Remarks on Deterministic Orbits in Fuzzy Dynamical Systems

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Abstract

In this work we compare the solutions of a classical Cauchy problem on differential equations with the solution of its Zadeh's extension, and the orbits of related fuzzy and deterministic cascades.

1 Introduction

In this work, we use the theory of fuzzy differential equations and the related Cauchy problem as developed by Kaleva and Seikalla [5, 9] to establish relations with deterministic flows of differential equations and the related discrete problems of Fuzzy Iterated Systems. The main conclusion is that the deterministic solutions could be seen as preferred solutions of the fuzzy extensions in the sense that will be precised.

Given a time dependent vector field $f(t,x)$ in $\mathbb{R}^n$ one can use the Zadeh's extension for produce a new vector field in $\mathcal{F}(\mathbb{R}^n)$. Thinking in the associated fuzzy differential equations we propose to compare the solutions of the both systems. After that we found a similar result for the case of the Zadeh's extension of diffeomorphisms, or even continuous transformation in $\mathbb{R}^n$.

Here we fix some notations and recall known results. The family of all compact nonempty subsets of $\mathbb{R}^n$ will be denoted as $Q(\mathbb{R}^n)$, while $Q_c(\mathbb{R}^n)$is for the subset of $Q(\mathbb{R}^n)$, whose elements are convex set in $\mathbb{R}^n$.

We also set $\mathcal{F}(\mathbb{R}^n)$for the family of fuzzy sets $u : \mathbb{R}^n \to [0,1]$ whose α-level:

$$[u]^\alpha = \{x \in \mathbb{R}^n : u(x) \geq \alpha\} 0 < \alpha \leq 1$$

and

$$[u]^0 = \mathrm{cl}\{x \in \mathbb{R}^n : u(x) > 0\}$$

are in $Q(\mathbb{R}^n)$. Finally $\mathcal{E}^n$denotes the family of fuzzy sets whose α-level are in $Q_c(\mathbb{R}^n)$.

It is known that the metric

$$D(u,v) = \sup_{0 \leq \alpha \leq 1} h([u]^\alpha, [v]^\alpha)$$

where h is the Hausdorff metric in $Q(\mathbb{R}^n)$, makes the spaces ($\mathcal{F}(\mathbb{R}^n)$,D) and ($\mathcal{E}^n$, D) into complete metric spaces [7].

In the remainder of this text T is the interval $[a,b] \subset \mathbb{R}$.

Definition 1. *Let $f : T \times \mathbb{R}^n \to \mathbb{R}^n$ be a mapping then for each fixed t we define the Zadeh's extension as:*

$$\hat{f}(t,u) = \begin{cases} \sup_{\tau \in f^{-1}(t,x)} u(\tau) \\ \quad if\ f^{-1}(t,x) \neq \emptyset \\ \\ 0\ if\ f^{-1}(t,x) = \emptyset \end{cases}$$

for all fuzzy set u.

The proof of the following results can be found in [1]

Theorem 1. *If $f : T \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous then $\hat{f} : T \times \mathcal{F}(\mathbb{R}^n) \to \mathcal{F}(\mathbb{R}^n)$ is well defined and for all $t \in T$ and $\alpha \in [0,1]$ we have*

$$[\hat{f}(t,u)]^\alpha = f(t,[u]^\alpha)$$

Theorem 2. *If $f : T \times \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz with constant K so is the Zadeh's extension $\hat{f} : T \times \mathcal{F}(\mathbb{R}^n) \to \mathcal{F}(\mathbb{R}^n)$, with the same Lipschitz constant with respect to the metric D.*

Note that if $x \in \mathbb{R}^n$ is a fixed point of a continuous function $f : \mathbb{R}^n \to \mathbb{R}^n$, the the fuzzy set χ_x which is one for x and zero for all other points, is a fixed point of $\hat{f}$, hence we have

Corollary 3. *If $f : \mathbb{R}^n \to \mathbb{R}^n$ is a contraction, then $\hat{f} : \mathcal{F}(\mathbb{R}^n) \to \mathcal{F}(\mathbb{R}^n)$ is also a contraction. And if x_0 is the only fixed point of f given by the Banach theorem, then χ_{x_0} is the only fixed point of $\hat{f}$.*

2 Fuzzy differential equations

The existence and uniqueness of solution to the Cauchy problem in fuzzy differential equations, has been studied by many authors, among them we cite Kaleva [5] and Seikalla [9].

It is our aim here to give some relations between the solutions a of a deterministic vector field in $\mathbb{R}^n$ and the solutions in the sense of Kaleva of the corresponding Zadeh's extension.

We consider the classical deterministic Cauchy problem in $\mathbb{R}^n$

$$x' = f(t,x) \quad x(a) = x_0 \tag{1}$$

and its fuzzy counterpart lifting the vector field $f(t,x)$ via Zadeh's extension:

$$u'(t) = \hat{f}(t,u) \quad u(a) = u_0 \in \mathcal{E}^n \tag{2}$$

where $a > 0$ and $\hat{f} : T \times \mathcal{E}^n \to \mathcal{E}^n$. We have the following theorem due to Kaleva [5].

Theorem 4. *Suppose that $\hat{f} : T \times \mathcal{E}^n \to \mathcal{E}^n$ is continuous and Lipschitz in the second variable. Then the problem (2) has a unique solution in T.*

The next result gives us a sufficient condition for the existence and uniqueness of problem (2) as well as gives a relation between this solution and a solution of a deterministic system (1).

Theorem 5. *If $f : T \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous and Lipschitz in the second variable, and if $\hat{f} : T \times \mathcal{E}^n \to \mathcal{E}^n$ is well defined, i.e. $f(t,[u]^\alpha)$ are convex for all $t \in T$ and $\alpha \in [0,1]$, then*

(a) Both problems (1) and (2) have unique solutions in T.

(b) Denoting respectively by $x(t,x_0)$ and $u(t,u_0)$ these solutions then

$$u(t,u_0)(x(t,x_0)) = 1$$

for all $t \in T$ if also $u_0(x_0) = 1$

Proof. For item (a) we recall that $\hat{f}$ is Lipschitz by theorem **2**

For (b) it is sufficient to show that for all $t \in T$ we have

$$x(t,x_0) \in [u(t,u_0)]^1 \text{ if } x_0 \in [u_0]^1$$

Fix $x_0 \in [u_0]^1$ and define $v_0 \in \mathcal{E}^n$ as

$$[v_0]^\alpha = \begin{cases} \{x_0\} & \text{if} \quad 1/2 \leq \alpha \leq 1 \\ [u_0]^1 & \text{if} \quad 0 < \alpha < 1/2 \end{cases}$$

the problem (2) with initial condition v_0 has a unique solution $v(t)$. Since we have $[\hat{f}(t,v(t))]^\alpha = f(t,[v(t)]^\alpha)$ we have that the α-level $[v(t)]^\alpha$ is the unique solution of

$$DX(t) = f(t,X(t)) \qquad X(a) = [v_0]^\alpha \tag{3}$$

where $DX(t)$ denote the Hukuhara derivative of the multi-valuated function $X(t)$, see Kaleva [5] .

In this way, taking $\alpha = 1$ by uniqueness we have $\{x(t,x_0)\} = [v(t,v_0)]^1$ for all $t \in T$

On the other hand for $\alpha = 1/2$ we conclude that $[u(t,u_0)]^1$ and $[v(t,v_0)]^{\frac{1}{2}}$ must coincide in T, being solutions of the same problem.

As $[v(t,v_0)]^1 \subset [v(t,v_0)]^{\frac{1}{2}}$ we conclude that $x(t,x_0) \in [u(t,u_0)]^1$ for all $t \in T$. □

Observe that the conclusion (b) of the theorem remain valid as long as we can assure the existence and uniqueness of solutions to the problem (2). In this case one can easily see that the solution of (2) with initial condition $u_0 \in \mathbb{R}^n$, that is the fuzzy set χ_{u_0} is in fact the deterministic solution.

Concerning the problem (3) one can prove in analogous way as in the theorem that if $A, B \in \mathcal{Q}_c(\mathbb{R}^n)$ with $A \subset B$ then $X(t,A) \subset X(t,B)$ where these are the solutions of the problem with initial conditions A and B respectively.

If $n = 1$ the continuity of f is sufficient to ensure that $\hat{f}$ take convex sets in convex sets.

3 Fuzzy cascades

We are going to consider the discrete time system in $\mathcal{F}(\mathbb{R}^n)$:

$$u_{n+1} = F(u_n) \tag{4}$$

Given $u_0 \in \mathcal{F}(\mathbb{R}^n)$ we call positive orbit the sequence $\{F^n(u_0)\}$, where n is a positive integer and F^n means the n-step composition of F and the convention $F^0(u) = u$ is in force.

From now on we assume that F is the Zadeh's extension of some map defined in $\mathbb{R}^n$, although some results remain valid in more general context. These systems were also used in Cabrelli et Alli [3] and Forte et Alli [4] to study fuzzy iterative function system related to image construction, in this case the functions were contractions in $\mathcal{F}(\mathbb{R}^n)$.

Given the iterative system in $\mathbb{R}^n$

$$x_{n+1} = f(x_n) \tag{5}$$

we would like to study its lift to $\mathcal{F}(\mathbb{R}^n)$via Zadeh's extension:

$$u_{n+1} = \hat{f}(u_n) \tag{6}$$

Theorem 6. *Suppose that x_n and u_n are respectively solutions of (5) and (6) with the initial condition x_0 and u_0 such that $u_0(x_0) = 1$. then we have that $u_n(x_n) = 1$ for all $n \geq 1$.*

Proof. As x_n and u_n are solutions we have

$$u_{n+1}(x_{n+1}) = \hat{f}(u_n)(x_{n+1}) =$$

$$= \sup_{x_{n+1}=f(s)} u_n(s) \geq u_n(x_n)$$

hence

$$u_{n+1}(x_{n+1}) \geq u_n(x_n) \geq \cdots \geq u_0(x_0) = 1$$

which concludes the proof. □

Note that for this proof we don't need the continuity of f. If f is continuous then the result above follows from Theorem **1** and in this case we have also that $\hat{f}^n(u_0) \in \mathcal{F}(\mathbb{R}^n)$ if $u_0 \in \mathcal{F}(\mathbb{R}^n)$, and if $\hat{f}^n(u_0)$ converges to u then $u \in \mathcal{F}(\mathbb{R}^n)$ since the space $(\mathcal{F}(\mathbb{R}^n), D)$ is complete.

Example 7. *Consider $F : \mathcal{E}^1 \to \mathcal{E}^1$ given by $F(u) = \lambda u$ where also $\lambda \in \mathcal{E}^1$ and the product is taken in the arithmetic of fuzzy numbers [7, 5, 9]*

Consider now the system

$$u_{n+1} = F(u_n) \tag{7}$$

Then we have according [9]

$$[F(u)]^\alpha = [\lambda_1^\alpha u_1^\alpha, \lambda_2^\alpha u_2^\alpha] \quad \forall \alpha \in [0,1]$$

where $[\lambda]^\alpha = [\lambda_1^\alpha, \lambda_2^\alpha] \subset \mathbb{R}^+$ and $[u]^\alpha = [u_1^\alpha, u_2^\alpha] \subset \mathbb{R}^+$.

Given an initial condition u_0 with u_{01}^α positive then the α-levels of the solution is

$$\begin{aligned} [F^n(u_0)]^\alpha &= [\lambda^n u_0]^\alpha \\ &= [\lambda_1^{\alpha n} u_{01}^\alpha, \lambda_2^{\alpha n} u_{02}^\alpha] \\ &\quad \forall \alpha \in [0,1] \end{aligned} \tag{8}$$

Although the function F isn't a Zadeh's extension of any function, we have that for any deterministic iterative linear system with coefficient λ such that $\lambda \in [\lambda_1^\alpha, \lambda_2^\alpha]$ and initial condition $x_0 \in [u_{01}^\alpha, u_{02}^\alpha]$ the following relation is valid

$$(\lambda_1^\alpha)^n u_{01}^\alpha \leq \lambda^n x_0 \leq (\lambda_2^\alpha)^n u_{02}^\alpha$$

and this means according equation (8) that

$$\lambda^n x_0 \in [F^n(u_0)]^\alpha$$

or in other words $u_n(x_n) = 1$ *where* x_n *is the solution of*

$$x_{n+1} = \lambda x_n$$

In this example one could also verify that the diameter of the fuzzy solution given by

$$\lambda_2^{\alpha n} u_{02}^\alpha - \lambda_1^{\alpha n} u_{01}^\alpha$$

is decreasing if $[\lambda]^\alpha \subset [0,1]$ as n goes to infinity which cannot happen for the solutions of fuzzy differential equations.

4 Final remarks

This work is part of a program toward a mathematical theory of fuzzy flows. We do not have yet clear how appropriated could be to consider fuzzy flows coming from fuzzy differential equations, since it is known that the solution of these equations enlarge the fuzziness of the system as time increases (see Kaleva [5]) and this makes more difficult to find the proper concepts of stability and Liapunov exponents. This difficulty doesn't appear in the case of fuzzy cascades, and one could study the asymptotic properties in the space $(\mathcal{F}(\mathbb{R}^n), D)$, ([2]). In the present work we have shown that in case of Zadeh's extension of deterministic systems we have also compatibilities between deterministic solutions and its corresponding fuzzy solutions.

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