) = 2(a+b)h + h^2$
   (b) $f(tx, ty) = (tx)^2 + 2(tx)(ty) + (ty)^2 = t^2(x^2 + 2xy + y^2) = t^2 f(x, y)$
5. (a) $F(K+1, L, M) - F(K, L, M)$ is the increase in output from increasing capital input by one unit. (b) $F(K, L, M) = AK^a L^b M^c$, where A, a, b, and c are positive constants. (c) $F(tK, tL, tM) = t^{a+b+c} F(K, L, M)$
7. (a) $y \neq x - 2$ (b) $x^2 + y^2 \leq 2$ (c) $1 \leq x^2 + y^2 \leq 4$. The sets in cases (b) and (c) are shown in Figs. 40 and 41.

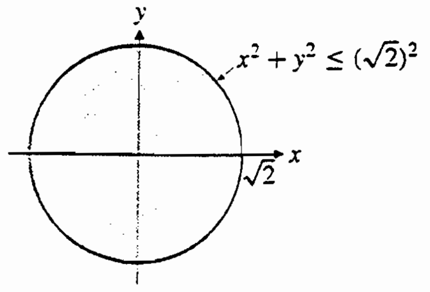


FIGURE 40

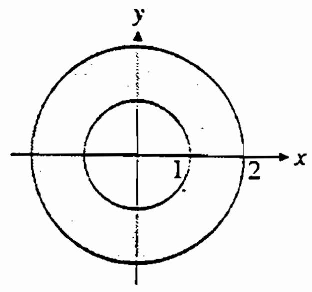


FIGURE 41

9. You drive $(5/60) \cdot 0 + (10/60) \cdot 30 + (20/60) \cdot 60 + (15/60) \cdot 80 = 45$ kilometers in $5+10+20+15 = 50$ minutes, so the average speed is $45 \times 60/50 = 54$ kph.
11. (a) Immediate. (b) In this case, (∗) becomes: $\ln[(1/n)\ (x_1 + \cdots + x_n)] \geq (1/n) \ln x_1 + \cdots + (1/n) \ln x_n = \ln x_1^{1/n} + \cdots + \ln x_n^{1/n} = \ln(x_1^{1/n} \ldots x_n^{1/n}) = \ln \sqrt[n]{x_1 \ldots x_n}$, and the conclusion follows.
   (c) The suggested replacements yield the inequality $\sqrt[n]{(1/x_1) \cdots (1/x_n)} \leq (1/n)\ (1/x_1 + \cdots + 1/x_n)$. This inequality says that $1/\bar{x}_G \leq 1/\bar{x}_H$, so $\bar{x}_H \leq \bar{x}_G$.

15.2

1. See Figs. 42 and 43. (Note that only a portion of the graph is indicated in each case.)

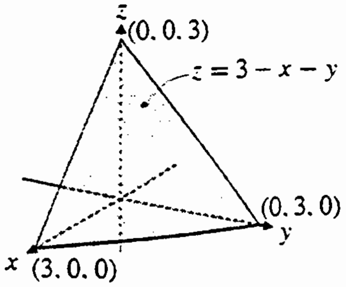
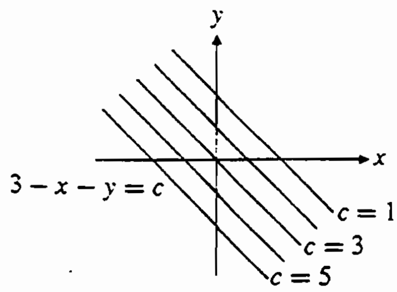


FIGURE 42



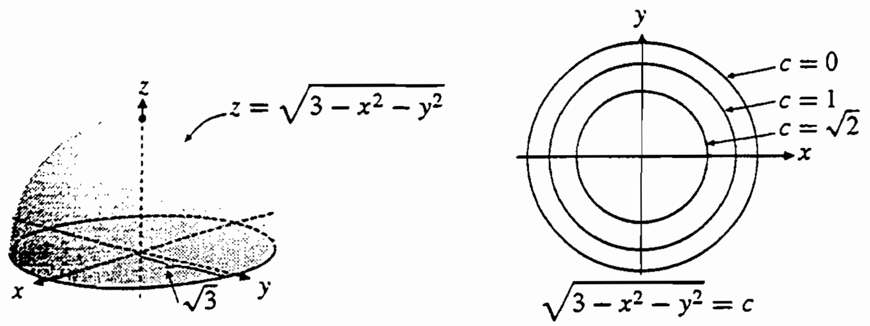


FIGURE 43

3. Note that $f(x, y) = e^{x^2-y^2} + (x^2 - y^2)^2$. Hence, for all pairs (x, y) where $x^2 - y^2 = c$, we have $f(x, y) = e^c + c^2$, so $x^2 - y^2 = c$ lies on a level curve at the height $e^c + c^2$.
5. At the point of intersection, f would have two different values, which is impossible when f is a function.

15.3

1. (a) $\partial z/\partial x = 2x$, $\partial z/\partial y = 6y$ (b) $\partial z/\partial x = y$, $\partial z/\partial y = x$
(c) $\partial z/\partial x = 20x^3y^2 - 2y^5$, $\partial z/\partial y = 10x^4y - 10xy^4$
(d) $\partial z/\partial x = \partial z/\partial y = e^{x+y}$
(e) $\partial z/\partial x = ye^{xy}$, $\partial z/\partial y = xe^{xy}$
(f) $\partial z/\partial x = e^x/y$, $\partial z/\partial y = -e^x/y^2$
(g) $\partial z/\partial x = \partial z/\partial y = 1/(x + y)$ (h) $\partial z/\partial x = 1/x$, $\partial z/\partial y = 1/y$
3. (a) $z'_x = 3$, $z'_y = 4$, and $z''_{xx} = z''_{xy} = z''_{yx} = z''_{yy} = 0$
(b) $z'_x = 3x^2y^2$, $z'_y = 2x^3y$, $z''_{xx} = 6xy^2$, $z''_{yy} = 2x^3$, and $z''_{xy} = z''_{yx} = 6x^2y$
(c) $z'_x = 5x^4 - 6xy$, $z'_y = -3x^2 + 6y^5$, $z''_{xx} = 20x^3 - 6y$, $z''_{yy} = 30y^4$, and $z''_{xy} = z''_{yx} = -6x$
(d) $z'_x = 1/y$, $z'_y = -x/y^2$, $z''_{xx} = 0$, $z''_{yy} = 2x/y^3$, and $z''_{xy} = z''_{yx} = -1/y^2$
(e) $z'_x = 2y(x + y)^{-2}$, $z'_y = -2x(x + y)^{-2}$, $z''_{xx} = -4y(x + y)^{-3}$, $z''_{yy} = 4x(x + y)^{-3}$, and $z''_{xy} = z''_{yx} = 2(x - y)(x + y)^{-3}$
(f) $z'_x = x(x^2 + y^2)^{-1/2}$, $z'_y = y(x^2 + y^2)^{-1/2}$, $z''_{xx} = y^2(x^2 + y^2)^{-3/2}$, $z''_{yy} = x^2(x^2 + y^2)^{-3/2}$, and $z''_{xy} = z''_{yx} = -xy(x^2 + y^2)^{-3/2}$
5. $xz'_x + yz'_y = x[2(ax + by)a] + y[2(ax + by)b] = 2a^2x^2 + 2abxy + 2abxy + 2b^2y^2 = 2(ax + by)^2$
7. $f'_1 = \ln y - y^3 2^{xy} \ln 2$ and $f'_2 = (x/y) - 2y2^{xy} - xy^2 2^{xy} \ln 2$, so $f'_1(1, 1) = -2\ln 2$, $f'_2(1, 1) = -3 - 2\ln 2$, $f''_{11}(1, 1) = -2(\ln 2)^2$, $f''_{22}(1, 1) = -5 - 8\ln 2 - 2(\ln 2)^2$, and $f''_{12}(1, 1) = f''_{21}(1, 1) = 1 - 6\ln 2 - 2(\ln 2)^2$
9. (a) $\partial^{p+q} z/\partial y^q \partial x^p = (-1)^{q-1}(q - 1)!\, e^x(1 + y)^{-q}$, which, when evaluated at $(x, y) = (0, 0)$, is equal to $(-1)^{q-1}(q - 1)!$ (Begin by differentiating w.r.t. x. Clearly, $\partial^p z/\partial x^p = e^x \ln(1 + y)$. Next, differentiating once w.r.t. y yields

$\partial^{p+1}z/\partial y\partial x^p = e^x(1+y)^{-1}$, then $\partial^{p+2}z/\partial y^2\partial x^p = e^x(-1)(1+y)^{-2}$, and so on.)

(b) $(p+1)q-1$. ($\partial^p z/\partial x^p = e^{x+y}[xy+(p+1)y-1]$ by induction on p. Then $\partial^{p+q}z/\partial y^q\partial x^p = e^{x+y}[xy+(p+1)y+qx+(p+1)q-1]$ by induction on q.)

15.4

1. (a) $f'_x > 0$ and $f'_y < 0$ at P, whereas $f'_x < 0$ and $f'_y > 0$ at Q.
   (b) (i) No solutions. (ii) $x \approx 2$ and $x \approx 6$ (c) 3
3. (a) $z = 2x + 4y - 5$ (b) $z = -10x + 3y + 3$

15.5

1. (a) $f'_1 = 2x$, $f'_2 = 3y^2$, and $f'_3 = 4z^3$ (b) $f'_1 = 10x$, $f'_2 = -9y^2$, and $f'_3 = 12z^3$
   (c) $f'_1 = yz$, $f'_2 = xz$, and $f'_3 = xy$ (d) $f'_1 = 4x^3/yz$, $f'_2 = -x^4/y^2z$, and $f'_3 = -x^4/yz^2$
   (e) $f'_1 = 12x(x^2+y^3+z^4)^5$, $f'_2 = 18y^2(x^2+y^3+z^4)^5$, and $f'_3 = 24z^3(x^2+y^3+z^4)^5$
   (f) $f'_1 = yze^{xyz}$, $f'_2 = xze^{xyz}$, and $f'_3 = xye^{xyz}$
3. $\partial T/\partial x = ky/d^n$ and $\partial T/\partial y = kx/d^n$ are both positive, so that the number of travelers increases if the size of either city increases, which is reasonable. $\partial T/\partial d = -nkxy/d^{n+1}$ is negative, so that the number of travelers decreases if the distance between the cities increases, which is also reasonable.
5. (a) $E'_p = 2ape^{bq}$ and $E'_q = abp^2e^{bq}$ (b) $\partial R/\partial p_1 = \alpha\beta p_1^{\beta-1} + \gamma p_2 e^{p_1p_2}$ and $\partial R/\partial p_2 = \gamma p_1 e^{p_1p_2}$ (c) $\partial x/\partial v_i = a_i$, $i = 1, 2, \ldots, n$. (For example, when $n = 3$ and $i = 2$, then $(\partial/\partial v_2)(a_1v_1 + a_2v_2 + a_3v_3) = a_2$.)
7. $f'_u = v^w \cdot u^{v^w-1}$, $f'_v = u^{v^w} \cdot w \cdot v^{w-1} \cdot \ln u$, and $f'_w = u^{v^w} \cdot v^w \cdot \ln u \cdot \ln v$.

15.6

1. (a) We get $\partial M/\partial Y = 0.14$ and $\partial M/\partial r = -0.84 \cdot 76.03(r-2)^{-1.84} = -63.8652(r-2)^{-1.84}$, so $\partial M/\partial Y$ is positive and $\partial M/\partial r$ is negative, which accords with standard economic intuition.
3. $F'_K = aF/K$, $F'_L = bF/L$, and $F'_M = cF/M$, so $KF'_K + LF'_L + MF'_M = (a+b+c)F$.
5. $\partial U/\partial x_i = e^{-x_i}$, for $i = 1, \ldots, n$

15.7

1. Profit $= (100 - Q_1)Q_1 + (80 - Q_2)Q_2 - 6(Q_1 + Q_2) = 94Q_1 - Q_1^2 + 74Q_2 - Q_2^2$. This is maximized when $Q_1 = 47$ and $Q_2 = 37$, so $P_1 = 53$ and $P_2 = 43$. Then profit is 3578. If price discrimination is illegal, then $P_1 = P_2 = P$, so $Q_1 = 100 - P$, $Q_2 = 80 - P$, and total demand is

$Q = 180 - 2P$. Thus, inverse demand is $P = 90 - \frac{1}{2}Q$, so profit $= (90 - \frac{1}{2}Q)Q - 6Q = 84Q - \frac{1}{2}Q^2$, which is maximized when $Q = 84$, and so $P = 48$. Then profit is 3528, so lost profit is 50.

3. $w = w_1 = w_2 = \alpha_1 + \beta_1 L_1 = \alpha_2 + \beta_2 L_2$, so that $L_1 = (w - \alpha_1)/\beta_1$ and $L_2 = (w - \alpha_2)/\beta_2$. Thus, total labor supply equals $L = L_1 + L_2 = [(\beta_1 + \beta_2)w - (\alpha_1\beta_2 + \alpha_2\beta_1).]/\beta_1\beta_2$. The inverse labor supply function is $w = (\alpha_1\beta_2 + \alpha_2\beta_1 + \beta_1\beta_2 L)/(\beta_1 + \beta_2)$. The firm's profit is

$$\pi(L) = (P - w)L = \left(P - \frac{\alpha_1\beta_2 + \alpha_2\beta_1}{\beta_1 + \beta_2}\right) L - \frac{\beta_1\beta_2}{\beta_1 + \beta_2} L^2$$

This is maximized at

$$L^* = \frac{\beta_1 + \beta_2}{2\beta_1\beta_2}\left(P - \frac{\alpha_1\beta_2 + \alpha_2\beta_1}{\beta_1 + \beta_2}\right)$$

with

$$\pi(L^*) = \frac{[\beta_1(P - \alpha_2) + \beta_2(P - \alpha_1)]^2}{4\beta_1\beta_2(\beta_1 + \beta_2)}$$

After some manipulation, the loss of profit compared with π^* given in Example 15.26 of Sec. 15.7 can be calculated as $(\alpha_1 - \alpha_2)^2/4(\beta_1 + \beta_2)$. (Note that the loss is zero when $\alpha_1 = \alpha_2$, which is as it should be because then the monopsonist does not want to discriminate anyway.)

15.8

1. (a) The associated symmetric matrix is $\begin{pmatrix} 4 & 4 \\ 4 & 5 \end{pmatrix}$, whose determinant is positive. The quadratic form satisfies [15.17], so is positive definite.
(b) $\begin{pmatrix} -1 & \frac{1}{2} \\ \frac{1}{2} & -3 \end{pmatrix}$ satisfies [15.19], so the quadratic form is negative definite.
(c) $\begin{pmatrix} 1 & -3 \\ -3 & 9 \end{pmatrix}$ satisfies [15.18], but not [15.17], so the quadratic form is positive semidefinite. (d) $\begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix}$ satisfies [15.21], so the quadratic form is indefinite. (e) Indefinite. (f) Negative semidefinite.

3. (a) $\begin{vmatrix} 0 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{vmatrix} = -4 < 0$, so positive definite subject to the constraint.

(b) $\begin{vmatrix} 0 & 3 & 4 \\ 3 & 2 & -2 \\ 4 & -2 & 1 \end{vmatrix} = -89 < 0$, so positive definite subject to the constraint.

(c) $\begin{vmatrix} 0 & 5 & -2 \\ 5 & -1 & \frac{1}{2} \\ -2 & \frac{1}{2} & -1 \end{vmatrix} = 19 > 0$, so *negative* definite subject to the constraint.

15.9

1. $a_{11}x_1^2+a_{12}x_1x_2+a_{13}x_1x_3+a_{21}x_2x_1+a_{22}x_2^2+a_{23}x_2x_3+a_{31}x_3x_1+a_{32}x_3x_2+a_{33}x_3^2$

3. (a) Positive definite. (b) Positive semidefinite. (c) Negative definite. (The associated symmetric matrix is

$$\begin{pmatrix} -3 & 1 & 0 \\ 1 & -1 & 2 \\ 0 & 2 & -8 \end{pmatrix}$$

whose leading principle minors are $D_1 = -3$, $D_2 = 2$, and $D_3 = -4$. Thus, the quadratic form is negative definite.)

Chapter 16

16.1

1. (a) $dz/dt = F_1'(x,y)dx/dt + F_2'(x,y)dy/dt = 1 \cdot 2t + 2y \cdot 3t^2 = 2t + 6t^5$

(b) $\dfrac{dz}{dt} = \left(\ln y + \dfrac{y}{x}\right) \cdot 1 + \left(\dfrac{x}{y} + \ln x\right) \cdot \dfrac{1}{t} = \ln(\ln t) + \dfrac{\ln t}{t+1} + \dfrac{t+1}{t \ln t} + \dfrac{\ln(t+1)}{t}$

3. $dY/dt = (10L - \frac{1}{2}K^{-1/2})\, dK/dt + (10K - \frac{1}{2}L^{-1/2})\, dL/dt = 35 - 7\sqrt{5}/100$ when $t = 0$.

5. $u = A\ln[1 + x^{\alpha}(ax^4+b)^{-\alpha/3}]$. It is sufficient to maximize with respect to x either $x^{\alpha}(ax^4+b)^{-\alpha/3}$, or $\ln[x^{\alpha}(ax^4+b)^{-\alpha/3}] = \alpha[\ln x - \frac{1}{3}\ln(ax^4+b)]$. The first-order condition $1/x - 4ax^3/3(ax^4+b) = 0$ is satisfied at $x^* = \sqrt[4]{3b/a}$, and $h(x^*) = \sqrt[3]{4b}$.

7. (a) $(2,1)\cdot(1/\sqrt{2}, 1/\sqrt{2}) = 3\sqrt{2}/2$ (b) $(2e-1, e-1)\cdot(3/5, 4/5) = 2e - 7/5$

16.2

1. (a) $\partial z/\partial t = y^2 + 2xy \cdot 2ts = 5t^4s^2 + 4t^3s^4$, and $\partial z/\partial s = y^2 \cdot 2s + 2xyt^2 = 2t^5s + 4t^4s^3$

(b) $\dfrac{\partial z}{\partial t} = \dfrac{2(1-s)e^{ts+t+s}}{(e^{t+s}+e^{ts})^2}$ and $\dfrac{\partial z}{\partial s} = \dfrac{2(1-t)e^{ts+t+s}}{(e^{t+s}+e^{ts})^2}$

3. (a) $\partial u/\partial x_i = F'(U) \cdot \partial U/\partial x_i$

(b) $\partial u/\partial x_i = \delta A_i \alpha_i x_i^{\alpha_i - 1}\left(\sum_{j=1}^{n} A_j x_j^{\alpha_j}\right)^{\delta-1}$

5. (a) $\dfrac{\partial u}{\partial r} = \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial r} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial r} + \dfrac{\partial f}{\partial z}\dfrac{\partial z}{\partial r} + \dfrac{\partial f}{\partial w}\dfrac{\partial w}{\partial r}$ (b) 28

7. (a) $F'(t) = (2t)^2 \cdot 2 - t^2 \cdot 1 = 7t^2$ (b) $F'(t) = \int_1^2 e^{tx}\,dx = (e^{2t} - e^t)/t$
(c) $F'(t) = \dfrac{2}{t}(e^{2t^2} - e^{t^2})$. $\left(F'(t) = \dfrac{e^{2t^2}}{2t} \cdot 2 - \dfrac{e^{t^2}}{t} + \int_t^{2t} \dfrac{\partial}{\partial t}\left(\dfrac{e^{tx}}{x}\right) dx = \right.$
$\left.\dfrac{e^{2t^2}}{t} - \dfrac{e^{t^2}}{t} + \int_t^{2t} e^{tx}\,dx = \dfrac{e^{2t^2}}{t} - \dfrac{e^{t^2}}{t} + \Big|_t^{2t} \dfrac{e^{tx}}{t} = \dfrac{2}{t}(e^{2t^2} - e^{t^2}).\right)$

9. $e^{-\rho g(\rho)} f(g(\rho))g'(\rho) - \int_0^{g(\rho)} te^{-\rho t} f(t)\,dt$

11. Using the general chain rule [16.6], the results are easy generalizations of [2] and [5] in the discussion of directional derivatives in Section 16.1.

16.3

1. [16.9] yields $y' = -(4x + 6y)/(6x + 2y) = -(2x + 3y)/(3x + y)$.

3. (a) $y' = x/y = \pm 1$. The origin must be excluded. See Fig. 44.
(b) $y' = 2x/3y^2 = (2/3)x^{-1/3}$. The origin must be excluded. See Fig. 45.

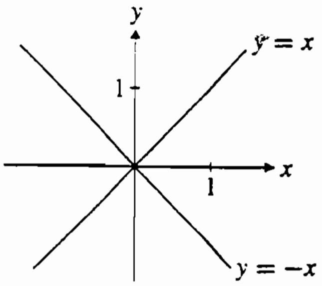


FIGURE 44

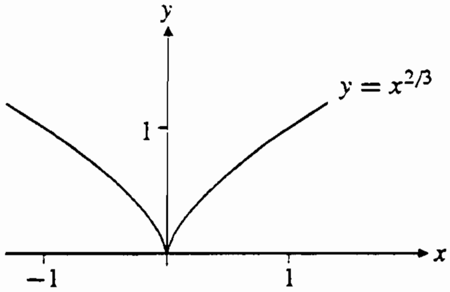


FIGURE 45

5. (a) $\dfrac{dx}{dt} = \dfrac{-[f'(x)]^2}{f(x)f''(x)} > 0$ for all t. (b) $\text{El}_t f'(x) = \dfrac{t}{f'(x)} \dfrac{df'(x)}{dt}$
$= \dfrac{t}{f'(x)} f''(x)\dfrac{dx}{dt} = \dfrac{t}{f'(x)} f''(x) \dfrac{-[f'(x)]^2}{f(x)f''(x)} = -\dfrac{tf'(x)}{f(x)} = -\dfrac{t}{t+x}$

7. (a) $U'_x(x, y) = F'\big(f(x)+g(y)\big) f'(x)$, and $U'_y(x, y) = F'\big(f(x)+g(y)\big) g'(y)$, so $\ln\left[U'_x(x, y)/U'_y(x, y)\right] = \ln\left[f'(x)/g'(y)\right] = \ln f'(x) - \ln g'(y)$. Now the result follows easily. (b) Confirming [2] is routine. Then $Ax^a y^b = e^{f(x)+g(y)}$, where $f(x) = \ln A + a \ln x$ and $g(y) = b \ln y$.

16.4

1. (a) $\text{El}_x z = 1$ and $\text{El}_y z = 1$ (b) $\text{El}_x z = 2$ and $\text{El}_y z = 5$ (c) $\text{El}_x z = n + x$ and $\text{El}_y z = n + y$ (d) $\text{El}_x z = x/(x + y)$ and $\text{El}_y z = y/(x + y)$

3. $\text{El}_i z = p + a_i x_i$, for $i = 1, \ldots, n$.

5. Differentiate each side of the identity $y/x = \ln(xy)$ w.r.t. x, thus obtaining $\frac{y'}{x} - \frac{y}{x^2} = \frac{1}{x} + \frac{y'}{y}$. Solving for y' gives $y' = \frac{(x+y)y}{x(y-x)}$, and $\mathrm{El}_x y = \frac{xy'}{y} = \frac{x+y}{y-x}$.

7. (a) $R_{yx} = (x/y)^{a-1} = (y/x)^{1-a}$ (b) $\sigma_{yx} = 1/(1-a)$

9. $F(K, N) = \gamma_1(K^{-\alpha} + \gamma_2 N^{-\alpha})^{-v/\alpha}$, so $F'_K/F'_N = K^{-\alpha-1}/\gamma_2 N^{-\alpha-1}$, implying that $\sigma_{KN} = 1/(1+\alpha)$.

11. $z = -\ln[aK^{-\varrho} + (1-a)L^{-\varrho}]/\varrho \to$ "0/0" as $\varrho \to 0$. By l'Hôpital's rule,

$$\lim_{\varrho\to 0} z = \lim_{\varrho\to 0}\left[\frac{aK^{-\varrho}\ln K + (1-a)L^{-\varrho}\ln L}{aK^{-\varrho} + (1-a)L^{-\varrho}}\right]/1$$
$$= a\ln K + (1-a)\ln L = \ln K^a L^{1-a}$$

16.5

1. $f(tx, ty) = (tx)^4 + (tx)^2(ty)^2 = t^4x^4 + t^2x^2t^2y^2 = t^4(x^4 + x^2y^2) = t^4 f(x, y)$, so f is homogeneous of degree 4.

3. $f(tx, ty) = (tx)(ty)^2 + (tx)^3 = t^3(xy^2 + x^3) = t^3 f(x, y)$, so f is homogeneous of degree 3. [16.19]: $xf_1'(x, y) + yf_2'(x, y) = x(y^2 + 3x^2) + y2xy = 3x^3 + 3xy^2 = 3(x^3 + xy^2) = 3f(x, y)$. [16.20]: It is easy to see that $f_1'(x, y) = y^2 + 3x^2$ and $f_2'(x, y) = 2xy$ are homogeneous of degree 2. [16.21]: $f(x, y) = x^3 + xy^2 = x^3[1 + (y/x)^2] = y^3[(x/y)^3 + x/y]$. [16.22]: $x^2 f''_{11} + 2xy f''_{12} + y^2 f''_{22} = x^2(6x) + 2xy(2y) + y^2(2x) = 6x^3 + 4xy^2 + 2xy^2 = 3\cdot 2f(x, y)$.

5. [16.18] requires that $t^3x^3 + t^2xy = t^k(x^3 + xy)$ for all $t > 0$. In particular, for $x = y = 1$, we must have $t^3 + t^2 = 2t^k$. For $t = 2$, we get $12 = 2\cdot 2^k$, or $2^k = 6$. For $t = 4$, we get $80 = 2\cdot 4^k$, or $4^k = 40$. But 4^k should be the square of 2^k. So the two values of k must be different, implying that f is not homogeneous of any degree.

7. From [*] with $k = 1$, we get $f''_{11} = (-y/x)f''_{12}$ and $f''_{22} = (-x/y)f''_{21}$. But $f''_{12} = f''_{21}$, so $f''_{11}f''_{22} - (f''_{12})^2 = (-y/x)f''_{12}(-x/y)f''_{12} - (f''_{12})^2 = 0$.

16.6

1. (a) Homogeneous of degree 1. (b) Not homogeneous. (c) Homogeneous of degree $-1/2$. (d) Homogeneous of degree 1. (e) Not homogeneous. (f) Homogeneous of degree n.

3. Homogeneous of degree $0.136 + (-0.727) + 0.914 + 0.816 = 1.139$.

5. $v_i' = u_i' - a/(x_1 + \cdots + x_n)$, so $\sum_{i=1}^n x_i v_i' = \sum_{i=1}^n x_i u_i' - \sum_{i=1}^n ax_i/(x_1 + \cdots + x_n) = a - a\sum_{i=1}^n x_i/(x_1 + \cdots + x_n) = a - a = 0$. Hence, by Euler's theorem, v is homogeneous of degree 0.

7. Let C and D denote the numerator and the denominator respectively in the expression for σ_{yx} that is given in Problem 10 of Sec. 16.4. Then, by Euler's theorem, $C = -F_1'F_2'(xF_1' + yF_2') = -F_1'F_2'F$. Using the facts that $xF_{11}'' = -yF_{12}''$ and $yF_{22}'' = -xF_{21}'' = -xF_{12}''$, it follows that D equals $xy\left[(F_2')^2F_{11}''-2F_1'F_2'F_{12}''+(F_1')^2F_{22}''\right] = -F_{12}''\left[(yF_2')^2+2xyF_1'F_2'+(xF_1')^2\right] = -F_{12}''(xF_1'+yF_2')^2 = -F_{12}''F^2$. Thus $\sigma_{xy} = C/D = (-F_1'F_2'F)/(-F_{12}''F^2) = F_1'F_2'/FF_{12}''$.

9. Differentiate $f(tx_1,\dots,tx_n) = g(t)f(x_1,\dots,x_n)$ w.r.t. t and put $t = 1$, as in the proof of Theorem 16.1. This yields $\sum_{i=1}^n x_i f_i'(tx_1,\dots,tx_n) = g'(1)f(x_1,\dots,x_n)$. Thus, by Euler's theorem, f must be homogeneous of degree $g'(1)$. In fact, $g(t) = t^k$ where $k = g'(1)$.

16.7

1. (a) $\dfrac{\partial z}{\partial x} = 3$ (b) $\dfrac{\partial z}{\partial x} = -\dfrac{yz + z^3 - y^2z^5}{xy + 3xz^2 - 5xy^2z^4}$

(c) $\dfrac{\partial z}{\partial x} = -\dfrac{yze^{xyz} - 3yz}{xye^{xyz} - 3xy} = -\dfrac{z}{x}$

3. $z_x' = -\dfrac{yx^{y-1} + z^x \ln z}{y^z \ln y + xz^{x-1}}$ and $z_y' = -\dfrac{x^y \ln x + zy^{z-1}}{y^z \ln y + xz^{x-1}}$

5. $z_x' = f(x/y) + xf'(x/y)(1/y)$ and $z_y' = xf'(x/y)(-x/y^2)$, so we have $xz_x'+yz_y' = xf(x/y)+(x^2/y)f'(x/y)+yx(-x/y^2)f'(x/y) = xf(x/y) = z$. Alternatively, note that z is homogeneous of degree 1, so the equality follows from Euler's theorem.

16.8

1. (a) $f(x,y) \approx 1+x/2+y/2$ (b) $f(x,y) \approx y$ (c) $f(x,y) \approx A(1+ax+by)$

3. (a) $f(1.02, 1.99) = 1.1909$ (b) $f(1.02, 1.99) \approx f(1,2) + (0.02)\cdot 8 + (-0.01)\cdot(-3) = 1.19$. The error is 0.0009.

5. $v(1.01, 0.02) \approx v(1,0) + v_1'(1,0)\cdot 0.01 + v_2'(1,0)\cdot 0.02 \approx -1.00667$.

7. (a) $dz = 3x^2\,dx + 3y^2\,dy$ (b) $dz = e^{y^2}(dx + 2xy\,dy)$

(c) $dz = \dfrac{2(x\,dx - y\,dy)}{x^2 - y^2}$

9. (a) $dz = 2xu\,dx + x^2(u_x'\,dx + u_y'\,dy)$ (b) $dz = 2u(u_x'\,dx + u_y'\,dy)$

(c) $dz = \left[(y + yu_x')\,dx + (x + u + yu_y')\,dy\right]/(xy + yu)$

11. $dU = \dfrac{\sqrt{y}}{e^U + Ue^U}dx + \dfrac{x}{2\sqrt{y}(e^U + Ue^U)}dy$

13. (a) $d^2z = 2\,dx\,dy + 2(dy)^2$ (b) $dz = (3t^2 + 4t^3)dt$ and then $d^2z = (6t + 12t^2)(dt)^2$. On the other hand, the expression for d^2z derived from (a) is equal to $(4t + 8t^2)(dt)^2$.

16.9

1. (a) $u^3dx+x3u^2du+dv = 2y\,dy$, $3v\,du+3u\,dv-dx = 0$. Solving for du and dv yields, with $D = 9xu^3-3v$, $du = D^{-1}(-3u^4-1)\,dx+D^{-1}6yu\,dy$, $dv = D^{-1}(3xu^2+3u^3v)\,dx + D^{-1}(-6yv)\,dy$ (b) $u'_x = D^{-1}(-3u^4-1)$, $v'_x = D^{-1}(3xu^2+3u^3v)$ (c) $u'_x = 283/81$ and $v'_x = -64/27$
3. With y fixed, $F'_x\,dx + F'_u\,du + F'_v\,dv = 0$ and $G'_x\,dx + G'_u\,du + G'_v\,dv = 0$. Eliminating dv and solving for du in terms of dx gives the answer $u'_x = -(F'_xG'_v - F'_vG'_x)/(F'_uG'_v - F'_vG'_u)$.
5. With a fixed, $I'(r)\,dr = S'(Y)\,dY$ and $a\,dY + L'(r)\,dr = dM$. Solving for dY and dr in terms of dM gives

$$\frac{\partial Y}{\partial M} = \frac{I'(r)}{aI'(r)+L'(r)S'(Y)} \quad \text{and} \quad \frac{\partial r}{\partial M} = \frac{S'(Y)}{aI'(r)+L'(r)S'(Y)}$$

7. $\dfrac{\partial x_1}{\partial p_1} = \dfrac{\lambda p_2^2 + x_1(p_2U''_{12} - p_1U''_{22})}{p_1^2U''_{22} - 2p_1p_2U''_{12} + p_2^2U''_{11}}$. (Differentiating with $dp_2 = dm = 0$ yields: [1′] $U''_{11}\,dx_1+U''_{12}\,dx_2 = p_1d\lambda+\lambda dp_1$; [2′] $U''_{21}\,dx_1+U''_{22}\,dx_2 = p_2d\lambda$; and [3′] $p_1\,dx_1 + dp_1x_1 + p_2\,dx_2 = 0$. Solve for dx_1.)

16.10

1. (a) Differentiation yields: (1′) $dY = dC + dI + dG$, (2′) $dC = f'_Y\,dY + f'_T\,dT + f'_r\,dr$, (3′) $dI = h'_Y\,dY + h'_r\,dr$. Hence, $dY = \big(f'_T\,dT + dG + (f'_r + h'_r)\,dr\big)/(1 - f'_Y - h'_Y)$.
(b) Because $\partial Y/\partial T = f'_T/(1 - f'_Y - h'_Y) < 0$, so Y decreases as T increases. But if $dT = dG$ with $dr = 0$, then $dY = \big(1 + f'_T\big)dT/(1 - f'_Y - h'_Y)$, which is positive provided that $f'_T > -1$.

Chapter 17

17.1

1. $x = 1$, $y = 2$ ($f'_1(x, y) = -4x + 4 = 0$, $f'_2(x, y) = -2y + 4 = 0$ for $x = 1$, $y = 2$).
3. (a) $x = 3$, $y = -4$ (b) $f(x, y) = x^2-6x+3^2+y^2+8y+4^2+35-3^2-4^2 = (x-3)^2 + (y+4)^2 + 10 \geq 10$ for all (x, y), whereas $f(3, -4) = 10$.
5. $Q_1 = (a_1 - \alpha)/2b_1$ and $Q_2 = (a_2 - \alpha)/2b_2$. (Solve the following equations for Q_1 and Q_2: $\partial\pi(Q_1, Q_2)/\partial Q_1 = a_1 - \alpha - 2b_1Q_1 = 0$ and $\partial\pi(Q_1, Q_2)/\partial Q_2 = a_2 - \alpha - 2b_2Q_2 = 0$.)
7. $L_1 = (P - \alpha_1)/2\beta_1$ and $L_2 = (P - \alpha_2)/2\beta_2$
9. P has maximum 3888 for $x = 36$, $y = 12$, and $z = 9$. (Note that $P = (108 - 3y - 4z)yz$. Then $\partial P/\partial y = 108z - 6yz - 4z^2 = 0$ and

$\partial P/\partial z = 108y - 3y^2 - 8yz = 0$. Because y and z are assumed to be positive, the first-order conditions reduce to $6y + 4z = 108$ and $3y + 8z = 108$, with solution $y = 12$ and $z = 9$.)

11. $x = mp^{-k}/Q$, $y = mq^{-k}/Q$, and $z = mr^{-k}/Q$, where $k = 1/(1-a)$, and $Q = p^{-ak} + q^{-ak} + r^{-ak}$. (The constraint yields $z = (m - px - qy)/r$, so we maximize $P = x^a + y^a + z^a$ w.r.t. x and y. The first-order conditions are $P'_x = ax^{a-1} + az^{a-1}(-p/r) = 0$, and $P'_y = ay^{a-1} + az^{a-1}(-q/r) = 0$. These give [1] $x = p^{-k}r^k z$, [2] $y = q^{-k}r^k z$. It follows that [3] $y = p^k q^{-k} x$. Then [1] implies $x = p^{-k}r^{ak}m - p^{1-k}r^{ak}x - q^{1-k}r^{ak}x$. Solving for x gives $x = mp^{-k}/(p^{-ak} + q^{-ak} + r^{-ak})$. Then [3] gives the correct expression for y, and that for z emerges from [1] or [2].)

17.2

1. (a) Minimum -10 at $(-1, 3)$, because $f(x, y)$ is clearly ≥ -10 for all (x, y), and $f(-1, 3) = -10$. No maximum exists.
(b) Maximum 3 for all (x, y) satisfying $x^2 + y^2 = 2$. Minimum $3 - \sqrt{2}$ at $(0, 0)$.

3. Because $F(u) = \frac{1}{2}(e^u - e^{-u})$ is strictly increasing, the problem is equivalent to max $(x^2 + y^2 - 2x)$ when $(x, y) \in S$.

5. Let $g(x) = 1$ in $[0, 1)$, $g(x) = 2$ in $[1, 2]$. Then g is discontinuous at $x = 1$ and $\{x : g(x) \leq 1\} = [0, 1)$, which is not closed. (Draw your own graph of g.)

7. (a) Yes. (b) No. (Because F is strictly increasing, f and g must have maxima at the same point in the domain. (c) Yes. (d) No. (Because f is a constant, $F(f(x))$ must be a constant.)

17.3

1. (a) $f'_1(x, y) = 4 - 4x$ and $f'_2(x, y) = -4y$. The only stationary point is $(1, 0)$.
(b) $f(x, y)$ has maximum 2 at $(1, 0)$ and minimum -70 at $(-5, 0)$. (A maximum and minimum exist, by the extreme-value theorem. At the stationary point, $f(1, 0) = 2$. Along the boundary, the function value is $4x - 50$, with $x \in [-5, 5]$. So its maximum along the boundary is -30 at $x = 5$ and its minimum is -70 at $x = -5$.)

3. A maximum and minimum exist, because of Theorem 17.3. The maximum is 1 at $(2, 1/2)$. The minimum is 0 at $(x, 0)$ and $(x, x - 1)$, with x arbitrary in $[1, 2]$.

5. For $k \in (0, 1)$, F has a maximum $\sqrt{k}$ at $(0, k)$. For $k \in (1, \infty)$, F has a maximum $k^{3/4}$ at $(k, 0)$. For $k = 1$, F has a maximum 1 at $(0, 1)$ and at $(1, 0)$.

17.4

1. (a) $f_1' = 2x + 2y^2$, $f_2' = 4xy + 4y$, $f_{11}'' = 2$, $f_{12}'' = f_{21}'' = 4y$, and $f_{22}'' = 4x + 4$.
(b) The stationary points satisfy $x + y^2 = 0$ and $(x + 1)y = 0$. Hence, $y = 0$ and $x = 0$, or $x = -1$ and $y = \pm 1$. Now Theorem 17.5 implies that $(0, 0)$ is a local minimum, whereas $(-1, 1)$ and $(-1, -1)$ are saddle points.

3. (a) The first-order conditions $2axy + by + 2y^2 = 0$ and $ax^2 + bx + 4xy = 0$ must have $(x, y) = (2/3, 1/3)$ as a solution. So $a = 1$ and $b = -2$. Also $c = 1/27$, so that $f(2/3, 1/3) = -1/9$. Applying Theorem 17.5 shows that this is a local minimum. (b) Maximum 193/27 at $(2/3, 8/3)$. Minimum $-1/9$ at $(2/3, 1/3)$.

5. (a) $(1, 2)$ is a local minimum; $(0, 0)$ and $(0, 4)$ are saddle points. (b) Study $f(x, 1)$ as $x \to -\infty$, and $f(-1, y)$ as $y \to \infty$. (c) A maximum and minimum exist because of Theorem 17.3. f has a minimum $-4/e$ at $(1, 2)$, and a maximum 0 at all $(x, 0)$ and $(x, 4)$ satisfying $x \in [0, 5]$, and at all $(0, y)$ satisfying $y \in [0, 4]$.
(d) $y' = -f_x'/f_y' = (x - 1)(y^2 - 4y)/x(2y - 4) = 0$ when $x = 1$ and $y = 4 - e$.

7. (a) $K = w^2L/r^2$ and $L = 2^{-8/3}p^{4/3}w^{-4/3}r^{2/3}(r + w)^{-2/3}$. For $p = 32\sqrt{2}$, $r = w = 1$, we have $K = L = 16$. (b) Value added per worker is

$$\frac{\pi + wL}{L} = \frac{p(\sqrt{\sqrt{kL} + \sqrt{L}} - rkL}{L} = \frac{p\sqrt{1 + \sqrt{k}}}{L^{3/4}} - rk$$

(c) $L = 16$. The maximum of h is $h(16, 1) = 7$.

9. Stationary points are: $(0, 0)$, $(a, -a)$, (a, a), $(-a, a)$, and $(-a, -a)$, where $a = \sqrt{u_0}$, and u_0 is the unique positive solution of the equation $u^3 + u - 1 = 0$. (The first-order conditions are $4x(x^2y^4 + y^2 - 1) = 0$ and $4y(x^4y^2 + x^2 - 1) = 0$. One possibility is $(x, y) = (0, 0)$. Otherwise, $x \neq 0 \iff y \neq 0$. If $(x, y) \neq (0, 0)$, we find that $x^2y^4 + y^2 = 1 = x^4y^2 + x^2$, so $x^2 = y^2$. Therefore, $y = \pm x$, and so x must satisfy $x^6 + x^2 - 1 = 0$. Let $u = x^2$. The equation $g(u) = u^3 + u - 1 = 0$ has a solution in $(0, 1)$ by the intermediate-value theorem. The solution is unique because $g'(u) = 3u^2 + 1 > 0$ for all u.) Global extreme points do not exist. (Consider $h(x, 0)$ and $h(x, x)$, as x tends to infinity.) $(0, 0)$ gives a local maximum; the others are saddle points.

17.5

1. Only (a) and (d) are convex.

3. At most one point. (If the set had two distinct points, any of the infinitely many points on the line segment between the points would have to belong to the set.)

5. Suppose $(s_1, t_1), (s_2, t_2) \in S \times T$, with $s_1, s_2 \in S$ and $t_1, t_2 \in T$. With $\lambda \in [0, 1]$, we have $(1-\lambda)(s_1, t_1) + \lambda(s_2, t_2) = ((1-\lambda)s_1 + \lambda s_2, (1-\lambda)t_1 + \lambda t_2)$. This belongs to $S \times T$ because $(1-\lambda)s_1 + \lambda s_2 \in S$ and $(1-\lambda)t_1 + \lambda t_2 \in T$, by the convexity of S and T, respectively. Hence, $S \times T$ is convex.

17.6

1. (a) Strictly convex. (b) Concave, but not strictly concave. (c) Strictly concave.
3. If $\mathbf{x}^0, \mathbf{x} \in R^n$ and $\lambda \in [0, 1]$, we have $f((1-\lambda)\mathbf{x}^0+\lambda\mathbf{x}) = ||(1-\lambda)\mathbf{x}^0+\lambda\mathbf{x}|| \leq ||(1-\lambda)\mathbf{x}^0|| + ||\lambda\mathbf{x}|| = (1-\lambda)||\mathbf{x}^0|| + \lambda||\mathbf{x}|| = (1-\lambda)f(\mathbf{x}^0) + \lambda f(\mathbf{x})$, so f is convex.

17.7

1. $f = u^2$, where $u = x + 2y + 3z$. So f is a convex function of a linear function, hence convex.
3. Because $\lambda(t) \geq 0$, $\lambda(t)f(x(t)) - \lambda(t)f(z) \leq f'(z)\lambda(t)x(t) - zf'(z)\lambda(t)$. Integrating each side w.r.t. t gives $\int_a^b \lambda(t)f(x(t))\,dt - f(z)\int_a^b \lambda(t)\,dt \leq f'(z)\int_a^b \lambda(t)x(t)\,dt - zf'(z)\int_a^b \lambda(t)\,dt$. But $\int_a^b \lambda(t)\,dt = 1$ and also $z = \int_a^b \lambda(t)x(t)\,dt$, so $\int_a^b \lambda(t)f(x(t))\,dt - f\left(\int_a^b \lambda(t)x(t)\,dt\right) \leq 0$.

17.8

1. (a) (i) $f''_{11} = -2 \leq 0$, $f''_{22} = 0 \leq 0$, and $f''_{11}f''_{22} - (f''_{12})^2 = 0 \geq 0$, so f is concave.
   (ii) $f(x) = (x - y) + (-x^2)$ is a sum of two concave functions, hence concave.
   (b) $F(u) = -e^{-u}$ is strictly increasing and concave ($F'(u) = e^{-u} > 0$ and $F''(u) = -e^{-u} < 0$). By Theorem 17.6, part (c), $z = -e^{-f(x,y)}$ is concave.
3. $f''_{11} = -12$, $f''_{22} = -2$, and $f''_{11}f''_{22} - (f''_{12})^2 = 24 - (2a+4)^2 = -4a^2 - 16a + 8$. Because $f''_{11} < 0$, the function is never convex. It is concave iff $-4a^2 - 16a + 8 \geq 0$, that is, iff $-2 - \sqrt{6} \leq a \leq -2 + \sqrt{6}$.
5. Draw your own figure and use definition [17.14] of Section 17.6 to show that f is convex.
7. (100, 300) is a stationary point for π. Moreover, $\pi''_{11} = -0.08 \leq 0$, $\pi''_{22} = -0.02 \leq 0$, and $\pi''_{11}\pi''_{22} - (\pi''_{12})^2 = (-0.08)(-0.02) - (0.01)^2 = 0.015 \geq 0$, so (100, 300) is a (global) maximum point for π.
9. Direct verification of second-order conditions.

11. The Hessian matrix $\mathbf{H} = \begin{pmatrix} f''_{11} & f''_{12} \\ f''_{21} & f''_{22} \end{pmatrix}$ is

$$\begin{pmatrix} \frac{a}{x^2}(\ln x)^{a-2}(\ln y)^b(a-1-\ln x) & \frac{ab}{xy}(\ln x)^{a-1}(\ln y)^{b-1} \\ \frac{ab}{xy}(\ln x)^{a-1}(\ln y)^{b-1} & \frac{b}{y^2}(\ln y)^{b-2}(\ln x)^a(b-1-\ln y) \end{pmatrix}$$

Here $f''_{11} < 0$ because $0 < a < 1$. Also $|\mathbf{H}|$ is equal to $f''_{11}f''_{22} - (f''_{12})^2$, or to

$$\frac{ab}{x^2y^2}(\ln x)^{2a-2}(\ln y)^{2b-2}\cdot\Big[\underbrace{1-(a+b)}_{>0}+\underbrace{(1-a)\ln y}_{>0}+\underbrace{(1-b)\ln x}_{>0}+\underbrace{\ln x\ln y}_{>0}\Big]$$

This is positive, so f is strictly concave.

17.9

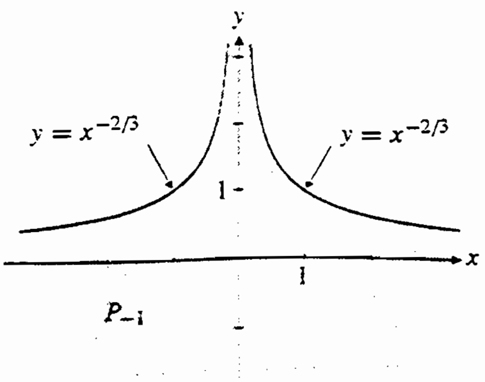
1. The only stationary point is $(0, 0, 0)$. The leading principal minors of the Hessian have the values $D_1 = 2$, $D_2 = 3$, and $D_3 = 4$, so $(0, 0, 0)$ is a local minimum point by Theorem 17.12, part (d).

3. When the Hessian is indefinite at $\mathbf{x}^0$, this point is neither a local maximum nor a local minimum, so it is a saddle point.

17.10

1. (a) f is linear, so quasi-concave. (b) $\ln f(x, y) = x + \ln y$, which is the sum of concave functions. Because e^u is strictly increasing, f must be quasi-concave (Theorem 17.16 (b)). (c) The set of points for which $f(x, y) \geq -1$ is $P_{-1} = \{(x, y) : y \leq x^{-2/3}\}$, which is not a convex set (see Fig. 46), so f is not quasi-concave. (It is quasi-convex in the first quadrant because of [17.32] (b).)
(d) The polynomial $x^3 + x^2 + 1$ is increasing in the interval $(-\infty, -2/3]$, and decreasing in $[-2/3, 0]$. So f is increasing in $(-\infty, -2/3]$ and decreasing

FIGURE 46



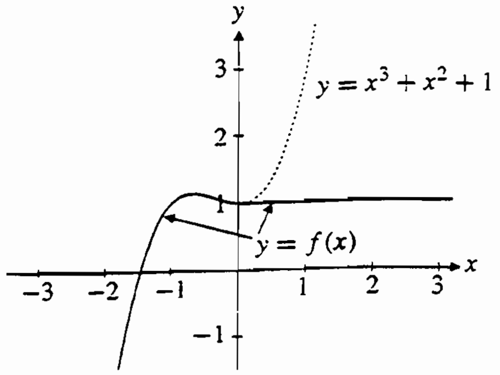


FIGURE 47

in $[-2/3, \infty)$. (See Fig. 47.) Then the level sets must be intervals, and it follows that f is quasi-concave.

3. Let $f_1(x) = 1 - x$ for $|x| \leq 1$, 0 for $|x| > 1$; $f_2(x) = -x - 1$ for $|x+2| \leq 1$, 0 for $|x + 2| > 1$. Then f_1 and f_2 are quasi-concave, but $f_1 + f_2$ is not. (See Fig. 48.)

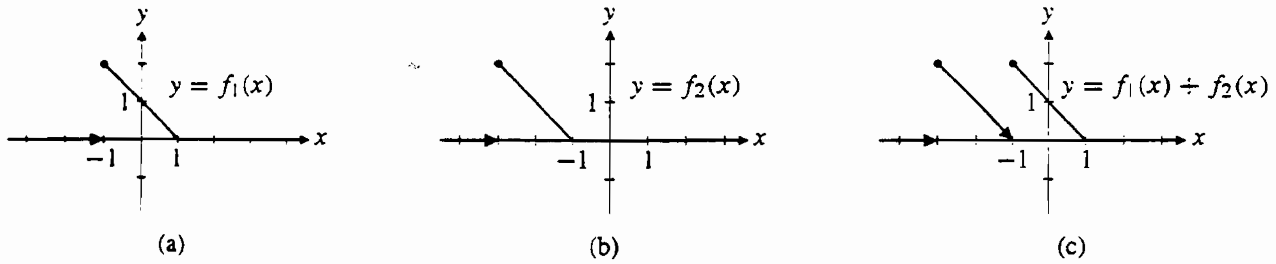


FIGURE 48

5. $f'(x) \neq 0$ for all x implies that f is quasi-concave. (In fact, $f'(x) \neq 0$ means that either f is (strictly) increasing or (strictly) decreasing. Then see Problem 2.)

7. From [16.13] in Section 16.3, $\phi''(x) = D_2(x, y)/(F_2')^3$ with $D_2(x, y)$ defined in Theorem 17.17. By part (a) of that theorem, $D_2(x, y) \geq 0$, so $\phi''(x) \geq 0$ and ϕ is convex.

Chapter 18

18.1

1. (a) $x = 1/2$ and $y = 3/4$ ($f_1'(x, y) = 1$, $f_2'(x, y) = 1$, $g_1'(x, y) = 2x$, $g_2'(x, y) = 1$, so [18.3] yields the equation $1/1 = 2x/1$. Then $x = 1/2$ and $y = 1 - (1/2)^2 = 3/4$.)
(b) $x = 4/5$ and $y = 8/5$

3. $x = 27/10$ and $y = 9/10$. ([18.3] yields $5x^{-1/2}y^{1/3}/(10/3)x^{1/2}y^{-2/3} = 2/4 = 1/2$, so $y = x/3$. Then $y = 9/10$, so $x = 27/10$.)

18.2

1. (a) $\mathcal{L}(x, y) = x + y - \lambda(x^2 + y - 1)$. The equations $\mathcal{L}_1' = 1 - 2\lambda x = 0$, $\mathcal{L}_2' = 1 - \lambda = 0$, and $x^2 + y = 1$ have the solution $x = 1/2$, $y = 3/4$, and $\lambda = 1$. (b) The solution is illustrated in Fig. 49. The minimization problem has no solution.
(c) $x = 0.5$ and $y = 0.85$. The change in the value function is $f^*(1.1) - f^*(1) = (0.5 + 0.85) - (0.5 + 0.75) = 0.1$. Because $\lambda = 1$, $\lambda \cdot dc = 1 \cdot 0.1 = 0.1$. So, in this case, [18.8] is satisfied with equality.

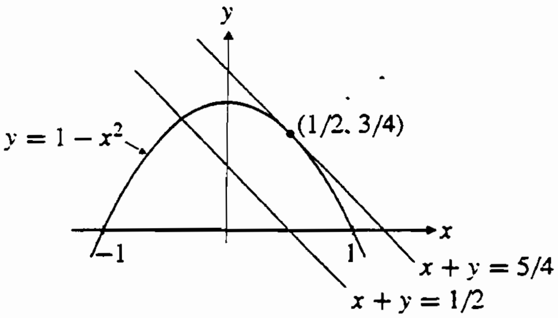


FIGURE 49

3. (a) Solution: $x = 50$ and $y = 50$, with $\lambda = 250$. To see that this solves the problem, let $x = 50 + h$ and $y = 50 + k$. Inserting these values of x and y into the constraint yields $50 + h + 50 + k = 100$, that is, $k = -h$. Then $x^2 + 3xy + y^2 = (50 + h)^2 + 3(50 + h)(50 + k) + (50 + k)^2 = 12{,}500 - h^2$, using $k = -h$ and simplifying. Now, $12{,}500 - h^2$ has a maximum for $h = 0$, that is, for $x = 50$, and then $y = 50$. (b) Solution: $x = 8/3$ and $y = 1$, with $\lambda = 4$. From the constraint equation, $y = 3 - 3x/4$. Because we must have $y \geq 0$, so $x \leq 4$. Now, if we let $h(x) = 12x\sqrt{3 - 3x/4}$, then $h'(x) = (72 - 27x)/2\sqrt{3 - 3x/4}$, and the sign variation of this derivative reveals that h is maximized at $x = 8/3$.

5. (a) $\mathcal{L}(x, y) = 10x^{1/2}y^{1/3} - \lambda(2x + 4y - m)$. The two first-order conditions $\mathcal{L}_1' = 5x^{-1/2}y^{1/3} - 2\lambda = 0$ and $\mathcal{L}_2' = (10/3)x^{1/2}y^{-2/3} - 4\lambda = 0$ imply that $(10/3)x^{1/2}y^{-2/3} = 10x^{-1/2}y^{1/3}$, so $x = 3y$. Substituting this into the constraint $2x + 4y = m$ gives $y = m/10$ and so $x = 3m/10$, with $\lambda = 2.5(10/27m)^{1/6}$.
(b) $f^*(m) = 10^{1/6}3^{1/2}m^{5/6}$ and $df^*(m)/dm = \lambda = 2.5 \cdot 10^{1/6}3^{-1/2}m^{-1/6}$.

7. (a) (2, 1) (b) With $\mathcal{L}(x, y) = x + 2y - \lambda[p(x^2 + y^2) + x^2y^2 - 4]$, equating the first-order partials to 0 yields $\mathcal{L}_1' = 1 - 2\lambda px - 2\lambda xy^2 = 0$ and $\mathcal{L}_2' = 2 - 2\lambda py - 2\lambda x^2 y = 0$. Hence, $2\lambda x(p + y^2) = 1$ and $2\lambda y(p + x^2) = 2$. Eliminating λ yields the first equality in [**]. The second is just the constraint.
(c) Differentiating [**] w.r.t. p, with x and y as functions of p, yields
[1] $2x + 2px' - y - py' + 2x'y^2 + 4xyy' - 2xx'y - x^2y' = 0$ and
[2] $x^2 + 2pxx' + y^2 + 2pyy' + 2xx'y^2 + 2x^2yy' = 0$. Letting $p = 0$, recalling that $x(0) = 2$ and $y(0) = 1$, yields the equations $-2x'(0) + 4y'(0) = -3$ and

$4x'(0) + 8y'(0) = -5$, with the solution $x'(0) = 1/8$ and $y'(0) = -11/16$. (d) $h(p) = x(p) + 2y(p)$, so $h'(0) = x'(0) + 2y'(0) = -5/4$.

18.3

1. The problem with systems of three equations and two unknowns is that they are usually inconsistent (have no solutions), not that they are difficult to solve. The equations $f'_x(x, y) = f'_y(x, y) = 0$ are *not* valid at the optimal point.

3. $x = -1$ and $y = 0$ solves the problem, with $f(-1, 0) = 1$. (Actually, this problem is quite tricky. The only stationary point of the Lagrangean is $(0, 0)$, with $\lambda = -4$, and with $f(0, 0) = 4$. The point is that at $(-1, 0)$ both $g'_1(-1, 0)$ and $g'_2(-1, 0)$ are 0, so the Lagrangean is not necessarily stationary at this point. The problem is to minimize the (square of the) distance from $(-2, 0)$ to a point on the graph of $g(x, y) = 0$. But the graph consists of the single point $(-1, 0)$ and a nice curve, as illustrated in Fig. 50.)

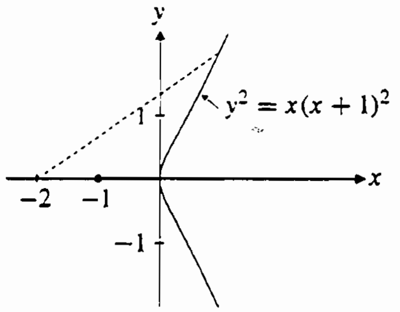


FIGURE 50

18.4

1. The Lagrangean $\mathcal{L}(x, y) = 10x^{1/2}y^{1/3} - \lambda(2x + 4y - m)$ is concave as a sum of two concave functions (see Example 17.18 in Section 17.8), so Theorem 18.2 applies.

18.5

1. (a) $\mathcal{L}(x, y, z) = x^2 + y^2 + z^2 - \lambda(x + y + z - 1)$. The only solution of the necessary conditions is $(1/3, 1/3, 1/3)$. (b) The problem is to find the shortest distance from the origin to a point in the plane $x + y + z = 1$. The corresponding maximization problem has no solution.

3. (a) $x = \alpha(wL+m)/p$, $y = \beta(wL+m)/q$, and $l = (\alpha+\beta)L - m(1-\alpha-\beta)/w$. The condition given is equivalent to $l \geq 0$. (b) The solution is $l = 0$, with $x = \alpha m/(\alpha + \beta)p$ and $y = \beta m/(\alpha + \beta)q$. (In this case, unearned income is so high that it is optimal for the individual not to work at all.)

5. With linear equality constraints, we shall not resist the temptation to eliminate variables. In fact, adding the constraints gives $3x = 6$ and so $x = 2$. Thus $y = -(1 + z)$. Substituting for x and y in the objective function

reduces it to $2(1+z)^2 + z^2 + z = 3z^2 + 5z + 2$. This quadratic polynomial has a minimum when $z = -5/6$. Then $y = -1/6$. The solution is $(x, y, z) = (2, -1/6, -5/6)$.

7. The Lagrangean is $\mathcal{L} = x + y - \lambda(x^2+2y^2+z^2-1) - \mu(x+y+z-1)$, which is stationary when [1] $\mathcal{L}'_x = 1 - 2\lambda x - \mu = 0$; [2] $\mathcal{L}'_y = 1 - 4\lambda y - \mu = 0$; [3] $\mathcal{L}'_z = -2\lambda z - \mu = 0$. From [1] and [2], $2\lambda(x - 2y) = 0$. If $\lambda = 0$, then [2] and [3] yield $\mu = 1$ and $\mu = 0$. Therefore $x = 2y$ instead. Substituting this value for x in the constraints gives $6y^2 + z^2 = 1$, $3y + z = 1$. Thus, $z = 1 - 3y$ and $1 = 6y^2 + (1-3y)^2 = 15y^2 - 6y + 1$. Hence $y = 0$ or $y = 2/5$, implying that $x = 0$ or $4/5$, and that $z = 1$ or $-1/5$. The only two solution candidates are $(x, y, z) = (0, 0, 1)$ with $\lambda = -1/2$, $\mu = 1$, and $(x, y, z) = (4/5, 2/5, -1/5)$ with $\lambda = 1/2$, $\mu = 1/5$. Because $x + y$ is 0 at $(0, 0, 1)$ and $6/5$ at $(4/5, 2/5, -1/5)$, these are respectively the minimum and the maximum. (The constraints determine geometrically the curve which is the intersection of an ellipsoid (see Fig. 15.12 in Section 15.2) and a plane. The continuous function $x + y$ does attain a maximum and a minimum over this closed bounded set.)

9. (a) Here $\mathcal{L} = (y+z-3)^2 - \lambda(x^2+y+z-2) - \mu(x+y^2+2z-2)$, which is stationary when [1] $\mathcal{L}'_x = -2\lambda x - \mu = 0$; [2] $\mathcal{L}'_y = 2(y+z-3) - \lambda - 2\mu y = 0$; [3] $\mathcal{L}'_z = 2(y + z - 3) - \lambda - 2\mu = 0$. From (2) and (3), $\lambda + 2\mu y = \lambda + 2\mu$, so $\mu(y - 1) = 0$. If $\mu = 0$, then from [1] and [2], $\lambda x = 0$ and $2(y + z - 3) = \lambda$. Now $\lambda = 0$ would imply that $y + z = 3$ and so $x^2 = -1$ from the first constraint. Thus $\lambda \neq 0$ and so $\mu = 0 \Rightarrow x = 0$. Then the two constraints yield $y = 2 - z$ and $y^2 = 2(1 - z)$, so that $y^2 - 2y + 2 = 0$, which also has no real root. We conclude that $\mu \neq 0$ and so $y = 1$. The two constraints now imply that $x^2 + z = 1$ and $x + 2z = 1$. Hence $x^2 + \frac{1}{2}(1 - x) = 1$, which has roots $x = 1$ and $x = -1/2$. The only two solution candidates are $(x, y, z) = (1, 1, 0)$ and $(x, y, z) = (-1/2, 1, 3/4)$. The corresponding values of (λ, μ) are $(4/3, -8/3)$ and $(-5/6, -5/6)$ respectively. Because $(y + z - 3)^2$ is 4 at $(1, 1, 0)$ and $25/16$ at $(-1/2, 1, 3/4)$, the latter is the appropriate solution. (The method used in Problem 4 can be applied to show that this gives the minimum.) (b) The second solution, which is $(1, 1, 0)$, gives $f(1, 1, 0) = 4$. But $(-2, -2, 0)$, for example, satisfies both constraints, and gives $f(-2, -2, 0) = 25$.

11. Differentiating the constraint w.r.t. x_1 yields $g'_1 + g'_3(\partial x_3/\partial x_1) = 0$, or (1) $\partial x_3/\partial x_1 = -g'_1/g'_3$. Similarly, (2) $\partial x_3/\partial x_2 = -g'_2/g'_3$. Now, the first-order conditions for the maximization of $z = f(x_1, x_2, x_3)$, where x_3 is a function of (x_1, x_2), are (3) $\partial z/\partial x_1 = f'_1 + f'_3(\partial x_3/\partial x_1) = 0$ and (4) $\partial z/\partial x_2 = f'_2 + f'_3(\partial x_3/\partial x_2) = 0$. Substitute from (1) and (2) into (3) and (4), letting $\lambda = f'_3/g'_3$. This yields the equations $f'_1 - \lambda g'_1 = 0$ and $f'_2 - \lambda g'_2 = 0$, and by the definition of λ, $f'_3 - \lambda g'_3 = 0$. These are the conditions in (18.17) for $n = 3$.

18.6

1. $x = -\frac{1}{6}\sqrt{b}$, $y = -\frac{1}{3}\sqrt{b}$, $z = -\frac{3}{2}\sqrt{b}$, $\lambda = -3/\sqrt{b}$, $f^*(b) = -6\sqrt{b}$, $df^*/db = -3/\sqrt{b} = \lambda$.

3. (a) [4] $1 - \frac{1}{3}\lambda = \mu x_2x_3x_4$; [5] $1 - \frac{1}{3}\lambda = \mu x_1x_3x_4$; [6] $1 - \frac{1}{8}\lambda = \mu x_1x_2x_4$; [7] $1 - \frac{1}{8}\lambda = \mu x_1x_2x_3$, together with [1], [2], and [3]. (b) Note that $\mu = 0$ would give $1 - \frac{1}{3}\lambda = 1 - \frac{1}{8}\lambda = 0$, which is impossible. Because $x_1, \dots, x_4$ are all nonzero, it follows from [4] and [5] that $x_1 = x_2$, and from [6] and [7] that $x_3 = x_4$. Then [4] and [5] imply that $1 - \frac{1}{3}\lambda = \mu x_1x_3^2$, whereas [6] and [7] imply that $1 - \frac{1}{8}\lambda = \mu x_1^2x_3$, and also $x_1^2x_3^2 = 144$ by [2]. Hence, $x_1x_3 = 12$ and also $\frac{2}{3}x_1 + \frac{1}{4}x_3 = 3$, from the first constraint. These last two equations have two solutions, which are $(x_1, x_3) = (3, 4)$ and $(3/2, 8)$. Maximizing $x_1+x_2+x_3+x_4 = 2(x_1+x_3)$ requires choosing the latter. Thus, $(x_1, x_2, x_3, x_4) = (3/2, 3/2, 8, 8)$ must solve the problem, with $\lambda = 13$ and $\mu = -5/144$.
(c) The change is approximately $\mu = -5/144$.

18.7

1. $x^* = m/2p$ and $y^* = m/2q$, with $\lambda = 5p^{-1/2}q^{-1/2}$, solves the problem. The optimal value function is $U^*(p, q, m) = 5p^{-1/2}q^{-1/2}m$, and [18.31] says that $\partial U^*/\partial p = -\lambda x^*$, $\partial U^*/\partial q = -\lambda y^*$ and $\partial U^*/\partial m = \lambda$. The correctness of these equalities is now easily checked.

3. (a) The first order conditions can be expressed as

$$\frac{au}{x_1} - \frac{bu}{x_1 + b - a} = \lambda p_1, \qquad \frac{(b-a)u}{x_2} = \lambda p_2$$

where $u = x_1^a x_2^{b-a}(x_1 + b - a)^{-b}$. Thus

$$\lambda p_1 x_1 = au - \frac{bux_1}{x_1 + b - a} = \frac{(b-a)(a-x_1)u}{x_1 + b - a}, \qquad \lambda p_2 x_2 = (b-a)u$$

From the first of these equations, $(b-a)u/\lambda = p_1x_1(x_1+b-a)/(a-x_1)$, and hence from the second, $p_2x_2 = (b-a)u/\lambda = p_1x_1(x_1+b-a)/(a-x_1)$. Substituting in the budget constraint and solving for x_1 eventually yields $x_1 = am/(m + bp_1)$ and $x_2 = m(m + bp_1 - ap_1)/p_2(m + bp_1)$. These are the required demand functions. They are both positive.
(b) Use logarithmic differentiation to obtain $\partial x_1/\partial p_1 = -bx_1/(m + p_1b) < 0$, $\partial x_1/\partial p_2 = 0$, $\partial x_1/\partial m = x_1/m - x_1/(m + p_1b) > 0$, $\partial x_2/\partial p_1 = -amx_2/[m + p_1(b-a)](m + p_1b) < 0$, $\partial x_2/\partial p_2 = -x_2/p_2 < 0$, and $\partial x_2/\partial m = x_2/m + p_1ax_2/[m + p_1(b-a)](m + p_1b) > 0$. (c) Routine.

18.8

1. (a) $\mathcal{L}(x, y) = x^2+2y^2-x-\lambda(x^2+y^2-1)$, and [18.35] gives [1] $2x-1-2\lambda x = 0$; [2] $4y - 2\lambda y = 0$. (b) $\lambda \geq 0$ ($= 0$ if $x^2 + y^2 < 1$)
(c) Candidates: $(1/2, 0)$ with $\lambda = 0$; $(1, 0)$ with $\lambda = 1/2$; $(-1, 0)$ with $\lambda = 3/2$; and $(-1/2, \pm\sqrt{3}/2)$ with $\lambda = 2$. Maximum 9/4 at $(-1/2, \sqrt{3}/2)$ and at $(-1/2, -\sqrt{3}/2)$.

3. (a) The Lagrangean is $\mathcal{L} = y - x^2 + \lambda y + \mu(y - x + 2) - \nu(y^2 - x)$, which is stationary when: [1] $-2x - \mu + \nu = 0$; [2] $1 + \lambda + \mu - 2\nu y = 0$. Also [3] $\lambda \geq 0 (= 0$ if $y > 0)$; [4] $\mu \geq 0 (= 0$ if $y - x + 2 > 0)$; [5] $\nu \geq 0 (= 0$ if $y^2 < x)$.
From [2], $2\nu y = 1 + \lambda + \mu > 0$, so $y > 0$. Then [3] implies $\lambda = 0$ and $2\nu y = 1 + \mu$. From [1], $x = \frac{1}{2}(\nu - \mu)$. But $x \geq y^2 > 0$, so $\nu > \mu \geq 0$, and from [5], $y^2 = x$.
Suppose $\mu > 0$. Then $y - x + 2 = y - y^2 + 2 = 0$ with roots $y = -1$ and $y = 2$. Only $y = 2$ is a candidate. Then $x = y^2 = 4$. Because $\lambda = 0$, the first-order conditions become $-\mu + \nu = 8$ and $\mu - 4\nu = -1$, so $\nu = -7/3$, which contradicts $\nu \geq 0$, so $(x, y) = (4, 2)$ is not a candidate. Therefore $\mu = 0$ after all. Thus $x = \frac{1}{2}\nu = y^2$ and $1 = 2\nu y = 4y^3$. Hence $y = 4^{-1/3}$, $x = 4^{-2/3}$. This is the only candidate, with $\lambda = 0$, $\mu = 0$, and $\nu = 2 \cdot 4^{-2/3} = 4^{1/6}$. (b) $x = 1$ and $y = 0$ with $\lambda = 0$, $\mu = 2e - e^{-1}$, $v = 0$. ($\mathcal{L} = xe^{y-x} - 2ey + \lambda x + \mu y - v(y - 1 - x/z)$)

5. (a) With $\mathcal{L} = \ln x_1 + x_2 + x_3 - \lambda_1(x_1 + x_2 + x_3 - 1) - \lambda_2(-x_1 + 1) - \lambda_3(x_1^2 + x_2^2 - 2)$, the necessary conditions are [1] $1/x_1 - \lambda_1 + \lambda_2 - 2\lambda_3 x_1 = 0$; [2] $1 - \lambda_1 - 2\lambda_3 x_2 = 0$; [3] $1 - \lambda_1 = 0$; [4] $\lambda_1 \geq 0$ ($= 0$ if $x_1 + x_2 + x_3 < 1$); [5] $\lambda_2 \geq 0$ ($= 0$ if $x_1 > 1$); [6] $\lambda_3 \geq 0$ ($= 0$ if $x_1^2 + x_2^2 < 2$).
(b) From [3] $\lambda_1 = 1$. Hence $x_1+x_2+x_3 = 1$, by [4]. Also $\lambda_3 x_2 = 0$, because of [2]. Suppose $\lambda_3 > 0$. Then $x_2 = 0$ and, by [6], $x_1 = \sqrt{2}$, so $\lambda_2 = 0$. But substituting in [1] gives $1/\sqrt{2} - 1 - 2\sqrt{2}\lambda_3 = 0$, which is impossible when $\lambda_3 > 0$. So $\lambda_3 = 0$ after all. With $\lambda_3 = 0$, condition [1] implies that $x_1 = 1/(1 - \lambda_2)$. This would imply $x_1 > 1$ if $\lambda_2 > 0$, contradicting [5]. Hence $x_1 = 1$ and $\lambda_2 = 0$. Also $x_2 + x_3 = 1 - x_1 = 0$. To summarize, all triples (x_1, x_2, x_3) with $x_1 = 1$, $x_2 + x_3 = 0$, and $x_2^2 \leq 2 - x_1^2 = 1$, and with $(\lambda_1, \lambda_2, \lambda_2) = (1, 0, 0)$, satisfy all the necessary conditions, and they all give the same value 0 to the objective function $\ln x_1 + x_2 + x_3$. So there are infinitely many solutions. (Using Theorem 18.4 in Section 18.10, one can show that all are optimal.)

18.9

1. (a) $1 - x^2 - y^2 \leq 1$ for all $x \geq 0$, $y \geq 0$, so the optimal solution must be $x = y = 0$. (b) With $\mathcal{L} = 1 - x^2 - y^2 - \lambda(-x) - \mu(-y)$, the Kuhn–Tucker conditions are [1] $\partial\mathcal{L}/\partial x = -2x + \lambda = 0$; [2] $\partial\mathcal{L}/\partial y = -2y + \mu = 0$;

[3] $\lambda \geq 0$ ($= 0$ if $x > 0$); and [4] $\mu \geq 0$ ($= 0$ if $y > 0$). From [1] and [2], $\lambda = 2x$ and $\mu = 2y$. If $\lambda > 0$, then [3] implies $x = 0$, which contradicts $\lambda = 2x$. Therefore, $\lambda = x = 0$. Similarly, $\mu = y = 0$.

3. A feasible triple (x_1^0, x_2^0, k^0) solves the problem iff there exist numbers λ and μ such that

$$1 - 2x_1^0 - \lambda \leq 0 \ (= 0 \text{ if } x_1^0 > 0)$$
$$3 - 2x_2^0 - \mu \leq 0 \ (= 0 \text{ if } x_2^0 > 0)$$
$$-2k^0 + \lambda + \mu \leq 0 \ (= 0 \text{ if } k^0 > 0)$$
$$\lambda \geq 0 \ (= 0 \text{ if } x_1^0 < k^0)$$
$$\mu \geq 0 \ (= 0 \text{ if } x_2^0 < k^0)$$

If $k^0 = 0$, then feasibility requires $x_1^0 = 0$ and $x_2^0 = 0$, and so the first and second statements in the display imply that $\lambda \geq 1$ and $\mu \geq 3$, which contradicts the third statement. Thus, $k^0 > 0$. Next, if $\mu = 0$, then $x_2^0 = 3/2$ and $\lambda = 2k^0 > 0$. So $x_1^0 = k^0 = 1/4$, contradicting $x_2^0 \leq k^0$. So $\mu > 0$, which implies that $x_2^0 = k^0$. Now, if $x_1^0 = 0 < k^0$, then $\lambda = 0$, which contradicts the first statement in the display. So $0 < x_1^0 = \frac{1}{2}(1 - \lambda)$. Next, if $\lambda > 0$, then $x_1^0 = k^0 = x_2^0 = \frac{1}{2}(1 - \lambda) = \frac{1}{2}(3 - \mu) = \frac{1}{2}(\lambda + \mu)$. But the last two equalities are only satisfied when $\lambda = -1/3$ and $\mu = 5/3$, which contradicts $\lambda \geq 0$. So $\lambda = 0$ after all, with $x_2^0 = k^0 > 0$, $\mu > 0$, $x_1^0 = \frac{1}{2}(1 - \lambda) = \frac{1}{2}$. Now, from the third equation in the display, it follows that $\mu = 2k^0$ and so, from the second equation, that $3 = 2x_2^0 + \mu = 4k^0$. The only possible solution is, therefore, $(x_1^0, x_2^0, k^0) = (1/2, 3/4, 3/4)$, with $\lambda = 0$ and $\mu = 3/2$.

5. The proof is quite similar to the proof of property 3.

18.10

1. (a) Use Theorem 17.9 in Section 17.8. (b) The Kuhn–Tucker conditions are: [1] $-2(x - 1) - 2\lambda x = 0$; [2] $-2ye^{y^2} - 2\lambda y = 0$; [3] $\lambda \geq 0$ ($= 0$ if $x^2 + y^2 < a$). So $x = (1 + \lambda)^{-1}$ and $y = 0$. The constraint $x^2 + y^2 \leq a$ implies $(1 + \lambda)^{-2} \leq a$, or $1 + \lambda \geq a^{-1/2}$. For $0 < a < 1$, the solution is $x = \sqrt{a}$, $y = 0$, and $\lambda = a^{-1/2} - 1$; for $a \geq 1$, it is $x = 1$, $y = 0$, and $\lambda = 0$. Because $x^2 + y^2$ is convex, this is the optimum.

3. Consider the problem: max $U(x, y) = x^\alpha y^\beta$ s.t. $g(x, y) = px + y \leq m$. Here $(x_0, y_0) = \big(\alpha m/p(\alpha + \beta), \beta m/(\alpha + \beta)\big)$, with $\lambda = (\alpha/p)^\alpha \beta^\beta [m/(\alpha + \beta)]^{\alpha+\beta-1}$ is feasible and satisfies (a) and (b) in Theorem 18.4. Moreover, $U(x, y)$ is quasi-concave and $g(x, y)$ is linear, so condition (c) in Theorem 18.5 is satisfied. Also, $(U_1'(x_0, y_0), U_2'(x_0, y_0)) = \alpha^\alpha \beta^\beta [m/(\alpha + \beta)]^{\alpha+\beta-1}(p^{1-\alpha}, p^{-\alpha}) \neq (0, 0)$. Hence, (x_0, y_0) solves the problem. When

$\alpha = \beta = 1$ and $p = 2$, the pair $(x_0, y_0) = (m/4, m/2)$, with $\lambda = m/4$, solves the problem in Example 18.1 (see also Example 18.3 in Section 18.2).

5. (a) The solution is $x = 2$. (b) Condition (b) is not satisfied. $x^0 = 1$ is not an optimum.

Chapter 19

19.1

1. (a) From Fig. 51, we see that the solution is at the intersection P of the two lines $3x_1 + 2x_2 = 6$ and $x_1 + 4x_2 = 4$. Solution: max $= 36/5$ for $(x_1, x_2) = (8/5, 3/5)$. (b) From Fig. 52, we see that the solution is at the intersection P of the two lines $u_1 + 3u_2 = 11$ and $2u_1 + 5u_2 = 20$. Solution: min $= 104$ for $(u_1, u_2) = (5, 2)$.
(c) From a graph it can be seen that the solution is at the intersection of the lines $-2x_1 + 3x_2 = 6$ and $x_1 + x_2 = 5$. Hence max $= 98/5$ for $(x_1, x_2) = (9/5, 16/5)$. (d) max $= 49$ for $(x_1, x_2) = (5, 1)$ (e) max $= -10/3$ for $(x_1, x_2) = (2, 2/3)$

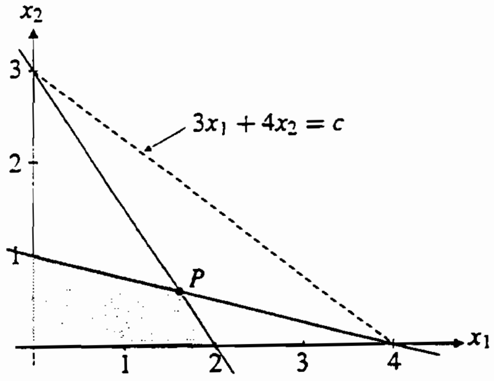


FIGURE 51

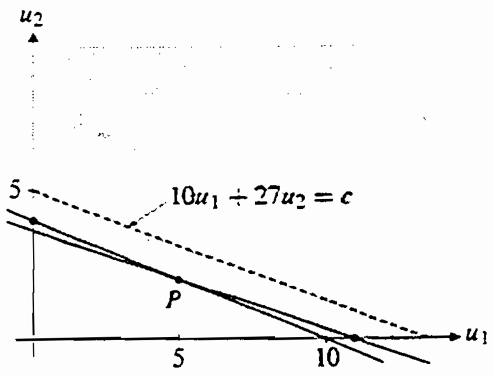


FIGURE 52

3. (a) max $= 18/5$ for $(x_1, x_2) = (4/5, 18/5)$.
(b) max $= 8$ for $(x_1, x_2) = (8, 0)$.
(c) max $= 24$ for $(x_1, x_2) = (8, 0)$.
(d) min $= -28/5$ for $(x_1, x_2) = (4/5, 18/5)$.
(e) max $= 16$ for all (x_1, x_2) of the form $(x_1, 4 - \frac{1}{2}x_1)$ where $x_1 \in [4/5, 8]$.
(f) min $= -24$ for $(x_1, x_2) = (8, 0)$ (follows from the answer to (c)).

5. The slope of the line $20x_1 + tx_2 = c$ must lie between $-1/2$ (the slope of the flour border) and -1 (the slope of the butter border). For $t = 0$, the line is vertical and the solution is the point D in Fig. 19.2. For $t \neq 0$, the slope of the line is $-20/t$. Thus, $-1 \leq -20/t \leq -1/2$, which implies that $t \in [20, 40]$.

19.2

1. (a) $(x_1, x_2) = (2, 1/2)$ and $u_1^* = 4/5$ (b) $(x_1, x_2) = (7/5, 9/10)$ and $u_2^* = 3/5$ (c) Multiplying the two $\leq$ constraints by 4/5 and 3/5, respectively, and then adding yields $(4/5)(3x_1 + 2x_2) + (3/5)(x_1 + 4x_2) \leq 6 \cdot 4/5 + 4 \cdot 3/5$, which reduces to $3x_1 + 4x_2 \leq 36/5$.

3. $\min 8u_1 + 13u_2 + 6u_3$ subject to $\begin{cases} u_1 + 2u_2 + u_3 \geq 8 \\ 2u_1 + 3u_2 + u_3 \geq 9 \end{cases}$ $u_1 \geq 0, u_2 \geq 0, u_3 \geq 0$

19.3

1. (a) $x = 0$ and $y = 3$ gives max $= 21$. See Fig. 53, where the optimum is P. (b) $\min 20u_1 + 21u_2$ subject to $\begin{cases} 4u_1 + 3u_2 \geq 2 \\ 5u_1 + 7u_2 \geq 7 \end{cases}$ $u_1 \geq 0,\ u_2 \geq 0$ has the solution $u_1 = 0$ and $u_2 = 1$, which gives min $= 21$. See Fig. 54. (c) Yes.

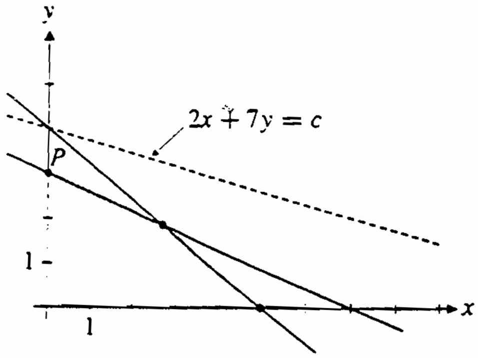


FIGURE 53

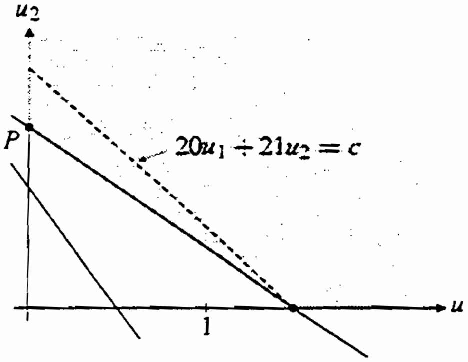


FIGURE 54

3. (a) The profit from selling x_1 small and x_2 medium television sets is $400x_1 + 500x_2$. The first constraint, $2x_1 + x_2 \leq 16$, says that we cannot use more hours in Division 1 than the hours available. The second constraint, $x_1 + 4x_2 \leq 16$, says that we cannot use more hours in Division 2 than the hours available. The third constraint, $x_1 + 2x_2 \leq 11$, says that we cannot use more hours in Division 3 than the hours available. (b) The solution is max $= 3800$ for $x_1 = 7$ and $x_2 = 2$. (Supply your own graph.) (c) Relaxing the first constraint to $2x_1 + x_2 \leq 17$ allows the new solution $x_1 = 23/3$, $x_2 = 5/3$, with an extra profit of 100. Relaxing the second constraint to $x_1 + 4x_2 \leq 17$ makes no difference, because some capacity in division 2 remainded unused anyway. Relaxing the third constraint to $x_1 + 2x_2 \leq 12$ allows the solution $x_1 = 20/3$, $x_2 = 8/3$, with an extra profit of 200. So division 3 should be the first to have its capacity increased.

19.4

1. According to formula [19.12], $\Delta z^* = u_1^* \Delta b_1 + u_2^* \Delta b_2 = 0 \cdot 0.1 + 1 \cdot (-0.2) = -0.2$.

19.5

1. $4u_1^* + 3u_2^* = 3 > 2$ and $x^* = 0$; $5u_1^* + 7u_2^* = 7$ and $y^* = 3 > 0$. Also $4x^* + 5y^* = 15 < 20$ and $u_1^* = 0$; $3x^* + 7y^* = 21$ and $u_2^* = 1 > 0$. So we see that [19.13] and [19.14] are satisfied.

3. (a) min $10{,}000y_1 + 8000y_2 + 11{,}000y_3$ s.t. $\begin{cases} 10y_1 + 20y_2 + 20y_3 \geq 300 \\ 20y_1 + 10y_2 + 20y_3 \geq 500 \end{cases}$
$y_1 \geq 0$, $y_2 \geq 0$, $y_3 \geq 0$.
(b) Solution of the dual: max $= 255{,}000$ for $x_1 = 100$ and $x_2 = 450$. Solution of the primal: min $= 255{,}000$ for $(y_1, y_2, y_3) = (20, 0, 5)$.
(c) The minimum cost will increase by 2000.

5. (a) For $x_3 = 0$, the solution is $x_1 = x_2 = 1/3$. For $x_3 = 3$, the solution is $x_1 = 1$ and $x_2 = 2$. (b) If $0 \leq x_3 \leq 7/3$, then $z_{\max}(x_3) = 2x_3 + 5/3$ for $x_1 = 1/3$ and $x_2 = x_3 + 1/3$. If $7/3 < x_3 \leq 5$, then $z_{\max}(x_3) = x_3 + 4$ for $x_1 = x_3 - 2$ and $x_2 = 5 - x_3$. If $x_3 > 5$, then $z_{\max}(x_3) = 9$ for $x_1 = 3$ and $x_2 = 0$. (c) The solution to the original problem is $x_1 = 3$ and $x_2 = 0$, with x_3 as an arbitrary number ≥ 5.

Chapter 20

20.1

1. (a) According to [20.4], $x_t = 2^t[1 - 4/(1-2)] + 4/(1-2) = 5 \cdot 2^t - 4$
(b) $x_t = (1/3)^t + 1$ (c) $x_t = (-3/5)(-3/2)^t - 2/5$ (d) $x_t = -3t + 3$

3. (a) $x_1 = ax_0 + A_1$, $x_2 = ax_1 + A_2 = a(ax_0 + A_1) + A_2 = a^2x_0 + (aA_1 + A_2)$, and so on. Formally, an induction proof is required. (b) If $A_t = bc^t$, then $x_t = a^t x_0 + bc(a^{t-1} + a^{t-2}c + \cdots + c^{t-1})$. So $x_t = a^t x_0 + \dfrac{c^t - a^t}{c - a} b$ for $a \neq c$, whereas $x_t = a^t x_0 + ta^t b$ for $a = c$.

5. $x_1 = \sqrt{x_0 - 1} = \sqrt{5 - 1} = 2$, $x_2 = \sqrt{x_1 - 1} = \sqrt{2 - 1} = 1$, and $x_3 = \sqrt{1 - 1} = \sqrt{0} = 0$. Then $x_4 = \sqrt{0 - 1} = \sqrt{-1}$, which is not a real number.

20.2

1. $w_t = 1250(1.2)^t - 250$

3. (a) Let the remaining principal after January 1 of year n be L_n. Then $L_0 = L$. Because the principal repayment in year n is $L_{n-1} - L_n$ and the interest payment is rL_{n-1}, one has $L_{n-1} - L_n = \frac{1}{2}rL_{n-1}$, $n = 1, 2, \ldots$, whose solution is $L_n = (1 - \frac{1}{2}r)^n L$. (b) $(1 - r/2)^{10}L = (1/2)L$ implies that $r = 2 - 2 \cdot 2^{-1/10} \approx 0.133934$. (c) The payment in the nth year will be $I_{n-1} - I_n + rI_{n-1}(3/2)r(1 - \frac{1}{2}r)^{n-1}L$. The loan will never be totally repaid, even though $L_n \to 0$ as $n \to \infty$.

20.3

1. Using the notation defined by [3] and [5], the repayment each period is

$$z = \frac{\prod_{s=1}^{T}(1+r_s)B}{\sum_{k=1}^{T}\left(\prod_{s=k+1}^{T}(1+r_s)\right)}$$

which can also be written as

$$z = \frac{B}{\dfrac{1}{1+r_1} + \dfrac{1}{(1+r_1)(1+r_2)} + \cdots + \dfrac{1}{(1+r_1)(1+r_2)\cdots(1+r_T)}}$$

We see that z increases as any r_k increases. If interest rates rise enough without any increase in repayments, the debt will never be paid off, and could even go on increasing indefinitely. This happens when $z_t < r_t b_t$, where b_t denotes the outstanding balance at time t.

20.4

1. (a) If $x_t = A + 2t$, then $x_{t+1} = A + 2(t+1) = A + 2t + 2$ and $x_t + 2 = (A+2t) + 2 = A + 2t + 2$, so $x_{t+1} = x_t + 2$. (b) If $x_t = A\,3^t + B\,4^t$, then $x_{t+2} - 7x_{t+1} + 12x_t = A\,3^{t+2} + B\,4^{t+2} - 7(A\,3^{t+1} + B\,4^{t+1}) + 12(A\,3^t + B\,4^t) = A\,3^t \cdot 3^2 + B\,4^t \cdot 4^2 - 7A\,3^t \cdot 3 - 7B\,4^t \cdot 4 + 12A\,3^t + 12B\,4^t = A\,3^t(9 - 21 + 12) + B\,4^t(16 - 28 + 12) = 0$

3. $x_t = A\,3^t + B\,4^t$ *is* a solution. Substituting $t = 0$ and $t = 1$ yields $A + B = x_0$ and $3A + 4B = x_1$, with solution $A = 4x_0 - x_1$ and $B = -3x_0 + x_1$. So $x_t = A\,3^t + B\,4^t$ is the general solution of the given equation.

5. (a) The homogeneous equation is $x_{t+1} = b^t x_t$, where $b = e^{-2a}$. Now, $x_1 = b^0 x_0 = x_0$, $x_2 = bx_1 = bx_0$, $x_3 = b^2 x_2 = b^2 b x_0 = b^{2+1}x_0$, $x_4 = b^3 x_3 = b^3 b^{2+1} x_0 = b^{3+2+1}$, and so on. In general, $x_t = b^{(t-1)+(t-2)+\cdots+3+2+1}x_0$. Now $(t-1) + (t-2) + \cdots + 3 + 2 + 1 = \frac{1}{2}(t-1)t$ (see Equation [B.5] in Section B.2), so $\tilde{x}_t = \tilde{x}_0 e^{-at(t-1)}$.
(b) $u^*_{t+1} - e^{-2at}u^*_t = [e^{-at^2} - e^{-2at}e^{-a(t-1)^2}]/(1 - e^{-a}) = e^{-at^2}$. The general solution is $x_t = \tilde{x}_t + u^*_t$ for arbitrary $\tilde{x}_0$.

20.5

1. (a) $x_t = C_1 2^t + C_2 4^t$ (b) $x_t = C_1 4^t + C_2 t 4^t$ (c) $x_t = C_1\sqrt{3}^t\cos\theta t + C_2\sqrt{3}^t\sin\theta t$, where $\cos\theta = -\sqrt{3}/3$ and $\sin\theta = \sqrt{6}/3$.
(d) $x_t = C_1(2/3)^{t/2}\cos\pi t/2 + C_2(2/3)^{t/2}\sin\pi t/2 + 4/5$

3. (a) $Y^*_t = b/(1-a)$ (b) $m^2 - a(1+c)m + ac = 0$. Two different real roots, a multiple real root, or two complex roots according as $a(1+c)^2 - 4c > 0$, $= 0$, or < 0.

5. If $a \neq -2$, then $D = c/(a+2)$. If $a = -2$, then $D = c/(a+4) = \frac{1}{2}c$.

7. (a) Stable. (b) Not stable. (c) Stable. (d) Not stable.

9. (a) $Y_t^* = \dfrac{a(1+g)^t}{(1+g)^2 - b(1+g) - kg}$ (when the denominator is not $\neq 0$).
(b) $(b+k)^2 < 4k$ (c) $r = \sqrt{k}$. Damped oscillations when $k < 1$.

11. (a) $\sigma\beta < 4\alpha(1-\alpha)$ (b) $\sigma\beta < 4\alpha$ and $\alpha < 1$.

Chapter 21

21.1

1. If $x(t) = Ce^{-t} + \frac{1}{2}e^t$, then $\dot{x}(t) + x(t) = -Ce^{-t} + \frac{1}{2}e^t + Ce^{-t} + \frac{1}{2}e^t = \frac{1}{2}e^t + \frac{1}{2}e^t = e^t$.

3. Differentiate $xe^{tx} = C$ implicitly to obtain $\dot{x}e^{tx} + x[e^{tx}(x + t\dot{x})] = 0$. Canceling e^{tx} and rearranging gives $(1 + tx)\dot{x} = -x^2$.

5. If $x = Ct - C^2$, then $\dot{x} = C$, so $\dot{x}^2 = C^2$ and $t\dot{x} - x = tC - Ct + C^2 = C^2$. If $x = \frac{1}{4}t^2$, then $\dot{x} = \frac{1}{2}t$, so $\dot{x}^2 = \frac{1}{4}t^2$, and $t\dot{x} - x = \frac{1}{4}t^2$. We conclude that $x = Ct - C^2$ is not the general solution.

21.2

1. The solutions are $x = Ct$, for $t \neq 0$, with C an arbitrary constant. See Fig. 55.

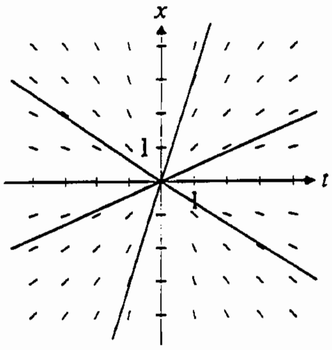


FIGURE 55

21.3

1. Separate: $x^2\,dx = (t+1)\,dt$. Integration yields $\frac{1}{3}x^3 = \frac{1}{2}t^2 + t + C$, or $x = \sqrt[3]{\frac{3}{2}t^2 + 3t + 3C}$. For $C = -7/6$, the integral curve passes through $(1, 1)$.

3. (a) $x = Ce^{t/2}$. For $C = 1$, the integral curve passes through (0, 1).
(b) $x = Ce^{at}$. For $C = x_0e^{-at_0}$, the integral curve passes through (t_0, x_0).

5. $x = Ce^{-\int a(t)\,dt}$. If $a(t) = a + bc^t$, then $\int a(t)\,dt = at + (b/\ln c)c^t$. This implies that $x = Ce^{-at}e^{(-b/\ln c)c^t} = C(e^{-a})^t(e^{-b/\ln c})^{c^t} = Cp^tq^{c^t}$, with $p = e^{-a}$ and $q = e^{-b/\ln c}$.

7. (a) $K = \left[\dfrac{An_0^\alpha a^b}{\alpha v + \varepsilon}(1 - b + c)e^{(\alpha v + \varepsilon)t} + C\right]^{1/(1-b+c)}$
(b) $|\alpha x - \beta|^{(\beta/\alpha)}|x - a|^{-a} = Ce^{(\alpha a - \beta)t}$

21.4

1. (a) $x = Cte^{-t}$; $C = 1$. (Separate: $dx/x = [(1/t) - 1]\,dt$. Integrate: $\ln|x| = \ln|t| - t + C_1$. Hence, $|x| = e^{\ln|t| - t + C_1} = e^{\ln|t|}e^{-t}e^{C_1} = C_2|t|e^{-t} = Cte^{-t}$, where $C = \pm C_2 = \pm e^{C_1}$.) (b) $x = C\sqrt[3]{1 + t^3}$; $C = 2$.

3. The equation is equivalent to $f'(t) = (-r/K)(f(t) - 0)(f(t) - K)$, so formula [4] in Example 21.10 yields $f(t) = K/(1 - Ce^{-rt})$.

5. (a) $\dfrac{K}{L} = \left[\dfrac{K_0^\alpha}{L_0^\alpha e^{\alpha\lambda t}} + (sA/\lambda)(1 - e^{-\alpha\lambda t})\right]^{1/\alpha} \to (sA/\lambda)^{1/\alpha}$ as $t \to \infty$. Also $X/L = A(K/L)^{1-\alpha} \to A(sA/\lambda)^{(1-\alpha)/\alpha}$.
(b) $\dot K = sAb^\alpha(t + a)^{p\alpha}K^{1-\alpha}$, so $K(t) = \{K_0^\alpha + s\alpha Ab^\alpha\left[(t + a)^{p\alpha+1} - a^{p\alpha+1}\right]/(p\alpha+1)\}^{1/\alpha}$. Hence, $\dfrac{K}{L} = \left[\dfrac{K_0^\alpha + s\alpha Ab^\alpha\left[(t + a)^{p\alpha+1} - a^{p\alpha+1}\right]}{(p\alpha + 1)b^\alpha(t + a)^{p\alpha}}\right]^{1/\alpha}$ $\to \infty$ as $t \to \infty$.

21.5

1. Applying (21.7) with $a = -1$ and $b(t) = t$ yields $x = e^{-(-t)}[C + \int e^{-t}\,t\,dt] = e^t[C + \int t\,e^{-t}\,dt]$. Integrating by parts, $\int t\,e^{-t}\,dt = -te^{-t} + \int e^{-t}\,dt = -te^{-t} - e^{-t}$, and so the solution is $x = Ce^t - t - 1$.

3. (a) Since $C = aY + b$ and $I = k\dot C = ka\dot Y$, equation (1) implies that $Y = C + I = aY + b + ka\dot Y$. Solving for $\dot Y$ yields the specified equation.
(b) $Y(t) = [Y_0 - b/(1 - a)]e^{(1-a)t/ka} + b/(1 - a)$ and $I(t) = (1 - a)Y(t) - b$
(c) $1/(1 - a)$

5. (a) $x = Ce^{3t} - 5/3$. For $C = 8/3$, the integral curve passes through (0, 1).
(b) $x = Ce^{-2t/3} - 8$. For $C = 9$, the integral curve passes through (0, 1).
(c) $x = Ce^{-2t} + \frac{1}{2}t^2 - \frac{1}{2}t + \frac{1}{4}$. For $C = 3/4$, the integral curve passes through (0, 1).

7. (a) $x(t) = X(t)/N(t)$ increases with t if $\alpha\sigma \geq \rho$. When $\sigma = 0.3$ and $\rho = 0.03$, this implies that $\alpha \geq 0.1 (= 10\%)$. (b) It is enough to note that $(1 - e^{-\xi t})/\xi > 0$ whenever $\xi \neq 0$, then apply this with $\xi = \alpha\sigma - \mu$. Faster

growth per head is to be expected because, in this model, foreign aid contributes positively.
(c) Using Equation [6], note that

$$x(t) = \left[x(0) + \left(\frac{\sigma}{\alpha\sigma - \mu}\right)\frac{H_0}{N_0}\right] e^{-(\rho-\alpha\sigma)t} + \left(\frac{\sigma}{\mu - \alpha\sigma}\right)\frac{H_0}{N_0}e^{(\mu-\rho)t}$$

Even if $\alpha\sigma < \rho$, this is positive and growing for large t as long as $\mu > \rho$. So foreign aid must grow faster than the population.

21.6

1. (a) $\dot{x} + (2/t)x = -1$. Apply [21.11] with $a(t) = 2/t$ and $b = -1$. Then $\int a(t)\,dt = \int (2/t)\,dt = 2\ln|t| = \ln|t|^2 = \ln t^2$ and so $\exp(\int a(t)\,dt) = \exp(\ln t^2) = t^2$. Then, $x = (1/t^2)[C + \int t^2(-1)\,dt] = Ct^{-2} - \frac{1}{3}t$. (b) Here $\int a(t)\,dt = -\int (1/t)\,dt = -\ln t$, and [21.11] yields the solution $x = Ct + t^2$. (c) In this case, $\int a(t)\,dt = -\frac{1}{2}\ln(t^2 - 1)$, and [21.11] yields the solution $x = C\sqrt{t^2 - 1} + t^2 - 1$. (d) $x = Ct^2 + 2a^2/3t$

3. $p = Ce^{1/t} + e^{1/t}\int t^{-3}e^{-1/t}\,dt = Ce^{1/t} + 1/t + 1$, after integrating by parts. For $p(1) = 0$ one needs $C = -2/e$.

5. (a) Putting $z = x^{1-3} = x^{-2}$ gives the linear equation $\dot{z} = 2tz - 2t^3$ whose solution is $z = e^{t^2}(C - \int 2t^3 e^{-t^2}\,dt)$. After integrating by parts, this gives $x^{-2} = Ce^{t^2} + t^2 + 1$. (b) Putting $z = x^{-1}$ leads to $\dot{z} = 2(z/t) - 1$ whose solution implies that $x = 1/t(1 + Ct)$. (c) $x = (Ce^{2t} - e^t)^2$

7. If $x = u + 1/z$, then $\dot{x} = \dot{u} - \dot{z}/z^2$ and the Riccati equation is converted into the linear form $\dot{z} + [Q(t) + 2u(t)R(t)]z = -R(t)$ in case $u = u(t)$ is a special solution. For the equation $t\dot{x} = x - (x - t)^2$ and the special solution $u = t$, the transformation $x = t + 1/z$ creates the linear equation $t\dot{z} + z = 1$, whose solution is $tz = C + t$. Thus $x = t + t/(t + C)$ is the general solution.

21.7

1. (a) $x = 1$ is unstable. See Fig. 56. (b) $x = 12$ is stable. See Fig. 57. (c) $x = -3$ is stable; $x = 3$ is unstable. See Fig. 58.

FIGURE 56

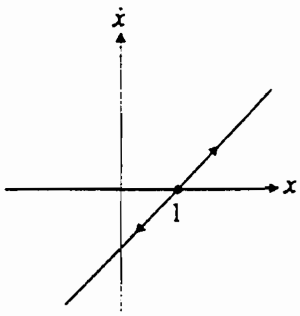


FIGURE 57

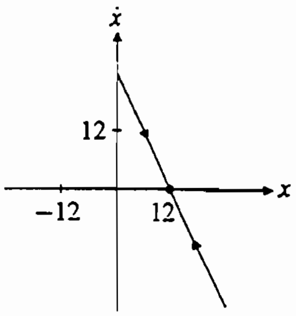
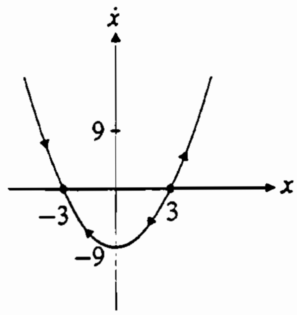


FIGURE 58



3. Note how $F(K, L) = AK^{1-\alpha}L^{\alpha}$ implies $f(k) = Ak^{1-\alpha}$. Then $sf(k) = \lambda k$ at $k^* = (sA/\lambda)^{1/\alpha}$, the same value as the limit of K/L in the cited problem.

5. (a) $\partial k^*/\partial s = f(k^*)/[\lambda - sf'(k^*)] > 0$ and $\partial k^*/\partial\lambda = -k^*/[\lambda - sf'(k^*)] < 0$ when $\lambda > sf'(k^*)$. At the equilibrium state, capital per worker increases as the savings rate increases, and decreases as the growth rate of the work force increases.
(b) From Equations [1] to [4], $c = (X - \dot{K})/L = (1-s)X/L = (1-s)f(k)$. But $sf(k^*) = \lambda k^*$, so when $k = k^*$ we have $c = f(k^*) - \lambda k^*$. The necessary first-order condition for this to be maximized w.r.t. k^* is that $f'(k^*) = \lambda$. But $F(K, L) = Lf(k)$ and so $F'_K = Lf'(k)dk/dK = f'(k)$ because $k = K/L$ with L fixed. Thus $\partial F/\partial K = \lambda$. (c) $0 = \dot{k}/k = \dot{K}/K - \dot{L}/L = \dot{K}/K - \lambda$ in the stationary state.

21.8

1. (a) $x = \frac{1}{6}t^3 + At + B$ (b) $x = -\sin t + At + B$ (c) $x = e^t + \frac{1}{12}t^4 + At + B$

3. (a) Direct verification. General solution: $x = Ae^{2t} + Be^{-3t}$
(b) $x = Ae^{2t} + Be^{-3t} - t - 1/6$

5. Substituting $x = (t + k)^{-1}$ eventually yields $k = a$ and $k = b$. General solution: $x = A(t + a)^{-1} + B(t + b)^{-1}$.

21.9

1. (a) $x = C_1e^{\sqrt{3}t} + C_2e^{-\sqrt{3}t}$; unstable. (b) $x = e^{-2t}(C_1 \cos 2t + C_2 \sin 2t)$; stable. (c) $x = C_1 + C_2e^{-8t/3}$; stable. (d) $x = e^{-t/2}(C_1 + C_2t)$; stable. (e) $x = C_1e^{-3t} + C_2e^{2t} - 4/3$; unstable. (f) $x = C_1e^{-t} + C_2e^{-2t} + (1/42)e^{5t}$; stable.

3. (a) $x = -(6 + t)e^{-t} + t^2 - 4t + 6$
(b) $x = \frac{1}{2}\sin 2t + (\pi/2 + 1/4)\cos 2t + t + 1/4$

5. $u^* = kt + L_0 + [\beta + \alpha(1-\beta)]k/\delta^*$ is a particular solution. Oscillations occur iff $(\gamma^2/4)[\beta + \alpha(1-\beta)]^2 + \gamma\delta^* < 0$.

7. By Problem 3 of Sec. C.1, setting $C = A\cos B$ and $D = -A\sin B$ gives $A\cos(\beta t + B) = A\cos\beta t \cos B - A\sin\beta t \sin B = C\cos\beta t + D\sin\beta t$.

9. $\ddot{p} + \lambda^2 p = a(d_0 - s_0)$. Solution: $p = C_1\cos\lambda t + C_2\sin\lambda t + (d_0 - s_0)/(s_1 - d_1)$, where $\lambda = [a(s_1 - d_1)]^{1/2}$.

Appendix A

Answers are given to all the problems in Appendix A.

A.1

1. (a) 216 (b) 4/9 (c) -1 (d) 0.09 (e) $(2.0)^4 = 16$ (f) $2^6 = 64$
(g) $(2 \cdot 3 \cdot 4)^2 = (24)^2 = 576$ (h) $6^6 = 46656$

2. (a) 15^3 (b) $(-1/3)^3$ (c) 10^{-1} (d) 10^{-7} (e) t^6 (f) $(a-b)^3$ (g) a^2b^4 (h) $(-a)^3$

3. (a) a^6 (b) a^8 (c) x^3 (d) b^{-3} (e) x^6y^9 (f) x^2 (g) z^{-2} (h) $3^{-6} = 1/729$

4. (a) $2^6 = 64$ (b) $64/27$ (c) $8/3$ (d) x^9 (e) y^{12} (f) $8x^3y^3$ (g) $10^{-2} = 1/100$ (h) k^4 (i) $(x+1)^2$

5. (a) 0 (b) Undefined. (c) 0 (d) 0 (e) 1 (f) Undefined. (g) 1 (h) 1

6. (a) $x = 5$ (b) $x = 0$ (c) $x = 3$ (d) $x = 4$ (e) $x = 8$ (f) $x = 0$

7. (a) False. $3^5 = 243$, $5^3 = 125$ (b) False. $(5^2)^3 = 5^6$, whereas $5^{2^3} = 5^8$. (c) True. $(a^p)^q = (a^q)^p$ (d) True. $0^3 \cdot 4^0 = 0 \cdot 1 = 0$ (e) False. 0^{-2} is not defined. (f) False. $(5+7)^2 = (12)^2 = 144$, $5^2 + 7^2 = 25 + 49 = 74$ (g) False. The correct value of the ratio is $(2x+4)/2 = x+2$. (h) True. Both equal $2x - 2y$. (i) True.

8. (a) False. $a^0 = 1$. (b) True. $c^{-n} = 1/c^n$ for all $c \neq 0$. (c) True. $a^m \cdot a^m = a^{m+m} = a^{2m}$. (d) False, unless $m = 0$. $a^m b^m = (ab)^m$. (e) False (unless $m = 1$). For example, $(a+b)^2$ is equal to $a^2 + 2ab + b^2$. (f) False (unless $a^m b^n = 1$). For example, a^2b^3 is not equal to $(ab)^{2+3} = (ab)^6 = a^6b^6$.

9. (a) $x^3y^3 = (xy)^3 = 3^3 = 27$ (b) $(ab)^4 = (-2)^4 = 16$ (c) $(a^{20})^0 = 1$, for all $a \neq 0$. (d) $2n$ is $0, \pm 2, \pm 4, \ldots$, so $(-1)^{2n} = [(-1)^2]^n = 1^n = 1$ (e) $x^3y^3 = (x^{-1}y^{-1})^{-3} = 3^{-3} = 1/27$ (f) $(x^{-3})^6(x^2)^2 = x^{-18}x^4 = x^{-14} = (x^7)^{-2} = 2^{-2} = 1/4$ (g) $(z/xy)^6 = (xy/z)^{-6} = [(xy/z)^{-2}]^3 = 3^3 = 27$ (h) $(abc)^4 = (a^{-1}b^{-1}c^{-1})^{-4} = (1/4)^{-4} = 4^4 = 256$

10. (a) $16x^4$ (b) 4 (c) $6xyz$ (d) $a^{27}b^9$ (e) a^{3^k} (f) x^{-15} (g) a^4 (h) 5^{-q}

11. (a) 19.5 (b) 144 (c) 11

12. 15%

13. (a) Given a constant interest rate of 11% per year, then in 8 years, an initial investment of 50 francs will be worth $50 \cdot (1.11)^8 \approx 115.23$ francs. (b) Given a constant interest rate of 12% per year, then in 20 years, an initial investment of 10,000 rand will be worth $10{,}000 \cdot (1.12)^{20} \approx 96{,}462.93$ rand. (c) $5000 \cdot (1.07)^{-10} \approx 2541.75$ crowns is what you should have deposited 10 years ago in order to have 5000 crowns today, given the constant interest rate of 7%.

14. $2^{10} = 1024$ and $10^3 = 1000$. So $2^{30} = (2^{10})^3$ is bigger than $(10^3)^3 = 10^9$. A calculator should say that $2^{30} = (1024)^3 = 1{,}073{,}741{,}824$.

A.2

1. (a) 3 (b) 40 (c) 10 (d) 5 (e) 1/6 (f) 0.7 (g) 1/10 (h) 1/5

2. (a) 81 (b) 4 (c) 623 (d) 15 (e) -1 (f) 3

3. (a) $\frac{6}{7}\sqrt{7}$ (b) 4 (c) $\frac{1}{8}\sqrt{6}$ (d) 1 (e) $\frac{1}{6}\sqrt{6}$ (f) $2\sqrt{2y}/y$ (g) $\sqrt{2x}/2$ (h) $x+\sqrt{x}$

4. (a) =, by [A.6]. Also, both expressions = 20. (b) ≠. In fact, $\sqrt{25+16} = \sqrt{41} \neq 9 = \sqrt{25}+\sqrt{16}$ (c) ≠. In fact, $(a^{1/2}+b^{1/2})^2 = a+2a^{1/2}b^{1/2}+b = a+b$ only when $ab=0$. (d) =. In fact, $(\sqrt{a+b})^{-1} = [(a+b)^{1/2}]^{-1} = (a+b)^{-1/2}$

A.3

1. (a) 1 (b) 6 (c) −18 (d) −18 (e) $3x+12$ (f) $45x-27y$ (g) 3 (h) 0 (i) −1

2. (a) $3a^2-5b$ (b) $-2x^2+3x+4y$ (c) t (d) $2r^3-6r^2s+3rs^2+2s^3$

3. (a) $-3n^2+6n-9$ (b) x^5+x^2 (c) $4n^2-11n+6$ (d) $-18a^3b^3+30a^3b^2$ (e) a^3b-ab^3 (f) $x^3-6x^2y+11xy^2-6y^3$

4. (a) a^2-a (b) $x^2+4x-21$ (c) $-3+3\sqrt{2}$ (d) $3-2\sqrt{2}$ (e) x^3-3x^2+3x-1 (f) $1-b^4$ (g) $1-x^4$ (h) $x^4+4x^3+6x^2+4x+1$

5. (a) $2x$ (b) $a^2-4ab+4b^2$ (c) $\frac{1}{4}x^2-\frac{1}{9}y^2$ (d) $-x^2y-3x-2$ (e) $x^2+(a+b)x+ab$ (f) $x^3-6x^2y+12xy^2-8y^3$

6. (a) $2t^3-5t^2+4t-1$ (b) 4 (c) $x^2+2xy+2xz+y^2+2yz+z^2$ (d) $4xy+4xz$

7. (a) $9x^2+12xy+4y^2$ (b) $5+2\sqrt{6}$ (c) $9u^2-48uv+64v^2$ (d) u^2-25v^2

8. 500 (Note that $(252)^2-(248)^2 = (252+248)(252-248) = 500\cdot 4 = 2000$.)

9. (a) $x^4-2x^2y^2+y^4$ (b) 1/2 (c) $a^2-2ab+b^2+2a-2b+1$ (d) $a-2\sqrt{ab}+b$ (e) 1 (f) $n^4-4n^3+6n^2-4n+1$

10. (a) $acx^2+(ad+bc)x+bd$ (b) $4-t^2$ (c) $a^2+b^2+c^2+2ab+2ac+2bc$ (d) $a^{10}-b^{10}$ (e) $2\sqrt{15}+1$ (f) $u^4-2u^2v^2+v^4$

11. In the first figure, the (big) square has sides of length $a+b$, so its area is $(a+b)^2$. The four rectangular parts have a combined area $a^2+ab+ab+b^2 = a^2+2ab+b^2$. The two ways of computing the area must give the same result, so $(a+b)^2 = a^2+2ab+b^2$. The interpretation of the second figure is similar.

12. $(a^{10}-b^{10})/(a-b) = a^9+a^8b+a^7b^2+a^6b^3+a^5b^4+a^4b^5+a^3b^6+a^2b^7+ab^8+b^9$

A.4

1. (a) $2\cdot 2\cdot 7\cdot a\cdot a\cdot b\cdot b\cdot b$ (b) $2\cdot 2(x+2y-6z)$ (c) $2x(x-3y)$ (d) $2a\cdot a\cdot b\cdot b(3a+2b)$ (e) $7x(x-7y)$ (f) $5x\cdot y\cdot y(1-3x)(1+3x)$ (g) $(4+b)(4-b)$ (h) $3(x+2)(x-2)$

2. (a) $(x-2)(x-2)$ (b) $2\cdot 2ts(t-2s)$ (c) $2\cdot 2(2a+b)(2a+b)$ (d) $5xx(x+\sqrt{2}y)(x-\sqrt{2}y)$

2. (a) $x = 3$ (b) $x = -7$ (c) $x = -28/11$ (d) $x = 5/11$ (e) $x = 1$ (f) $x = 121$

3. (a) $x = 0$ (b) $x = -6$ (c) $x = 5$

4. (a) $2x+5 = x-3$. Solution: $x = -8$. (b) $x+(x+1)+(x+2) = 10+2x$. Solution: $x = 7$, so that the numbers are 7, 8, and 9. (c) If x is Ann's regular hourly wage, then $38x + (48-38)2x = 812$. Solution: $x = \$14$. (d) $15{,}000 \cdot 10\% + x \cdot 12\% = 2100$. Solution: $x = 5000$. (e) $\frac{2}{3}x + \frac{1}{4}x + 1000 = x$. Solution: $x = 12{,}000$.

5. (a) $y = 17/23$ (b) $x = -4$ (c) $z = 4$ (d) $p = 15/16$

6. 10 minutes. (If x is the number of liters per minute from the first hosepipe, the two others give $2x/3$ and $x/3$ liters per minute. The number of minutes needed to fill the pool with all three hosepipes in use is given by the expression $20x/(x + 2x/3 + x/3)$, which is 10.)

A.7

1. (a), (b), (d), (f), and (h) are true; (c), (e), and (g) are false.

2. (a) $x < -9$ (b) Satisfied for all x. (c) $x \le 25/2$ (d) $x \le 19/7$ (e) $t > -1/4$ (f) $x \le -5$ or $x > -4$

3. (a) $-2 < x < 1$ (b) $x < -4$ or $x > 3$ (c) $-5 \le a \le 5$ (d) $-7 < x < -2$ (e) $n \ge 160$ or $n < 0$ (f) $0 \le g \le 2$ (g) $p \ge -1$ and $p \ne 2$ (h) $-4 < n < -10/3$ (i) $-1 < x < 0$ or $0 < x < 1$. (*Hint:* $x^4 - x^2 = x^2(x+1)(x-1)$.)

4. (a) $x > 1$ or $x < -4$ (b) $x > -4$ and $x \ne 1$ (c) $x \le 1$ or $2 \le x \le 3$ (d) $x < 1$ and $x \ne 1/5$ (e) $1/5 < x < 1$ (f) $x < 0$ (g) $-3 < x < -2$ or $x > 0$ (h) $-5 \le x \le 1$ (*Hint:* $x^2 + 4x - 5 = (x+5)(x-1)$.) (i) $-6 \le x \le 0$ or $x \ge 3$. (*Hint:* $-\frac{1}{3}x^3 - x^2 + 6x = -\frac{1}{3}x(x+6)(x-3)$.)

5. (a) $-41/6 < x \le 2/3$ (b) $x < -1/5$ (c) $-1 < x < 0$

6. (a) Yes. (b) No, put $x = 1/2$, for example. (c) No, not for $x \le 0$. (d) Yes, because the inequality is equivalent to $x^2 - 2xy + y^2 \ge 0$, or $(x-y)^2 \ge 0$, and this inequality *is* satisfied for all x and y.

7. (a) $\$120 + 0.167x$ (b) Smallest number of calls: 300. Largest number of calls: 400.

8. (a) Between 39.2°F and 42.8°F (b) Between 2.2°C and 4.4°C, approximately.

A.8

1. (a) $x = 0$ and $x = 15$ (b) $p = \pm 4$ (c) $q = 3$ and $q = -4$ (d) No solution. (e) $x = 0$ and $x = 3$ (f) $x = 2$

2. (a) $x^2 - 5x + 6 = (x-2)(x-3) = 0$ for $x = 2$ and $x = 3$
(b) $y^2 - y - 12 = (y-4)(y+3) = 0$ for $y = 4$ and $y = -3$

(c) No solutions and no factorization.
(d) $-\frac{1}{4}x^2+\frac{1}{2}x+\frac{1}{2} = -\frac{1}{4}[x-(1+\sqrt{3})][x-(1-\sqrt{3})] = 0$ for $x = 1 \pm \sqrt{3}$
(e) $m^2-5m-3 = [m-\frac{1}{2}(5+\sqrt{37})][m-\frac{1}{2}(5-\sqrt{37})] = 0$ for $m = \frac{1}{2}(5\pm\sqrt{37})$
(f) $0.1p^2+p-2.4 = 0.1(p-2)(p+12) = 0$ for $p = 2$ and $p = -12$

3. (a) $r = -13$, $r = 2$ (b) $p = -16$, $p = 1$ (c) $K = 100$, $K = 200$
(d) $r = -\sqrt{3}$, $r = \sqrt{2}$ (e) $x = -0.5$, $x = 0.8$ (f) $p = -1/6$, $p = 1/4$

4. (a) $x = 1$, $x = 2$ (b) $t = \frac{1}{10}(1 \pm \sqrt{61})$ (c) $x = \frac{1}{4}(3 \pm \sqrt{13})$ (d) $x = \frac{1}{3}(-7 \pm \sqrt{5})$ (e) $x = -300$, $x = 100$ (f) $x = \frac{1}{6}(5 \pm \sqrt{13})$

5. (a) The rectangle has sides 5 cm and 15 cm. (If the sides have length x and y, then $2x + 2y = 40$, that is, $x + y = 20$, and $xy = 75$. So x and y are the roots of $r^2 - 20r + 75 = 0$, because the sum of the roots is 20 and the product is 75.)
(b) $x^2 + (x+1)^2 = 13 \Leftrightarrow x = -3$, $x = 2$. Thus, the two natural numbers must be 2 and 3. (c) $x^2 + (x+14)^2 = (34)^2$. The shortest side is x and is 16 cm long. The longest side is 30 cm long. (d) 50 km/h. (If the usual driving speed is x km/h and the usual time spent is t hours, then $xt = 80$ and $(x+10)(t - 16/60) = 80$. From the first equation, $t = 80/x$. Inserting this into the second equation, and then rearranging, we obtain $x^2 + 10x - 3000 = 0$, whose positive solution is $x = 50$.)

6. (a) $x = -2$, $x = 0$, $x = 2$ ($x(x^2-4) = 0$ or $x(x+2)(x-2) = 0$)
(b) $x = -2$, $x = -1$, $x = 1$, $x = 2$ (Let $x^2 = u$.) (c) $x = -1/3$, $x = 1/5$ (Let $z^{-1} = u$.)

7. $4a^2x^2 + 4abx + b^2 = b^2 - 4ac$, that is, $(2ax + b)^2 = b^2 - 4ac$, and so on.

A.9

1. (a) $x = 8$, $y = 3$ (b) $x = 1/2$, $y = 1/3$ (c) $x = 1.1$, $y = -0.3$

2. (a) $K = 2.8$, $L = 5.75$ (b) $p = 2$, $q = 3$ (c) $r = 2.1$, $s = 0.1$

3. (a) 39 and 13 (b) \$120 for a table and \$60 for a chair. (c) 30 of quality A and 20 of quality B. (d) \$8000 at 7.2% and \$2000 at 5% interest.

Appendix B

B.1

1. (a) 55 (b) 585 (c) 36 (d) 22 (e) 20 (f) 73/12

3. (a) $\sum_{k=1}^{n} 4k$ (b) $\sum_{k=1}^{n} k^3$ (c) $\sum_{k=0}^{n} (-1)^k \frac{1}{2k+1}$ (d) $\sum_{k=1}^{n} a_{ik}b_{kj}$

(e) $\sum_{n=1}^{6} 3^n x^n$ (f) $\sum_{j=3}^{p} a_i^j b_{i+j}$ (g) $\sum_{k=0}^{p} a_{i+k}^{k+3} b_{i+k+3}$

(h) $\sum_{k=0}^{3}(81{,}297 + 198k)$

5. (a) $\sum_{i=1}^{5}(x_i - \bar{x}) = \sum_{i=1}^{5} x_i - \sum_{i=1}^{5} \bar{x} = \sum_{i=1}^{5} x_i - 5\bar{x}$
(b) $\sum_{i=1}^{n}(x_i - \bar{x}) = \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} \bar{x} = \sum_{i=1}^{n} x_i - n\bar{x}$

7. (a), (c), (d), and (e) are always true; (b) and (f) are generally not true.

B.2

1. $\sum_{k=1}^{n}(k^2 + 3k + 2) = \sum_{k=1}^{n} k^2 + 3\sum_{k=1}^{n} k + \sum_{k=1}^{n} 2 =$
$\frac{1}{6}n(n+1)(2n+1) + 3[\frac{1}{2}n(n+1)] + 2n = \frac{1}{3}n(n^2 + 6n + 11)$

3. (a) In both sums, all terms cancel pairwise, except $-a_1$, the last term within the first parentheses, and a_9 (or, generally, a_n), the first term within the last parentheses. (b) (i) $1 - (1/51) = 50/51$ (ii) $3^{13} - 3$ (iii) $ar(r^n - 1)$

5. $\sum_{k=0}^{n-1} \frac{n}{x}\left(\frac{kx}{n}\right)^2 = \frac{x}{n}\sum_{k=0}^{n-1} k^2 = \frac{x}{n}\frac{1}{6}(n-1)n[2(n-1)+1] =$
$\frac{1}{6}x(2n^2 - 3n + 1).$

B.3

1. (a) $\sum_{i=1}^{3}\sum_{j=1}^{4} i \cdot 3^j = \sum_{i=1}^{3}(i \cdot 3 + i \cdot 9 + i \cdot 27 + i \cdot 81) = \sum_{i=1}^{3} 120i = 720$
(b) $\sum_{s=0}^{2}\sum_{r=2}^{4}\left(\frac{rs}{r+s}\right)^2 = \sum_{s=0}^{2}\left[\left(\frac{2s}{2+s}\right)^2 + \left(\frac{3s}{3+s}\right)^2 + \left(\frac{4s}{4+s}\right)^2\right]$
$= \left(\frac{2}{3}\right)^2 + \left(\frac{3}{4}\right)^2 + \left(\frac{4}{5}\right)^2 + \left(\frac{4}{4}\right)^2 + \left(\frac{6}{5}\right)^2 + \left(\frac{8}{6}\right)^2 = 5 + \frac{3113}{3600}$
(c) $\frac{1}{2}m(m+1)k\frac{k^n - 1}{k - 1}$

3. (a) The total number of units of good i.
(b) The total number of units of all commodities owned by person j.
(c) The total number of units of goods owned by the group as a whole.

B.4

1. (a) $2^{-21} = 1/2^{21}$ (b) $(360)^3 = 46{,}656{,}000$ (c) 0
(d) $1/(1 + r_1)(1 + r_2)$

3. (a) $\prod_{i=1}^{n} ka_i = (ka_1)(ka_2)\cdots(ka_n) = k^n \prod_{i=1}^{n} a_i$, so the given equality is only satisfied for $n = 1$.
(b) True: $\prod_{i=1}^{n} y_i^3 = y_1^3 y_2^3 \cdots y_n^3 = (y_1 y_2 \cdots y_n)^3 = \left(\prod_{i=1}^{n} y_i\right)^3$

(c) True: $\prod_{i=1}^{n} x_i y_i = (x_1y_1)(x_2y_2)\cdots(x_ny_n) = (x_1x_2\cdots x_n)(y_1y_2\cdots y_n) = \left(\prod_{i=1}^{n} x_i\right)\left(\prod_{i=1}^{n} y_i\right)$.

(d) True: $\prod_{i=1}^{n}\left(\prod_{j=1}^{i} a_{ij}\right) = \left(\prod_{j=1}^{1} a_{1j}\right)\left(\prod_{j=1}^{2} a_{2j}\right)\cdots\left(\prod_{j=1}^{n} a_{nj}\right) = (a_{11})(a_{21}a_{22})\cdots(a_{n1}a_{n2}\cdots a_{nn}) = (a_{11}a_{21}\cdots a_{n1})(a_{22}a_{32}\cdots a_{n2})\cdots a_{nn} = \prod_{i=1}^{n} a_{i1}\prod_{i=2}^{n} a_{i2}\cdots\prod_{i=n}^{n} a_{in} = \prod_{j=1}^{n}\left(\prod_{i=j}^{n} a_{ij}\right)$

B.5

1. For $n = 1$, both sides are 1. Suppose [$*$] is true for $n = k$. Then $1+2+3+\cdots+k+(k+1) = \frac{1}{2}k(k+1)+(k+1) = \frac{1}{2}(k+1)(k+2)$, which is [$*$] for $n = k+1$. Thus, by induction, [$*$] is true for all n.

3. (a) For $n = 1$, both sides are 1/2. Suppose (a) is true for $n = k$. Then

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}$$
$$= \frac{k}{k+1}+\frac{1}{(k+1)(k+2)} = \frac{k(k+2)+1}{(k+1)(k+2)} = \frac{(k+1)^2}{(k+1)(k+2)}$$
$$= \frac{k+1}{k+2}$$ which is [a] for $n = k+1$. Thus, by induction, [a] is true for all n.

(b) For $n = 1$, both sides are 3. Suppose [b] is true for $n = k$. Then $3+3^2+3^3+3^4+\cdots+3^k+3^{k+1} = \frac{1}{2}(3^{k+1}-3)+3^{k+1} = \frac{1}{2}(3^{k+2}-3)$, which is [b] for $n = k+1$. Thus, by induction, [b] is true for all n.

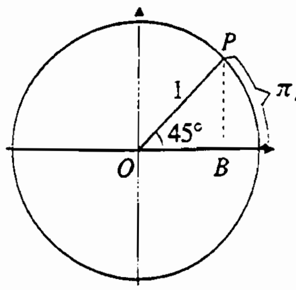
5. For $n = 1$, both sides are a. Suppose the formula is true for $n = m$. Then $a+ak+\cdots+ak^{m-1}+ak^m = a\dfrac{1-k^m}{1-k}+ak^m = (a)\dfrac{1-k^{m+1}}{1-k}$ which is the given formula for $n = k+1$. By induction, the formula is true for all n.

Appendix C

C.1

1. See Fig. 59. $OB = BP = \frac{1}{2}\sqrt{2}$, by Pythagoras's Theorem. Hence, $\sin 45° = BP/OP = \frac{1}{2}\sqrt{2} = \cos 45°$, whereas $\tan 45° = \sin 45°/\cos 45° = 1$.

FIGURE 59



3. $\cos(x+y) = \cos[x-(-y)] = \cos x \cos(-y) + \sin x \sin(-y) = \cos x \cos y - \sin x \sin y$

5. $\tan(x+\pi) = \dfrac{\sin(x+\pi)}{\cos(x+\pi)} = \dfrac{\sin x \cos\pi + \cos x \sin\pi}{\cos x \cos\pi - \sin x \sin\pi} = \dfrac{-\sin x}{-\cos x} = \tan x$
$\sin(x+\pi/2) = \sin x \cos\frac{1}{2}\pi + \cos x \sin\frac{1}{2}\pi = (\sin x)\cdot 0 + (\cos x)\cdot 1 = \cos x$

7. (a) $\sqrt{2}\sin(x+\pi/4) - \cos x = \sqrt{2}(\sin x \cos\pi/4 + \cos x \sin\pi/4) - \cos x = \sqrt{2}(\sin x \cdot 1/\sqrt{2} + \cos x \cdot 1/\sqrt{2}) - \cos x = \sin x$
(b) $\tan(\alpha+\beta)$ ($\sin[\pi-(\alpha+\beta)] = \sin(\alpha+\beta)$, $\cos[2\pi-(\alpha+\beta)] = \cos(\alpha+\beta)$.)
(c) $-\cos a/\sin a$

9. $\sin(x+y)\sin(x-y) = (\sin x\cos y + \cos x \sin y)(\sin x \cos y - \cos x \sin y) = \sin^2 x\cos^2 y - \cos^2 x \sin^2 y = \sin^2 x(1-\sin^2 y) - (1-\sin^2 x)\sin^2 y = \sin^2 x - \sin^2 y$

11. (a) Because $|f(x)| = |(1/2)^x \sin x| \le (1/2)^x$ for all x, and $(1/2)^x \to 0$ as $x \to \infty$, the oscillations die out. (b) Because $2^x \to \infty$ as $x \to \infty$, the oscillations explode.

13. Because $(AB)^2 = (b - a\cos x)^2 + (0 - a\sin x)^2 = c^2$, we have the identity $b^2 - 2ab\cos x + a^2\cos^2 x + a^2 \sin^2 x = c^2$. Because $\sin^2 x + \cos^2 x = 1$, [C.10] follows immediately.

C.2

1. (a) $y' = \frac{1}{2}\cos\frac{1}{2}x$ (b) $y' = \cos x - x\sin x$ (c) $y' = \dfrac{2x}{\cos^2 x^2}$
(d) $y' = (-\sin x \sin x - \cos x \cos x)/\sin^2 x = -1/\sin^2 x$

3. (a) $\cos x - \sin x$ (b) $5x^4 \sin x + x^5 \cos x + (\frac{1}{2\sqrt{x}})\cos x - \sqrt{x}\sin x$
(c) $\dfrac{1}{(x^2+1)^2}\left[\left(\dfrac{1}{2\sqrt{x}} - \dfrac{3x\sqrt{x}}{2}\right)\cos x - \sqrt{x}\,(1+x^2)\sin x\right]$

5. (a) 2 (b) m/n (c) 1/2 (d) 0 (l'Hôpital's rule does not apply.)

7. $p'(t) = -\lambda C_1 \sin\lambda t + \lambda C_2 \cos\lambda t$ and $p''(t) = -\lambda^2 C_1 \cos\lambda t - \lambda^2 C_2 \sin\lambda t$, so $p''(t) + \lambda^2 p(t) = C_0\lambda^2$. Thus, $K = C_0\lambda^2$.

9. $f'(x) = 6\cos 2x - 16\sin 4x$

11. (a) $-\ln|\cos x| + C$ (b) $e^{\sin x} + C$ (c) $-\frac{1}{6}\cos^6 x + C$

13. (a) $\dfrac{2}{\sqrt{1-4x^2}}$ (b) $\dfrac{2x}{1+(x^2+1)^2}$ (c) $-\dfrac{1}{2\sqrt{x}\sqrt{1-x}}$

15. Let Q and R be the nearest points on the shoreline PP' to A and B. Let $a = AQ$, $b = BR$, $QR = l$, and $QC = x$. The time needed to reach the drowning swimmer is

$$T(x) = \frac{AC}{v_1} + \frac{BC}{v_2} = \frac{\sqrt{a^2+x^2}}{v_1} + \frac{\sqrt{b^2+(l-x)^2}}{v_2}, \quad x \in [0, l]$$

The first-order condition for minimum w.r.t. x yields

$$\frac{x}{v_1\sqrt{a^2+x^2}} = \frac{l-x}{v_2\sqrt{b^2+(l-x)^2}}$$

which is equivalent to $\sin\alpha_1/v_1 = \sin\alpha_2/v_2$. This condition is also sufficient for a minimum because we find that $T''(x) > 0$ in $[0, l]$.

C.3

1. (a) $z+w = 5-2i$ (b) $zw = 21-9i$ (c) $z/w = (-3-7i)/6$
(d) $|z| = \sqrt{2^2+(-5)^2} = \sqrt{29}$

3. (a) $\frac{1}{2}(1+5i)$ (b) $-3-4i$ (c) $(1/26)(31+27i)$ (d) i

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