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Operators with analytic orbit for the torus action

by

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Abstract. The class of bounded operators on $L^2(\mathbb{T}^n)$ which have an analytic orbit under the action of $\mathbb{T}^n$ by conjugation with the translation operators is shown to coincide with the class of zero-order pseudodifferential operators whose discrete symbol $(a_j)_{j\in\mathbb{Z}^n}$ is *uniformly analytic*, in the sense that there exists $C>1$ such that the derivatives of a_j satisfy $|\partial^\alpha a_j(x)| \le C^{1+|\alpha|}\alpha!$ for all $x\in\mathbb{T}^n$, all $j\in\mathbb{Z}^n$ and all $\alpha\in\mathbb{N}^n$. It then follows that this class of analytic pseudodifferential operators is a spectrally invariant *-subalgebra of the algebra of bounded operators on $L^2(\mathbb{T}^n)$, dense (in norm topology) in the algebra of $\rho=\delta=0$ Hörmander-type operators.

Introduction. Let us consider the unitary representations $y\mapsto T_y$ and $\eta\mapsto M_\eta$ of $\mathbb{R}^n$ on $L^2(\mathbb{R}^n)$ defined by $T_yu(x)=u(x-y)$ and $M_\eta u(x)=e^{ix\cdot\eta}u(x)$, $x\in\mathbb{R}$. Cordes [3] proved that a bounded linear operator A on $L^2(\mathbb{R}^n)$ is such that

$$\mathbb{R}^{2n}\ni(y,\eta)\mapsto M_\eta T_y A T_{-y} M_{-\eta}\in\mathcal{L}(L^2(\mathbb{R}^n)) \tag{1}$$

is a smooth function with values in the Banach space of all bounded operators on $L^2(\mathbb{R}^n)$ if and only if there is a $p\in C^\infty(\mathbb{R}^{2n})$, bounded and with all its partial derivatives also bounded, such that, for all smooth and rapidly decreasing u and all $x\in\mathbb{R}^n$, one has

$$Au(x)=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix\cdot\xi}p(x,\xi)\widehat{u}(\xi)\,d\xi,\qquad \widehat{u}(\xi)=\int_{\mathbb{R}^n} e^{-is\cdot\xi}u(s)\,ds. \tag{2}$$

This class of operators is often denoted by $\mathrm{OPS}^0_{0,0}(\mathbb{R}^n)$, they are the pseudodifferential operators of order zero with symbols satisfying $\rho=\delta=0$ Hörmander-type global estimates. Cordes' result can thus be regarded as the characterization of $\mathrm{OPS}^0_{0,0}(\mathbb{R}^n)$ as the operators with smooth orbit under the canonical (not everywhere strongly continuous) action of the Heisenberg

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group on the C*-algebra $\mathcal{L}(L^2(\mathbb{R}^n))$. It implies, in particular, that this class is a spectrally invariant *-subalgebra of $\mathcal{L}(L^2(\mathbb{R}^n))$.

The class $\mathrm{OPS}^0_{0,0}(\mathbb{R}^n)$ is not invariant under diffeomorphisms. Hence, in general, it does not make sense to define it on manifolds. But since $\mathrm{OPS}^0_{0,0}(\mathbb{R}^n)$ is invariant under translations, it does make sense to define $\mathrm{OPS}^0_{0,0}(\mathbb{T}^n)$ as the class of operators acting on functions defined on the torus $\mathbb{T}^n$ which are "locally" (the quotation marks indicate that only the canonical charts of the torus are considered) given by operators in $\mathrm{OPS}^0_{0,0}(\mathbb{R}^n)$. It follows from Cordes' result on $\mathbb{R}^n$ that $\mathrm{OPS}^0_{0,0}(\mathbb{T}^n)$ can be characterized as those bounded operators on $L^2(\mathbb{T}^n)$ which have smooth orbits when acted on by the group $\mathbb{T}^n$ via conjugation with translations. This is stated, using the discrete symbol representation of pseudodifferential operators on $\mathbb{T}^n$, in [6, Theorem 2] for $n = 1$ and here in our Theorem 2.

The main purpose of this paper is to address the question of which bounded operators on $L^2(\mathbb{T}^n)$ have real-analytic orbits. We prove in Theorem 3 that these are precisely the operators in $\mathrm{OPS}^0_{0,0}(\mathbb{T}^n)$ whose discrete symbols, which are sequences (a_j), are "uniformly analytic", meaning that each a_j is analytic and the coefficients of their Taylor series satisfy estimates uniform in j. Following [2], we denote by $\mathfrak{Op}_0(\mathbb{T}^n)$ this class of pseudodifferential operators.

The global representation of pseudodifferential operators on the torus, in which the discrete Fourier transform and discrete symbols replace, respectively, the Fourier transform on $\mathbb{R}^n$ and localization, goes back to Volevich in the 1970s [1]. A complete treatment of this subject, including analogues of the Fourier transform for noncommutative Lie groups, is given in [11].

It is an easy corollary of our Theorem 3 that $\mathfrak{Op}_0(\mathbb{T}^n)$ is a spectrally invariant *-subalgebra of $\mathcal{L}(L^2(\mathbb{T}^n))$. Chinni and Cordaro [2] give a more concrete proof that it is a *-algebra, but we do not know if there is a proof of the spectral invariance directly using local or discrete symbols. It also follows that $\mathfrak{Op}_0(\mathbb{T}^n)$ is dense in $\mathrm{OPS}^0_{0,0}(\mathbb{T}^n)$.

Characterizations of classes of pseudodifferential operators as the smooth operators for actions of Lie groups on C*-algebras have also been considered [7, 8, 10, 12], in connection with questions arising from deformation quantization and nonlinear PDEs.

1. Smoothness result. Let $\mathbb{T}^n$ denote the torus $\mathbb{R}^n/(2\pi\mathbb{Z})^n$. For each $j \in \mathbb{Z}^n$, let $e_j \in C^\infty(\mathbb{T}^n)$ be defined by $e_j(x) = e^{ij\cdot x}$. We have just denoted, as we often will, by the same letter x both an element x of $\mathbb{R}^n$ and its class $[x] \in \mathbb{T}^n$. We equip $\mathbb{T}^n$ with the measure induced by the Lebesgue measure of $\mathbb{R}^n$. If $g \in L^1(]-\pi,\pi]^n)$ and $f([x]) = g(x)$, then

$$\int_{\mathbb{T}^n} f = \int_{]-\pi,\pi]^n} g(x)\,dx.$$

Given a linear map $A : C^\infty(\mathbb{T}^n) \to C^\infty(\mathbb{T}^n)$, the sequence $(a_j)_{j\in\mathbb{Z}^n}$, $a_j = e_{-j}(Ae_j) \in C^\infty(\mathbb{T}^n)$, is called the *discrete symbol* of A. Using Fourier series, one may then write

$$Au(x) = \frac{1}{(2\pi)^n} \sum_{j\in\mathbb{Z}^n} a_j(x)e_j(x)\widehat{u}_j, \qquad \widehat{u}_j = \int_{\mathbb{T}^n} e_{-j}u, \tag{3}$$

for all $u \in C^\infty(\mathbb{T}^n)$ and $x \in \mathbb{T}^n$.

We say that $(a_j)_{j\in\mathbb{Z}^n}$, $a_j \in C^\infty(\mathbb{T}^n)$, is a *symbol of order* $m \in \mathbb{R}$ if, for every multiindex $\alpha \in \mathbb{N}^n$,

$$\sup\{(1+|j|)^{-m}|\partial^\alpha a_j(x)|;\ j \in \mathbb{Z}^n,\ x \in \mathbb{T}^n\} < \infty,$$

where $|j| = (j_1^2 + \cdots + j_n^2)^{1/2}$, $j = (j_1, \ldots, j_n)$. For $n = 1$, the following theorem is [6, Proposition 1]. To adapt that proof to the case $n > 1$, one uses $e^{ij\cdot x} = (1+|j|^2)^{-p}|(1-\Delta)^p e^{ij\cdot x}|$, $j \in \mathbb{Z}^n$, $p \in \mathbb{N}$, where Δ denotes the Laplace operator. See also [2, Theorem 3.4].

THEOREM 1. *If $(a_j)_{j\in\mathbb{Z}^n}$ is a symbol of order m, then* (3) *defines a linear operator $A : C^\infty(\mathbb{T}^n) \to C^\infty(\mathbb{T}^n)$. If $m \le 0$, then A extends to a bounded linear operator $A : L^2(\mathbb{T}^n) \to L^2(\mathbb{T}^n)$, whose norm satisfies, for any integer $p > n/2$,*

$$\|A\| \le C_p \sup\{|(1-\Delta)^p a_j(x)|;\ j \in \mathbb{Z}^n,\ x \in \mathbb{T}^n\}, \quad C_p = \sum_{l\in\mathbb{Z}^n} (1+|l|^2)^{-p}. \tag{4}$$

We then write $A = \mathrm{Op}((a_j)_{j\in\mathbb{Z}^n})$, or simply $A = \mathrm{Op}(a_j)$.

For each $y \in \mathbb{T}^n$, let T_y denote the operator, unitary on $L^2(\mathbb{T}^n)$, defined by $(T_y u)(x) = u(x-y)$. For $A = \mathrm{Op}(a_j)$, one has $T_y A T_{-y} = \mathrm{Op}((T_y a_j)_{j\in\mathbb{Z}^n})$. For each $\alpha \in \mathbb{N}^n$, denote $A^\alpha = \mathrm{Op}((\partial^\alpha a_j)_{j\in\mathbb{Z}^n})$. It follows from the group property $T_y T_z = T_{y+z}$ and from (4) that $y \mapsto T_y A T_{-y}$ is a smooth function on $\mathbb{T}^n$ with values in the Banach space $\mathcal{L}(L^2(\mathbb{T}^n))$, and moreover $\partial_y^\alpha(T_y A T_{-y}) = T_y A^\alpha T_{-y}$. This proves the "if" statement in the following theorem.

THEOREM 2. *Let $A \in \mathcal{L}(L^2(\mathbb{T}^n))$ be a bounded operator. The map*

$$\mathbb{T}^n \ni y \mapsto T_y A T_{-y} \tag{5}$$

is smooth with respect to the norm topology of $\mathcal{L}(L^2(\mathbb{T}^n))$ if and only if $A = \mathrm{Op}(a_j)$ for some symbol $(a_j)_{j\in\mathbb{Z}^n}$ of order zero.

Proof. If a bounded operator A on $L^2(\mathbb{R}^n)$ is such that the map (1) is smooth, we call it a *Heisenberg smooth* operator. Analogously, if $A \in \mathcal{L}(L^2(\mathbb{T}^n))$ is such that (5) is smooth, we say that it is *translation smooth.*

For each $\theta = (\theta_1, \ldots, \theta_n) \in \mathbb{R}^n$, let $\tilde{U}_\theta \subset \mathbb{R}^n$ denote the open rectangle

$$\tilde{U}_\theta =]\theta_1 - \pi, \theta_1 + \pi[\times \cdots \times]\theta_n - \pi, \theta_n + \pi[,$$

let $U_\theta \subset \mathbb{R}^n/(2\pi\mathbb{Z})^n = \mathbb{T}^n$ denote the set of the equivalence classes of all $x \in \tilde{U}_\theta$ and let $\chi_\theta : U_\theta \to \tilde{U}_\theta$ be the chart that sends a class in U_θ to its unique representative in $\tilde{U}_\theta$. Let then $I_\theta : L^2(\mathbb{T}^n) \to L^2(\tilde{U}_\theta)$ denote the unitary map $I_\theta u = u \circ \chi_\theta^{-1}$, $E_\theta : L^2(\tilde{U}_\theta) \to L^2(\mathbb{R}^n)$ denote the extension-by-zero embedding and $R_\theta : L^2(\mathbb{R}^n) \to L^2(\tilde{U}_\theta)$ denote the restriction map. Finally, let $\varXi_\theta : \mathcal{L}(L^2(\mathbb{T}^n)) \to \mathcal{L}(L^2(\mathbb{R}^n))$ denote the isometric *-homomorphism $\varXi_\theta(A) = E_\theta I_\theta A I_\theta^{-1} R_\theta$.

There exists a finite collection $\{\phi_1, \dots, \phi_q\}$ of smooth functions (we may choose $q = 3^n$) such that $\sum_{r=1}^q \phi_r \equiv 1$ and, for each pair of integers $1 \le m, p \le q$, there is a $\theta_{mp} \in \mathbb{R}^n$ such that the union of the supports of ϕ_m and ϕ_p is contained in $U_{\theta_{mp}}$. A translation smooth operator $A \in \mathcal{L}(L^2(\mathbb{T}^n))$ can thus be written as a finite sum of operators of the form $\phi(M)A\psi(M)$, where $\phi(M)$ and $\psi(M)$ are the operators of multiplication by, respectively, the smooth functions ϕ and ψ, whose supports are both contained in U_θ for some $\theta \in \mathbb{R}^n$. It is then enough to show that such a $\phi(M)A\psi(M)$ equals $\mathrm{Op}(a_j)$ for some symbol $(a_j)_{j\in\mathbb{Z}^n}$ of order zero.

Noting that $T_y\phi(M)T_{-y} = \phi_y(M)$, where $\phi_y = T_y\phi$, we see that the operators $\phi(M)$ and $\psi(M)$ are translation smooth, and hence so is $\phi(M)A\psi(M)$. Next we prove that $\tilde{A} = \varXi_\theta[\phi(M)A\psi(M)]$ is Heisenberg smooth. For that it suffices to show (see [4, Section 8.1]) that both maps

$$\mathbb{R}^n \ni y \mapsto T_y\tilde{A}T_{-y} \quad \text{and} \quad \mathbb{R}^n \ni \eta \mapsto M_\eta \tilde{A} M_{-\eta} \tag{6}$$

are smooth (abusing notation, we are denoting by T_y translation operators both on $L^2(\mathbb{T}^n)$ and on $L^2(\mathbb{R}^n)$). It suffices to prove that the partial derivatives exist at $y = 0$ and at $\eta = 0$, respectively, because $y \mapsto T_y$ and $\eta \mapsto M_\eta$ are representations.

For $i = 1, \dots, n$, let A_i denote the partial derivative with respect to y_i at $y = 0$ of $y \mapsto T_yAT_{-y}$,

$$A_i = \lim_{h\to 0} \frac{T_{(0,\dots,0,h,0,\dots 0)}\phi(M)A\psi(M)T_{(0,\dots,0,-h,0,\dots,0)} - \phi(M)A\psi(M)}{h}. \tag{7}$$

Since

$$\varXi_\theta[T_y\phi(M)A\psi(M)T_{-y}] = T_y\varXi_\theta[\phi(M)A\psi(M)]T_{-y}$$

for all sufficiently small y, we conclude from (7) that the derivative with respect to y_i at $y = 0$ of $y \mapsto T_y\tilde{A}T_{-y}$ equals $\varXi_\theta(A_i)$.

Let $\bar{\phi}$ and $\bar{\psi}$ be smooth functions with support contained in U_θ such that, for all sufficiently small y, $\bar{\phi}_y\phi \equiv \phi$ and $\bar{\psi}_y\psi \equiv \psi$, and hence

$$\bar{\phi}(M)[T_y\phi(M)A\psi(M)T_{-y}]\bar{\psi}(M) = T_y\phi(M)A\psi(M)T_{-y}.$$

Multiplying both sides of (7) on the left by $\bar{\phi}(M)$ and on the right by $\bar{\psi}(M)$, we then conclude that $\bar{\phi}(M)A_i\bar{\psi}(M) = A_i$. The argument we have used to find the first-order derivatives can next be used to find the second-order derivatives, and successively to prove that $\mathbb{R}^n \ni y \mapsto T_y\tilde{A}T_{-y}$ is C^∞.

Take $\rho \in C_c^\infty(\tilde{U}_\theta)$ such that $\rho(M)\tilde{A}\rho(M) = \tilde{A}$. The second map in (6) is smooth because

$$M_\eta \tilde{A} M_{-\eta} = [M_\eta \rho(M)]\, \tilde{A}\, [\rho(M) M_{-\eta}]$$

and the maps $\eta \mapsto M_\eta \rho(M)$ and $\eta \mapsto \rho(M) M_{-\eta}$ are smooth.

By Cordes's result [3] (see also [4, Theorem 8.2.1] and [8, Theorem 1]), there exists $p \in C^\infty(\mathbb{R}^{2n})$, bounded and with all its partial derivatives also bounded, such that (2) holds for $\tilde{A}$. The discrete symbol (a_j) of $\phi(M)A\psi(M)$ can then be expressed in terms of p by the iterated integral

$$a_j(x) = e_{-j}(x)\phi(x)A(\psi e_j)(x) = \frac{1}{(2\pi)^n} \int\limits_{\mathbb{R}^n} p(x,\xi) \int\limits_{\tilde{U}_\theta} e^{i(x-s)\cdot(\xi-j)} \rho(s)\, ds\, d\xi.$$

Using a standard technique of pseudodifferential calculus (for $n = 1$, what we need is [6, estimate (10) for $l = 0$]), it follows that (a_j) is a zero-order discrete symbol. ∎

2. Analytic operators. As in the case of complex-valued functions (see, for example, [5, Section II.3] or [13, Section V.1]), a smooth function $f : \mathbb{T}^n \to \mathcal{L}(L^2(\mathbb{T}^n))$ is analytic, in the sense that at each point its Taylor series converges to f, if and only if there is $C > 1$ such that, for every multiindex $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$,

$$\sup\{\|\partial^\alpha f(x)\|;\ x \in \mathbb{T}^n\} \le C^{1+|\alpha|}\alpha! \tag{8}$$

(as usual, we denote $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $\alpha! = \alpha_1! \ldots \alpha_n!$).

We say that $A \in \mathcal{L}(L^2(\mathbb{T}^n))$ is *translation analytic* if the function $f(y) = T_y A T_{-y}$, $y \in \mathbb{T}^n$, is analytic. Being translation smooth, a translation analytic operator is, by Theorem 2, necessarily of the form $A = \mathrm{Op}(a_j)$ for some symbol $(a_j)_{j\in\mathbb{Z}^n}$ of order zero. The following theorem gives a necessary and sufficient condition on (a_j) for $A = \mathrm{Op}(a_j)$ to be translation analytic.

For each multiindex $\beta = (\beta_1, \ldots, \beta_n)$, $L^\beta = (1 + \partial_1)^{\beta_1} \ldots (1 + \partial_n)^{\beta_n}$ defines an invertible operator $L^\beta : C^\infty(\mathbb{T}^n) \to C^\infty(\mathbb{T}^n)$. Its inverse can be expressed using Fourier series:

$$(L^\beta)^{-1} u(x) = \frac{1}{(2\pi)^n} \sum_{l\in\mathbb{Z}^n} \frac{e^{il\cdot x}\widehat{u}_l}{(1 + il_1)^{\beta_1} \ldots (1 + il_n)^{\beta_n}}. \tag{9}$$

For each positive integer m, define a seminorm ρ_m on $C^\infty(\mathbb{T}^n)$ by

$$\rho_m(a) = \sup\{|\partial^\beta a(x)|;\ |\beta| \le m,\ x \in \mathbb{T}^n\}.$$

THEOREM 3. *Let $A \in \mathcal{L}(L^2(\mathbb{T}^n))$ be a bounded operator. Then A is translation analytic if and only if $A = \mathrm{Op}(a_j)$ for some symbol $(a_j)_{j\in\mathbb{Z}^n}$ of order zero satisfying, for some constant $C > 1$ and for every multiindex α,*

$$\sup\{|\partial^\alpha a_j(x)|;\ x \in \mathbb{T}^n,\ j \in \mathbb{Z}^n\} \le C^{1+|\alpha|}\alpha!. \tag{10}$$

Proof. Suppose first that $A = \mathrm{Op}(a_j)$ for some (a_j) satisfying (10), and consider $f(y) = T_y A T_{-y}$, $y \in \mathbb{T}^n$. Using the remarks and notation of the paragraph right after Theorem 1, and by the norm estimate (4), for $p > n/2$ we have

$$\sup\{\|\partial^\alpha f(x)\|;\, x \in \mathbb{T}^n\} \le \|A^\alpha\| \le 2^p C_p \sup_{j\in\mathbb{Z}^n} \rho_{2p}(\partial^\alpha a_j).$$

It then follows from (10) that

$$\begin{aligned}\sup\{\|\partial^\alpha f(x)\|;\, x \in \mathbb{T}^n\} &\le 2^p C_p C^{1+2p+|\alpha|}[(\alpha_1+2p)!\dots(\alpha_n+2p)!]\\ &\le \mu^n 2^p C_p C^{2p+1}(2C)^{|\alpha|}\alpha!,\end{aligned}$$

where $\mu = \sup_{t>0} 2^{-t}(t+2p)(t+2p-1)\dots(t+1)$. Therefore A is translation analytic.

Conversely, let A be translation analytic. By Theorem 2, $A = \mathrm{Op}(a_j)$ for some zero-order discrete symbol (a_j). For each nonzero multiindex β, define

$$B^\beta = \mathrm{Op}[(L^\beta a_j)_{j\in\mathbb{Z}^n}].$$

Then $T_y B^\beta T_{-y} = L^\beta f(y)$, where $f(y) = T_y A T_{-y}$, $y \in \mathbb{T}^n$, is analytic by hypothesis. Since $C > 1$, from (8) we get

$$\|B^\beta\| \le 2^{|\beta|} C^{1+|\beta|}\beta!.$$

On the other hand, the discrete symbol of A can be recovered from the discrete symbol $(b_j^\beta)_{j\in\mathbb{Z}^n}$ of B^β, $b_j^\beta = e_j(B^\beta e_{-j}) = L^\beta a_j$, using (9):

$$a_j(x) = \frac{1}{(2\pi)^n}\sum_{l\in\mathbb{Z}^n}\frac{e^{ilx}[e_j(B^\beta e_{-j})]^\wedge_l}{(1+il_1)^{\beta_1}\dots(1+il_n)^{\beta_n}}.$$

Given now a multiindex α, let $\beta = \alpha + (2,\dots,2)$. Using $|[e_j(B^\beta e_{-j})]^\wedge_l| \le (2\pi)^n\|B^\beta\|$ for all j and l, we get

$$\begin{aligned}|\partial^\alpha a_j(x)| &= \frac{1}{(2\pi)^n}\left|\sum_{l\in\mathbb{Z}^n}\frac{(il)^\alpha e^{il\cdot x}[e_j(B^\beta e_{-j})]^\wedge_l}{(1+il_1)^{\beta_1}\dots(1+il_n)^{\beta_n}}\right|\\ &\le \|B^\beta\|\sum_{l\in\mathbb{Z}^n}\frac{|l_1|^{\alpha_1}\dots|l_n|^{\alpha_n}}{(1+l_1^2)^{\beta_1/2}\dots(1+l_n^2)^{\beta_n/2}}\\ &\le \left(\sum_{p=-\infty}^{\infty}\frac{1}{1+p^2}\right)^n 2^{|\alpha|+2n}C^{1+2n+|\alpha|}\beta!.\end{aligned}$$

We may now bound $\beta!$ by a constant times $\alpha!$, much as we did in the first part of this proof, which will then prove (10). ∎

As in [2], we denote by $\mathfrak{Op}_0(\mathbb{T}^n)$ the class of all $A = \mathrm{Op}(a_j)$ with $(a_j)_{j\in\mathbb{Z}^n}$ satisfying (10).

COROLLARY 1. *The class* $\mathfrak{Op}_0(\mathbb{T}^n)$ *is a* **-subalgebra of* $\mathcal{L}(L^2(\mathbb{T}^n))$ *which is spectrally invariant in the sense that if an* $A \in \mathfrak{Op}_0(\mathbb{T}^n)$ *is invertible in* $\mathcal{L}(L^2(\mathbb{T}^n))$, *then its inverse* A^{-1} *belongs to* $\mathfrak{Op}_0(\mathbb{T}^n)$.

Proof. If $f : \mathbb{T}^n \to \mathcal{L}(L^2(\mathbb{T}^n))$ and $g : \mathbb{T}^n \to \mathcal{L}(L^2(\mathbb{T}^n))$ are real-analytic, their pointwise product is analytic. It follows that the product of two translation analytic operators is translation analytic. The map $y \mapsto f(y)^*$ is also analytic. Hence, the adjoint of a translation analytic operator is translation analytic. Moreover if $f(y)$ is invertible for every $y \in \mathbb{T}^n$, then $y \mapsto f(y)^{-1}$ is also analytic (because $T_{-y} = T_y^{-1}$ and a pointwise invertible analytic function has an analytic inverse). Then, if $A \in \mathcal{L}(L^2(\mathbb{T}^n))$ is invertible and translation analytic, so is A^{-1}. ∎

COROLLARY 2. $\mathfrak{Op}_0(\mathbb{T}^n)$ *is* L^2*-operator-norm dense in* $\mathrm{OPS}^0_{0,0}(\mathbb{T}^n)$.

Proof. It is the content of Nelson's [9, Theorem 4] that the set of analytic vectors of a strongly continuous representation of a Lie group on a Banach space $\mathfrak{X}$ is dense in $\mathfrak{X}$. We apply that theorem to the representation $y \mapsto T_y A T_{-y}$ of $\mathbb{T}^n$ on the Banach space $\mathfrak{X} = \{A \in \mathcal{L}(L^2(\mathbb{T}^n));\ y \mapsto T_y A T_{-y}$ is continuous$\}$. By our Theorem 3, the set of analytic vectors for this representation is equal to $\mathfrak{Op}_0(\mathbb{T}^n)$, which is dense in $\mathfrak{X}$ by Nelson's result. On the other hand, $\mathfrak{X}$ contains the set of smooth vectors, which is equal to $\mathrm{OPS}^0_{0,0}(\mathbb{T}^n)$ by our Theorem 2. ∎

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