On the Continuity of The Zadeh's Extension

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Abstract

Let $f:\mathbb{R}^n \to \mathbb{R}^n$ be a continuous map, we study the continuity of the Zadeh's extension of f to the space of fuzzy sets $\mathcal{F}(\mathbb{R}^n)$, with respect to two different metrics, the usual metric derived from the Hausdorff metric on the family of compact sets and the endograph metric defined by Kloeden [4]. Finally we determine a class of fuzzy sets where these metrics are equivalent.

1 Introduction

The Zadeh extension is the way we produce a fuzzy transformation $\hat{f}: \mathcal{F}(\mathbb{R}^n) \to \mathcal{F}(\mathbb{R}^n)$ from a given function $f:\mathbb{R}^n \to \mathbb{R}^n$. In the next section we will give briefly the precise definition. Now we want just to point that, although this extension is well known since several years, the study of its properties in the topological spaces was almost always neglected. Among the few works we found on the subject we mention that of Cabrelli et alli [1], who was motivated by the study of Fuzzy Iterated Function Systems and that of Nguyen [5] working in the context of fuzzy numbers. We are interested in the study of Discrete Fuzzy Dynamical Systems applied to biological populational dynamics, and our approach is indeed very close to that of the papers mentioned above, containing some improvements and generalizations; more specifically, it is our objective in the future, to model some biological phenomenal using difference equations where the subjectivity affects the states and parameters involved, playing a decisive role in the dynamics of the process. In this sense the Zadeh extension will appear as a fundamental tool relating the classical and fuzzy models.

Our goal is to present the continuity properties of the Zadeh extension to the space $\mathcal{F}(\mathbb{R}^n)$ with the different metrics D [7] and H [4]. Let us also observe that all the results here are true taking a complete metric space X instead of $\mathbb{R}^n$, being the generalizations straightforward.

2 Preliminaries

Let $\mathcal{Q}(\mathbb{R}^n)$ be the family of nonempty, compact subsets of $\mathbb{R}^n$. The distance of Hausdorff is defined as:

$$h(A,B) = \max\{h_1(A,B), h_2(A,B)\}$$

where

$$h_1(A,B) = \sup_{x\in A}\inf_{y\in B} ||x-y||$$

and

$$h_2(A,B) = \sup_{y\in B}\inf_{x\in A} ||x-y||$$

It is well known that the metric space $(\mathcal{Q}(\mathbb{R}^n), h)$ is complete and separable.

We say that a sequence $A_p \in \mathcal{Q}(\mathbb{R}^n)$ converges to A in the sense of Kuratowski if

$$A = \liminf_{p\to+\infty} A_p = \limsup_{p\to+\infty} A_p$$

where

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$$\liminf_{p\to+\infty} A_p = \{x : x = \lim_{p\to\infty} x_p, x_p \in A_p\}$$

$$\limsup_{p\to+\infty} A_p = \{x : x = \lim_{p_j\to\infty} x_{p_j}, x_{p_j} \in A_{p_j}\}$$

or

$$\limsup_{p\to+\infty} A_p = \bigcap_{p=1}^{\infty} \overline{\bigcup_{n\geq p} A_p}$$

The following theorem can be found in Hausdorff [3]

Theorem 1. *Let $A_p, A \in Q(\mathbb{R}^n)$, then the following are equivalents:*
(1) A_p converges to A in the Hausdorff metric h.
(2) A and A_p are contained in a compact set K, and A_p converges to A in the Kuratowski sense.

We will need also the following proposition:

Proposition 2. *If $f : \mathbb{R}^n \to \mathbb{R}^n$ is a uniformly continuous function, the transformation in the space $(Q(\mathbb{R}^n), h)$ assigning to each compact set K, the compact set $f(K)$ is also uniformly continuous.*

We give here a proof for sake of completeness.

Proof. From now on we denote $||x-y|| = d(x,y)$ and given $\varepsilon > 0$ take the $\delta > 0$ such that

$$d(x,y) < \delta \implies d(f(x), f(y)) < \varepsilon$$

If $h(K,C) < \delta$, then we have by definition

$$\forall x \in K \implies \inf_{y\in C} d(x,y) < \delta$$

then there is $\bar{y}$ such that $d(x, \bar{y}) < \delta$, and

$$d(f(x), f(\bar{y})) < \varepsilon$$

which implies that $\inf_{y\in C} d(f(x), f(y)) < \varepsilon$ and

$$\sup_{x\in K} \inf_{y\in C} d(f(x), f(y)) < \varepsilon$$

A symmetric argument shows the other inequality. □

The above proposition 2 and the theorem 1 immediately implies the following proposition

Proposition 3. *$f : \mathbb{R}^n \to \mathbb{R}^n$ is continuous if and only if the transformation in the Hausdorff space $(Q(\mathbb{R}^n), h)$ assigning to each compact set K, the compact set $f(K)$ is also continuous.*

3 The fuzzy metric spaces

We define the fuzzy sets as:

$$\mathcal{F}(\mathbb{R}^n) = \{u : \mathbb{R}^n \to [0,1]\}$$

with $[u]^\alpha$ compact nonempty and where for each $0 < \alpha \leq 1$

$$[u]^\alpha = \{x \in \mathbb{R}^n : u(x) \geq \alpha\}$$

and

$$[u]^0 = \overline{\{x \in \mathbb{R}^n : u(x) > 0\}}$$

In this set we have the metrics:

$$D(u,v) = \sup_{0\leq\alpha\leq 1} h([u]^\alpha, [v]^\alpha)$$

that we call normal metric, and we have also the endograph metric

$$H(u,v) = h(\mathrm{send}(u), \mathrm{send}(v))$$

where

$$\mathrm{send}(u) = ([u]^0 \times [0,1]) \cap \mathrm{end}(u)$$

with

$$\mathrm{end}(u) = \{(x,\alpha) \in \mathbb{R}^n \times [0,1] : u(x) \geq \alpha\}$$

this set is called the endograph [4] and here h means the Hausdorff metric in the corresponding space.
It is well known that the space$(\mathcal{F}(\mathbb{R}^n), D)$ is complete but not separable [7] whereas $\mathcal{F}(\mathbb{R}^n)$ with the metric H is separable but not complete [4].
In the following we will need a proposition which can be found in [6]

Proposition 4. *Let u_p be a sequence and u an element in $\mathcal{F}(\mathbb{R}^n)$. Then the sequence u_p converges in the endograph metric to u if and only if*

$$\begin{aligned}\{u > \alpha\} &\subset \lim_{p\to\infty} \inf\{u_p \geq \alpha\} \\ &\subset \lim_{p\to\infty} \sup\{u_p \geq \alpha\} \\ &\subset \{u \geq \alpha\}, \forall\alpha \in [0,1]\end{aligned} \tag{1}$$

and

$$\lim_{p\to\infty} h([u_p]^0, [u]^0) = 0 \tag{2}$$

4 On the Zadeh extension

Let $f:\mathbb{R}^n \to \mathbb{R}^n$ be any function. We define the Zadeh extension $\hat{f}$ as

$$\hat{f}(u)(x) = \begin{cases} \sup_{\tau \in f^{-1}(x)} u(\tau) \text{ if } f^{-1}(x) \neq \emptyset \\ 0 \text{ otherwise} \end{cases}$$

for all $u \in \mathcal{F}(\mathbb{R}^n)$. Note that in the next theorem we give a sufficient condition for $\hat{f}$ to be a well defined function in $\mathcal{F}(\mathbb{R}^n)$

Obs.: 1. *It follows from this definition that if f is a constant function $f(x) = c$ then its extension is $\hat{f}(u) = \hat{c} = \chi_{\{c\}}$ the characteristic function.*

Obs.: 2. *Identifying $\mathbb{R}^n$with the subset*

$$\{\chi_{\{x\}} : x \in \mathbb{R}^n\}$$

then we have

$$\hat{f}(\chi_{\{x\}}) = \chi_{\{f(x)\}}, \forall x \in \mathbb{R}^n$$

which shows that the Zadeh extension is in fact an extension.

Obs.: 3. *If f is bijective then*

$$\hat{f}(u)(x) = u(f^{-1}(x)).$$

Obs.: 4. *For the case where*

$$f(x) = Ax + b,$$

the extension is given by

$$\hat{f}(u)(x) = u(A^{-1}(x-b))$$

if A is invertible. If $a \neq 0$ and $f(x) = ax$ then $\hat{f}(u)(x) = u(a^{-1}x) = au(x)$ according the rules of multiplication on the fuzzy sets (see, for instance, [8]).

The next theorem was first proved in Nguyen [5], some generalizations of these results appeared latter [2] and more recently Cabrelli et alli [1] works on it in the context of a Fuzzy Iterated Function System. We present here an improved version which fits better to our needs and is indeed very close to the one given in Cabrelli et alli [1].

Theorem 5. *If $f:\mathbb{R}^n \to \mathbb{R}^n$ is continuous, then the Zadeh extension is well defined and we have:*

$$[\hat{f}(u)]^\alpha = f([u]^\alpha)$$

Proof. Note that to show the fact $\hat{f}(u) \in \mathcal{F}(\mathbb{R}^n)$, it is enough to verify that the levels $[\hat{f}(u)]^\alpha$ are compact and nonempty for each α, then the second assertion of the theorem will imply the first one. Let us show the second part.
First off all note that $f^{-1}(x)$ being closed and $[u]^0$ compact then $f^{-1}(x) \cap [u]^0$ is a compact subset of $\mathbb{R}^n$. We divide the proof in two:
For $\alpha > 0$ let $x \in [\hat{f}(u)]^\alpha$, then $\hat{f}(u)(x) \geq \alpha$ which implies that $f^{-1}(x) \neq \emptyset$ and $f^{-1}(x) \cap [u]^0 \neq \emptyset$ then

$$\begin{aligned} \hat{f}(u)(x) &= \sup_{\tau \in f^{-1}(x)} u(\tau) \\ &= \sup_{\tau \in f^{-1}(x) \cap [u]^0} u(\tau) \qquad (3) \\ &= u(y) \end{aligned}$$

For some $y \in f^{-1}(x) \cap [u]^0$, since u is upper semicontinuous (c.f. Rudin [10, pg 195]); then $x = f(y)$, which means that $x \in f([u]^\alpha)$.
On the other hand $f([u]^\alpha) \subset [\hat{f}(u)]^\alpha$ is always true since for $x \in f([u]^\alpha)$ there is a $y \in [u]^\alpha$ and $x = f(y)$, then $\hat{f}(u)(x) \geq u(y) \geq \alpha$ or $x \in [\hat{f}(u)]^\alpha$.
For $\alpha = 0$, we note that

$$A = \{x : \hat{f}(u)(x) > 0\} = f(\{x : u(x) > 0\}) = f(B)$$

since if $x \in A$ then $\sup_{\tau \in f^{-1}(x)} u(\tau) > 0$ which implies that there exists a y with $f(y) = x$ and $u(y) > 0$ that is $x \in B$; if $x \in f(B)$ then there exists $y \in B$ and $f(y) = x$ hence

$$\sup_{\tau \in f^{-1}(x)} u(\tau) \geq u(y)$$

that is $x \in A$.
Now using the compactness of $\overline{B}$ and the continuity of f we get

$$\overline{A} = \overline{f(B)} = f(\overline{B})$$

from what we conclude

$$[\hat{f}(u)]^0 = f([u]^0)$$

□

We observe that there is a one-to-one correspondence between the fuzzy sets and the family of compact sets $\{[u]^\alpha : 0 \leq \alpha \leq 1\}$, in particular the family $\{f([u]^\alpha) : 0 \leq \alpha \leq 1\}$ gives rise to a unique fuzzy set in $\mathcal{F}(\mathbb{R}^n)$.
A direct consequence of this theorem is

Corollary 6. *If $f:\mathbb{R}^n \to \mathbb{R}^n$ is continuous then $\hat{f}$ is monotone in the following sense:*

$$\hat{f}(u) \leq \hat{f}(v) \text{ if } u \leq v$$

where $u \leq v$ means $u(x) \leq v(x)$ for all $x \in \mathbb{R}^n$.

Proof. The relation $u \leq v$ implies that $[u]^\alpha \subset [v]^\alpha$ and from the theorem 5 we have $[\hat{f}(u)]^\alpha = f([u]^\alpha)$ the result follows immediately. ☐

Theorem 7. *Let $f:\mathbb{R}^n \to \mathbb{R}^n$ be a uniformly continuous function, then the Zadeh extension*

$$\hat{f}:(\mathcal{F}(\mathbb{R}^n), D) \to (\mathcal{F}(\mathbb{R}^n), D)$$

is also uniformly continuous.

Proof. Given $\varepsilon > 0$ take $\delta > 0$ such that

$$h([u]^\alpha, [v]^\alpha) < \delta$$

implies $h(f([u]^\alpha), f([v]^\alpha)) < \varepsilon$ see (theorem 1). Hence taking $D(u,v) < \delta$ then $h([u]^\alpha, [v]^\alpha) < \delta$ which implies

$$h(f([u]^\alpha), f([v]^\alpha)) < \varepsilon$$

and from above

$$h([\hat{f}(u)]^\alpha, [\hat{f}(v)]^\alpha) < \varepsilon$$

for all $\alpha \in [0,1]$, that is $D(\hat{f}(u), \hat{f}(v)) < \varepsilon$ this concludes the proof. ☐

Note that if $\hat{f}:(\mathcal{F}(\mathbb{R}^n), D) \to (\mathcal{F}(\mathbb{R}^n), D)$ is continuous and is the Zadeh extension of some function f, then f must be continuous because

$$\begin{aligned} D(\hat{f}(\chi_{\{x\}}), \hat{f}(\chi_{\{y\}})) &= ||f(x) - f(y)|| \\ &= d(f(x), f(y)), \forall x, y \in \mathbb{R}^n \end{aligned}$$

Proposition 8. *Let $f:\mathbb{R}^n \to \mathbb{R}^n$ be a function and $\hat{f}:\mathcal{F}(\mathbb{R}^n) \to \mathcal{F}(\mathbb{R}^n)$ its Zadeh extension, then f is Lipschitz with constant K if and only if $\hat{f}$ is Lipschtz with the metric D with the same constant K.*

Proof. Since f is Lipschitz, it is also continuous and then

$$D(\hat{f}(u), \hat{f}(v)) = \sup_{0\leq\alpha\leq 1} h(f([u]^\alpha), f([v]^\alpha))$$

Now

$$\begin{aligned} & h(f([u]^\alpha), f([v]^\alpha)) \\ &= \max\{\sup_{x\in[u]^\alpha} \inf_{y\in[v]^\alpha} d(f(x), f(y)), \\ &\quad \sup_{y\in[v]^\alpha} \inf_{x\in[u]^\alpha} d(f(x), f(y))\} \\ &\leq \max\{\sup_{x\in[u]^\alpha} \inf_{y\in[v]^\alpha} Kd(x,y), \\ &\quad \sup_{y\in[v]^\alpha} \inf_{x\in[u]^\alpha} Kd(x,y)\} \\ &= K\max\{\sup_{x\in[u]^\alpha} \inf_{y\in[v]^\alpha} d(x,y), \sup_{y\in[v]^\alpha} \inf_{x\in[u]^\alpha} d(x,y)\} \\ &= Kh([u]^\alpha, [v]^\alpha) \end{aligned}$$

hence

$$D(\hat{f}(u), \hat{f}(v)) \leq KD(u,v)$$

On the other hand assuming that $\hat{f}$ is Lipschitz and taking in account the note before this proposition, we get that

$$D(\hat{f}(\chi_{\{x\}}, \hat{f}(\chi_{\{y\}}) = d(x,y), \forall x, y \in \mathbb{R}^n$$

it is easy to show the other side of the result. ☐

Proposition 9. *If $f:\mathbb{R}^n \to \mathbb{R}^n$ is a contraction then $\hat{f}:\mathcal{F}(\mathbb{R}^n) \to \mathcal{F}(\mathbb{R}^n)$ has only one fixed point in $\mathcal{F}(\mathbb{R}^n)$.*

Proof. By the proposition above $\hat{f}$ is a contraction and since $(\mathcal{F}(\mathbb{R}^n), D)$ is complete $\hat{f}$ has only one fixed point. ☐

Obs.: 5. *Calling $\overline{u}$ the unique fixed point from the proposition, then we have*

$$D(\hat{f}^n(u), \overline{u}) \leq \frac{K^n}{1-K} D(\hat{f}(u), \overline{u})$$

Obs.: 6. *Note that the fixed point $\overline{u}$ of $\hat{f}$ must be actually a characteristic function of a one-point set $\{x\}$, where x is the fixed point of f. This means that if we have a contraction*

$$F:(\mathcal{F}(\mathbb{R}^n), D) \to (\mathcal{F}(\mathbb{R}^n), D)$$

whose fixed point is not of this type, F cannot be a Zadeh extension of any contraction.

Now we have a similar result for the endograph metric in $\mathcal{F}(\mathbb{R}^n)$.

Theorem 10. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous function, then the Zadeh extension*

$$\hat{f} : (\mathcal{F}(\mathbb{R}^n), H) \to (\mathcal{F}(\mathbb{R}^n), H)$$

is also continuous.

Proof. We will show that if a sequence u_p converges to u in the metric H, this implies that $\hat{f}(u_p)$ converges to $\hat{f}(u)$ using the same metric H.
Using proposition 4 and continuity of f we have to prove

$$\lim_{p\to\infty} h(f([u_p]^0), f([u]^0)) = 0 \tag{4}$$

$$\begin{aligned} \{\hat{f}(u) > \alpha\} &\subset \lim_{p\to\infty}\inf\{\hat{f}(u_p) \geq \alpha\} \qquad (5) \\ &\subset \lim_{p\to\infty}\sup\{\hat{f}(u_p) \geq \alpha\} \\ &\subset \{\hat{f}(u) \geq \alpha\} \forall \alpha \in [0,1] \end{aligned}$$

Equation (4) follows immediately from the continuity of f and proposition 3
To prove (5); note first that the case where $\alpha = 0$ is trivial. For $\alpha \neq 0$ we have

$$\{\hat{f}(u) > \alpha\} = f(\{u > \alpha\}). \tag{6}$$

From the hypothesis that u_p converges to u in the endograph metric, it follows:

$$\{u > \alpha\} \subset \lim_{p\to\infty}\inf[u_p]^\alpha \subset \lim_{p\to\infty}\sup[u_p]^\alpha \subset [u]^\alpha \tag{7}$$

Applying f to each term we can prove (5) by showing that

$$f(\lim_{p\to\infty}\inf[u_p]^\alpha) \subset \lim_{p\to\infty}\inf f([u_p]^\alpha)$$

and

$$f(\lim_{p\to\infty}\sup[u_p]^\alpha) = \lim_{p\to\infty}\sup f([u_p]^\alpha)$$

To verify the first relation take

$$y \in f(\lim_{p\to\infty}\inf[u_p]^\alpha)$$

then, by definition, $y = f(x)$ where $x = \lim_{p\to\infty} x_p$ where each x_p was taken from $[u_p]^\alpha$ by continuity of f we get $y = \lim_{p\to\infty} f(x_p)$ and then

$$y \in \lim_{p\to\infty}\inf f([u_p]^\alpha).$$

For the second equality if

$$y \in f(\lim_{p\to\infty}\sup[u_p]^\alpha)$$

then $y = f(x)$ with $x = \lim_{j\to\infty} x_{p_j}$, and $x_{p_j} \in [u_{p_j}]^\alpha$ then

$$y = \lim_{j\to\infty} f(x_{p_j})$$

Suppose now we have $y \in \lim_{p\to\infty}\sup f([u_p]^\alpha)$ this implies $y = \lim_{j\to\infty} f(x_{p_j})$; again take a subsequence $x_{p_k} \to x \in \lim_{p\to\infty}\sup[u_p]^\alpha$ and hence

$$y = \lim_{k\to\infty} f(x_{p_k}) = f(x)$$

by continuity and uniqueness of the limit. This concludes the theorem. □

It is clear that the continuity in the endograph metric of $\hat{f}$ implies the continuity of f.
We have proved two theorems on the continuity of the Zadeh extension considering two different metrics on $\mathcal{F}(\mathbb{R}^n)$. Now we study in which measure are these metrics equivalents.

Definition 1. *We say that $u \in \mathcal{F}(\mathbb{R}^n)$ has no proper local maximal points if for all $x \in \mathbb{R}^n$ with $0 < u(x) < 1$, x isn't a local maximal point of u.*

Note that this definition is equivalent to

$$\overline{\{u > \alpha\}} = [u]^\alpha \text{ for } 0 < \alpha < 1$$

We quote another result from [6]

Proposition 11. *If $u_p \in \mathcal{F}(\mathbb{R}^n)$ and $u \in \mathcal{F}(\mathbb{R}^n)$ have no proper local maximal points, and $[u]^1$ has just one point then*

$$u_p \to u \text{ in } D \iff u_p \to u \text{ in } H$$

Proof. See Quelho or Rojas-Medar et alli [6, 9] for a proof. □

Definition 2. *We define the family of restrict fuzzy sets as the elements $u \in \mathcal{F}(\mathbb{R}^n)$ such that u has no proper local maximal points and $[u]^1$ has just one element. We denote this subset of $\mathcal{F}(\mathbb{R}^n)$ as $\mathcal{F}(\mathbb{R}^n)^*$*

Proposition 12. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous function then*

1. $\hat{f}(\mathcal{F}(\mathbb{R}^n)^*) \subset \mathcal{F}(\mathbb{R}^n)^*$
2. $\hat{f}/_{\mathcal{F}(\mathbb{R}^n)^*}$ *is continuous with the metric D*

Proof. Take $u \in \mathcal{F}(\mathbb{R}^n)^*$, then we need prove that

$$\overline{\{\hat{f}(u) > \alpha\}} = [\hat{f}(u)]^\alpha \text{ for } 0 < \alpha < 1$$

and

$$[\hat{f}(u)]^1 \text{ has just one element.}$$

By continuity of the function f and $\alpha > 0$ we proceed like in theorem 5 obtaining

$$\overline{\{\hat{f}(u) > \alpha\}} = f(\overline{\{u > \alpha\}} = f([u]^\alpha) = [\hat{f}(u)]^\alpha$$

The theorem 5 also ensures that

$$[\hat{f}(u)]^1 = f([u]^1)$$

and then $[\hat{f}(u)]^1$ has just one element. This proves assertion (1).
The second part follows immediately from the proposition 11, since taking $u_p, u \in \mathcal{F}(\mathbb{R}^n)^*$, we know that if u_p converges to u in the metric D then converges also in the metric H, and hence $\hat{f}(u_p)$ converges to $\hat{f}(u)$ in the metric H by theorem 10. To conclude the proof of this second part we use the first part and proposition 11 to show that $\hat{f}(u_p)$ converges to $\hat{f}(u)$ in the metric D. $\square$

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