Transitive Actions of Semigroups in Semi-simple Lie Groups

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Abstract

Let G be a connected semi-simple Lie group with finite center and $S \subset G$ a semigroup with interior points. It is proved that S is transitive on a homogeneous space G/L only if the action of L on B is topologically transitive and contracting, where $B = G/P$ is the flag manifold of G associated with S. In [4, Thm.6.4] the authors claimed another necessary condition in case G is simple, namely, that L is discrete. It is shown by means of an example that this condition is wrong without the further assumption that G/L is compact.

1 Introduction

Let G be a connected semi-simple Lie group with finite center and $S \subset G$ a sub-semigroup with nonvoid interior. Given a closed subgroup $L \subset G$, S is said to be

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transitive on the homogeneous space G/L if for every $x, y \in G/L$ there exists $g \in S$ such that $y = gx$. We look here at the possibilities for L in order that S is transitive on G/L. The main results are stated in Theorems 1.1 and 1.2 below.

We work here in the context of [4]. So we use freely the concepts and notations of that paper. In particular, let $\mathbf{g}$ be the Lie algebra of G and $\mathbf{g} = \mathbf{k} \oplus \mathbf{s}$ one of its Cartan decompositions with $\mathbf{k}$ standing for a maximal embedded subalgebra. Select a maximal abelian $\mathbf{a} \subset \mathbf{s}$, let Π be the set of roots of the pair $(\mathbf{g}, \mathbf{a})$ and $\Sigma \subset \Pi$ a simple system of roots. Denote by Π^+ the corresponding set of positive roots. Let $\mathbf{m}$ be the centralizer of $\mathbf{a}$ in $\mathbf{k}$. The standard minimal parabolic subalgebra of $\mathbf{g}$ is given by $\mathbf{p} = \mathbf{m} \oplus \mathbf{a} \oplus \mathbf{n}$ where

$$\mathbf{n} = \sum_{\alpha \in \Pi^+} \mathbf{g}_\alpha$$

is the direct sum of the root spaces associated to the positive roots. The normalizer P of $\mathbf{p}$ in G is a minimal parabolic subgroup and $B = G/P$ is a maximal flag manifold of G. It is well known that $\mathbf{p}$ is the Lie algebra of P. Given a subset $\Theta \subset \Sigma$, let $\mathbf{n}^-(\Theta)$ be the subalgebra generated by $\sum_\alpha \mathbf{g}_{-\alpha}$ with the sum extended to $\alpha \in \Theta$. We denote by $\mathbf{p}_\Theta$ the parabolic subalgebra

$$\mathbf{p}_\Theta = \mathbf{n}^-(\Theta) \oplus \mathbf{p}\,.$$

Its normalizer P_Θ in G is a parabolic subgroup whose Lie algebra is $\mathbf{p}_\Theta$. We put $B_\Theta = G/P_\Theta$ for the corresponding flag manifold.

Denote by W the Weyl group for $(\mathbf{g}, \mathbf{a})$, and by W_Θ the subgroup of W generated by the reflections with respect to the roots in $\Theta \subset \Sigma$. In [4, Section 4] it was associated with a semigroup $S \subset G$ with $\operatorname{int} S \neq \emptyset$ a subgroup $W(S) \subset W$ which accounts for the number of S-control sets on B. It was shown that $W(S) = W_{\Theta_S}$ for some subset Θ_S of the simple system of roots. We use the notation $B(S) = B_{\theta_S}$. The main property of $B(S)$ which will be used here is that if $C \subset B(S)$ stands for the invariant control set for S then C is contained in the stable manifold of the attractor in $B(S)$ of any $h \in \operatorname{int} S$ which is split regular. In particular, there are $b \in B(S)$ and a split regular element H in the Lie algebra $\mathbf{g}$ such that $\exp(tH)x \to b$ as $t \to +\infty$ for all $x \in C$.

The statement of the main result requires the notion of *contracting sequences* (see [2]): Let g_k be a sequence in G, and write the polar decomposition of its elements as $g_k = v_k h_k u_k$ with $v_k, u_k \in K$ and $h_k \in \operatorname{cl} A^+$. Here K is the compact subgroup appearing in a Cartan decomposition of G and $A^+ = \exp \mathbf{a}^+$, where $\mathbf{a}^+ \subset \mathbf{a}$ is a Weyl chamber. For a root $\alpha \in \Pi$ and $h \in \exp \mathbf{a}$, put $\phi_\alpha(h) = \exp(\alpha(\log h))$. The sequence g_k is said to be contracting if $\phi_\alpha(h_k) \to 0$ as $k \to +\infty$ for all negative root

α. Moreover, the sequence is said to be contracting with respect to a flag manifold B_Θ if $\phi_\alpha(h_k) \to 0$ for all negative root α which is not in the subset $\langle\Theta\rangle$ of roots spanned by Θ. It is known that if g_k is contracting with respect to B_Θ then there are a subsequence g_{k_n} and $b_0 \in B_\Theta$ such that $g_{k_n}x \to b_0$ for x in an open and dense subset of B_Θ (see Proposition 2.5 below).

The action of a group L on the topological space X is said to be topologically transitive if every orbit Lx, $x \in X$ is dense in X.

Theorem 1.1 *Let $S \subset G$ a semigroup with $\operatorname{int} S \neq \emptyset$ and $L \subset G$ a closed subgroup. In order that S is transitive on G/L it is necessary that*

1. *L is topologically transitive on $B(S)$ and*
2. *L admits a contractive sequence with respect to $B(S)$.*

It was claimed by the authors that in case G is simple a necessary condition for a proper semigroup to be transitive on G/L is that either $\dim L = 0$ or $L = G$ (see Theorem 6.4 in [4]). This result is wrong: As we show in Section 3 below there are proper semigroups which are transitive on the quotient of $Sl(2n, \mathbb{R})$ by the symplectic group.

Despite that example, Theorem 6.4 in [4] holds with the additional assumption that G/L is compact. We have

Theorem 1.2 *Suppose that G is simple and that $0 < \dim L < \dim G$. Suppose also that G/L is compact. Then S is not transitive on G/L unless $S = G$.*

2 Proofs

We start with the following useful criterion for deciding the transitivity of a semigroup.

Proposition 2.1 *Let G be a topological group, $L \subset G$ a closed subgroup and $S \subset G$ a semigroup with $\operatorname{int} S \neq \emptyset$. If S is transitive on G/L then $\operatorname{int} S \cap gLg^{-1} \neq \emptyset$ for all $g \in G$. Reciprocally, assume that G/L is connected. Then S is transitive on G/L if $\operatorname{int} S \cap gLg^{-1} \neq \emptyset$ for all $g \in G$.*

Proof: Suppose that S is transitive on G/L and take $x \in G/L$ and $g \in \operatorname{int} S$. Then there exists $h \in S$ such that $hgx = x$. Since $hg \in \operatorname{int} S$, this shows that $\operatorname{int} S$ intercepts any conjugate of L.

As to the converse, the condition ensures that $\operatorname{int} S$ intercepts the isotropy at any $x \in G/L$. This implies that $x \in (\operatorname{int} S)\, x \subset \operatorname{int}(Sx)$ which shows that Sx is open for all $x \in G/L$. The same statement holds with S^{-1} instead of S. Fixing x, set

$$\mathcal{O} = \bigcup_{y \notin Sx} S^{-1} y\,.$$

We have that $\mathcal{O} \cup Sx = G/L$. If $y \notin Sx$ then $S^{-1}y \cap Sx = \emptyset$ which shows that $\mathcal{O} \cap Sx = \emptyset$. Since G/L is connected, this shows that $\mathcal{O} = \emptyset$, and we conclude that $Sx = G/L$ and S is transitive. □

This proposition can be stated as:

Corollary 2.2 *With the same notations and assumptions,*

1. *if S is transitive on G/L then* $\operatorname{int}(gSg^{-1}) \cap L \neq \emptyset$ *for all $g \in G$.*
2. *If G/L is connected and* $\operatorname{int}(gSg^{-1}) \cap L \neq \emptyset$ *for all $g \in G$ then S is transitive on G/L.*

We shall need the following fact which is also of a general nature.

Proposition 2.3 *Suppose S is transitive on G/L. Then for every $h \in G$ there exists $g \in L$ such that $hg \in S$.*

Proof: Let x_0 be the origin in G/L. Then there exists $s \in S$ such that $sx_0 = hx_0$. This implies that $s^{-1}hx_0 = x_0$ and hence that $s^{-1}h \in L$. Putting $g = h^{-1}s$, we get the result. □

In order to start the proof of Theorem 1.1 let S be a semigroup transitive on G/L. Let also $C \subset B(S)$ be the invariant control set for S on $B(S)$ and denote by $C_0 \subset C$ its set of transitivity. This is an open an dense subset of C. Moreover, for any $x \in C_0$ there exists a split regular element $H \in \mathbf{g}$ such that x is the attractor of $\exp(tH)$, $t > 0$, and C is contained in its stable manifold (see [4, Prop.4.8]). Since C is compact, this implies that for any neighborhood $U \ni x$ there exists $t_0 > 0$ such that $\exp(tH)\, C \subset U$ for all $t > t_0$. These contractions will be exploited to show that the L-orbits on $B(S)$ are dense. We check first the density of the orbits inside the invariant control set.

Lemma 2.4 *Given $x, y \in C$ there exists a sequence $g_k \in L$ such that $g_k y \to x$ as $k \to \infty$.*

Proof: Take $x \in C_0$ and U a neighborhood of x. By the above comments there exists $h \in G$ such that $hC \subset U$. Apply Proposition 2.3 to h^{-1} to get $g \in L$ such that $h^{-1}g \in S$. Then $h^{-1}gC \subset C$ because C is S-invariant. This implies that $gC \subset hC \subset U$. This ensures the existence of a sequence converging to any $x \in C_0$. Using the density of C_0 in C we get the lemma. □

We can show now the density of the L-orbits on $B(S)$. The lemma above still holds with gC, $g \in G$ in place of C because gC is the invariant control set for gSg^{-1} and this semigroup is also transitive on G/L if S is transitive. Now, the family $\mathrm{int}(gC)$, $g \in G$ covers $B(S)$ so by compactness there exists a finite number $C_i = g_i C$, $i = 1, \ldots, k$ such that

$$B(S) = \mathrm{int}\, C_1 \cup \cdots \cup \mathrm{int}\, C_k .$$

Given $x, y \in B(S)$ we can find $1 \leq i_1, \ldots, i_l \leq k$ with $x \in \mathrm{int}\, C_{i_1}$ and $y \in \mathrm{int}\, C_{i_l}$, and such that $\mathrm{int}\, C_{i_j} \cap \mathrm{int}\, C_{i_{j+1}} \neq \emptyset$ for otherwise $B(S)$ would not be connected. This being so, pick $z_j \in \mathrm{int}\, C_{i_j} \cap \mathrm{int}\, C_{i_{j+1}}$, $j = 1, \ldots, l$ and a neighborhood $V \ni y$. By the lemma above, there exists $h_2 \in L$ such that $h_2 z_l \in V$. Hence $V_l = h_2^{-1}V$ is a neighborhood of z_l. Applying again the lemma, there exists $g_{l-1} \in L$ such that $g_{l-1}z_{l-1} \in V_l$ and thus we get the neighborhood $V_{l-1} = g_{l-1}^{-1}V_l$ of z_{l-1}. Applying successively the lemma, we get neighborhoods V_i of z_i such that $V_{i+1} = g_i V_i$ with $g_i \in L$. Since V_1 is a neighborhood of z_1, there exists $h_1 \in L$ with $h_1 x \in V_1$. This way,

$$h_2 g_l \cdots g_1 h_1 x \in V$$

which shows that there exists a sequence $h_k \in L$ with $h_k x \to y$ concluding the proof that L is topologically transitive on $B(S)$.

Now, we check that L satisfies the second condition of Theorem 1.1. For this we reproduce here the following well known description of the action on a flag manifold B_Θ of sequences $g_k \in G$ (see [2]).

Let $g_k = v_k h_k u_k$, $v_k, u_k \in K$, $h_k \in \mathrm{cl}\, A^+$ be the polar decomposition of the sequence. Denote by $b_0 \in B_\Theta$ the attractor of the elements in A^+ and let $\sigma = N^- b_0$ the corresponding open Bruhat component (stable manifold). Substituting g_k by a subsequence we can assume that $v_k \to v$ and $u_k \to u$. This being so, take $x \in u^{-1}\sigma$. Then $u_k x \to ux \in \sigma$ so that $y_k = u_k x$ belongs σ for large k, and $y_k \to y = ux$.

We can write $y_k = n_k b_0$ with $n_k = \exp(X_k)$, and $X_k \in \mathbf{n}^-(\Theta)$. The same way, $y = \exp(X) b_0$, $X \in \mathbf{n}^-(\Theta)$, and we have that $X_k \to X$.

With this notation, the action of h_k on y_k is

$$h_k y_k = h_k \exp(X_k) b_0 = \exp(\mathrm{Ad}(h_k) X_k) b_0 .$$

We decompose X_k as

$$X_k = \sum X_k^\alpha$$

with $X_k^\alpha \in \mathbf{g}_\alpha$, and α running over the negative roots which are not in $\langle\Theta\rangle$. A similar decomposition exists for X with components X^α. We have that

$$\mathrm{Ad}(h_k) X_k = \sum \phi_\alpha(h_k) X_k^\alpha$$

where $\phi_\alpha(h_k) = \exp(\alpha(\log h_k))$. Since $0 < \phi_\alpha(h_k) \leq 1$, we can take subsequences again and assume that $\lim \phi_\alpha(h_k) = a_\alpha \in [0,1]$ exists for all negative root α. Assuming this, we have that the restriction of $\mathrm{Ad}(h_k)$ to $\mathbf{n}^-(\Theta)$ converges to a linear mapping, say τ of $\mathbf{n}^-(\Theta)$. This τ is diagonal and its eigenvalues are a_α. Clearly, $\mathrm{Ad}(h_k) X \to \tau X$, and since $X_k \to X$ we have also that $\mathrm{Ad}(h_k) X_k \to \tau X$. We get thus the

Proposition 2.5 *Take a sequence $g_k \in G$. Then there are*

1. a subsequence g_{k_n},

2. elements $v, u \in K$, and

3. a linear mapping τ of $\mathbf{n}^-(\Theta)$

such that for every $Y \in \mathbf{n}^-(\Theta)$,

$$g_{k_n} u^{-1} \exp(Y) b_0 \to v \exp(\tau Y) b_0$$

as $n \to \infty$. The subsequence is contracting if and only if $\tau = 0$. □

Corollary 2.6 *Let $g_k \in G$ be a sequence, and suppose that for an open subset $U \in B_\Theta$, $g_k x \to b_0$ for all $x \in U$, where $b_0 \in B_\Theta$ is fixed. Then g_k admits a subsequence which is contracting with respect to B_Θ.*

Proof: Take the polar decomposition in such a way that b_0 is the attractor of the elements in the Weyl chamber A^+ and apply the proposition to the sequence. The subset

$$V = \{Y \in \mathbf{n}^- (\Theta) : u^{-1} \exp(Y) b_0 \in U\}$$

is open and not empty in $\mathbf{n}^- (\Theta)$. For $Y \in V$, we have by the proposition that

$$g_{k_n} u^{-1} \exp(Y) b_0 \to v \exp(\tau Y) b_0 ,$$

and since $u^{-1} \exp(Y) b_0 \in U$ we have also that

$$g_{k_n} u^{-1} \exp(Y) b_0 \to b_0 .$$

Comparing these limits we get that $v = 1$ and $\tau Y = 0$ for $Y \in V$. The fact that $V \neq \emptyset$ is open implies then that $\tau = 0$ and the subsequence is contracting with respect to B_Θ. □

With this corollary it becomes easy to get a contracting sequence in L. In fact, take x in C_0 and a sequence U_k of neighborhoods of x whose intersection is $\{x\}$. Take also a sequence h_k of split regular elements in G such that $h_k C \subset U_k$. By Proposition 2.3 there exists, for each k, $g_k \in L$ such that $h_k^{-1} g_k \in S$. Therefore $h_k^{-1} g_k C \subset C$ so that

$$g_k C \subset h_k C \subset U_k$$

and $g_k y \to x$ for every $y \in C$. Since $\operatorname{int} C \neq \emptyset$ the above corollary implies that g_k admits a contracting subsequence. Therefore L contains a sequence which is contracting with respect to $B(S)$ concluding the proof of Theorem 1.1.

Let us consider now Theorem 1.2. The proof of Theorem 6.4 in [4] works with the assumption that G/L is compact. Here is a modification of that proof which is based in Theorem 1.1: Let $\mathbf{l}$ be the Lie algebra of L and put $J = N(\mathbf{l})$ for its normalizer in G. The assumption on the dimension of L and the fact that $\mathbf{g}$ is simple imply that $0 < \dim J < \dim G$. We have that G/J is the orbit under G of $\mathbf{l}$ in the Grassmannian of $k = \dim \mathbf{l}$ subspaces of $\mathbf{g}$. This orbit is compact because $L \subset J$. Therefore, the result is a consequence of the following lemma.

Lemma 2.7 *Suppose that G is simple and let G/J be a compact projective orbit for some finite dimensional representation of G. Then J is not topologically transitive on any flag manifold unless $J = G$.*

Proof: Let $G = KAN$ be an Iwasawa decomposition. In any finite dimensional representation of G the elements of $T = AN$ are represented by upper triangular matrices. Therefore, the fact that G/J is compact implies that there exists $x \in G/J$ which is fixed by T (see [5]). Hence we can assume without loss of generality that $T \subset J$. This being so, put $U = J \cap K$. Then U is compact and $J = UT$. Now, suppose that J is topologically transitive on some boundary $B = G/Q$ with Q parabolic. We can assume that $T \subset Q$ hence the density of the orbit under J of the origin $b_0 \in B$ implies that the U-orbit of b_0 is also dense. From the compactness of U we then have that U is transitive on B.

Now we realize B as an adjoint orbit under K: let $\mathbf{k}$ be the Lie algebra of K and $\mathbf{g} = \mathbf{k} \oplus \mathbf{s}$ the corresponding Cartan decomposition. We have that the Lie algebra $\mathbf{a}$ of A is contained in $\mathbf{s}$, and there exists $H \in \mathbf{a}$ such that $\mathrm{Ad}\,(K)\,H$ coincides with B as a homogeneous space. Since $\mathbf{g}$ is simple the adjoint action of K on $\mathbf{s}$ is irreducible and hence the subspace spanned by the orbit $\mathrm{Ad}\,(K)\,H$ coincides with $\mathbf{s}$. Now, H belongs to the Lie algebra $\mathbf{j}$ of J so that $\mathrm{Ad}\,(U)\,H \subset \mathbf{j}$. However, $\mathrm{Ad}\,(U)\,H$ coincides with $\mathrm{Ad}\,(K)\,H$ because U is transitive on B. This shows that $\mathbf{s}$ is contained in $\mathbf{j}$ and since the Lie algebra generated by $\mathbf{s}$ is $\mathbf{g}$ we conclude that $\mathbf{j} = \mathbf{g}$ and hence that $J = G$. □

3 Counterexamples

Let W be a pointed generating cone in $\mathbb{R}^{2n}$ and define

$$S_W = \{g \in Sl\,(2n, \mathbb{R}) : gW \subset W\}\,.$$

This is a subsemigroup with nonempty interior of $G = Sl\,(2n, \mathbb{R})$ for which $B\,(S)$ is the projective space $\mathbb{R}P^{2n-1}$. Let L be the symplectic group $Sp\,(n, \mathbb{R})$. Its Lie algebra $\mathbf{sp}\,(n, \mathbb{R})$ is the algebra of matrices which are written in blocks $n \times n$ as

$$\begin{pmatrix} A & B \\ C & -A^t \end{pmatrix}$$

with B and C symmetric.

We shall prove that S_W is transitive on G/L.

Lemma 3.1 *Take $v \in \mathbb{R}^{2n}$ with $|v| = 1$ and put $V = v^\perp$ for the orthogonal complement of v. Then there exists $H \in \mathbf{sp}\,(n, \mathbb{R})$ which is diagonalizable and has a* *principal eigenvalue λ_m of multiplicity one, that is, $\lambda_m > \mu$ for any other eigenvalue, and moreover,*

1. *v spans the eigenspace associated with λ_m, and*

2. *the other eigenspaces are contained in V.*

Proof: If $v = e_1$, the first basic vector, take

$$H_0 = \mathrm{diag}\{\lambda_1, \ldots, \lambda_n, -\lambda_1, \ldots, -\lambda_n\}$$

with $\lambda_1 > \cdots > \lambda_n > 0$. This H_0 satisfies the requirements.

On the other hand, let K be the compact component of a Cartan decomposition of $Sp\,(n, \mathbb{R})$ contained in the orthogonal group. It is well known that K is transitive on the sphere S^{2n-1} (see e.g. [1]). Therefore for an arbitrary $v \in S^{2n-1}$, there exists $k \in K$ such that $ke_1 = v$. Then $H = kH_0k^{-1}$ is the required element in $\mathbf{sp}\,(n, \mathbb{R})$ because its eigenspaces are the images under k of the eigenspaces of H_0. □

Lemma 3.2 *Let $W \in \mathbb{R}^d$ be a pointed generating cone and consider its dual*

$$W^* = \{v \in \mathbb{R}^d : \langle v, w\rangle \geq 0 \text{ for all } w \in W\}.$$

Then $\mathrm{int}\, W \cap \mathrm{int}\, W^* \neq \emptyset$.

Proof: By induction on d. For $d = 1$ or 2 the result is trivial. Before proving the induction step, let $P : \mathbb{R}^d \to \mathbb{R}^d$ be an orthogonal projection. Then $P^t = P$ and since W is generating, $P\,(W)$ is generating in the image of P. Moreover, the dual $(P\,(W))^*$ in the image of P is contained in W^*. In fact, take $y \in (P\,(W))^*$. Then

$$\langle y, x\rangle = \langle P^t y, x\rangle = \langle y, Px\rangle \geq 0$$

for all $x \in W$.

This fact will be used in the following situation: If $\mathrm{int}\, W^* \subset W$ there is nothing to prove. Otherwise, let $x \in (\mathrm{int}\, W^*) - W$, and denote by P the orthogonal projection onto $x^\perp$. We claim that $P\,(W)$ is a pointed cone. In fact, suppose $0 \neq \pm y \in P\,(W)$. Then there are $a_\pm \in \mathbb{R}$ such that $z_\pm = \pm y + a_\pm x \in W$. Since $x \in \mathrm{int}\, W^*$, $a_\pm > 0$. However,

$$z_+ + z_- = (a_+ + a_-)\, x$$

with $a_+ + a_- > 0$ which implies that $x \in W$ contradicting the choice of x.

The induction hypothesis applies then to $P(W)$ so that

$$\operatorname{int} P(W) \cap \operatorname{int} P(W)^* \neq \emptyset$$

with the interior taken in $x^\perp$. By the previous comment, W^* contains the wedge

$$V = \mathbb{R}^+ x + (P(W))^*,$$

and it is clear that $\lambda x + \operatorname{int}(P(W))^*$ is contained in the interior of V if $\lambda > 0$. This being so, pick

$$z \in \operatorname{int} P(W) \cap \operatorname{int} P(W)^*.$$

Then $\lambda x + z \in \operatorname{int} V \subset \operatorname{int} W^*$ for all $\lambda > 0$. Moreover, there exists $a \in \mathbb{R}$ such that $ax + z \in W$ because $z \in P(W)$. Since $x \in \operatorname{int} W^*$, $a > 0$. This shows that $W \cap \operatorname{int} W^* \neq \emptyset$ concluding the proof of the lemma because if two pointed and generating wedges are such that one of them itercepts the interior of the other than they have a common interior point. □

We can show now that S_W is transitive on $Sl(2n, \mathbb{R}) / Sp(n, \mathbb{R})$. According to Corollary 2.2 we must show that $Sp(n, \mathbb{R})$ meets the interior of $gS_W g^{-1}$ for all g. Now, $gS_W g^{-1} = S_{gW}$, and of course, gW is pointed and generating if and only if the same happens to W. Also, $g \in \operatorname{int} S_W$ if and only if $gW \subset \operatorname{int} W$. Hence the transitivity of S_W follows if we show that there exists $g \in Sp(n, \mathbb{R})$ such that $gW \subset \operatorname{int} W$. For this, take

$$v \in \operatorname{int} W \cap \operatorname{int} W^*.$$

We have that $v^\perp \cap W = 0$ because $v \in \operatorname{int} W^*$. Let $H \in \mathbf{sp}(n, \mathbb{R})$ be as in Lemma 3.1 with v a principal eigenvector. Then v is an attractor for the spherical action of $\exp(tH)$, $t > 0$ with the stable manifold given by $\langle v, \cdot \rangle > 0$. From this we have that $\exp(tH) W \subset \operatorname{int} W$ for $t > 0$ big enough. This shows that $Sp(n, \mathbb{R})$ meets the interior of any S_W so that these semigroups are transitive on $Sl(2n, \mathbb{R}) / Sp(n, \mathbb{R})$.

The transitivity of S_W on $Sl(2n, \mathbb{R}) / Sp(2n, \mathbb{R})$ shows that Theorem 1.2 does not hold without the assumption that G/L is compact as was claimed in [4, Thm. 6.4].

The flaw in the proof offered in [4] for this fact comes from Lemma 1 in [3] which is wrong. That lemma claims that if a subsemigroup S, with nonvoid interior, of a linear group G is transitive in a projective orbit $\mathcal{O}$ of G then it is also transitive on the orbits which are in the closure of $\mathcal{O}$.

In order to provide a counterexemple for this statement we use again the semigroups $S_W \subset Sl\,(2n, \mathbb{R})$ and the symplectic group

$$Sp\,(n, \mathbb{R}) = \{g \in Sl\,(2n, \mathbb{R}) : gJg^t = J\}$$

where

$$J = \begin{pmatrix} 0 & -1_{n\times n} \\ 1_{n\times n} & 0 \end{pmatrix} .$$

Let $V = \bigwedge^2 \left(\mathbb{R}^{2n}\right)^*$ be the space of skew-symmetric bilinear forms on $\mathbb{R}^{2n}$. $Sl\,(2n, \mathbb{R})$ represents in V by

$$(g\beta)\,(u, v) = \beta\left(g^{-1}u, g^{-1}v\right) .$$

Taking the symplectic form $\omega \in V$, whose matrix is J, the isotropy of the action of $Sl\,(2n, \mathbb{R})$ is exactly the symplectic group. Therefore, S is transitive on the orbit of ω and thus in its projective orbit. On the other hand, on the closure of this projective orbit there is a Grassmannian. In fact, the matrix of $g\omega$, $g \in Sl\,(2n, \mathbb{R})$ is

$$\left(g^{-1}\right)^t Jg^{-1}$$

so that if $h^{-1} = \mathrm{diag}\{\lambda_1, \ldots, \lambda_{2n}\}$ with $\lambda_1 > \cdots > \lambda_{2n} > 0$ then the matrix of $h^k\omega$, $k \geq 1$ is

$$\begin{pmatrix} 0 & -\Lambda^k \\ \Lambda^k & 0 \end{pmatrix}$$

with $\Lambda = \mathrm{diag}\{\lambda_1\lambda_{n+1}, \ldots, \lambda_n\lambda_{2n}\}$. The eigenvalue $\lambda_1\lambda_{n+1}$ of Λ is strictly bigger than any other eigenvalue. This implies that

$$\frac{1}{\lambda_1^k\lambda_2^k}h^k\omega \longrightarrow \varepsilon_1 \wedge \varepsilon_{n+1}$$

as $k \to \infty$. Here ε_i, $i = 1, \ldots, 2n$ is the basis of $\left(\mathbb{R}^{2n}\right)^*$ dual to the basis of $\mathbb{R}^{2n}$. This shows that the orbit of the decomposable vector $\varepsilon_1 \wedge \varepsilon_{n+1}$ is in the closure of the orbit of ω. Now it is easily seen that the isotropy at $\varepsilon_1 \wedge \varepsilon_{n+1}$ is the subgroup Q of matrices of the form

$$\begin{pmatrix} x & 0 \\ * & * \end{pmatrix}$$

with x being a 2×2 matrix. In other words, the orbit of $\varepsilon_1 \wedge \varepsilon_{n+1}$ is the Grassmannian of $2n-2$ subspaces of $\mathbb{R}^{2n}$. None of the semigroups S_W is transitive on this

Grassmannian. This can be seen either by Theorem 6.2 in [4], or by Theorem 1.1 above (the isotropy Q is not transitive on the projective space) or even directly: The $(2n-2)$-subspaces which meet W is a proper subset of the Grassmannian which is invariant under S_W.

References

[1] Boothby, W. M. : *A transitivity problem from control theory*, J. Differential Equations **17** (1975) 296-307.

[2] Guivarc'h, Y. and A. Raugi: *Frontière de Furstenberg, propriétés de contraction e théoremes de convergence*. Z. Wahrscheinlichkeitstheorie Verw. Geb. **69** (1985), 187-242.

[3] San Martin, L.A.B.: *Nonreversibility of subsemigroups of semi-simple Lie groups.* Semigroup Forum **44** (1992), 376-387.

[4] San Martin, L. A. B. and P. A. Tonelli: *Semigroup actions on homogeneous spaces.* Semigroup Forum **50** (1995), 59-88.

[5] Vinberg, E.B.: *The Morosov-Borel theorem for real Lie groups.* Soviet Math. Dokl. **2** (1961), 1416-1419.