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On Rieffel's conjecture characterizing a deformed algebra as Heisenberg smooth operators

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Abstract. Let $\mathcal{A}$ be a unital C*-algebra and E_n be the Hilbert $\mathcal{A}$-module defined as the completion of the $\mathcal{A}$-valued Schwartz function space $\mathcal{S}^{\mathcal{A}}(\mathbb{R}^n)$ with respect to the norm $\|f\|_2 := \|\int_{\mathbb{R}^n} f(x)^* f(x)\, dx\|_{\mathcal{A}}^{1/2}$. Also, let $\mathrm{Ad}\,\mathcal{U}$ be the canonical action of the $(2n+1)$-dimensional Heisenberg group by conjugation on the algebra of adjointable operators on E_n, and let J be a skew-symmetric linear transformation on $\mathbb{R}^n$. We characterize the smooth vectors under $\mathrm{Ad}\,\mathcal{U}$ which commute with a certain algebra of right multiplication operators R_h, with $h \in \mathcal{S}^{\mathcal{A}}(\mathbb{R}^n)$, where the product is "twisted" with respect to J according to a deformation quantization procedure introduced by M.A. Rieffel. More precisely, we establish that they coincide with an algebra of left multiplication operators and show that this solves, in particular, a conjecture posed by Rieffel.

This paper is dedicated to the memory of Marcela I. Merklen (1969–2018)

1. Introduction

In his AMS Memoir [14], M.A. Rieffel defined an algebra of pseudodifferential operators with C*-algebra valued symbols and used it to construct a strict deformation quantization of a C*-algebra with an action of $\mathbb{R}^n$ with respect to a given skew-symmetric linear transformation. We start by describing Rieffel's algebra.

Given a C*-algebra $\mathcal{A}$ with C*-norm $\|\cdot\|_{\mathcal{A}}$ and unit $1_{\mathcal{A}}$, we denote by $\mathcal{S}^{\mathcal{A}}(\mathbb{R}^n)$ the space of all $\mathcal{A}$-valued smooth functions on $\mathbb{R}^n$ which, together with all their partial derivatives, decay to 0 at infinity more rapidly than the inverse of any polynomial on $\mathbb{R}^n$. The function space $\mathcal{S}^{\mathcal{A}}(\mathbb{R}^n)$ has a canonical Fréchet space structure defined by the norms

$$p_{\alpha,\beta}(f) := \sup_{x\in\mathbb{R}^n} \|x^\alpha\, \partial_x^\beta f(x)\|_{\mathcal{A}}, \quad f \in \mathcal{S}^{\mathcal{A}}(\mathbb{R}^n),\ \alpha,\beta \in \mathbb{N}^n. \tag{1.1}$$

We also define $\mathcal{B}^{\mathcal{A}}(\mathbb{R}^n)$ as the space of $\mathcal{A}$-valued bounded smooth functions on $\mathbb{R}^n$ whose partial derivatives of all orders are also bounded. When $\mathcal{A} = \mathbb{C}$, we will write simply

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$\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{B}(\mathbb{R}^n)$, respectively. Another space which plays a central role in [14] and in this paper is the Hilbert $\mathcal{A}$-module E_n, defined as the Banach space completion of $\mathcal{S}^{\mathcal{A}}(\mathbb{R}^n)$ with respect to the norm

$$\|f\|_2 := \left\| \int_{\mathbb{R}^n} f(x)^* f(x)\, dx \right\|_{\mathcal{A}}^{1/2}, \quad f \in \mathcal{S}^{\mathcal{A}}(\mathbb{R}^n). \tag{1.2}$$

This norm is induced by an $\mathcal{A}$-valued inner product [6, p. 2] $\langle \cdot, \cdot \rangle_{E_n}$ on E_n, defined as the unique continuous extension to $E_n \times E_n$ of the map

$$(f, g) \longmapsto \int_{\mathbb{R}^n} f(x)^* g(x)\, dx \tag{1.3}$$

on $\mathcal{S}^{\mathcal{A}}(\mathbb{R}^n) \times \mathcal{S}^{\mathcal{A}}(\mathbb{R}^n)$. The space $L^2(\mathbb{R}^n, \mathcal{A})$ of (equivalence classes of) square-integrable functions will also play a central role in Section 3. Finally, we denote the C*-algebra of (continuous) *adjointable* operators on the Hilbert $\mathcal{A}$-module E_n by $\mathcal{L}_{\mathcal{A}}(E_n)$ [6, p. 9] and its usual operator C*-norm by $\|\cdot\|$.

We will now define Rieffel's pseudodifferential operators (for more details, see Appendix A). Given any skew-symmetric linear transformation J on $\mathbb{R}^n$ and $f \in \mathcal{B}^{\mathcal{A}}(\mathbb{R}^n)$, the linear operator given by the iterated integral

$$L_f(g)(x) := \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} f(x + J\xi)\, g(x + y)\, e^{2\pi i \langle \xi, y \rangle}\, dy \right) d\xi, \qquad g \in \mathcal{S}^{\mathcal{A}}(\mathbb{R}^n),\ x \in \mathbb{R}^n \tag{1.4}$$

maps $\mathcal{S}^{\mathcal{A}}(\mathbb{R}^n)$ into $\mathcal{S}^{\mathcal{A}}(\mathbb{R}^n)$ [14, Proposition 3.3, p. 25], satisfies $\langle L_f(g), h \rangle_{E_n} = \langle g, L_{f^*}(h) \rangle_{E_n}$ for all $g, h \in \mathcal{S}^{\mathcal{A}}(\mathbb{R}^n)$ [14, Proposition 4.2, p. 30] and extends to a bounded operator on the Hilbert $\mathcal{A}$-module E_n [14, Theorem 4.6 and Corollary 4.7, p. 34]. By the continuity of the $\mathcal{A}$-valued inner product, we see that this extension, also denoted by L_f, is an adjointable operator on E_n satisfying $(L_f)^* = L_{f^*}$.

Given $f, g \in \mathcal{B}^{\mathcal{A}}(\mathbb{R}^n)$, we have the identity $L_f L_g = L_{f \times_J g}$, where $f \times_J g \in \mathcal{B}^{\mathcal{A}}(\mathbb{R}^n)$ is defined via *Rieffel's deformed product* [14, p. 23]

$$(f \times_J g)(x) := \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x + J\xi)\, g(x + y)\, e^{2\pi i \langle \xi, y \rangle}\, dy\, d\xi, \quad x \in \mathbb{R}^n. \tag{1.5}$$

We shall refer to $\{L_f : f \in \mathcal{B}^{\mathcal{A}}(\mathbb{R}^n)\}$ as *Rieffel's deformed algebra*. The integral signs, above, do not denote true integrals (not even iterated integrals), but rather *oscillatory integrals* (see [14, Chapter 1], [3, pp. 66–69]). If $f \in \mathcal{B}^{\mathcal{A}}(\mathbb{R}^n)$ and $g \in \mathcal{S}^{\mathcal{A}}(\mathbb{R}^n)$, then equation (1.5) coincides with (1.4), so that $L_f(g) = f \times_J g$. Moreover, the operator of right multiplication by $g \in \mathcal{B}^{\mathcal{A}}(\mathbb{R}^n)$, defined by $R_g(f) := f \times_J g$, $f \in \mathcal{S}^{\mathcal{A}}(\mathbb{R}^n)$, is also (the restriction of) an adjointable operator on E_n.

Let $C^\infty(\mathbb{R}^{2n}, \mathcal{A})$ be the space of smooth $\mathcal{A}$-valued functions on $\mathbb{R}^{2n}$. We call a linear operator $A\colon \mathcal{S}^{\mathcal{A}}(\mathbb{R}^n) \to \mathcal{S}^{\mathcal{A}}(\mathbb{R}^n)$ a *pseudodifferential operator* with symbol $a \in C^\infty(\mathbb{R}^{2n}, \mathcal{A})$ and write $A = \mathrm{Op}(a)$ if, for every $g \in \mathcal{S}^{\mathcal{A}}(\mathbb{R}^n)$, we have

$$A(g)(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} a(x, \xi)\, \mathcal{F}(g)(\xi)\, e^{i \langle \xi, x \rangle}\, d\xi, \quad x \in \mathbb{R}^n, \tag{1.6}$$

where $\mathcal{F}$ denotes the Fourier transform (see equation (2.4)). Therefore, the operator L_f is a pseudodifferential operator with symbol $(x, \xi) \mapsto f(x - J\xi/(2\pi))$ (see equation (3.2)).

In the scalar case $\mathcal{A} = \mathbb{C}$, when E_n is the usual Hilbert space $L^2(\mathbb{R}^n)$, H.O. Cordes proved ([2], [3, Chapter 8]) that a bounded operator A on $L^2(\mathbb{R}^n)$ is a smooth vector for the canonical action of the $(2n+1)$-dimensional Heisenberg group by conjugation if, and only if, $A = \mathrm{Op}(a)$ for some $a \in \mathcal{B}(\mathbb{R}^{2n})$. Motivated by Cordes' "lovely characterization", Rieffel conjectured, in the last paragraph of [14, Chapter 4], that an adjointable operator $A \in \mathcal{L}_{\mathcal{A}}(E_n)$ is smooth under the action $\mathrm{Ad}\,\mathcal{U}$ of the Heisenberg group (see equation (2.3)) *and commutes with every* R_g, $g \in \mathcal{B}^{\mathcal{A}}(\mathbb{R}^n)$ if, and only if, $A = L_f$ for some $f \in \mathcal{B}^{\mathcal{A}}(\mathbb{R}^n)$. That each L_f commutes with every R_g follows immediately from the associative property of Rieffel's product $\times_J$, and the fact that each L_f, $f \in \mathcal{B}^{\mathcal{A}}(\mathbb{R}^n)$, is smooth under the Heisenberg action can be proved using a generalized version of the Calderón–Vaillancourt inequality (see [1, Subsection "The algebra $\mathcal{B}^{\mathcal{C}}_J(\mathbb{R}^n)$"]). Therefore, in the present paper we are mainly interested in investigating the converse part of Rieffel's conjecture [14, Chapter 4, p. 39]:

> *"... and then let the Heisenberg group act on the whole algebra of bounded operators, B, by conjugation. Then the operators which one obtains from F's in $\mathcal{B}$ [via the Kohn-Nirenberg representation, in the scalar case] are exactly the smooth vectors for this action of the Heisenberg group on B by conjugation. Presumably a similar theorem holds in our present context, though the fact that we are using the left regular representation rather than an irreducible representation means that one must replace B by the algebra of operators which commute with the right regular representation..."*

The results obtained in [8, 9, 11] (see also [7]) imply that Rieffel's conjecture is true if $\mathcal{A}$ is a matrix algebra $M_k(\mathcal{C})$ for a commutative separable unital C*-algebra $\mathcal{C}$. The main goal of this paper is to prove Rieffel's conjecture for an arbitrary unital C*-algebra $\mathcal{A}$.

The partial results obtained in [7–9, 11] all depend on an interplay between Rieffel's conjecture and generalizations of Cordes' characterization. Here we take a different approach.

We begin by showing in Section 2 that $\mathcal{S}^{\mathcal{A}}(\mathbb{R}^n)$, which by definition is a dense subspace of E_n, is left invariant by the operators in the algebra $C^\infty(\mathrm{Ad}\,\mathcal{U})$ of smooth vectors for the canonical action of the $(2n+1)$-dimensional Heisenberg group by conjugation (the elements of $C^\infty(\mathrm{Ad}\,\mathcal{U})$ will be referred to as the *Heisenberg smooth operators* – see equation (2.3) and Proposition 2.3). General results from Lie group representation theory will be applied in order to obtain this conclusion, which will be an important step for the arguments in Section 4. In the process, we prove in Proposition 2.2 that $\mathcal{S}^{\mathcal{A}}(\mathbb{R}^n)$ coincides with the subspace of smooth vectors for the canonical unitary representation of the $(2n+1)$-dimensional Heisenberg group (see equation (2.2)), generalizing a result which is well known in the scalar case $\mathcal{A} = \mathbb{C}$.

In Section 3, we need the hypothesis of $\mathcal{A}$ being unital in order to use the definition and the properties of a certain "symbol map" $S\colon C^\infty(\mathrm{Ad}\,\mathcal{U}) \to \mathcal{B}^{\mathcal{A}}(\mathbb{R}^{2n})$, introduced in [8, equation (12)], which adapts a construction given by Cordes. More precisely, for every $A \in C^\infty(\mathrm{Ad}\,\mathcal{U})$, we show in Proposition 3.7 that $S(A)$ can be pointwise approximated by a sequence $(S(A \circ L_{\widetilde{e}_m}))_{m\in\mathbb{N}}$ of functions in $\mathcal{B}^{\mathcal{A}}(\mathbb{R}^{2n})$, where $(\widetilde{e}_m)_{m\in\mathbb{N}}$ is a specific sequence of functions in $\mathcal{S}^{\mathcal{A}}(\mathbb{R}^n)$. A careful interplay between the spaces E_n, $L^2(\mathbb{R}^n, \mathcal{A})$ and $L^2(\mathbb{R}^n) \cdot 1_{\mathcal{A}}$ (and their corresponding topologies) will be required in the proof. Moreover, once certain key estimates are established, the strategy will be to "tensor product" these estimates and go from dimension n to $2n$, so that uniform bounds on the norms of certain tensor product operators may be obtained.

Finally, Section 4 will be devoted to the proof of Theorem A, whose statement will be given next. Denote by R_n the algebra of the right multiplication operators R_g acting on E_n, with $g \in \mathcal{S}^{\mathcal{A}}(\mathbb{R}^n)$, and let R_n' be its commutant inside the C*-algebra $\mathcal{L}_{\mathcal{A}}(E_n)$, that is,

$$R_n' := \{A \in \mathcal{L}_{\mathcal{A}}(E_n) : A \circ R_g = R_g \circ A,\ g \in \mathcal{S}^{\mathcal{A}}(\mathbb{R}^n)\}.$$

Equation (3.11) shows that the map $\mathcal{R} \circ S|_{C^\infty(\mathrm{Ad}\,\mathcal{U})\cap R_n'}$ is a left inverse for $L\colon \mathcal{B}^{\mathcal{A}}(\mathbb{R}^n) \to C^\infty(\mathrm{Ad}\,\mathcal{U})$, $f \mapsto L_f$. We will show in Theorem A that $\mathcal{R} \circ S|_{C^\infty(\mathrm{Ad}\,\mathcal{U})\cap R_n'}$ is injective. More precisely, we will establish the equality

$$L \circ \mathcal{R} \circ S|_{C^\infty(\mathrm{Ad}\,\mathcal{U})\cap R_n'} = \mathrm{Id}_{C^\infty(\mathrm{Ad}\,\mathcal{U})\cap R_n'},$$

where $\mathrm{Id}_{C^\infty(\mathrm{Ad}\,\mathcal{U})\cap R_n'}$ denotes the identity operator on $C^\infty(\mathrm{Ad}\,\mathcal{U}) \cap R_n'$.

Theorem A. *Let $\mathcal{A}$ be a unital C*-algebra, and let R_n' be the commutant defined above. If $A \in C^\infty(\mathrm{Ad}\,\mathcal{U}) \cap R_n'$, then $A = L_{(\mathcal{R}\circ S)(A)}$.*

The reason we are interested in Theorem A is that it allows us to obtain, as an immediate corollary, a solution to Rieffel's conjecture.

Theorem (Rieffel's conjecture for a unital C*-algebra). *Let $\mathcal{A}$ be a unital C*-algebra. If a Heisenberg smooth operator $A \in \mathcal{L}_{\mathcal{A}}(E_n)$ commutes with every operator of the form R_h, with $h \in \mathcal{B}^{\mathcal{A}}(\mathbb{R}^n)$, then there exists $f \in \mathcal{B}^{\mathcal{A}}(\mathbb{R}^n)$ such that $A = L_f$.*

Proof. Let $A \in C^\infty(\mathrm{Ad}\,\mathcal{U})$. If A commutes with every operator R_h, with $h \in \mathcal{B}^{\mathcal{A}}(\mathbb{R}^n)$, then in particular it commutes with every R_h, with $h \in \mathcal{S}^{\mathcal{A}}(\mathbb{R}^n)$. But, as a consequence of Theorem A, we know that if A commutes with every operator of the form R_h, with $h \in \mathcal{S}^{\mathcal{A}}(\mathbb{R}^n)$, then $A = L_{(\mathcal{R}\circ S)(A)}$. Therefore, since $(\mathcal{R} \circ S)(A)$ belongs to $\mathcal{B}^{\mathcal{A}}(\mathbb{R}^n)$, the result follows immediately. ∎

The proof of Theorem A consists essentially of the computation of two limits. For the first one, we make use of Proposition 3.7, as well as of certain estimates which are obtained in its proof. For the second one, the hypothesis that $A \in C^\infty(\mathrm{Ad}\,\mathcal{U})$ commutes with every R_g, $g \in \mathcal{S}^{\mathcal{A}}(\mathbb{R}^n)$, is finally considered: we adapt a useful calculation performed in [10, Corolário 2.7] and combine it with the results of Proposition 2.3 and Lemma 3.4. Finally, a careful combination of the conclusions summarized in equations (4.3) and (4.4) will establish Theorem A.

2. A (reasonable) dense invariant subspace for the Heisenberg smooth operators

We begin this section by recalling a few basic concepts and fixing the notation which will be employed throughout the paper. For more information on Lie group representations, see [13].

If G is a (finite-dimensional) Lie group with unit e and $(\mathcal{X}, \|\cdot\|)$ is a Banach space, we say that a family $\{V_g\}_{g\in G}$ of continuous (everywhere defined) linear operators on $\mathcal{X}$ is a *strongly continuous representation of G on $\mathcal{X}$* if

$$V_e = I,\ V_{gh} = V_g V_h \quad \text{and} \quad \lim_{h'\to h} V_{h'}x = V_h x, \quad x \in \mathcal{X},\ g, h \in G,$$

with I denoting the identity operator on $\mathcal{X}$. If $G = \mathbb{R}$, then we call $\{V_t\}_{t\in\mathbb{R}}$ a *strongly continuous one-parameter group*. In this case, the subspace of vectors $x \in \mathcal{X}$ such that $\lim_{t\to 0} t^{-1}(V_t x - x)$ exists in $\mathcal{X}$ defines the domain $\operatorname{Dom} T$ of a linear operator T on $\mathcal{X}$, $T(x) := \lim_{t\to 0} t^{-1}(V_t x - x)$, which is called the *infinitesimal generator* of $\{V_t\}_{t\in\mathbb{R}}$. Moreover, if G is any Lie group and $\{V_g\}_{g\in G}$ is a strongly continuous representation of G on $\mathcal{X}$, then each X in its Lie algebra $\mathfrak{g}$ gives rise to a one-parameter group $t \mapsto V_{\exp tX}$ on $\mathcal{X}$ (exp denotes the exponential map of the Lie group G), and its infinitesimal generator is denoted by $dV(X)$. Finally, a vector $x \in \mathcal{X}$ is called a *smooth vector for V* if the map $G \ni g \mapsto V_g x$ is of class C^∞ (in other words, if the map $G \ni g \mapsto V_g x$ has continuous partial derivatives of all orders on an arbitrary chart). The subspace of smooth vectors for V will be denoted by $C^\infty(V)$. It turns out that $C^\infty(V)$ is a dense subspace of $\mathcal{X}$ which is invariant by the operators $\{dV(X)\}_{X\in\mathfrak{g}}$, and the map $\partial V : X \mapsto \partial V(X) := dV(X)|_{C^\infty(V)}$ is a Lie algebra representation on $C^\infty(V)$ which extends to a representation of (the complexification of) the universal enveloping algebra of $\mathfrak{g}$. Given a basis $\mathcal{B} := (X_k)_{1\le k\le d}$ for $\mathfrak{g}$, we may equip $C^\infty(V)$ with a Fréchet space topology defined by the family

$$\{\rho_n : n \in \mathbb{N}\} \tag{2.1}$$

of norms, with $\rho_0(x) := \|x\|$, $dV(X_0) := I$ and

$$\rho_n(x) := \max\{\|dV(X_{i_1}) \cdots dV(X_{i_n})x\| : 0 \le i_j \le d\}, \quad n \ge 1;$$

note, however, that this topology does not depend upon the fixed basis $\mathcal{B}$.

We recall that the Heisenberg group $H_{2n+1}(\mathbb{R}) = \{(\mathtt{a}, \mathtt{b}, c) : \mathtt{a}, \mathtt{b} \in \mathbb{R}^n,\ c \in \mathbb{R}\}$ (with multiplication given by $(\mathtt{a}, \mathtt{b}, c) \cdot (\mathtt{a}', \mathtt{b}', c') := (\mathtt{a} + \mathtt{a}', \mathtt{b} + \mathtt{b}', c + c' - \langle \mathtt{b}', \mathtt{a}\rangle)$) embeds as a subgroup of the group of invertible matrices $\mathrm{GL}_{n+2}(\mathbb{R})$ via the map

$$\iota\colon (\mathtt{a},\mathtt{b},c) \mapsto \begin{bmatrix} 1 & \mathtt{a}^T & c \\ 0 & I_n & -\mathtt{b} \\ 0 & 0 & 1 \end{bmatrix},$$

with I_n denoting the identity matrix of $M_n(\mathbb{R})$. A *unitary operator* on E_n is, by definition, an adjointable operator $u \in \mathcal{L}_{\mathcal{A}}(E_n)$ satisfying $u^*u = uu^* = I$ [6, p. 24] (some might denote "unitary" instead of unitary, only to stress that E_n is not a Hilbert space). One may then define a strongly continuous unitary representation $\mathcal{U}$ of $H_{2n+1}(\mathbb{R})$ on the Hilbert $\mathcal{A}$-module E_n: first, we define it on $\mathcal{S}^{\mathcal{A}}(\mathbb{R}^n)$ by

$$\mathcal{U}_{\mathtt{a},\mathtt{b},c}(f)(x) := e^{ic}e^{i\langle \mathtt{b},x\rangle} f(x-\mathtt{a}), \quad f \in \mathcal{S}^{\mathcal{A}}(\mathbb{R}^n),\ x \in \mathbb{R}^n \tag{2.2}$$

and then extend it by continuity to all of E_n. Conjugating with the representation $\mathcal{U}$ gives rise to a (not everywhere strongly continuous) representation of the Heisenberg group $H_{2n+1}(\mathbb{R})$ on the C*-algebra of adjointable operators $\mathcal{L}_{\mathcal{A}}(E_n)$ defined by

$$(\mathrm{Ad}\,\mathcal{U})_{\mathtt{a},\mathtt{b},c}(\cdot) := \mathcal{U}_{\mathtt{a},\mathtt{b},c}\,(\cdot)\,(\mathcal{U}_{\mathtt{a},\mathtt{b},c})^{-1}. \tag{2.3}$$

Note that $(\mathrm{Ad}\,\mathcal{U})_{\mathtt{a},\mathtt{b},c}$ does not depend on the real variable c, so we will simply write $(\mathrm{Ad}\,\mathcal{U})_{\mathtt{a},\mathtt{b}}$. We will restrict our attention to the $\mathrm{Ad}\,\mathcal{U}$-invariant C*-subalgebra $C(\mathrm{Ad}\,\mathcal{U})$ of elements $A \in \mathcal{L}_{\mathcal{A}}(E_n)$ for which $\mathrm{Ad}\,\mathcal{U}$ is strongly continuous or, in other words, the C*-subalgebra $C(\mathrm{Ad}\,\mathcal{U})$ of elements $A \in \mathcal{L}_{\mathcal{A}}(E_n)$ satisfying

$$\lim_{(\mathtt{a},\mathtt{b})\to(0_n,0_n)} (\mathrm{Ad}\,\mathcal{U})_{\mathtt{a},\mathtt{b}}(A) = A$$

(0_n denotes the zero vector in $\mathbb{R}^n$). The corresponding smooth vectors for this (slightly modified) representation will be denoted by $C^\infty(\mathrm{Ad}\,\mathcal{U})$, and its elements will be referred to as *Heisenberg smooth operators*.

Let $(f_k)_{1\le k\le n}$ be the canonical basis of $\mathbb{R}^n$, and let $(X_k)_{1\le k\le 2n+1}$ be the canonical basis for the Lie algebra of $H_{2n+1}(\mathbb{R})$, in which X_k is defined as

$$(f_k, 0_n, 0), \quad \text{if } 1 \le k \le n, \quad (0_n, f_{k-n}, 0), \quad \text{if } n+1 \le k \le 2n,$$

or

$$(0_n, 0_n, 1), \quad \text{if } k = 2n+1.$$

Then for terminological convenience the restrictions of the infinitesimal generators $d(\mathrm{Ad}\,\mathcal{U})(X_k)$ to $C^\infty(\mathrm{Ad}\,\mathcal{U})$, when $1 \le k \le 2n$, will be denoted simply by ∂_k. Since $C^\infty(\mathrm{Ad}\,\mathcal{U})$ is a core for each infinitesimal generator, we have $d(\mathrm{Ad}\,\mathcal{U})(X_k) = \overline{\partial_k}$ for all $1 \le k \le 2n$, where $\overline{(\cdot)}$, as usual, denotes the closure of the operator under consideration.

Before proceeding to the main results of this section, we make a few comments about the Fourier transform $\mathcal{F}$, a Fréchet space automorphism of $\mathcal{S}^{\mathcal{A}}(\mathbb{R}^n)$ defined by

$$\mathcal{F}(g)(\xi) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\langle s,\xi\rangle}\, g(s)\, ds, \quad g \in \mathcal{S}^{\mathcal{A}}(\mathbb{R}^n),\ \xi \in \mathbb{R}^n, \tag{2.4}$$

with inverse given by $\mathcal{F}^{-1}(g)(x) = \mathcal{F}(g)(-x)$ (see [4, Section 2.4, p. 105]). It extends to an isometry on E_n. In fact, for every $f, g \in \mathcal{S}^{\mathcal{A}}(\mathbb{R}^n)$, we have

$$\begin{aligned}(2\pi)^{n/2}\langle \mathcal{F}(f), g\rangle &= \int \Big(\int e^{-i\langle x,y\rangle} f(y)\, dy\Big)^* g(x)\, dx \\ &= \int f(y)^* \Big(\int e^{i\langle x,y\rangle} g(x)\, dx\Big) dy = (2\pi)^{n/2}\langle f, \mathcal{F}^{-1}(g)\rangle, \end{aligned} \quad (2.5)$$

so substituting $g = \mathcal{F}(f)$ in the above equality shows that $\mathcal{F}$ is an isometry onto $\mathcal{S}^{\mathcal{A}}(\mathbb{R}^n)$ (with respect to the norm $\|\cdot\|_2$). Therefore, since $\mathcal{S}^{\mathcal{A}}(\mathbb{R}^n)$ is dense in E_n, we see that $\mathcal{F}$ extends to an isometry onto E_n. Moreover, by the continuity of the $\mathcal{A}$-valued inner product, we obtain $\langle \mathcal{F}(f), g\rangle = \langle f, \mathcal{F}^{-1}(g)\rangle$ for $f, g \in E_n$. This shows that $\mathcal{F}$ is an adjointable operator on E_n, with $\mathcal{F}^* = \mathcal{F}^{-1}$, that is, a generalized form of Plancherel's theorem holds for the Hilbert $\mathcal{A}$-module E_n.

Now, we focus on the main objective of this section, which is to prove that every Heisenberg smooth operator leaves $\mathcal{S}^{\mathcal{A}}(\mathbb{R}^n)$ invariant. We begin with an auxiliary lemma, which proves that each such operator maps $\mathcal{S}^{\mathcal{A}}(\mathbb{R}^n)$ into the space of smooth vectors for the unitary representation $\mathcal{U}$.

Lemma 2.1. *Every Heisenberg smooth operator $A \in \mathcal{L}_{\mathcal{A}}(E_n)$ maps $\mathcal{S}^{\mathcal{A}}(\mathbb{R}^n)$ into $C^\infty(\mathcal{U})$.*

Proof. Let $(f_k)_{1\le k\le n}$ be the canonical basis of $\mathbb{R}^n$ and $f \in \mathcal{S}^{\mathcal{A}}(\mathbb{R}^n)$. Moreover, let $(X_k)_{1\le k\le 2n+1}$ be the canonical basis for the Lie algebra of $H_{2n+1}(\mathbb{R})$, as defined just before equation (2.4).

For each fixed $v \in \mathbb{R}^n$, let T_v denote the continuous extension to E_n of the operator on $\mathcal{S}^{\mathcal{A}}(\mathbb{R}^n)$ defined by $(T_v\, g)(\cdot) := g(\cdot - v)$. As a first step, let us check that the limit

$$\lim_{h\to 0} \frac{(T_{-hf_k} Af) - Af}{h} \quad (2.6)$$

exists in E_n for all $1 \le k \le n$; to simplify notation, let us write h for hf_k. Write the decomposition

$$\frac{T_{-h}(Af) - Af}{h} = \underbrace{\frac{(T_{-h}AT_h)f - Af}{h}}_{\text{(I)}} + \underbrace{\frac{(T_{-h}AT_h)(T_{-h}f) - (T_{-h}AT_h)f}{h}}_{\text{(II)}},$$

and let us calculate the limits of the expressions (I) and (II) when $h \to 0$. We begin with (I), which is simpler: by the definition of ∂_k, we have

$$\lim_{h\to 0}\Big\| \frac{(T_{-h}AT_h)f - Af}{h} - (-\partial_k A)f\Big\|_2 \le \lim_{h\to 0}\Big\|\frac{T_{-h}AT_h - A}{h} - (-\partial_k A)\Big\| \|f\|_2 = 0,$$

since A is a Heisenberg smooth operator and $T_{-h}AT_h = (\mathrm{Ad}\,\mathcal{U})_{-hf_k,0}(A)$. To analyze (II), we first write the decomposition

$$\begin{aligned}&\frac{(T_{-h}AT_h)(T_{-h}f) - (T_{-h}AT_h)f}{h}\\ &\quad= \underbrace{\left[\frac{(T_{-h}AT_h)(T_{-h}f) - (T_{-h}AT_h)f}{h} - (T_{-h}AT_h)\frac{\partial}{\partial x_k}f\right]}_{\text{(III)}}\\ &\qquad+ \underbrace{\left[(T_{-h}AT_h)\left(\frac{\partial}{\partial x_k}f\right) - A\left(\frac{\partial}{\partial x_k}f\right)\right]}_{\text{(IV)}} + A\left(\frac{\partial}{\partial x_k}f\right).\end{aligned}$$

The expression corresponding to (IV) goes to 0, as $h \to 0$, because A is a Heisenberg smooth operator so, in particular, $\lim_{h\to 0}\|T_{-h}AT_h - A\| = 0$. To see that the expression corresponding to (III) also goes to 0, as $h \to 0$, first note that the sum

$$\frac{T_{-h} - I}{h}f - \frac{\partial}{\partial x_k}f$$

converges to 0 in $(\mathcal{S}^{\mathcal{A}}(\mathbb{R}^n), \|\cdot\|_2)$ (because it does in the natural Fréchet topology of $\mathcal{S}^{\mathcal{A}}(\mathbb{R}^n)$), so

$$\begin{aligned}&\lim_{h\to 0}\left\|\frac{(T_{-h}AT_h)(T_{-h}f) - (T_{-h}AT_h)f}{h} - (T_{-h}AT_h)\frac{\partial}{\partial x_k}f\right\|_2\\ &\quad\le \|A\| \lim_{h\to 0}\left\|\frac{T_{-h} - I}{h}f - \frac{\partial}{\partial x_k}f\right\|_2 = 0.\end{aligned}$$

Therefore, we get

$$\lim_{h\to 0}\frac{(T_{-h}AT_h)(T_{-h}f) - (T_{-h}AT_h)f}{h} = A\left(\frac{\partial}{\partial x_k}f\right).$$

Combining the analyses of (I) and (II) proves that the limit (2.6) exists in E_n and satisfies

$$\lim_{h\to 0}\frac{(T_{-hf_k}Af) - Af}{h} = (-\partial_k A)f + A\left(\frac{\partial}{\partial x_k}f\right). \tag{2.7}$$

In particular, this shows that Af belongs to $\mathrm{Dom}\, d\mathcal{U}(X_k)$ for all $1 \le k \le n$.

Now, for each fixed $v \in \mathbb{R}^n$, denote by M_v the continuous extension to E_n of the operator on $\mathcal{S}^{\mathcal{A}}(\mathbb{R}^n)$ defined by $(M_v\, g)(\cdot) := e^{i\langle v,\cdot\rangle}\, g(\cdot)$. Then we prove with an analogous reasoning that the limit $\lim_{h\to 0} h^{-1}[(M_{hf_k}Af) - Af]$ exists in E_n and that

$$\lim_{h\to 0}\frac{(M_{hf_k}Af) - Af}{h} = (\partial_{n+k}A)f + A(ix_k\, f), \tag{2.8}$$

where ix_k denotes the multiplication operator on $\mathcal{S}^{\mathcal{A}}(\mathbb{R}^n)$ given by $(ix_k\, g)(x) := ix_k\, g(x)$. This shows that Af belongs to $\mathrm{Dom}\, d\mathcal{U}(X_{n+k})$ for all $f \in \mathcal{S}^{\mathcal{A}}(\mathbb{R}^n)$ and $1 \le k \le n$.

Clearly, Af also belongs to $\mathrm{Dom}\, d\mathcal{U}(X_{2n+1})$, hence we conclude that $Af \in \bigcap_{j=1}^{2n+1} \mathrm{Dom}\, d\mathcal{U}(X_j)$. Since the Heisenberg smooth operator $A \in \mathcal{L}_{\mathcal{A}}(E_n)$ and the function $f \in \mathcal{S}^{\mathcal{A}}(\mathbb{R}^n)$ were arbitrary, an inductive procedure on the above calculations shows that every smooth vector A sends $\mathcal{S}^{\mathcal{A}}(\mathbb{R}^n)$ into

$$\bigcap_{j=1}^{2n+1} \bigcap_{m\in\mathbb{N}} \mathrm{Dom}(d\mathcal{U}(X_j))^m = C^\infty(\mathcal{U})$$

(for this last equality see, e.g., [13, p. 90]). ∎

It is clear that the inclusion $\mathcal{S}^{\mathcal{A}}(\mathbb{R}^n) \subseteq C^\infty(\mathcal{U})$ holds. We will now prove that $C^\infty(\mathcal{U})$ actually coincides with the function space $\mathcal{S}^{\mathcal{A}}(\mathbb{R}^n)$.

Proposition 2.2. *The space of smooth vectors $C^\infty(\mathcal{U})$ for the unitary representation $\mathcal{U}$ coincides with $\mathcal{S}^{\mathcal{A}}(\mathbb{R}^n)$.*

Proof. In this first step of the proof, we show that the families $\{p_{\alpha,\beta}\}_{\alpha,\beta\in\mathbb{N}^n}$ and $\{q_{N_1,N_2}\}_{N_1,N_2\in\mathbb{N}}$ of seminorms defined by

$$\begin{aligned} p_{\alpha,\beta}(f) &:= \sup_{x\in\mathbb{R}^n} \|x^\alpha\, \partial_x^\beta f(x)\|_{\mathcal{A}}, && f \in \mathcal{S}^{\mathcal{A}}(\mathbb{R}^n),\ \alpha,\beta \in \mathbb{N}^n \\ q_{N_1,N_2}(f) &:= \Bigg(\sum_{|\alpha|\le N_1,\, |\beta|\le N_2} \|x^\alpha\, \partial_x^\beta f\|_2^2 \Bigg)^{1/2}, && f \in \mathcal{S}^{\mathcal{A}}(\mathbb{R}^n),\ N_1,N_2 \in \mathbb{N} \end{aligned} \tag{2.9}$$

generate equivalent topologies on $\mathcal{S}^{\mathcal{A}}(\mathbb{R}^n)$. The topology generated by the family $\{p_{\alpha,\beta}\}_{\alpha,\beta\in\mathbb{N}^n}$ is finer than the one generated by $\{q_{N_1,N_2}\}_{N_1,N_2\in\mathbb{N}}$, because using the identity

$$(1+|x|^2)^m = \sum_{\gamma\in\mathbb{N}^n,\, |\gamma|\le m} C_{m,\gamma}\, x^{2\gamma}, \quad C_{m,\gamma} := \frac{m!}{(m-|\gamma|)!\,|\gamma|!} \cdot \frac{|\gamma|!}{\gamma_1!\cdots\gamma_n!}$$

we see that, for all $m \in \mathbb{N}$, $m > n/2$, there exists $C_m > 0$ such that

$$\begin{aligned} \|x^\alpha\, \partial_x^\beta f\|_2^2 &\le \int_{\mathbb{R}^n} \frac{(1+|x|^2)^m\, \|x^\alpha\, \partial_x^\beta f(x)\|_{\mathcal{A}}^2}{(1+|x|^2)^m}\, dx \\ &\le \sum_{|\gamma|\le m} \sup_{x\in\mathbb{R}^n} \|x^{\alpha+2\gamma}\, \partial_x^\beta f(x)\|_{\mathcal{A}}^2 \int_{\mathbb{R}^n} \frac{C_{m,\gamma}}{(1+|x|^2)^m}\, dx \\ &\le C_m \Bigg(\sum_{|\gamma|\le m} \sup_{x\in\mathbb{R}^n} \|x^{\alpha+2\gamma}\, \partial_x^\beta f(x)\|_{\mathcal{A}} \Bigg)^2 \end{aligned} \tag{2.10}$$

for every $f \in \mathcal{S}^{\mathcal{A}}(\mathbb{R}^n)$ and $\alpha,\beta \in \mathbb{N}^n$.

On the other hand, applying Fourier's inversion formula combined with the Cauchy–Schwarz inequality for Hilbert C*-modules [6, Proposition 1.1, p. 3] and equation (2.5), we obtain, for each $m \in \mathbb{N}$ satisfying $m > n/2$, a constant $D_m > 0$ such that the following estimate holds:

$$\begin{aligned}\|x^\alpha \partial_x^\beta f(x)\|_{\mathcal{A}} &= \|(\mathcal{F}^{-1} \circ \mathcal{F})(y^\alpha \partial_y^\beta f)(x)\|_{\mathcal{A}} \\ &= \frac{1}{(2\pi)^{n/2}} \left\| \int_{\mathbb{R}^n} \left[\frac{e^{i\langle \xi, x\rangle}}{(1+|\xi|^2)^m} 1_{\mathcal{A}} \right] (1+|\xi|^2)^m \, [\mathcal{F}(y^\alpha \partial_y^\beta f)](\xi)\, d\xi \right\|_{\mathcal{A}} \\ &\leq \frac{1}{(2\pi)^{n/2}} \left\| \frac{e^{i\langle \xi, x\rangle}}{(1+|\xi|^2)^m} 1_{\mathcal{A}} \right\|_2 \|(1+|\xi|^2)^m \, [\mathcal{F}(y^\alpha \partial_y^\beta f)](\xi)\|_2 \\ &\overset{(2.5)}{=} D_m \, \|(1-\Delta_\xi)^m \, [\xi^\alpha \partial_\xi^\beta f](\xi)\|_2, \quad x \in \mathbb{R}^n \end{aligned} \tag{2.11}$$

(note that, in order to guarantee that the function $\xi \mapsto (e^{i\langle \xi,x\rangle}/(1+|\xi|^2)^m)\, 1_{\mathcal{A}}$ belongs to E_n, we have used the fact that $L^2(\mathbb{R}^n)$ is isometrically embedded in E_n; see [1, Remark D.2, Appendix D]). Taking the supremum over all $x \in \mathbb{R}^n$ in (2.11) shows that the topology generated by the family $\{p_{\alpha,\beta}\}_{\alpha,\beta\in\mathbb{N}^n}$ is coarser than the one generated by $\{q_{N_1,N_2}\}_{N_1,N_2\in\mathbb{N}}$. This proves our first claim: the topologies generated by the two sets of seminorms in (2.9) are equivalent.

Note that $\mathcal{S}^{\mathcal{A}}(\mathbb{R}^n) \subseteq C^\infty(\mathcal{U})$ is dense in E_n and $\mathcal{U}_{\mathsf{a},\mathsf{b},c}[\mathcal{S}^{\mathcal{A}}(\mathbb{R}^n)] \subseteq \mathcal{S}^{\mathcal{A}}(\mathbb{R}^n)$ for all $\mathsf{a}, \mathsf{b} \in \mathbb{R}^n$ and $c \in \mathbb{R}$. Therefore, it follows from [13, Theorem 1.3] that $\mathcal{S}^{\mathcal{A}}(\mathbb{R}^n)$ is dense in $C^\infty(\mathcal{U})$ with respect to the Fréchet topology generated by the family (2.1) of norms on $C^\infty(\mathcal{U})$ (with $\|\cdot\|$ and $\{V_g\}_{g\in G}$ replaced by $\|\cdot\|_2$ and $\{\mathcal{U}_g\}_{g\in H_{2n+1}(\mathbb{R})}$, respectively). The topology induced on $\mathcal{S}^{\mathcal{A}}(\mathbb{R}^n)$ by the above family of norms on $C^\infty(\mathcal{U})$ coincides with the one generated by the family $\{q_{N_1,N_2}\}_{N_1,N_2\in\mathbb{N}}$ of norms. But, in our first claim, we have proved, in particular, that equipping $\mathcal{S}^{\mathcal{A}}(\mathbb{R}^n)$ with the family $\{q_{N_1,N_2}\}_{N_1,N_2\in\mathbb{N}}$ of norms turns it into a Fréchet space, because we already know that $\mathcal{S}^{\mathcal{A}}(\mathbb{R}^n)$, when equipped with the family $\{p_{\alpha,\beta}\}_{\alpha,\beta\in\mathbb{N}^n}$ of seminorms, becomes a Fréchet space. This shows that $\mathcal{S}^{\mathcal{A}}(\mathbb{R}^n)$ is a closed and dense subspace of the Fréchet space $C^\infty(\mathcal{U})$, which forces the equality $\mathcal{S}^{\mathcal{A}}(\mathbb{R}^n) = C^\infty(\mathcal{U})$. ∎

The main theorem of this section is an immediate corollary of Lemma 2.1 combined with Proposition 2.2.

Proposition 2.3. *Every Heisenberg smooth operator $A \in \mathcal{L}_{\mathcal{A}}(E_n)$ maps $\mathcal{S}^{\mathcal{A}}(\mathbb{R}^n)$ into $\mathcal{S}^{\mathcal{A}}(\mathbb{R}^n)$.*

3. An approximation theorem for Cordes' symbol map

Throughout the rest of the manuscript, we will denote $\mathbb{N}^* := \mathbb{N}\backslash\{0\}$. Also, since we will need to invoke certain results about vector-valued integration from reference [4], we shall adopt some of its notations and definitions. Let μ be the Lebesgue measure on $\mathbb{R}^n$. Then we shall say that a function $f\colon \mathbb{R}^n \to \mathcal{A}$ is *μ-simple* if $f(x) = \sum_{j=1}^N 1_{B_j}(x)\, a_j$ for some fixed natural number $N > 0$ and all $x \in \mathbb{R}^n$, with a_j being certain elements of $\mathcal{A}$ and 1_{B_j}

the indicator functions of Lebesgue-measurable subsets B_j of $\mathbb{R}^n$ satisfying $\mu(B_j) < +\infty$ for all $1 \leq j \leq N$ [4, Definition 1.1.13, p. 8]. Moreover, we shall say that a function $f : \mathbb{R}^n \to \mathcal{A}$ is *strongly μ-measurable* if it is the μ-almost everywhere pointwise limit of a sequence of μ-simple functions [4, Definition 1.1.14, p. 8]. Finally, the space of equivalence classes of strongly μ-measurable square-integrable $\mathcal{A}$-valued functions on $\mathbb{R}^n$ will be denoted by $L^2(\mathbb{R}^n, \mathcal{A})$ [4, Definition 1.2.15, p. 21]. It will be customary to write simply dx, instead of $d\mu(x)$, when integrating a strongly μ-measurable function $f : \mathbb{R}^n \to \mathcal{A}$, $x \mapsto f(x)$, with respect to the Lebesgue measure μ. Hence, if $g \in L^2(\mathbb{R}^n, \mathcal{A})$, then

$$\|g\|_{L^2} := \left(\int_{\mathbb{R}^n} \|g(x)\|_{\mathcal{A}}^2 \, dx \right)^{1/2} = \left(\int_{\mathbb{R}^n} \|g(x)^* g(x)\|_{\mathcal{A}} \, dx \right)^{1/2} < +\infty. \tag{3.1}$$

Comparing with (1.2), it is clear that $\|g\|_2 \leq \|g\|_{L^2} < +\infty$ for all $g \in \mathcal{S}^{\mathcal{A}}(\mathbb{R}^n)$. Also, $\mathcal{S}^{\mathcal{A}}(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n, \mathcal{A})$ with respect to the norm $\|\cdot\|_{L^2}$ (this follows from a standard argument combining [4, Lemma 1.2.31, p. 29] and [4, Proposition 1.2.32, p. 29]). Another fact which we shall frequently use throughout the paper is that $L^2(\mathbb{R}^n, \mathcal{A})$ is continuously embedded in E_n as a dense subspace [1, Appendix D], with $L^2(\mathbb{R}^n)$ being isometrically embedded in E_n [1, Remark D.2, Appendix D].

Let J be a skew-symmetric linear transformation on $\mathbb{R}^n$, $f, g \in \mathcal{S}^{\mathcal{A}}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$. Then using the relations $\langle \xi, J\xi \rangle = \langle w, Jw \rangle = 0$, $\xi, w \in \mathbb{R}^n$ and equation (1.4), we obtain

$$\begin{aligned} L_f(g)(x) &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f\left(x - \frac{1}{2\pi} J\xi\right) g(y)\, e^{i\langle \xi, x-y \rangle} \, dy \, d\xi \\ &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f\left(x - \frac{1}{2\pi} J\xi\right) \mathcal{F}(g)(\xi)\, e^{i\langle \xi, x \rangle} \, d\xi \\ &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \mathcal{F}(f)(w)\, \mathcal{F}(g)(\xi)\, e^{i\langle w+\xi, x-(2\pi)^{-1}J\xi \rangle} \, dw \, d\xi \\ &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \mathcal{F}(f)(w)\, \mathcal{F}(g)(\xi - w)\, e^{i\langle \xi, x-(2\pi)^{-1}J(\xi-w) \rangle} \, d\xi \, dw \\ &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \mathcal{F}(f)(w)\, \mathcal{F}(g)(\xi - w)\, e^{i\langle \xi, x+(2\pi)^{-1}Jw \rangle} \, d\xi \, dw \\ &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \mathcal{F}(f)(w)\, g\left(x + \frac{1}{2\pi} Jw\right) e^{i\langle w, x \rangle} \, dw. \end{aligned} \tag{3.2}$$

Note that the integral in

$$L_f(g)(\cdot) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \mathcal{F}(f)(w)\, g\left(\cdot + \frac{1}{2\pi} Jw\right) e^{i\langle w, \cdot \rangle} \, dw$$

remains absolutely convergent in the L^2-sense even if $g \in L^2(\mathbb{R}^n, \mathcal{A})$, with

$$\begin{aligned} \|L_f(g)\|_{L^2} &\leq \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \|\mathcal{F}(f)(w)\, \mathcal{U}_{-\frac{1}{2\pi}Jw, w, 0}(g)\|_{L^2} \, dw \\ &\leq \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \|\mathcal{F}(f)(w)\|_{\mathcal{A}} \, \|\mathcal{U}_{-\frac{1}{2\pi}Jw, w, 0}(g)\|_{L^2} \, dw \\ &= \frac{1}{(2\pi)^{n/2}} \|\mathcal{F}(f)\|_1 \, \|g\|_{L^2}, \end{aligned} \tag{3.3}$$

where $\|\cdot\|_1$ denotes the usual L^1-norm. This proves the following useful result.

Lemma 3.1. *Let $f \in \mathcal{S}^{\mathcal{A}}(\mathbb{R}^n)$. Then L_f extends to a continuous operator on $L^2(\mathbb{R}^n, \mathcal{A})$ such that the evaluation of L_f on an element $g \in L^2(\mathbb{R}^n, \mathcal{A})$ is given by*

$$L_f(g)(\cdot) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \mathcal{F}(f)(\xi)\, g\left(\cdot + \frac{1}{2\pi} J\xi\right) e^{i\langle \xi, \cdot\rangle}\, d\xi, \quad \textit{with} \tag{3.4}$$

$$\|L_f(g)\|_{L^2} \leq \frac{1}{(2\pi)^{n/2}} \|\mathcal{F}(f)\|_1\, \|g\|_{L^2}. \tag{3.5}$$

Now consider a function $\psi \in C_c^\infty(\mathbb{R}^n)$ satisfying $\psi \geq 0$ and $\int_{\mathbb{R}^n} \psi(\xi)\, d\xi = 1$ such that its support $\operatorname{supp}\psi$ is contained in the open ball $B(0,1)$ of radius 1, centered at the origin. For each $m \in \mathbb{N}^*$, define $\psi_m(\xi) := m^n\, \psi(m\xi)$, so that $\operatorname{supp}\psi_m \subseteq B(0, 1/m)$ and $\int_{\mathbb{R}^n} \psi_m(\xi)\, d\xi = 1$, and define

$$e_m := (2\pi)^{n/2}\, \mathcal{F}^{-1}(\psi_m), \quad \widetilde{e}_m := e_m \cdot 1_{\mathcal{A}}. \tag{3.6}$$

Lemma 3.2. *For each fixed $g \in L^2(\mathbb{R}^n, \mathcal{A})$ and each polynomial function p on $\mathbb{R}^n$ satisfying $p(0) = 0$, we have*

$$\lim_{m\to+\infty} \int_{\mathbb{R}^n} p(\xi)\, \psi_m(\xi)\, g\left(\cdot + \frac{1}{2\pi} J\xi\right) e^{i\langle \xi, \cdot\rangle}\, d\xi = 0 \textit{ in } L^2(\mathbb{R}^n, \mathcal{A}). \tag{3.7}$$

Proof. If $p(\xi) := \sum_{0<|\alpha|\leq d} c_\alpha\, \xi^\alpha$, with $c_\alpha \in \mathbb{C}$, $\alpha \in \mathbb{N}^n$, then

$$p(\xi)\, \mathcal{F}(\widetilde{e}_m)(\xi) = \mathcal{F}\left(p\left(\frac{1}{i^{\alpha_1}} \frac{\partial^{\alpha_1}}{\partial \xi_1^{\alpha_1}}, \ldots, \frac{1}{i^{\alpha_n}} \frac{\partial^{\alpha_n}}{\partial \xi_n^{\alpha_n}}\right) \widetilde{e}_m\right)(\xi), \quad m \in \mathbb{N}^*.$$

Therefore,

$$p(\xi)\, \psi_m(\xi) \cdot 1_{\mathcal{A}} = \mathcal{F}\Bigg(\underbrace{\frac{1}{(2\pi)^{n/2}}\, p\left(\frac{1}{i^{\alpha_1}} \frac{\partial^{\alpha_1}}{\partial \xi_1^{\alpha_1}}, \ldots, \frac{1}{i^{\alpha_n}} \frac{\partial^{\alpha_n}}{\partial \xi_n^{\alpha_n}}\right) \widetilde{e}_m}_{:= f_m}\Bigg)(\xi), \quad m \in \mathbb{N}^*, \tag{3.8}$$

so, as noted in Lemma 3.1, the integral in equation (3.7) is absolutely convergent in $L^2(\mathbb{R}^n, \mathcal{A})$ and equals $L_{f_m}(g)$. Fix $\varepsilon > 0$ and let $\delta > 0$ be a real number such that $|p(\xi)| < \varepsilon/(\|g(\cdot)\|_{L^2} + 1)$ for all $\xi \in \mathbb{R}^n$ satisfying $|\xi| < \delta$. Also, fix $m_0 \in \mathbb{N}^*$ such that $1/m_0 < \delta$. Then reasoning similarly as in equation (3.3) shows that

$$\begin{aligned}\Big\| \int_{\mathbb{R}^n} p(\xi)\,\psi_m(\xi)\,g\Big(\cdot + \frac{1}{2\pi}J\xi\Big)e^{i\langle\xi,\cdot\rangle}\,d\xi\Big\|_{L^2} &\le \int_{\mathbb{R}^n}\psi_m(\xi)\,|p(\xi)|\|g(\cdot)\|_{L^2}\,d\xi \\ &= \underbrace{\int_{|\xi|<\delta}\psi_m(\xi)\,|p(\xi)|\|g(\cdot)\|_{L^2}\,d\xi}_{(I)} \\ &\quad + \underbrace{\int_{|\xi|\ge\delta}\psi_m(\xi)\,|p(\xi)|\|g(\cdot)\|_{L^2}\,d\xi}_{(II)}, \quad m\in\mathbb{N}^*. \end{aligned} \tag{3.9}$$

Since $\int_{\mathbb{R}^n}\psi_m(\xi)\,d\xi = 1$ for all $m \in \mathbb{N}^*$, we conclude that $(I) < \varepsilon$. Furthermore, for all $m \in \mathbb{N}$ satisfying $m \ge m_0$, we have $(II) = 0$, since $\operatorname{supp}\psi_m \subseteq B(0,1/m) \subseteq B(0,\delta)$. This proves the result. ∎

Lemma 3.3. *Let $D_0 := \sum_{0<|\alpha|\le d} c_\alpha\,\partial^\alpha$ be a constant coefficient differential operator of order d, where $c_\alpha \in \mathbb{C}$, $\alpha \in \mathbb{N}^{2n}$ and $\partial^\alpha := \partial_1^{\alpha_1}\cdots\partial_{2n}^{\alpha_{2n}}$ is a monomial in the generators of the adjoint representation* $\operatorname{Ad}\mathcal{U}$ *(note that $c_0 = 0$). For each fixed $g \in L^2(\mathbb{R}^n,\mathcal{A})$, we have the equality*

$$\lim_{m\to+\infty} D_0(L_{\widetilde{e}_m})(g) = 0 \quad \textit{in } L^2(\mathbb{R}^n,\mathcal{A}).$$

Proof. Denote by $(f_k)_{1\le k\le n}$ the canonical basis of $\mathbb{R}^n$ and, for each $v \in \mathbb{R}^n$, denote by ∂_v the operator on $\mathcal{S}^{\mathcal{A}}(\mathbb{R}^n)$ which associates the corresponding directional derivative $\partial_v\phi$ of a function $\phi \in \mathcal{S}^{\mathcal{A}}(\mathbb{R}^n)$. We note that, as shown in [1, Subsection "The algebra $\mathcal{B}_J^{\mathbb{C}}(\mathbb{R}^n)$", equation (3.22)], the evaluation of the differential operator D_0 on $L_{\widetilde{e}_m}$ is legitimate, with

$$\partial_1^{\beta_1}\cdots\partial_{2n}^{\beta_{2n}}(L_{\widetilde{e}_m}) = (-1)^{|\beta|}\,L_{\partial_{v_1}^{\beta_1}\cdots\partial_{v_{2n}}^{\beta_{2n}}\widetilde{e}_m},$$

where $|\beta| := \sum_{k=1}^{2n}\beta_k$, $v_k = f_k$, if $1 \le k \le n$ and $v_k = J(f_{k-n})/2\pi$, if $n+1 \le k \le 2n$ (note that ∂_k is acting on operators in $C^\infty(\operatorname{Ad}\mathcal{U})$, while ∂_{v_k} acts on functions in $\mathcal{S}^{\mathcal{A}}(\mathbb{R}^n)$).

Let T_v be the translation operator on E_n defined in Lemma 2.1. Then since $L^2(\mathbb{R}^n,\mathcal{A})$ is continuously embedded in E_n as a dense subspace, we may apply the operators T_v to elements of $L^2(\mathbb{R}^n,\mathcal{A})$. In the following calculations, we will use the simplified notation $\mathcal{U}_{hf_k}$ to denote both the operators $\mathcal{U}_{hf_k,0,0}$ and $\mathcal{U}_{0,hf_k,0}$. Using (3.4) and (3.5), we obtain, for a fixed $1 \le k \le 2n$ and every $0 \ne h \in \mathbb{R}$, $\phi \in \mathcal{S}^{\mathcal{A}}(\mathbb{R}^n)$,

$$\Big\| \Big[\frac{\mathcal{U}_{hf_k}\,L_\phi\,\mathcal{U}_{-hf_k} - L_\phi}{h}\Big](g)(\cdot) - \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}\mathcal{F}(-\partial_{v_k}\phi)(\xi)\,g\Big(\cdot + \frac{1}{2\pi}J\xi\Big)\,e^{i\langle\xi,\cdot\rangle}\,d\xi\Big\|_{L^2}$$

$$= \left\| \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \mathcal{F}\left(\frac{T_{hv_k}\phi - \phi}{h} - (-\partial_{v_k})\phi \right)(\xi)\, g\left(\cdot + \frac{1}{2\pi} J\xi \right) e^{i\langle \xi, \cdot \rangle}\, d\xi \right\|_{L^2}$$
$$\leq \frac{1}{(2\pi)^{n/2}} \left\| \mathcal{F}\left(\frac{T_{hv_k}\phi - \phi}{h} - (-\partial_{v_k})\phi \right) \right\|_1 \|g\|_{L^2}.$$

Since

$$\mathcal{F}\left(\frac{T_{hv_k}\phi - \phi}{h} - (-\partial_{v_k})\phi \right) \longrightarrow 0 \quad \text{in } \mathcal{S}^{\mathcal{A}}(\mathbb{R}^n) \text{ (hence, in } L^1(\mathbb{R}^n, \mathcal{A}))$$

when $h \to 0$, we see that

$$\partial_k(L_\phi)(g) = -\frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \mathcal{F}(\partial_{v_k}\phi)(\xi)\, g\left(\cdot + \frac{1}{2\pi} J\xi \right) e^{i\langle \xi, \cdot \rangle}\, d\xi \; (= -L_{\partial_{v_k}\phi}(g)),$$
$$\phi \in \mathcal{S}^{\mathcal{A}}(\mathbb{R}^n).$$

Iterating this procedure and using the identity $\mathcal{F}((\partial/\partial_{\xi_l})(\phi))(\xi) = i\xi_l\, (\mathcal{F}(\phi))(\xi)$, $1 \leq l \leq n$, show that, if $\alpha \in \mathbb{N}^{2n}$, $\alpha \neq 0$, then

$$\partial^\alpha(L_\phi)(g) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} p_\alpha(\xi)\, \mathcal{F}(\phi)(\xi)\, g\left(\cdot + \frac{1}{2\pi} J\xi \right) e^{i\langle \xi, \cdot \rangle}\, d\xi, \quad \phi \in \mathcal{S}^{\mathcal{A}}(\mathbb{R}^n),$$

where p_α is a polynomial of degree $\deg p_\alpha \geq 1$ satisfying $p_\alpha(0) = 0$. Substituting $\phi = \widetilde{e}_m$ and using that $\mathcal{F}(\widetilde{e}_m) = (2\pi)^{n/2}\, \psi_m \cdot 1_{\mathcal{A}}$, we get from the definition of D_0 the relation

$$D_0(L_{\widetilde{e}_m})(g) = \int_{\mathbb{R}^n} p(\xi)\, \psi_m(\xi)\, g\left(\cdot + \frac{1}{2\pi} J\xi \right) e^{i\langle \xi, \cdot \rangle}\, d\xi, \quad m \in \mathbb{N}^*,$$

where p is a linear combination of polynomials q of degree $\deg q \geq 1$ satisfying $q(0) = 0$ (so p also satisfies $p(0) = 0$). Therefore, by Lemma 3.2, the limit $\lim_{m\to+\infty} D_0(L_{\widetilde{e}_m})(g)$ exists in $L^2(\mathbb{R}^n, \mathcal{A})$ and equals zero. ■

Now consider the symbol map (see [8, equation (12)])

$$S\colon C^\infty(\mathrm{Ad}\,\mathcal{U}) \longrightarrow \mathcal{B}^{\mathcal{A}}(\mathbb{R}^{2n})$$

given by

$$S(A)(x,\xi)$$
$$:= (2\pi)^{n/2} \langle u \cdot 1_{\mathcal{A}}, \{(D\,[(\mathrm{Ad}\,\mathcal{U})_{-x,-\xi}(A)] \circ \mathcal{F}^{-1}) \otimes I_{E_n}\}\, v \cdot 1_{\mathcal{A}} \rangle_{E_{2n}} \qquad (3.10)$$

for all $A \in C^\infty(\mathrm{Ad}\,\mathcal{U})$ and $(x,\xi) \in \mathbb{R}^{2n}$, where $D := \prod_{j=1}^n (1 + \partial_{x_j})^2 (1 + \partial_{\xi_j})^2$ and u and v are (fixed) suitable scalar-valued functions belonging to $L^2(\mathbb{R}^{2n}) \cap L^1(\mathbb{R}^{2n})$ (for more information on the tensor product operator in equation (3.10), see [1, Appendix C, Lemma C.1]). We make the trivial, but important, observation that the formula defining the map S remains unchanged if we substitute $\{(D\,[(\mathrm{Ad}\,\mathcal{U})_{-x,-\xi}(A)] \circ \mathcal{F}^{-1}) \otimes I_{E_n}\}$ by the restricted map

$$\{(D\,[(\mathrm{Ad}\,\mathcal{U})_{-x,-\xi}(A)] \circ \mathcal{F}^{-1}) \otimes I_{E_n}\}|_{L^2(\mathbb{R}^{2n})\cdot 1_{\mathcal{A}}} \colon L^2(\mathbb{R}^{2n}) \cdot 1_{\mathcal{A}} \longrightarrow E_{2n}.$$

Define the restriction map $\mathcal{R} \colon \mathcal{B}^{\mathcal{A}}(\mathbb{R}^{2n}) \to \mathcal{B}^{\mathcal{A}}(\mathbb{R}^{n})$ which takes the function $f \colon (x,\xi) \mapsto f(x,\xi)$ in $\mathcal{B}^{\mathcal{A}}(\mathbb{R}^{2n})$ to the function $\mathcal{R}f(x) := f(x,0)$, and let

$$L \colon \mathcal{B}^{\mathcal{A}}(\mathbb{R}^{n}) \longrightarrow C^{\infty}(\mathrm{Ad}\,\mathcal{U})$$

be the map which sends a function $f \in \mathcal{B}^{\mathcal{A}}(\mathbb{R}^{n})$ to the operator L_f. Using [8, Theorem 1] (which remains valid even if $\mathcal{A}$ is non-separable – see [1, Appendix C]), we see that this map satisfies

$$\mathcal{R} \circ S \circ L = \mathrm{Id}_{\mathcal{B}^{\mathcal{A}}(\mathbb{R}^{n})}, \tag{3.11}$$

where $\mathrm{Id}_{\mathcal{B}^{\mathcal{A}}(\mathbb{R}^{n})}$ is the identity operator on $\mathcal{B}^{\mathcal{A}}(\mathbb{R}^{n})$.

Before we can establish the main result of this section (Proposition 3.7), we need two auxiliary lemmas (the first one, below, is an adaptation of [10, Proposição 2.5]).

Lemma 3.4. *Let $(\widetilde{e}_m)_{m\in\mathbb{N}^*}$ be the sequence in $\mathcal{S}^{\mathcal{A}}(\mathbb{R}^{n})$ introduced in equation (3.6). Then for every $g \in \mathcal{S}^{\mathcal{A}}(\mathbb{R}^{n})$, we have the equality*

$$\lim_{m\to+\infty} L_{\widetilde{e}_m}(g) = g \quad \text{in } E_n. \tag{3.12}$$

Proof. Fix a skew-symmetric linear transformation J on $\mathbb{R}^n$ and $0 \neq g \in \mathcal{S}^{\mathcal{A}}(\mathbb{R}^{n})$. We begin the proof by showing that for any $\varepsilon > 0$ there exists $\delta > 0$ with the property that for all $x \in \mathbb{R}^n$ and all $\xi \in \mathbb{R}^n$ satisfying $|\xi| < \delta$, we have

$$\|(1+|x|^2)^{n/2}\,[e^{i\langle x,\xi\rangle}\, g(x+J\xi) - g(x)]\|_{\mathcal{A}} < \varepsilon. \tag{3.13}$$

Take $\varepsilon > 0$ and define $K := \|g\|_\infty := \sup_{y\in\mathbb{R}^n}\|g(y)\|_{\mathcal{A}}$. Since g belongs to $\mathcal{S}^{\mathcal{A}}(\mathbb{R}^{n})$, there exists $R_1 > 1$ such that

$$(1+|y|^2)^{n/2}\,\|g(y)\|_{\mathcal{A}} \le \frac{\varepsilon}{1+4^{n/2}}, \quad \text{whenever } |y| > R_1. \tag{3.14}$$

Moreover, choose $R_2 > 0$ such that $|J\xi| < 1$ for all $\xi \in \mathbb{R}^n$ satisfying $|\xi| < R_2$.

By the uniform continuity of the maps

$$(x,\xi) \longmapsto (1+|x+J\xi|^2)^{n/2} \quad \text{and} \quad (x,\xi) \longmapsto (1+|x+J\xi|^2)^{n/2}\, e^{i\langle x,\xi\rangle}\, g(x+J\xi)$$

on the compact set $S := \{(x,\xi) \in \mathbb{R}^{2n} : |x| \le R_1+1, |\xi| \le R_2\}$, we may find $0 < R_3 < R_2$ such that if $(x,\xi) \in S$ and $|(x,\xi)-(x,0)| = |\xi| < R_3$, then

$$|(1+|x+J\xi|^2)^{n/2} - (1+|x|^2)^{n/2}| < \frac{\varepsilon}{2K}$$

and

$$\|(1+|x+J\xi|^2)^{n/2}\, e^{i\langle x,\xi\rangle}\, g(x+J\xi)-(1+|x|^2)^{n/2}\, g(x)\|_{\mathcal{A}}<\frac{\varepsilon}{2}.$$

Hence, if $|x|\le R_1+1$ and $|\xi|<R_3$, we have

$$\begin{aligned}
&\|(1+|x|^2)^{n/2}\, e^{i\langle x,\xi\rangle}\, g(x+J\xi)-(1+|x|^2)^{n/2}\, g(x)\|_{\mathcal{A}}\\
&\quad\le \|(1+|x+J\xi|^2)^{n/2}\, e^{i\langle x,\xi\rangle}\, g(x+J\xi)-(1+|x|^2)^{n/2}\, g(x)\|_{\mathcal{A}}\\
&\qquad+|(1+|x+J\xi|^2)^{n/2}-(1+|x|^2)^{n/2}|\,\|g(x+J\xi)\|_{\mathcal{A}}<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.
\end{aligned}$$

On the other hand, if $|x|>R_1+1$ and $|\xi|<R_3$, then $|x+J\xi|\ge|x|-|J\xi|>R_1>1$, so $|J\xi|<1<|x+J\xi|$. Therefore, $|x|\le|x+J\xi|+|J\xi|<2|x+J\xi|$ which, when combined with (3.14), implies the estimates

$$\begin{aligned}
&\|(1+|x|^2)^{n/2}\, e^{i\langle x,\xi\rangle}\, g(x+J\xi)-(1+|x|^2)^{n/2}\, g(x)\|_{\mathcal{A}}\\
&\quad\le(1+4|x+J\xi|^2)^{n/2}\,\|g(x+J\xi)\|_{\mathcal{A}}+(1+|x|^2)^{n/2}\,\|g(x)\|_{\mathcal{A}}\\
&\quad\le 4^{n/2}\,(1+|x+J\xi|^2)^{n/2}\,\|g(x+J\xi)\|_{\mathcal{A}}+(1+|x|^2)^{n/2}\,\|g(x)\|_{\mathcal{A}}\\
&\quad\le(1+4^{n/2})\sup_{|y|>R_1}\,[(1+|y|^2)^{n/2}\,\|g(y)\|_{\mathcal{A}}]\le(1+4^{n/2})\cdot\frac{\varepsilon}{1+4^{n/2}}=\varepsilon
\end{aligned}$$

(note that we have used the fact that $|x+J\xi|>R_1$). This establishes (3.13) with $\delta:=R_3$.

Now, we apply what was just proved for the skew-symmetric linear transformation $J'=\frac{J}{2\pi}$: if we fix $\varepsilon_0>0$, then we may use the relation (see equation (3.2))

$$L_f(g)(x)=\frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}\mathcal{F}(f)(w)\,g\left(x+\frac{1}{2\pi}Jw\right)e^{i\langle w,x\rangle}\,dw,\quad f\in\mathcal{S}^{\mathcal{A}}(\mathbb{R}^n),\ x\in\mathbb{R}^n$$

to obtain $\delta_0>0$ such that, for all $m\in\mathbb{N}$ satisfying $m>1/\delta_0$ and any fixed $x\in\mathbb{R}^n$,

$$\begin{aligned}
&(1+|x|^2)^{n/2}\,\|L_{\tilde{e}_m}(g)(x)-g(x)\|_{\mathcal{A}}\\
&\quad\le\int_{\mathbb{R}^n}\psi_m(\xi)\,(1+|x|^2)^{n/2}\left\|e^{i\langle x,\xi\rangle}g\left(x+\frac{1}{2\pi}J\xi\right)-g(x)\right\|_{\mathcal{A}}d\xi<\varepsilon_0.
\end{aligned}$$

Define $L_0:=\int_{\mathbb{R}^n}(1+|x|^2)^{-n}\,dx$. Then the above inequality allows us to conclude, for all $m\in\mathbb{N}$ satisfying $m>1/\delta_0$, the estimates

$$\begin{aligned}
\|L_{\tilde{e}_m}(g)-g\|_2^2&\le\int_{\mathbb{R}^n}\|L_{\tilde{e}_m}(g)(x)-g(x)\|_{\mathcal{A}}^2\,dx\\
&=\int_{\mathbb{R}^n}(1+|x|^2)^{-n}\,[(1+|x|^2)^{n/2}\,\|L_{\tilde{e}_m}(g)(x)-g(x)\|_{\mathcal{A}}]^2\,dx\\
&<L_0\,\varepsilon_0^2.
\end{aligned}\tag{3.15}$$

This establishes (3.12). ∎

Corollary 3.5. *Let $(\tilde{e}_m)_{m\in\mathbb{N}^*}$ be the sequence in $\mathcal{S}^{\mathcal{A}}(\mathbb{R}^n)$ introduced in equation (3.6). Then for every $\tilde{f} \in L^2(\mathbb{R}^n)\cdot 1_{\mathcal{A}}$, we have the equality*

$$\lim_{m\to+\infty} L_{\tilde{e}_m}(\tilde{f}) = \tilde{f} \quad \text{in } E_n. \tag{3.16}$$

Proof. By making use of equation (3.5) from Lemma 3.1 and the definition of $\tilde{e}_m := (2\pi)^{n/2}\,\mathcal{F}^{-1}(\psi_m)\cdot 1_{\mathcal{A}}$, we obtain

$$\|L_{\tilde{e}_m}(\tilde{h})\|_{L^2} \le \|\psi_m\cdot 1_{\mathcal{A}}\|_1\,\|\tilde{h}\|_{L^2} = \|\tilde{h}\|_{L^2}, \quad m\in\mathbb{N}^*,\ \tilde{h}\in L^2(\mathbb{R}^n)\cdot 1_{\mathcal{A}}.$$

Therefore, the result follows from Lemma 3.4 by noting in particular that (1) each $L_{\tilde{e}_m}$ leaves $L^2(\mathbb{R}^n)\cdot 1_{\mathcal{A}}$ invariant, (2) restricting the topologies of $L^2(\mathbb{R}^n,\mathcal{A})$ and of E_n to $L^2(\mathbb{R}^n)\cdot 1_{\mathcal{A}}$ yield the same canonical topology, (3) $\mathcal{S}(\mathbb{R}^n)\cdot 1_{\mathcal{A}}$ is dense in $L^2(\mathbb{R}^n)\cdot 1_{\mathcal{A}}$ with respect to this topology and that (4) the estimates

$$\|L_{\tilde{e}_m}(\tilde{f})-\tilde{f}\|_{L^2} \le \|L_{\tilde{e}_m}(\tilde{f}-\tilde{g})\|_{L^2} + \|L_{\tilde{e}_m}(\tilde{g})-\tilde{g}\|_{L^2} + \|\tilde{g}-\tilde{f}\|_{L^2}, \quad m\in\mathbb{N}^*$$

hold for every $\tilde{f}\in L^2(\mathbb{R}^n)\cdot 1_{\mathcal{A}}$ and $\tilde{g}\in\mathcal{S}(\mathbb{R}^n)\cdot 1_{\mathcal{A}}$. ∎

Lemma 3.6. *For every $A\in C^\infty(\operatorname{Ad}\mathcal{U})$, $\tilde{g}\in L^2(\mathbb{R}^n)\cdot 1_{\mathcal{A}}$ and $(x,\xi)\in\mathbb{R}^{2n}$, we have*

$$\lim_{m\to+\infty}\{D\,[(\operatorname{Ad}\mathcal{U})_{-x,-\xi}(A\circ L_{\tilde{e}_m}-A)]\circ\mathcal{F}^{-1}\}(\tilde{g}) = 0 \quad \text{in } E_n. \tag{3.17}$$

Proof. First, note that

$$\begin{aligned}
&D\,[(\operatorname{Ad}\mathcal{U})_{-x,-\xi}(A\circ L_{\tilde{e}_m})]\circ\mathcal{F}^{-1}\\
&\quad = \{[D\,((\operatorname{Ad}\mathcal{U})_{-x,-\xi}(A))]\circ L_{\tilde{e}_m}\}\circ\mathcal{F}^{-1} + \text{linear combination of terms of the form}\\
&\qquad\cdot\{\partial^\alpha_{x,\xi}[(\operatorname{Ad}\mathcal{U})_{-x,-\xi}(A)]\circ[\mathcal{U}_{-x,-\xi}\,\partial^\beta\,(L_{\tilde{e}_m})\,(\mathcal{U}_{-x,-\xi})^{-1}]\}\circ\mathcal{F}^{-1},
\end{aligned}\tag{3.18}$$

where the ∂^β's, $\beta\neq 0$, are monomials in the generators $(\partial_k)_{1\le k\le 2n}$ of the adjoint representation $\operatorname{Ad}\mathcal{U}$. Therefore, for all $\tilde{g}\in L^2(\mathbb{R}^n)\cdot 1_{\mathcal{A}}$, we have as a consequence of Lemma 3.3 and Corollary 3.5 that

$$\begin{aligned}
&\lim_{m\to+\infty}\{D\,[(\operatorname{Ad}\mathcal{U})_{-x,-\xi}(A\circ L_{\tilde{e}_m})]\circ\mathcal{F}^{-1}\}(\tilde{g})\\
&\quad = \lim_{m\to+\infty}\{[D\,((\operatorname{Ad}\mathcal{U})_{-x,-\xi}(A))]\circ L_{\tilde{e}_m}\circ\mathcal{F}^{-1}\}(\tilde{g})\\
&\quad = \{[D\,((\operatorname{Ad}\mathcal{U})_{-x,-\xi}(A))]\circ\mathcal{F}^{-1}\}(\tilde{g}),
\end{aligned}\tag{3.19}$$

where the limit is performed in the Hilbert C*-module E_n (for the nonzero order terms, we have used that the L^2-topology is finer than the topology of E_n). This proves equation (3.17). ∎

Proposition 3.7. *For every $A\in C^\infty(\operatorname{Ad}\mathcal{U})$ and each fixed $(x,\xi)\in\mathbb{R}^{2n}$, we have*

$$\lim_{m\to+\infty} S(A\circ L_{\tilde{e}_m})(x,\xi) = S(A)(x,\xi) \quad \text{in } \mathcal{A}. \tag{3.20}$$

Proof. Let us begin by majorizing the norms of the (restricted) linear maps (from $L^2(\mathbb{R}^{2n}) \cdot 1_{\mathcal{A}}$ to E_{2n})

$$\{(D\,[(\mathrm{Ad}\,\mathcal{U})_{-x,-\xi}(A \circ L_{\widetilde{e}_m})] \circ \mathcal{F}^{-1}) \otimes I_{E_n}\}|_{L^2(\mathbb{R}^{2n})\cdot 1_{\mathcal{A}}}, \quad m \in \mathbb{N}^*,\ (x,\xi) \in \mathbb{R}^{2n},$$

uniformly in m and in (x,ξ), by establishing norm bounds on each of the summands in (3.18).

Using again, just as in Lemma 3.3, the fact that

$$\partial_1^{\beta_1} \cdots \partial_{2n}^{\beta_{2n}}(L_{\widetilde{e}_m}) = (-1)^{|\beta|}\, L_{\partial_{v_1}^{\beta_1} \cdots \partial_{v_{2n}}^{\beta_{2n}} \widetilde{e}_m},$$

where $|\beta| := \sum_{k=1}^{2n} \beta_k$, $v_k = f_k$, if $1 \le k \le n$ and $v_k = J(f_{k-n})/2\pi$, if $n+1 \le k \le 2n$, we see, in particular, that each $\partial^\beta\,(L_{\widetilde{e}_m})$ sends $L^2(\mathbb{R}^n) \cdot 1_{\mathcal{A}}$ into itself, as a result of Lemma 3.1. This conclusion will be useful soon, when we have to write a certain operator (restricted to $L^2(\mathbb{R}^{2n}) \cdot 1_{\mathcal{A}}$) as the composition of two tensor product operators.

Applying Lemma 3.3 and the uniform boundedness principle for the restricted bounded operators $[\partial^\beta\,(L_{\widetilde{e}_m})]|_{L^2(\mathbb{R}^n)\cdot 1_{\mathcal{A}}}$, $\beta \neq 0$, we conclude that there exists $M_\beta > 0$ such that

$$\sup_{m \in \mathbb{N}^*} \|[\partial^\beta\,(L_{\widetilde{e}_m})]|_{L^2(\mathbb{R}^n)\cdot 1_{\mathcal{A}}}\| \le M_\beta,$$

so

$$\sup_{m \in \mathbb{N}^*} \|\{[\mathcal{U}_{-x,-\xi}\,\partial^\beta\,(L_{\widetilde{e}_m})\,(\mathcal{U}_{-x,-\xi})^{-1}] \circ \mathcal{F}^{-1}\}|_{L^2(\mathbb{R}^n)\cdot 1_{\mathcal{A}}}\| \le M_\beta \tag{3.21}$$

(the norm, above, is the usual operator norm on $L^2(\mathbb{R}^n) \cdot 1_{\mathcal{A}}$; note that the constant M_β does not depend on the fixed $(x,\xi) \in \mathbb{R}^{2n}$). Hence, the norm of each operator

$$\{[\mathcal{U}_{-x,-\xi}\,\partial^\beta\,(L_{\widetilde{e}_m})\,(\mathcal{U}_{-x,-\xi})^{-1}] \circ \mathcal{F}^{-1}\}|_{L^2(\mathbb{R}^n)\cdot 1_{\mathcal{A}}} \otimes I_{L^2(\mathbb{R}^n)\cdot 1_{\mathcal{A}}}$$

on the L^2-completion

$$[L^2(\mathbb{R}^n) \cdot 1_{\mathcal{A}}] \otimes [L^2(\mathbb{R}^n) \cdot 1_{\mathcal{A}}] \simeq L^2(\mathbb{R}^{2n}) \cdot 1_{\mathcal{A}}$$

of the algebraic tensor product $[L^2(\mathbb{R}^n) \cdot 1_{\mathcal{A}}] \otimes_{\mathrm{alg}} [L^2(\mathbb{R}^n) \cdot 1_{\mathcal{A}}]$ is bounded by M_β, independently of m and of (x,ξ). Combining this observation with the identity

$$\begin{aligned}
&\{\{\partial^\alpha_{x,\xi}[(\mathrm{Ad}\,\mathcal{U})_{-x,-\xi}(A)] \circ [\mathcal{U}_{-x,-\xi}\,\partial^\beta\,(L_{\widetilde{e}_m})\,(\mathcal{U}_{-x,-\xi})^{-1}] \circ \mathcal{F}^{-1}\} \otimes I_{E_n}\}|_{L^2(\mathbb{R}^{2n})\cdot 1_{\mathcal{A}}} \\
&\quad = \{\partial^\alpha_{x,\xi}[(\mathrm{Ad}\,\mathcal{U})_{-x,-\xi}(A)] \otimes I_{E_n}\}|_{L^2(\mathbb{R}^{2n})\cdot 1_{\mathcal{A}}} \\
&\qquad \circ \{\{[\mathcal{U}_{-x,-\xi}\,\partial^\beta\,(L_{\widetilde{e}_m})\,(\mathcal{U}_{-x,-\xi})^{-1}] \circ \mathcal{F}^{-1}\}|_{L^2(\mathbb{R}^n)\cdot 1_{\mathcal{A}}} \otimes I_{L^2(\mathbb{R}^n)\cdot 1_{\mathcal{A}}}\},
\end{aligned}$$

we obtain for each term (corresponding to $\beta \neq 0$) in the linear combination appearing in (3.18) the estimates

$$\begin{aligned}&\|\{\{\partial^{\alpha}_{x,\xi}[(\mathrm{Ad}\,\mathcal{U})_{-x,-\xi}(A)] \circ [\mathcal{U}_{-x,-\xi}\,\partial^{\beta}\,(L_{\tilde{e}_m})\,(\mathcal{U}_{-x,-\xi})^{-1}] \circ \mathcal{F}^{-1}\}\\ &\qquad \otimes I_{E_n}\}|_{L^2(\mathbb{R}^{2n})\cdot 1_{\mathcal{A}}}\|\\ &\quad\le \|\partial^{\alpha}_{x,\xi}[(\mathrm{Ad}\,\mathcal{U})_{-x,-\xi}(A)] \otimes I_{E_n}\| \cdot \|\{[\mathcal{U}_{-x,-\xi}\,\partial^{\beta}\,(L_{\tilde{e}_m})\,(\mathcal{U}_{-x,-\xi})^{-1}]\\ &\qquad \circ \mathcal{F}^{-1}\}|_{L^2(\mathbb{R}^n)\cdot 1_{\mathcal{A}}} \otimes I_{L^2(\mathbb{R}^n)\cdot 1_{\mathcal{A}}}\|\\ &\quad\le M_{\beta}\,\|\partial^{\alpha}_{x,\xi}[(\mathrm{Ad}\,\mathcal{U})_{-x,-\xi}(A)]\| = M_{\beta}\,\|\mathcal{U}_{-x,-\xi}\,\partial^{\alpha}(A)\,(\mathcal{U}_{-x,-\xi})^{-1}\|\\ &\quad\le M_{\beta}\|\partial^{\alpha}(A)\|,\end{aligned}$$

where (1) the norm in the first line is just the usual one of a bounded linear map from $L^2(\mathbb{R}^{2n}) \cdot 1_{\mathcal{A}}$ to E_{2n}; (2′) the second norm in the second line (from left to right) is the usual operator norm on $L^2(\mathbb{R}^{2n}) \cdot 1_{\mathcal{A}}$; (2″) the first norm in the second line (from left to right) is the operator norm on E_{2n} (we have implicitly used that, if $T\colon E_{2n} \to E_{2n}$ is a bounded operator, if $\|T|_{L^2(\mathbb{R}^{2n})\cdot 1_{\mathcal{A}}}\|_1$ denotes the usual norm of $T|_{L^2(\mathbb{R}^{2n})\cdot 1_{\mathcal{A}}}$ as a bounded linear map from $L^2(\mathbb{R}^{2n}) \cdot 1_{\mathcal{A}}$ to E_{2n}, and $\|T\|_2$ denotes the usual operator norm on E_{2n}, then $\|T|_{L^2(\mathbb{R}^{2n})\cdot 1_{\mathcal{A}}}\|_1 \le \|T\|_2$); (3) the norms in the third line are all operator norms on E_{2n}.

Note that, in order to obtain the above identity, we have used the fact that

$$[\mathcal{U}_{-x,-\xi}\,\partial^{\beta}\,(L_{\tilde{e}_m})\,(\mathcal{U}_{-x,-\xi})^{-1}] \circ \mathcal{F}^{-1}$$

leaves $L^2(\mathbb{R}^n) \cdot 1_{\mathcal{A}}$ invariant, so the equality

$$\begin{aligned}&\{\{[\mathcal{U}_{-x,-\xi}\,\partial^{\beta}\,(L_{\tilde{e}_m})\,(\mathcal{U}_{-x,-\xi})^{-1}] \circ \mathcal{F}^{-1}\} \otimes I_{E_n}\}|_{L^2(\mathbb{R}^{2n})\cdot 1_{\mathcal{A}}}\\ &\quad = \{\{[\mathcal{U}_{-x,-\xi}\,\partial^{\beta}\,(L_{\tilde{e}_m})\,(\mathcal{U}_{-x,-\xi})^{-1}] \circ \mathcal{F}^{-1}\}|_{L^2(\mathbb{R}^n)\cdot 1_{\mathcal{A}}}\} \otimes I_{L^2(\mathbb{R}^n)\cdot 1_{\mathcal{A}}}\end{aligned}$$

holds. We also make the observation that one of the tensor products is performed between adjointable operators on E_n, while the other one is performed between bounded operators on $L^2(\mathbb{R}^n) \cdot 1_{\mathcal{A}}$ (both of them are denoted simply by "$\otimes$").

Finally, to deal with the first summand in equation (3.18), we note that, defining $\widetilde{D} := \prod_{j=1}^{n}(1 + \partial_j)^2(1 + \partial_{j+n})^2$, we have the equality

$$D\,[(\mathrm{Ad}\,\mathcal{U})_{-x,-\xi}(A)] = (\mathrm{Ad}\,\mathcal{U})_{-x,-\xi}(\widetilde{D}(A)), \quad (x,\xi) \in \mathbb{R}^{2n},$$

so adapting the argument contained in (2″), above, to E_n, we obtain the estimate

$$\begin{aligned}&\|\{[D\,((\mathrm{Ad}\,\mathcal{U})_{-x,-\xi}(A))] \circ L_{\tilde{e}_m} \circ \mathcal{F}^{-1}\}|_{L^2(\mathbb{R}^n)\cdot 1_{\mathcal{A}}}\|\\ &\qquad \le \|\widetilde{D}(A)\|\,\|L_{\tilde{e}_m}|_{L^2(\mathbb{R}^n)\cdot 1_{\mathcal{A}}}\|\,\|\mathcal{F}^{-1}|_{L^2(\mathbb{R}^n)\cdot 1_{\mathcal{A}}}\|,\end{aligned}$$

where (i) $\|\widetilde{D}(A)\|$ is the operator norm of $\mathcal{L}_{\mathcal{A}}(E_n)$ evaluated on $\widetilde{D}(A)$, and (ii) the other two norms involved are just the usual ones of bounded linear operators on

$L^2(\mathbb{R}^n)\cdot 1_{\mathcal{A}}$. Therefore, since as a consequence of the first estimate obtained in the proof of Corollary 3.5, we have $\|L_{\widetilde{e}_m}|_{L^2(\mathbb{R}^n)\cdot 1_{\mathcal{A}}}\| \leq 1$ for all $m \in \mathbb{N}$, we get

$$\|\{[D\,((\mathrm{Ad}\,\mathcal{U})_{-x,-\xi}(A))] \circ L_{\widetilde{e}_m} \circ \mathcal{F}^{-1}\}|_{L^2(\mathbb{R}^n)\cdot 1_{\mathcal{A}}}\| \leq \|\widetilde{D}(A)\|.$$

In order to finish the proof of the desired uniform boundedness for the norms of

$$\{([D\,((\mathrm{Ad}\,\mathcal{U})_{-x,-\xi}(A))] \circ L_{\widetilde{e}_m} \circ \mathcal{F}^{-1}) \otimes I_{E_n}\}|_{L^2(\mathbb{R}^{2n})\cdot 1_{\mathcal{A}}},$$

we need to “tensor product” this last estimate and go from dimension n to $2n$. This can be done exactly as we did for the terms depending on a $\beta \neq 0$.

Therefore, we have just proved the existence of a constant $M > 0$, independent of $(x,\xi) \in \mathbb{R}^{2n}$, such that

$$\sup_{m\in\mathbb{N}^*} \|\{(D\,[(\mathrm{Ad}\,\mathcal{U})_{-x,-\xi}(A \circ L_{\widetilde{e}_m})] \circ \mathcal{F}^{-1}) \otimes I_{E_n}\}|_{L^2(\mathbb{R}^{2n})\cdot 1_{\mathcal{A}}}\| \leq M. \tag{3.22}$$

Now we will show how to prove equation (3.20) using (3.17) and (3.22). By the Cauchy–Schwarz inequality applied to equation (3.10) (with A substituted by $A \circ L_{\widetilde{e}_m} - A$),

$$\begin{aligned}\|S(A \circ L_{\widetilde{e}_m} - A)(x,\xi)\|_{\mathcal{A}} \leq (2\pi)^{n/2}\,&\|u\cdot 1_{\mathcal{A}}\|_{E_{2n}}\\ &\cdot \|\{(D\,[(\mathrm{Ad}\,\mathcal{U})_{-x,-\xi}(A \circ L_{\widetilde{e}_m} - A)] \circ \mathcal{F}^{-1}) \otimes I_{E_n}\}\\ &\cdot v\cdot 1_{\mathcal{A}}\|_{E_{2n}},\end{aligned}$$

hence it suffices to show that

$$\lim_{m\to+\infty} \|\{(D\,[(\mathrm{Ad}\,\mathcal{U})_{-x,-\xi}(A \circ L_{\widetilde{e}_m} - A)] \circ \mathcal{F}^{-1}) \otimes I_{E_n}\}\,v\cdot 1_{\mathcal{A}}\|_{E_{2n}} = 0.$$

Fix $\varepsilon > 0$ and define $K := \|\widetilde{D}(A)\| = \|(\mathrm{Ad}\,\mathcal{U})_{-x,-\xi}(\widetilde{D}(A))\| = \|D\,[(\mathrm{Ad}\,\mathcal{U})_{-x,-\xi}(A)]\|$, $(x,\xi) \in \mathbb{R}^{2n}$. Since the algebraic tensor product $L^2(\mathbb{R}^n) \otimes_{\mathrm{alg}} L^2(\mathbb{R}^n)$ can be naturally identified as a dense subspace of $L^2(\mathbb{R}^{2n})$, we can find $f \in L^2(\mathbb{R}^n) \otimes_{\mathrm{alg}} L^2(\mathbb{R}^n)$ such that

$$\|(f - v)\cdot 1_{\mathcal{A}}\|_{L^2} < \frac{\varepsilon}{3(M+1)(K+1)}.$$

Moreover, by (3.17) there exists $m_0 \in \mathbb{N}$ such that $m \geq m_0$ implies

$$\|\{(D\,[(\mathrm{Ad}\,\mathcal{U})_{-x,-\xi}(A \circ L_{\widetilde{e}_m} - A)] \circ \mathcal{F}^{-1}) \otimes I_{E_n}\}\,f\cdot 1_{\mathcal{A}}\|_{E_{2n}} < \frac{\varepsilon}{3}.$$

Therefore, combining these approximations with (3.22) yields for every $m \geq m_0$ the estimates

$$\begin{aligned}&\|\{(D\,[(\mathrm{Ad}\,\mathcal{U})_{-x,-\xi}(A \circ L_{\widetilde{e}_m} - A)] \circ \mathcal{F}^{-1}) \otimes I_{E_n}\}\,v\cdot 1_{\mathcal{A}}\|_{E_{2n}}\\ &\quad\leq \|\{(D\,[(\mathrm{Ad}\,\mathcal{U})_{-x,-\xi}(A \circ L_{\widetilde{e}_m})] \circ \mathcal{F}^{-1}) \otimes I_{E_n}\}\,(f - v)\cdot 1_{\mathcal{A}}\|_{E_{2n}}\\ &\qquad+ \|\{(D\,[(\mathrm{Ad}\,\mathcal{U})_{-x,-\xi}(A \circ L_{\widetilde{e}_m} - A)] \circ \mathcal{F}^{-1}) \otimes I_{E_n}\}\,f\cdot 1_{\mathcal{A}}\|_{E_{2n}}\\ &\qquad+ \|\{(D\,[(\mathrm{Ad}\,\mathcal{U})_{-x,-\xi}(A)] \circ \mathcal{F}^{-1}) \otimes I_{E_n}\}\,(f - v)\cdot 1_{\mathcal{A}}\|_{E_{2n}}\\ &\quad\leq \frac{M\,\varepsilon}{3(M+1)(K+1)} + \frac{\varepsilon}{3} + \frac{K\,\varepsilon}{3(M+1)(K+1)} < \varepsilon.\end{aligned}$$

This completes the proof. ∎

4. Proof of Theorem A

Fix $A \in C^{\infty}(\mathrm{Ad}\,\mathcal{U}) \cap R'_n$ and $g \in \mathcal{S}^{\mathcal{A}}(\mathbb{R}^n)$. Equation (3.22) guarantees the existence of a constant $M > 0$ such that

$$\sup_{m \in \mathbb{N}^*} \|\{(D\,[(\mathrm{Ad}\,\mathcal{U})_{-x,-\xi}(A \circ L_{\tilde{e}_m})] \circ \mathcal{F}^{-1}) \otimes I_{E_n}\}|_{L^2(\mathbb{R}^{2n})\cdot 1_{\mathcal{A}}}\| \leq M.$$

Therefore, applying the Cauchy–Schwarz inequality to the expression (3.10) defining the symbol map S gives, for every fixed $(x, \xi) \in \mathbb{R}^{2n}$, the estimates

$$\begin{aligned}\|S(A \circ L_{\tilde{e}_m})(x,\xi)\|_{\mathcal{A}} \leq (2\pi)^{n/2}\|u \cdot 1_{\mathcal{A}}\|_{L^2} \|\{\{D\,[(\mathrm{Ad}\,\mathcal{U})_{-x,-\xi}(A \circ L_{\tilde{e}_m})] \circ \mathcal{F}^{-1}\}\\ \otimes I_{E_n}\}|_{L^2(\mathbb{R}^{2n})\cdot 1_{\mathcal{A}}}\|\ \|v \cdot 1_{\mathcal{A}}\|_{L^2}\\ \leq M\,(2\pi)^{n/2}\,\|u \cdot 1_{\mathcal{A}}\|_{L^2}\,\|v \cdot 1_{\mathcal{A}}\|_{L^2}, \quad m \in \mathbb{N}^*.\end{aligned}$$

In particular,

$$\|(\mathcal{R} \circ S)(A \circ L_{\tilde{e}_m})(x)\|_{\mathcal{A}} = \|S(A \circ L_{\tilde{e}_m})(x,0)\|_{\mathcal{A}} \leq M\,(2\pi)^{n/2}\,\|u \cdot 1_{\mathcal{A}}\|_{L^2}\,\|v \cdot 1_{\mathcal{A}}\|_{L^2}$$

for every $x \in \mathbb{R}^n$ and $m \in \mathbb{N}^*$. Substituting x by $x - (2\pi)^{-1}J\xi$ in the above equation and then multiplying both sides by the number $\|\mathcal{F}(g)(\xi)\|_{\mathcal{A}}$, we conclude, using the submultiplicative property of the C*-norm $\|\cdot\|_{\mathcal{A}}$, that the estimate

$$\begin{aligned}\|(\mathcal{R} \circ S)(A \circ L_{\tilde{e}_m})\left(x - \frac{1}{2\pi}J\xi\right)\mathcal{F}(g)(\xi)\|_{\mathcal{A}}\\ \leq M\,(2\pi)^{n/2}\,\|u \cdot 1_{\mathcal{A}}\|_{L^2}\,\|v \cdot 1_{\mathcal{A}}\|_{L^2}\,\|\mathcal{F}(g)(\xi)\|_{\mathcal{A}}\end{aligned} \tag{4.1}$$

holds for every $(x, \xi) \in \mathbb{R}^{2n}$ and $m \in \mathbb{N}^*$. As a consequence of Proposition 3.7, we have the following pointwise convergence (in $\mathcal{A}$):

$$\lim_{m \to +\infty} S(A \circ L_{\tilde{e}_m})(x,\xi) = S(A)(x,\xi), \quad (x,\xi) \in \mathbb{R}^{2n}.$$

Hence, using the definition of $\mathcal{R}$,

$$\begin{aligned}\lim_{m \to +\infty} (\mathcal{R} \circ S)(A \circ L_{\tilde{e}_m})\left(x - \frac{1}{2\pi}J\xi\right)\mathcal{F}(g)(\xi)\\ = (\mathcal{R} \circ S)(A)\left(x - \frac{1}{2\pi}J\xi\right)\mathcal{F}(g)(\xi), \quad (x,\xi) \in \mathbb{R}^{2n},\end{aligned}$$

which when combined with the estimate in equation (4.1) allows an application of the dominated convergence theorem [4, Proposition 1.2.5, p. 16], yielding, for every fixed $x \in \mathbb{R}^n$, the equality

$$\lim_{m \to +\infty} \int_{\mathbb{R}^n} \left\|[(\mathcal{R} \circ S)(A \circ L_{\tilde{e}_m}) - (\mathcal{R} \circ S)(A)]\left(x - \frac{1}{2\pi}J\xi\right)\mathcal{F}(g)(\xi)\right\|_{\mathcal{A}} d\xi = 0. \tag{4.2}$$

But looking at the second equality in (3.2) (which also holds for $f \in \mathcal{B}^{\mathcal{A}}(\mathbb{R}^n)$ – see Appendix A), we see that (4.2) actually implies that, for every fixed $x \in \mathbb{R}^n$,

$$\begin{aligned} &\lim_{m\to+\infty} L_{(\mathcal{R}\circ S)(A\circ L_{\tilde{e}_m})}(g)(x) \\ &\quad = \lim_{m\to+\infty} \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} (\mathcal{R}\circ S)(A\circ L_{\tilde{e}_m})\Big(x - \frac{1}{2\pi} J\xi\Big) \mathcal{F}(g)(\xi)\, e^{i\langle \xi, x\rangle}\, d\xi \\ &\quad = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} (\mathcal{R}\circ S)(A)\Big(x - \frac{1}{2\pi} J\xi\Big) \mathcal{F}(g)(\xi)\, e^{i\langle \xi, x\rangle}\, d\xi \\ &\quad = L_{(\mathcal{R}\circ S)(A)}(g)(x). \end{aligned} \tag{4.3}$$

On the other hand, it is the content of Proposition 2.3 that A sends $\mathcal{S}^{\mathcal{A}}(\mathbb{R}^n)$ into $\mathcal{S}^{\mathcal{A}}(\mathbb{R}^n)$, so the set $\{L_{A(\tilde{e}_m)} : m \in \mathbb{N}^*\}$ consists of well-defined operators on $\mathcal{S}^{\mathcal{A}}(\mathbb{R}^n)$ (see (1.4)). Moreover, since by hypothesis A commutes with the operator R_h, for every $h \in \mathcal{S}^{\mathcal{A}}(\mathbb{R}^n)$, we obtain

$$(A\circ L_{\tilde{e}_m})(h) = A(L_{\tilde{e}_m}(h)) = (A\circ R_h)(\tilde{e}_m) = (R_h\circ A)(\tilde{e}_m) = R_h(A(\tilde{e}_m)) = L_{A(\tilde{e}_m)}(h)$$

for all $m \in \mathbb{N}^*$ and $h \in \mathcal{S}^{\mathcal{A}}(\mathbb{R}^n)$, so

$$(\mathcal{R}\circ S)(A\circ L_{\tilde{e}_m}) = (\mathcal{R}\circ S)(L_{A(\tilde{e}_m)}) = A(\tilde{e}_m), \quad m \in \mathbb{N}^*$$

(see equation (3.11)). Hence, $L_{(\mathcal{R}\circ S)(A\circ L_{\tilde{e}_m})} = L_{A(\tilde{e}_m)} = A\circ L_{\tilde{e}_m}$ for all $m \in \mathbb{N}^*$, so by Lemma 3.4 the equality

$$\lim_{m\to+\infty} L_{(\mathcal{R}\circ S)(A\circ L_{\tilde{e}_m})}(g) = \lim_{m\to+\infty} (A\circ L_{\tilde{e}_m})(g) = A(g) \tag{4.4}$$

holds in E_n.

Now we must find an argument to combine equations (4.3) and (4.4) and conclude that $A = L_{(\mathcal{R}\circ S)(A)}$. Since $A(g)$ belongs to $\mathcal{S}^{\mathcal{A}}(\mathbb{R}^n)$, equation (4.4) can be translated in terms of integrals:

$$\begin{aligned} \lim_{m\to+\infty} \Big\| \int_{\mathbb{R}^n} &[L_{(\mathcal{R}\circ S)(A\circ L_{\tilde{e}_m})}(g)(x) - A(g)(x)]^* \\ &\cdot [L_{(\mathcal{R}\circ S)(A\circ L_{\tilde{e}_m})}(g)(x) - A(g)(x)]\, dx \Big\|_{\mathcal{A}}^{1/2} = 0. \end{aligned}$$

Fix a positive linear functional ρ on $\mathcal{A}$. Then

$$\begin{aligned} &\lim_{m\to+\infty} \int_{\mathbb{R}^n} |\rho([L_{(\mathcal{R}\circ S)(A\circ L_{\tilde{e}_m})}(g)(x) - A(g)(x)]^* \\ &\qquad\qquad \cdot [L_{(\mathcal{R}\circ S)(A\circ L_{\tilde{e}_m})}(g)(x) - A(g)(x)])| dx \\ &\quad = \lim_{m\to+\infty} \int_{\mathbb{R}^n} \rho([L_{(\mathcal{R}\circ S)(A\circ L_{\tilde{e}_m})}(g)(x) - A(g)(x)]^* \\ &\qquad\qquad \cdot [L_{(\mathcal{R}\circ S)(A\circ L_{\tilde{e}_m})}(g)(x) - A(g)(x)]) dx = 0, \end{aligned}$$

so, by a standard result in measure theory [15, Theorem 3.12, p. 68], we may extract a subsequence $(L_{(\mathcal{R}\circ S)(A\circ L_{\tilde{e}_{m_k}})}(g))_{k\in\mathbb{N}}$ (which depends on ρ) such that

$$(\rho \circ ([L_{(\mathcal{R}\circ S)(A\circ L_{\tilde{e}_{m_k}})}(g) - A(g)]^*[L_{(\mathcal{R}\circ S)(A\circ L_{\tilde{e}_{m_k}})}(g) - A(g)]))_{k\in\mathbb{N}} \\ \text{is pointwise convergent to } 0$$

on a subset $S_\rho \subseteq \mathbb{R}^n$ whose complement has Lebesgue measure equal to zero. Combining equation (4.3) with the continuity of ρ, we obtain

$$\begin{aligned} &\rho([L_{(\mathcal{R}\circ S)(A)}(g)(x) - A(g)(x)]^*[L_{(\mathcal{R}\circ S)(A)}(g)(x) - A(g)(x)]) \\ &\quad= \lim_{k\to+\infty} \rho([L_{(\mathcal{R}\circ S)(A\circ L_{\tilde{e}_{m_k}})}(g)(x) - A(g)(x)]^* \\ &\qquad\cdot [L_{(\mathcal{R}\circ S)(A\circ L_{\tilde{e}_{m_k}})}(g)(x) - A(g)(x)]) = 0 \end{aligned}$$

for all $x \in S_\rho$. But $L_{(\mathcal{R}\circ S)(A)}(g)$ and $A(g)$ both belong to $\mathcal{S}^{\mathcal{A}}(\mathbb{R}^n)$, which establishes that the above equality actually holds for all $x \in \mathbb{R}^n$. Moreover, since ρ is arbitrary, we get [12, Theorem 3.3.6, p. 90]

$$[L_{(\mathcal{R}\circ S)(A)}(g)(x) - A(g)(x)]^*[L_{(\mathcal{R}\circ S)(A)}(g)(x) - A(g)(x)] = 0, \quad x \in \mathbb{R}^n.$$

Hence, using the C*-identity for the norm $\|\cdot\|_{\mathcal{A}}$, we conclude that $L_{(\mathcal{R}\circ S)(A)}(g)(x) = A(g)(x)$ for all $x \in \mathbb{R}^n$. By the arbitrariness of g, we conclude that $A = L_{(\mathcal{R}\circ S)(A)}$, which is exactly what we wanted to prove.

A. A few remarks on Rieffel's deformed algebra

As noted in the Introduction, given a skew-symmetric linear transformation J on $\mathbb{R}^n$ and $f \in \mathcal{B}^{\mathcal{A}}(\mathbb{R}^n)$, we may define a linear operator via the iterated integral

$$L_f(g)(x) := \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} f(x + J\xi)\, g(x + y)\, e^{2\pi i\langle\xi,y\rangle}\, dy\right) d\xi, \\ g \in \mathcal{S}^{\mathcal{A}}(\mathbb{R}^n),\ x \in \mathbb{R}^n, \tag{A.1}$$

in the sense that integration in the variable y has to be performed before integration over ξ. In fact, since the Fourier transform maps $\mathcal{S}^{\mathcal{A}}(\mathbb{R}^n)$ continuously into itself [4, p. 117], we see that the map $\xi \mapsto \int_{\mathbb{R}^n} f(x + J\xi)\, g(x + y)\, e^{2\pi i\langle\xi,y\rangle}\, dy$ belongs to $\mathcal{S}^{\mathcal{A}}(\mathbb{R}^n)$, so

$$\int_{\mathbb{R}^n}\left\|\int_{\mathbb{R}^n} f(x + J\xi)\, g(x + y)\, e^{2\pi i\langle\xi,y\rangle}\, dy\right\|_{\mathcal{A}} d\xi < +\infty.$$

Using well-known techniques in the theory of pseudodifferential operators, one can write the iterated integral in equation (A.1) as a Bochner integral of an $\mathcal{A}$-valued function on $\mathbb{R}^{2n}$ and, as a consequence, conclude that L_f maps $\mathcal{S}^{\mathcal{A}}(\mathbb{R}^n)$ into itself [14, Proposition 3.3, p. 25], a fact which we will now show in detail. After successive integration by parts we get, for every fixed $\xi \in \mathbb{R}^n$ and $N \geq 1$, the equality

$$\begin{aligned}
&\int_{\mathbb{R}^n} f(x + J\xi)\, g(x + y)\, e^{2\pi i\langle \xi, y\rangle}\, dy \\
&\quad = \int_{\mathbb{R}^n} f(x + J\xi)\, g(x + y)\, \frac{(1 - \Delta_y)^N e^{2\pi i\langle \xi, y\rangle}}{(1 + 4\pi^2|\xi|^2)^N}\, dy \\
&\quad = \int_{\mathbb{R}^n} f(x + J\xi)\, [(1 - \Delta_y)^N g](x + y)\, \frac{e^{2\pi i\langle \xi, y\rangle}}{(1 + 4\pi^2|\xi|^2)^N}\, dy.
\end{aligned}$$

Hence, choosing $N > n/2$ and using equation (A.1), we see that $L_f(g)(x)$ equals

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x + J\xi)\, [(1 - \Delta_y)^N g](x + y)\, \frac{e^{2\pi i\langle \xi, y\rangle}}{(1 + 4\pi^2|\xi|^2)^N}\, dy\, d\xi,$$

where the integrand is absolutely convergent in the variable (y, ξ).

Repeating the procedure of integrating by parts, but with respect to the ξ variable, instead, we get

$$\begin{aligned}
L_f(g)(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} & e^{2\pi i\langle \xi, y\rangle}\, (1 - \Delta_\xi)^M \left\{ \frac{f(x + J\xi)}{(1 + 4\pi^2|\xi|^2)^N} \right\} \\
& \cdot [(1 - \Delta_y)^N g](x + y)\, \frac{1}{(1 + 4\pi^2|y|^2)^M}\, dy\, d\xi
\end{aligned} \tag{A.2}$$

for all $N > n/2$ and $M \geq 1$.

Differentiating the above formula under the integral sign, we see that for each fixed $\alpha, \beta \in \mathbb{N}^n$, the expression $x^\alpha\, \partial^\beta L_f(g)(x)$ equals the sum

$$\begin{aligned}
\sum_{|\gamma| \leq |\beta|} \binom{\beta}{\gamma} \cdot x^\alpha \cdot \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} & e^{2\pi i\langle \xi, y\rangle}\, (1 - \Delta_\xi)^M \left\{ \frac{\partial^\gamma f(x + J\xi)}{(1 + 4\pi^2|\xi|^2)^N} \right\} \\
& \cdot [(1 - \Delta_y)^N \partial^{\beta - \gamma} g](x + y)\, \frac{1}{(1 + 4\pi^2|y|^2)^M}\, dy\, d\xi
\end{aligned}$$

for sufficiently large M and N. Therefore,

$$\begin{aligned}
\|x^\alpha\, \partial^\beta L_f(g)(x)\|_{\mathcal{A}} \leq \sum_{|\gamma| \leq |\beta|} \binom{\beta}{\gamma} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} & \left\| (1 - \Delta_\xi)^M \left\{ \frac{\partial^\gamma f(x + J\xi)}{(1 + 4\pi^2|\xi|^2)^N} \right\} \right\|_{\mathcal{A}} \\
& \cdot \{(1 + |x|^2)^{|\alpha|/2}\, \|[(1 - \Delta_y)^N \partial^{\beta - \gamma} g](x + y)\|_{\mathcal{A}}\} \\
& \cdot \frac{1}{(1 + 4\pi^2|y|^2)^M}\, dy\, d\xi.
\end{aligned}$$

Using Peetre's inequality [5, (3.6)], we get

$$(1 + |x|^2)^{|\alpha|/2} \leq 2^{|\alpha|/2}\, (1 + |y|^2)^{|\alpha|/2}\, (1 + |x + y|^2)^{|\alpha|/2},$$

so the sum above may be majorized by the expression

$$2^{|\alpha|/2} \sum_{|\gamma|\le|\beta|} \binom{\beta}{\gamma} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \left\| (1-\Delta_\xi)^M \left\{ \frac{\partial^\gamma f(x+J\xi)}{(1+4\pi^2|\xi|^2)^N} \right\} \right\|_{\mathcal{A}}$$
$$\cdot \{(1+|x+y|^2)^{|\alpha|/2} \, \|[(1-\Delta_y)^N \partial^{\beta-\gamma} g](x+y)\|_{\mathcal{A}}\}$$
$$\cdot \frac{(1+4\pi^2|y|^2)^{|\alpha|/2}}{(1+4\pi^2|y|^2)^M} \, dy \, d\xi,$$

which will be a real number as long as we choose $N > n/2$ and $M > (n+|\alpha|)/2$. Finally, this shows that the expression $\sup_{x\in\mathbb{R}^n} \|x^\alpha \, \partial^\beta L_f(g)(x)\|_{\mathcal{A}}$ may be majorized by a linear combination of terms of the form

$$\sup_{x\in\mathbb{R}^n} \{\|\partial^{\delta_1} f(x)\|_{\mathcal{A}}\} \sup_{x\in\mathbb{R}^n} \{(1+|x|^2)^{|\alpha|/2} \, \|\partial^{\delta_2} g(x)\|_{\mathcal{A}}\}$$
$$\cdot \int_{\mathbb{R}^n} \frac{1}{(1+4\pi^2|y|^2)^{M-|\alpha|/2}} \, dy \int_{\mathbb{R}^n} \frac{1}{(1+4\pi^2|\xi|^2)^Q} \, d\xi,$$

where $\delta_1, \delta_2 \in \mathbb{N}^n$ are multiindices and $Q \ge N$. Hence, since $f \in \mathcal{B}^{\mathcal{A}}(\mathbb{R}^n)$ and $g \in \mathcal{S}^{\mathcal{A}}(\mathbb{R}^n)$, these terms are all real numbers. This shows that L_f maps $\mathcal{S}^{\mathcal{A}}(\mathbb{R}^n)$ continuously into itself.

Making use of oscillatory integrals (see [14, Chapter 1], [3, pp. 66–69]), we can attribute meaning to the integral in equation (A.1) even when both f and g belong to $\mathcal{B}^{\mathcal{A}}(\mathbb{R}^n)$, defining Rieffel's deformed product [14, p. 23] to be

$$(f \times_J g)(x) := \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x+J\xi)\, g(x+y)\, e^{2\pi i\langle\xi,y\rangle} \, dy \, d\xi$$
$$:= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{2\pi i\langle\xi,y\rangle} (1-\Delta_\xi)^M \left\{ \frac{f(x+J\xi)}{(1+4\pi^2|\xi|^2)^N} \right\}$$
$$\cdot [(1-\Delta_y)^N g](x+y) \, \frac{1}{(1+4\pi^2|y|^2)^M} \, dy \, d\xi, \tag{A.3}$$

where $x \in \mathbb{R}^n$ and $M, N > n/2$. It can be shown that this definition is independent of the choices of M and N. Comparing the definition given in equation (A.3) with equation (A.2), we see that if $f \in \mathcal{B}^{\mathcal{A}}(\mathbb{R}^n)$ and $g \in \mathcal{S}^{\mathcal{A}}(\mathbb{R}^n)$, then $L_f(g) = f \times_J g$. Moreover, after differentiating under the integral sign, one sees that $f \times_J g$ also belongs to $\mathcal{B}^{\mathcal{A}}(\mathbb{R}^n)$.

As a consequence of a version of the Calderón–Vaillancourt inequality for Hilbert C*-modules [1, Theorem 3.2], every pseudodifferential operator $\mathrm{Op}(a)$ with symbol $a \in \mathcal{B}^{\mathcal{A}}(\mathbb{R}^{2n})$ (see equation (1.6)) can be extended by continuity to a bounded operator (which will also be denoted by $\mathrm{Op}(a)$) on the Hilbert $\mathcal{A}$-module E_n. Moreover, its operator norm satisfies

$$\|\mathrm{Op}(a)\| \le C \max_{\beta,\gamma\le\mathring{\alpha}} \sup\{\|\partial_x^\beta \partial_\xi^\gamma a(x,\xi)\|_{\mathcal{A}} : x,\xi \in \mathbb{R}^n\}, \quad \mathring{\alpha} = (1,\ldots,1) \in \mathbb{N}^n$$

for a certain constant $C > 0$. In particular, this conclusion holds for every L_f, $f \in \mathcal{B}^{\mathcal{A}}(\mathbb{R}^n)$, since L_f is a pseudodifferential operator $\mathrm{Op}(a)$ with symbol $a \in \mathcal{B}^{\mathcal{A}}(\mathbb{R}^{2n})$ given by $a(x,\xi) := f(x - J\xi/(2\pi))$.

Using oscillatory integrals, we can show that the set $\{\mathrm{Op}(a) : a \in \mathcal{B}^{\mathcal{A}}(\mathbb{R}^{2n})\}$ of pseudodifferential operators on E_n is a $*$-subalgebra of the C*-algebra $\mathcal{L}_{\mathcal{A}}(E_n)$ of adjointable operators on E_n (for the proof that each $\mathrm{Op}(a)$ is an adjointable operator on E_n, see [1, Proposition 3.3]). In fact, the restriction of the involution and composition maps to $\{\mathrm{Op}(a) : a \in \mathcal{B}^{\mathcal{A}}(\mathbb{R}^{2n})\}$ are defined, respectively, by $\mathrm{Op}(a) \mapsto \mathrm{Op}(a^\dagger)$ and $\mathrm{Op}(a) \circ \mathrm{Op}(b) \mapsto \mathrm{Op}(a \times b)$, with corresponding symbols given by [1, (3.20), (3.21)]

$$\begin{aligned} a^\dagger(x,\xi) &:= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-i\langle z,\eta\rangle}\, a(x-z,\xi-\eta)^*\, dz\, d\eta \\ &:= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-i\langle z,\eta\rangle}\, (1+|z|^2)^{-N}(1-\Delta_\eta)^N \\ &\qquad \cdot \{(1+|\eta|^2)^{-M}(1-\Delta_z)^M[a(x-z,\xi-\eta)^*]\} dz\, d\eta, \end{aligned} \tag{A.4}$$

and

$$\begin{aligned} (a\times b)(x,\xi) &:= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-i\langle z,\eta\rangle}\, a(x,\xi-\eta)\, b(x-z,\xi)\, dz\, d\eta \\ &:= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-i\langle z,\eta\rangle}\, (1+|z|^2)^{-N}(1-\Delta_\eta)^N \\ &\qquad \cdot \{(1+|\eta|^2)^{-M}(1-\Delta_z)^M[a(x,\xi-\eta)\, b(x-z,\xi)]\} dz\, d\eta \end{aligned} \tag{A.5}$$

for all $x, \xi \in \mathbb{R}^n$, where $M, N > n/2$. Moreover, the above definitions are independent of the choices of M and N. Specializing these formulas to the operators L_f, $f \in \mathcal{B}^{\mathcal{A}}(\mathbb{R}^n)$, shows that the product $\times_J$ is associative (see the last paragraph of [1, Subsection "Pseudodifferential operators with $\mathcal{C}$-valued symbols"]) and that Rieffel's deformed algebra $\{L_f : f \in \mathcal{B}^{\mathcal{A}}(\mathbb{R}^n)\}$ is also a $*$-subalgebra of $\mathcal{L}_{\mathcal{A}}(E_n)$, with the involution satisfying $(L_f)^* = L_{f^*}$. Indeed, if f belongs to $\mathcal{S}^{\mathcal{A}}(\mathbb{R}^n)$, then the function a defined by $a(x,\xi) := f(x - J\xi/(2\pi))$ belongs to $\mathcal{S}^{\mathcal{A}}(\mathbb{R}^{2n})$, so using equation (A.4) and the fact that $\langle J\eta, \eta\rangle = 0$, for all $\eta \in \mathbb{R}^n$, we can perform the transformation $z \mapsto z + (2\pi)^{-1}J\eta$ of the integration variable z to obtain

$$\begin{aligned} a^\dagger(x,\xi) &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-i\langle z,\eta\rangle}\, f\left(x - z - \frac{J(\xi-\eta)}{2\pi}\right)^* dz\, d\eta \\ &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-i\langle z+\frac{J\eta}{2\pi},\eta\rangle}\, f\left(x - z - \frac{J\xi}{2\pi}\right)^* dz\, d\eta \\ &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \mathcal{F}\left(f\left(x - \cdot - \frac{J\xi}{2\pi}\right)^*\right)(\eta)\, d\eta \\ &= f\left(x - \frac{J\xi}{2\pi}\right)^* =: f^*\left(x - \frac{J\xi}{2\pi}\right) \end{aligned}$$

for all $x, \xi \in \mathbb{R}^n$. For the general case in which f belongs to $\mathcal{B}^{\mathcal{A}}(\mathbb{R}^n)$, let ϕ be a compactly supported complex-valued smooth function on $\mathbb{R}^{2n}$ satisfying $0 \le \phi \le 1$ which equals 1 on a neighborhood of 0 and define, for each $m \in \mathbb{N}^*$, the functions $f_m(y) := \phi(y/m) f(y)$, $y \in \mathbb{R}^n$, and $a_m(x,\xi) := f_m(x - J\xi/(2\pi))$, $(x,\xi) \in \mathbb{R}^{2n}$. Then $(a_m)_{m\in\mathbb{N}^*}$ converges pointwise to $a\colon (x,\xi) \mapsto f(x - J\xi/(2\pi))$, so by the continuity of the involution operation on $\mathcal{A}$, we see that $((a_m)^\dagger(x,\xi) = f_m^*(x - J\xi/(2\pi)))_{m\in\mathbb{N}^*}$ converges to $f^*(x - J\xi/(2\pi))$ for each fixed $(x,\xi) \in \mathbb{R}^{2n}$. On the other hand, after successive integration by parts and applications of the Leibniz rule, we get

$$\begin{aligned}(a_m)^\dagger(x,\xi) &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-i\langle z,\eta\rangle}\, f_m\Big(x - z - \frac{J(\xi-\eta)}{2\pi}\Big)^* \,dz\,d\eta \\ &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-i\langle z,\eta\rangle}\,(1+|z|^2)^{-N}(1-\Delta_\eta)^N \\ &\qquad \cdot\Big\{(1+|\eta|^2)^{-M}(1-\Delta_z)^M\Big[f_m\Big(x - z - \frac{J(\xi-\eta)}{2\pi}\Big)^*\Big]\Big\}\,dz\,d\eta,\end{aligned}$$

which equals

$$\begin{aligned}&\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-i\langle z,\eta\rangle}\,(1+|z|^2)^{-N}(1-\Delta_\eta)^N \\ &\quad\cdot\Big\{(1+|\eta|^2)^{-M}(1-\Delta_z)^M\Big[f\Big(x - z - \frac{J(\xi-\eta)}{2\pi}\Big)^*\Big]\Big\} \\ &\quad\cdot\phi\bigg(\frac{x - z - \frac{J(\xi-\eta)}{2\pi}}{m}\bigg)dz\,d\eta\end{aligned}$$

plus a linear combination of terms of the form

$$\begin{aligned}&\frac{1}{m^{|\gamma_2|}}\cdot\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-i\langle z,\eta\rangle}(\partial^{\gamma_1} f)\Big(x - z - \frac{J(\xi-\eta)}{2\pi}\Big)^* \\ &\quad\cdot\ (\partial^{\gamma_2}\phi)\bigg(\frac{x - z - \frac{J(\xi-\eta)}{2\pi}}{m}\bigg)\, g_{\gamma_1,\gamma_2}(z,\eta)\,dz\,d\eta,\end{aligned}$$

with $\gamma_1 \in \mathbb{N}^n$, $0 \neq \gamma_2 \in \mathbb{N}^n$ and $g_{\gamma_1,\gamma_2} \in L^1(\mathbb{R}^{2n})$.

Therefore, applying Fubini's theorem and the dominated convergence theorem, we obtain

$$\begin{aligned}(a_m)^\dagger(x,\xi) &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-i\langle z,\eta\rangle}\, f_m\Big(x - z - \frac{J(\xi-\eta)}{2\pi}\Big)^* \,dz\,d\eta \\ &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-i\langle z,\eta\rangle}\,(1+|z|^2)^{-N}(1-\Delta_\eta)^N \\ &\qquad \cdot\Big\{(1+|\eta|^2)^{-M}(1-\Delta_z)^M\Big[f_m\Big(x - z - \frac{J(\xi-\eta)}{2\pi}\Big)^*\Big]\Big\}\,dz\,d\eta \\ &\qquad \xrightarrow{m\to+\infty} \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-i\langle z,\eta\rangle}\,(1+|z|^2)^{-N}(1-\Delta_\eta)^N\end{aligned}$$

$$\cdot \left\{ (1+|\eta|^2)^{-M} (1-\Delta_z)^M \left[f\left(x - z - \frac{J(\xi-\eta)}{2\pi} \right)^* \right] \right\}$$
$$dz\, d\eta =: a^\dagger(x,\xi),$$

allowing us to conclude that $a^\dagger(x,\xi) = f^*(x - J\xi/(2\pi))$ for all $(x,\xi) \in \mathbb{R}^{2n}$.

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References

[1] R. A. H. M. Cabral, M. Forger, and S. T. Melo, Differential norms and Rieffel algebras. *Math. Nachr.* **298** (2025), no. 7, 2177–2203 Zbl 08080336 MR 4939748

[2] H. O. Cordes, On pseudodifferential operators and smoothness of special Lie-group representations. *Manuscripta Math.* **28** (1979), no. 1–3, 51–69 Zbl 0415.35083 MR 0535694

[3] H. O. Cordes, *The technique of pseudodifferential operators*. London Math. Soc. Lecture Note Ser. 202, Cambridge University Press, Cambridge, 1995 Zbl 0828.35145 MR 1314815

[4] T. Hytönen, J. van Neerven, M. Veraar, and L. Weis, *Analysis in Banach spaces. Volume I: Martingales and Littlewood-Paley theory*. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge/ A Series of Modern Surveys in Mathematics. Translated from the French, Springer, Cham, 2016, 614 pp. Zbl 1366.46001 MR 3617205

[5] J. J. Kohn and L. Nirenberg, An algebra of pseudo-differential operators. *Comm. Pure Appl. Math.* **18** (1965), no. 1–2, 269–305 Zbl 0171.35101 MR 0176362

[6] E. C. Lance, *Hilbert C^*-modules – a toolkit for operator algebraists*. London Math. Soc. Lecture Note Ser. 210, Cambridge University Press, Cambridge, 1995 Zbl 0822.46080 MR 1325694

[7] S. T. Melo and M. I. Merklen, On a conjectured noncommutative Beals-Cordes-type characterization. *Proc. Amer. Math. Soc.* **130** (2002), no. 7, 1997–2000 Zbl 1026.47032 MR 1896033

[8] S. T. Melo and M. I. Merklen, Pseudodifferential operators with C^*-algebra-valued symbols: abstract characterizations. *Proc. Amer. Math. Soc.* **136** (2008), no. 1, 219–227 Zbl 1129.47038 MR 2350407

[9] S. T. Melo and M. I. Merklen, Cordes characterization for pseudodifferential operators with symbols valued in a noncommutative C^*-algebra. *C. R. Math. Acad. Sci. Soc. R. Can.* **31** (2009), no. 1, 24–32 Zbl 1191.47063 MR 2519019

[10] M. I. Merklen, *Resultados Motivados por uma Caracterização de Operadores Pseudo-Diferenciais Conjecturada por Rieffel*. Ph.D. thesis, Instituto de Matemática e Estatística, University of São Paulo, 2002

[11] M. I. Merklen, Boundedness of pseudodifferential operators of a C^*-algebra-valued symbol. *Proc. Roy. Soc. Edinburgh Sect. A* **135** (2005), no. 6, 1279–1286 Zbl 1097.47047 MR 2191899

[12] G. J. Murphy, *C^*-algebras and operator theory*. Academic Press, Boston, MA, 1990 Zbl 0714.46041 MR 1074574

[13] N. S. Poulsen, On C^∞-vectors and intertwining bilinear forms for representations of Lie groups. *J. Funct. Anal.* **9** (1972), no. 1, 87–120 Zbl 0237.22013 MR 0310137

[14] M. A. Rieffel, Deformation quantization for actions of $\mathbf{R}^d$. *Mem. Amer. Math. Soc.* **106** (1993), no. 506, 93 pp. Zbl 0798.46053 MR 1184061

[15] W. Rudin, *Real and complex analysis*. 3rd edn., McGraw-Hill Book, New York, 1987 Zbl 0925.00005 MR 0924157

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