CONLEY DECOMPOSITIONS AND ACTIONS ON FLAG MANIFOLDS

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Abstract. This work is concerned with the action of a semisimple element g of the semisimple connected Lie group $\mathrm{Sl}(d,R)$ on flag manifolds or F_Θ. These manifolds are obtained as left cosets of a parabolic group T_Θ.

We consider minimal sets of g on F_Θ, assigning to each minimal set M an index $K(M)$ which appears if one consider an special decomposition of the vector space $\mathbb{R}^d$ related to g called the Conley Decomposition.

The main result to be proved is:

Theorem 1. *Let M and N two minimal sets with respect to g in F_Θ. Then there is a continuous curve $\alpha : [0,1] \to F_\Theta$ joining M and N and such that each $\alpha(t)$ be almost periodic if and only if $K(M) = K(N)$.*

We give an application of the theorem in control theory.

1. Introduction

We present a brief resumée on flag manifolds to fix some notations. This approach follows closely that of the paper by Guivarch and Raugi [GR89]. The details can be found there and in the book by Rohlin and Fuchs [RF81], chapter 3.

By Θ is meant an ordered subset of the set $\{1,\dots,d\}$, that is $\Theta = \{i_1,\dots,i_s\}$ where $i_k < i_{k+1} < d$ for all $k \in \{1,\dots,s\}$. $\mathbb{R}^d$ is the d-dimensional vector space over $\mathbf{R}$.

Definition 1. The space of all s-tuple of subspaces $(E_{d-i_s},\dots,E_{d-i_1})$, where each E_i is a i-dimensional subspace of $\mathbb{R}^d$, and also a subspace of E_j for $j > i$, is called Θ-flag manifold and is denoted by F_Θ.

This set can be given directly a topology and a differentiable structure that will not be described here (see [RF81]), instead one looks to each F_Θ as homogeneous spaces of $\mathrm{Sl}(d,R)$.

As particular instances of flag manifolds one have to keep in mind

(1) The complete flag manifold that occurs when $\Theta = \{1,\dots,d\}$.

(2) The Grassmanian manifold obtained when $\Theta = \{i\}$ has just one element. In this case it would be used the traditional notation $\mathbf{Gr}(d-i,\mathbb{R}^d)$, this include yet the projective space $\mathbf{P}^{d-1}$ as particular case ($\Theta = \{d-1\}$).

With respect to the canonical basis there is a canonical and transitive action of $\mathrm{Sl}(d,R)$ on $\mathbb{R}^d$ and as consequence also on F_Θ, such that this space is actually an homogeneous space identified with the left coset space $\mathrm{Sl}(d,R)/T_\Theta$, where T_Θ is the isotropy group with respect to the action on F_Θ, this group is composed of all matrices with the following form:

$$T = \begin{pmatrix} G_{i_1} & 0 & 0 & 0 \\ * & G_{i_2-i_1} & 0 & 0 \\ * & * & \ddots & 0 \\ * & * & * & G_{d-i_s} \end{pmatrix}$$

Here G_i is an element of the group $\mathrm{Gl}(i, \mathbf{R})$ and the product of the $\det G_i$ must be one.

To obtain this representation we take in F_Θ a canonical element δ_Θ defined as following:

Let $f_1, \ldots, f_d$ be the canonical basis of $\mathbb{R}^d$ and set

$$F_{d-i_k} = [f_d, \ldots, f_{i_k+1}]_{vs}$$

where the rightside means the vector space generated by the vectors between brackets. Then δ_Θ is defined to be

$$\delta_\Theta = (F_{d-i_s}, \ldots, F_{d-i_1})$$

It will be computed the isotropy group for this element (which is the same for all elements). For this purpose, let $T \in T_\Theta$ be fixed. Then it is easy to compute that

$$T\delta_\Theta = \delta_\Theta \text{ if and only if } TF_i = F_i \forall i \in \{d - i_k : 1 \leq k \leq s\}$$

Being T a matrix (a_{ij}), then one can write this condition as $Tf_i \in F_{d-i_k}$ for $i \in \{i_k + 1, \ldots, d\}$, which means in this case that $a_{1i} = \cdots = a_{i_k i} = 0$, which gives the general form of the elements of T_Θ.

Since the special orthogonal group $\mathrm{SO}(d)$ also acts transitively on the Θ-flag manifolds, they are compact. Next we analyze the semi simple elements of $\mathrm{Sl}(d, R)$.

2. Conley Decomposition of the Vector Space

From now on in this paper $g \in \mathrm{Sl}(d, R)$ is a fixed semisimple element. One remind that g is semisimple if for each g-invariant subspace V of $\mathbb{R}^d$ there is another g-invariant subspace W such that $\mathbb{R}^d = V \oplus W$. (c.f. Tits [Ti83]).

Let $\{\lambda_1, \ldots, \lambda_r\}$ be the set of all different eigenvalues of g. Some of them are eventually complex numbers. To this spectrum there corresponds a decomposition of $\mathbb{R}^d$ in the eigenspaces

$$\mathbb{R}^d = V_1 \oplus \cdots \oplus V_r$$

This decomposition will be referred as Jordan decomposition. Since g is semisimple we know the structure of the 'blocks' $\Lambda_i = g|_{V_i}$. Its convenient to denote

$$g = \Lambda_1 \oplus \cdots \oplus \Lambda_r$$

where each $\Lambda_i : V_i \to V_i$ is an operator and they have one of the following forms

First case: The eigenvalue λ_i associated to Λ_i is real. In this case the dimension of V_i is the multiplicity of λ_i and each vector in V_i is an eigenvector. Then if I_i denote the identity in V_i then one can write

$$\Lambda_i = \lambda_i I_i$$

Second case: The eigenvalue λ_i is complex. Then there is a basis of the vector space V_i in which the operator Λ_i will take the form:

$$\Lambda_i = |\lambda_i| \begin{pmatrix} R_{\theta_i} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & R_{\theta_i} \end{pmatrix}$$

where each block R_{θ_i} denote the matrix $\begin{pmatrix} \cos\theta_i & -\sin\theta_i \\ \sin\theta_i & \cos\theta_i \end{pmatrix}$ with $\theta_i \in (0, 2\pi) \setminus \pi$.

Although the Jordan decomposition is the classical one related to the spectral theory it reveals to be too fine to study the dynamical features of the operator g acting as cascade on the compact homogeneous spaces. The difficulties will be clear after the simple example that follows the discussion. In contrast the Conley decomposition seems to have a good behavior with respect to isolated invariant sets on flag manifolds.

Put $\Sigma^* = \{\alpha \in \mathbf{R}$ such that there is a λ_i with $|\lambda_i| = \alpha\}$. It is clear that Σ^* has $s \leq r$ elements and they will be supposed enumerated as $\alpha_1 > \cdots > \alpha_s$. Now let W_j denote the vector subspace of $\mathbb{R}^d$ obtained as direct sum of all subspaces V_i from the jordan decomposition for which $|\lambda_i| = \alpha_j$.

Definition 2. The new decomposition of the vector space as

$$\mathbb{R}^d = W_1 \oplus \cdots \oplus W_s$$

will be called Conley decomposition.

One should note that the operator $g|_{W_j}$ has possibly different eigenvalues but all with the same absolute value (α_j). It will be denoted as $\boldsymbol{\pi}_j$ the projector $\boldsymbol{\pi}_j = \mathbb{R}^d \to \mathbb{R}^d$ such that $\mathrm{Im}\boldsymbol{\pi}_j = W_j$ and $\mathrm{Ker}\boldsymbol{\pi}_j = \oplus_{i \neq j} W_i$.

This section ends with a very simple but important lemma.

Lemma 1. *Let x be a k-dimensional subspace of $\mathbb{R}^d$. Then $\sum_{j=1}^{s} \dim \boldsymbol{\pi}_j(x) \geq k$ and the equality holds if and only if $x = \oplus_{j=1}^{s} \boldsymbol{\pi}_j(x)$*

Proof. Let $\{e_1, \ldots, e_k\}$ be a basis of x. Each element e_i can be uniquely decomposed as sum of vectors in W_j, this obviously implies that x is a subspace of $\oplus_{j=1}^{s} \boldsymbol{\pi}_j(x)$ and this implies the first inequality. The second assertion is immediate comparing the dimension of the subspaces x and $\oplus_{j=1}^{s} \boldsymbol{\pi}_j(x)$. $\square$

3. The action on grassmanian manifolds

It was already seen that the grassmanian manifolds are particular instances of flag manifolds. Here $\mathbf{Gr}(k, \mathbb{R}^d)$ will denote the space of all k-dimensional subspaces of $\mathbb{R}^d$ (Remember that this is the special case for $\Theta = \{d - k\}$).

Then a point $x \in \mathbf{Gr}(k, \mathbb{R}^d)$ is a k-dimensional subspace of $\mathbb{R}^d$. It will be denoted as $\omega(x)$ the omega limit set of x with respect to g. That is $\omega(x) = \{y \in \mathbf{Gr}(k, \mathbb{R}^d) : \lim_{i\to\infty} g^{n_i}(x) = y\}$.

Proposition 1. *Let x an arbitrary point in $\mathbf{Gr}(k, \mathbb{R}^d)$, and consider y a point in $\omega(x)$. Then $\sum_{j=1}^{s} \dim \boldsymbol{\pi}_j(y) = k$.*

Proof. Let $\{n_k\}$ denote the increasing sequence of integers such that $g^{n_k}(x)$ tends to y. Keep fixed a basis $\{f_1, \dots, f_k\}$ of the vector space x. For each integer n, one has that $g^n(x)$ is the k-dimensional subspace generated by $g^n(f_1), \dots, g^n(f_k)$, which will be denoted by $[g^n(f_1), \dots, g^n(f_k)]_{vs}$, one must note that this subspace is the same as $[\frac{g^n(f_1)}{\beta_1}, \dots, \frac{g^n(f_k)}{\beta_k}]_{vs}$ where β_i are non zero real numbers.
The elements of this basis can be decomposed as

$$f_i = \sum_{j=1}^{s} e_{ij}$$

Where each e_{ij} is in W_j, the subspace of the Conley decomposition.
It follows that:

$$g^n(f_i) = \sum_{j=1}^{s} \rho_j^n e_{ij}^{(n)}$$

where ρ_j is the absolute value of the eigenvalues of $g|_{W_j}$, and $e_{ij}^{(n)} = \frac{g^n|_{W_j}(e_{ij})}{{\rho_j}^n}$.
Let $j(i)$ be the least index in the decomposition of f_i for which e_{ij} is different of zero. As we are interested just in the vector space generated by the family $g^n(f_i)$ we have dividing both sides by $\rho_{j(i)}$ in the above equation

$$[g^n(f_i)]_{vs} = [e_{ij(i)}^{(n)} + \sum_{j=j(i)+1}^{s} \frac{\rho_j}{\rho_{j(i)}} e_{ij}^{(n)}]_{vs}$$

As n goes to infinity remain just the terms $e_{ij(i)}$ belonging to W_i. This implies that y has a basis with each element belonging to a W_i which completes the proof □

Corollary 1. *If $M \subset \mathbf{Gr}(k, \mathbb{R}^d)$ is a minimal set, and if $x \in M$ then*

$$\sum_{i=1}^{s} \dim \boldsymbol{\pi}_i(x) = k$$

.

Proof. In this case $\omega(x) = M$. □

These results were not true if we had taken the original Jordan decomposition of $\mathbb{R}^d$. The following proposition is the key of our construction, giving the main motivation to consider the Conley decomposition. It is convenient to see firstly a simple example which shows the differences.
Example: We consider in $\mathrm{Sl}(3, R)$ the following elements

$$g_1 = \begin{pmatrix} 1 & 0 \\ 0 & R_\theta \end{pmatrix} \text{ e } g_2 = \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} R_\theta \end{pmatrix}$$

Then one can easily see that they induces the same Jordan decomposition of $\mathbb{R}^3$ but different Conley decomposition. The whole space is the decomposition for g_1 and for g_2 the space is decomposed in direct sums of a subspace of dimension one and a subspace of dimension 2.

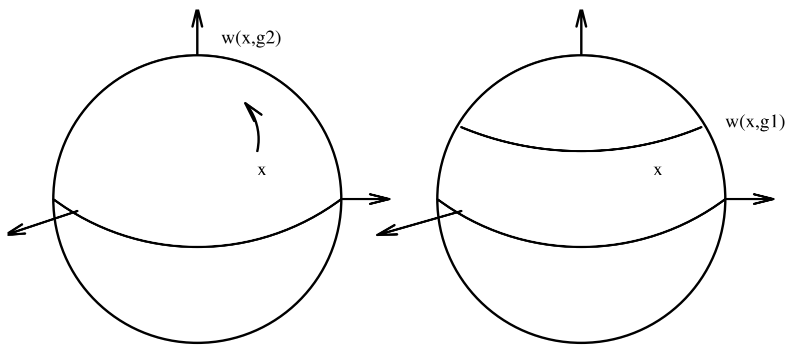

FIGURE 1. Conley decompositions and corresponding minimal sets

Proposition 2. *Let g be a semisimple element of $Sl(d, R)$, whose eigenvalues have all the same modulus (that is, 1 in this case), then for an arbitrary element $x \in \mathbf{Gr}(k, \mathbb{R}^d)$ the set $\omega(x)$ is a minimal set, and $x \in \omega(x)$.*

before we give the proof, let us recall the definition of an almost periodic point [Brw76, Bro79]

Definition 3. A point x is almost periodic, if the closure of its orbit is a compact minimal set.

Now is clear that we want to prove that all points in the proposition are almost periodic.

Proof. Suppose we have the complex eigenvalues $\lambda_1, \dots, \lambda_r$ each one with multiplicity m_i. We count apart the $1, -1$ which appear eventually and their multiplicity are respectivally α and β. then we can write

$$g = I_\alpha \oplus J_\beta \oplus R_{\theta_1} \oplus \cdots \oplus R_{\theta_r}$$

In this decomposition of g each member is related to the restriction of g to the proper eigenspace, and θ_i are the arguments of λ_i.
In this way one can see that

$$g^n = I_\alpha \oplus (-1)^n J_\beta \oplus R_{n\theta_1} \oplus \cdots \oplus R_{n\theta_r}$$

Taking a point x in $\mathbf{Gr}(k, \mathbb{R}^d)$ we write the action of g on it as

$$g^n(x) = I_\alpha \oplus (-1)^n J_\beta \oplus R_{n\theta_1} \oplus \cdots \oplus R_{n\theta_r}(x)$$

Now let T^r the r-dimensional torus and $D = \{-1, 1\}$ the discrete space and consider the following continuous mapping

$$i_x : D \times T^r \to \mathbf{Gr}(k, \mathbb{R}^d)$$

defined by

$$i_x(\gamma, \omega_1, \cdots, \omega_r) = I_\alpha \oplus \gamma . J_\beta \oplus R_{\omega_1} \oplus \cdots \oplus R_{\omega_r}(x)$$

It is immediate to see that $i_x(D \times T^r)$ is a compact set and g-invariant. Now we consider a flow in $D \times T^r$ giving by the homeomorphism

$$t(\gamma, \omega_1, \ldots, \omega_r) = (\gamma, \omega_1 + \theta_1, \ldots, \omega_r + \theta_r)$$

Then the transformation i_x provides a topological conjugation of this action with the action of g on $\mathbf{Gr}(k, \mathbb{R}^d)$. Now since each point in $D \times T^r$ is almost periodic and takes the orbit of $(1, 0, \ldots, 0)$ onto the orbit of x in $\mathbf{Gr}(k, \mathbb{R}^d)$. Hence the orbit of x is minimal and is his omega limit. The proposition is proved □

What this result says is that when the Conley decomposition is the whole space then each point of a grassmanian manifold belongs to a minimal set, and this grassmanian manifold itself is the only isolated invariant set. This also implies that any continuous curve in $\mathbf{Gr}(k, \mathbb{R}^d)$ has only almost periodic points. This is obvious but important.
Its our purpose to generalize this result for those semisimple operators that have eigenvalues of different modulus, that is whose Conley decomposition have more than one component.
We have to see a little more generalities on projections. Let

$$\mathbb{R}^d = V \oplus W$$

an arbitrary decomposition of $\mathbb{R}^d$. Associated to this decomposition there is a projection $\boldsymbol{\pi} : \mathbb{R}^d \to V$ with $\operatorname{Ker} \boldsymbol{\pi} = W$. In this way the projection associate to each element $x \in \mathbf{Gr}(k, \mathbb{R}^d)$ is a subspace of V. We are interested in some properties of this mapping.

Lemma 2. *Set $l = \dim \boldsymbol{\pi}(x)$, for $x \in \mathbf{Gr}(k, \mathbb{R}^d)$. There is a neighborhood U of x, where $\dim \boldsymbol{\pi}(y) \geq l$ for all $y \in U$.*

Proof. Consider the subspace $\boldsymbol{\pi}(x) \subset V$ and let be $e_1, \ldots e_l$ a basis of this subspace. We take also $f_1, \ldots, f_l$ in x such that $\boldsymbol{\pi}(f_i) = e_i$.
Since $\mathbf{Gr}(k, \mathbb{R}^d)$ is an homogeneous space of $\mathrm{Sl}(d, R)$ the lemma is equivalent to say that there is a neighborhood $\tilde{U}$ of the identity in $\mathrm{Sl}(d, R)$ such that for all $h \in \tilde{U}$ it holds $\dim \boldsymbol{\pi}(hx) \geq l$.
First of all we note that there exists a neighborhood U_1 of the identity such that for each $h \in U_1$ and all f_i one has $\boldsymbol{\pi}(hf_i) \neq 0$. In fact if this were not the case, we could find an f_i and a sequence h_n converging to the identity I such that $\boldsymbol{\pi}(h_n f_i) = 0$ and then $\boldsymbol{\pi}(h_n f_i) \to 0 = \boldsymbol{\pi}(f_i)$ by continuity contradicting the fact that $\boldsymbol{\pi}(f_i) = e_i$.

Now we prove that in an eventually smaller neighborhood $\boldsymbol{\pi}(hf_i)$ remains linearly independent.
We can write

$$\boldsymbol{\pi}(hf_i) = \sum_{j=1}^{l} a_{ij}(h)e_j$$

and then is clear that independence of $\boldsymbol{\pi}(hf_i)$ is determined by $\det a_{ij}(h)$. The assertion then follows from the continuity of det and from the fact that $\det a_{ij}(I) = 1$. Hence follows also the Lemma. □

Consider now a minimal set M in $Gr(k, \mathbb{R}^d)$ with respect to a semisimple element g, and suppose that the subspaces W and V given as in the last lemma are invariants then we have

Lemma 3. *Let $x \in Gr(k, \mathbb{R}^d)$, then one has $\boldsymbol{\pi}(g(x)) = g(\boldsymbol{\pi}(x))$*

Proof. Let $e_1, \ldots, e_k$ be a basis of x. $g(e_i)$ form also a basis of $g(x)$. It insufficient to prove the commutativity for the basis. In one hand one has $e_i = v + w \implies \boldsymbol{\pi}(e_i) = v \implies g(\boldsymbol{\pi}(e_i)) = g(v)$, in the other hand: $g(e_i) = g(v) + g(w) \implies \boldsymbol{\pi}(g(e_i)) = g(v)$ □

Proposition 3. $\dim \boldsymbol{\pi}(x)$ *is constant for all $x \in M$.*

Proof. Let x in M and U a neighborhood of x as given by lemma above.
Now take a point y in this neighborhood and in M , such that $\dim \boldsymbol{\pi}(y) > \dim \boldsymbol{\pi}(x)$. Apply again lemma 2 for the point y, we find a neighborhood of y such that the dimension of $\boldsymbol{\pi}(z)$ is not smaller than $\dim \boldsymbol{\pi}(y)$.
But since M is a minimal set, there is one element of the form $g^n(x)$ in each neighborhood of y. As $\boldsymbol{\pi}(x)$ and $\boldsymbol{\pi}(g^n(x))$ have the same dimension in consequence of the last lemma we have a contradiction. That is, there is no such y. □

This last proposition determines then a continuous application $\boldsymbol{\pi}$ from a minimal set M to a Grassmann manifold $Gr(k(\boldsymbol{\pi}, M), V)$, where $k(\boldsymbol{\pi}, M) = \dim \boldsymbol{\pi}(x)$ with x being an arbitrary point in M. We have morever that

Lemma 4. *$\boldsymbol{\pi}(M)$ is a minimal set in $Gr(k(\boldsymbol{\pi}, M), V)$ with respect to $g|_V$.*

Proof. The commutativity in lemma implies that in this case $\boldsymbol{\pi}$ is a conjugation between the flows

$$g : M \to M$$

and

$$g|_V : Gr(k(\boldsymbol{\pi}, M), V) \to Gr(k(\boldsymbol{\pi}, M), V)$$

As the first flow is minimal then $\boldsymbol{\pi}(M)$ is also minimal. □

In particular using this construction for the decomposition given by we get a family of l natural numbers $\{k_i\}$ and l applications

$$\pi_i : M \to Gr(k_i, W_i)$$

where $k_i = k(\pi_i, M)$.

Definition 4. The sequence given above $(k_1, \ldots, k_l)$ connected with a minimal set $M \subset Gr(k, \mathbb{R}^d)$ will be called the g-multigrade of the minimal set M. It will be denoted as $C(M)$

Note that by lemma 3.5 we have $\sum k_i = k$.

4. A theorem on the multigrade

The most important result in this section is that the multigrade of minimal sets on $Gr(k, \mathbb{R}^d)$ gives also information about the connection of two minimal sets.
Let $g \in SL(d, R)$ be a semisimple element. We say that two minimal sets M and N are g-connected if there are points $x \in M$ and $y \in N$ such that x and y are g-connected according definition 2.4.

Proposition 4. *Let M and N be two minimal sets in $Gr(k, \mathbb{R}^d)$, and suppose that $C(M) = C(N)$, then M and N are g-connected.*

Proof. We take for each minimal set the mappings defined above

$$\pi_i : M \to Gr(k_i, W_i) \text{ and } \pi_i : N \to Gr(k_i, W_i)$$

. Note that we keep the same notation although these mappings depends naturally on the sets M and N. Note also that as consequence from the hypothesis both mappings go into the same Grassmann manifold $Gr(k_i, W_i)$, since $k_i(M) = k_i(n)$.
Then we have that $\pi_i(M)$ and $\pi_i(N)$ are minimal sets in $Gr(k_i, W_i)$. We also know that the action of $g|_{W_i}$ on W_i has all eigenvalues with the same absolute values according the definition of W_i. Then we can apply the Proposition 3.6. That is to say, we take an arbitrary continuous curve from $\pi_i(M)$ to $\pi_i(N)$, then each point of this curve is a point in a minimal set according this proposition. We fix this curve and denote it as $K_i : [0, 1] \to Gr(k_i, W_i)$. The next task is to construct a continuous curve between M and N such that each point belongs to a minimal set. We produce this curve with help from the family of curves K_i. We define the continuous curve on $Gr(k, \mathbb{R}^d)$ as:

$$K(t) = K_1(t) \oplus \cdots \oplus K_l(t)$$

It is easy to see that this curve is well defined and connected N and M. We have just to check, whether each point $K(t)$ belongs to some minimal set.
First of all let us prove that $K(t)$ is a almost periodic point (see section 3.6), and then is the proof complete.
This a consequence of a general fact: If $x \in Gr(k, \mathbb{R}^d)$ can be decomposed as direct sum $x = x_1 \oplus \cdots \oplus x_l$, where x_i are subspace of W_i, then x is a almost periodic point for the action of g.
In fact we have:

$$g^n(x) = g^n(x_1) \oplus \cdots \oplus g^n(x_l)$$

And using again the method of proposition 3.6 we can write

$$g^n(x) = I_\alpha \oplus J^n_\beta \oplus R_{n\theta^1_1} \oplus \cdots \oplus R_{n\theta^1_{j_1}}(x_1) \oplus \cdots \oplus R_{n\theta^l_1} \oplus \cdots \oplus R_{n\theta^l_{j_l}}(x_l)]$$

Now by means of the conjugation already used we indentify this action with the action of $t_{(\theta^1_1 \dots, \theta^l_{j_l})}$ on $D \times T^{j_1+\cdots+j_l}$ which concludes the assertion.
Since the closure of the orbit of a almost periodic point is a minimal set (Auslander,[Au88, Th. 7 pg 11]) we are also ready with the proposition. $\square$

The most important consequence of this proposition is that we can give a boundary for the classes of g-connected minimal sets. The only possibility for two minimal sets being not g-connected is that they have differents multigrades. But the number of different multigrades are finite as is easy to see. In fact a family of natural numbers $(k_1, \dots, k_l)$ could be a multigrade for the action of g only if ,

(1) $\sum_1^l k_i = k$.
(2) $k_i \leq \dim W_i = n_i$

If we denote by $C(k; n_1, \cdots, n_l)$ the number of solutions of this system then we can formulate the following

Corollary 2. *There are at most $C(k, n_1, \dots, n_l)$ classes of g-connected minimal sets on $Gr(k, \mathbb{R}^d)$ for this semisimple g.*

The next step is the generalization of this result to an arbitrary flag manifold F_Θ.

5. Generalization for F_Θ

In the last section we proved a delimitation for the number of classes of g-connected minimal sets on the Grassmann manifolds in this section we give a general version of this result for general flag manifolds, the idea will be the same, but we have to care about some details.
As in the beginning of this paper let $\Theta = \{k_1, \dots, k_s\}$ denote a subset of $\{1, \dots, d\}$. To avoid complications with the notations we set $l_i = d - k_i$. Then a point f of F_Θ is composed by a family of subspaces $(F_{l_s}, \dots, F_{l_1})$, where $F_{l_{i+1}} \subset F_{l_i}$. We have also the canonical projections P_i given by

$$P_i : F_\Theta \to Gr(l_i, \mathbb{R}^d)$$

$$f \mapsto F_{l_i}$$

From the definition of a action of g on both spaces it is easy to see that P_i is a conjugation. We also note that the minimal sets on F_Θ are taken to minimal sets on $Gr(l_i, \mathbb{R}^d)$ (see section 3.6)
Let X and Y two minimal sets on F_Θ, then $P_i(X)$ and $P_i(Y)$ are minimal sets on the corresponding Grassmann manifold, each one with a multigrade $C_i(P_i(X))$ and $C_i(P_i(Y))$. Then we have the following proposition

Proposition 5. *Keep g a semisimple element as in the last section, and consider X and Y two miniaml sets of F_Θ, if $C_i(P_i(X)) = C_i(P_i(Y))$ for all $i \in \{1, \dots, s\}$ then X and Y are g-connected*

Proof. Recall that $C_i(P_i(X))$ denotes the multigrade of the minimal set $P_i(X)$ on $Gr(l_i,\mathbb{R}^d)$ under the action of g, and this means

$$C_i(P_i(X)) = \{k_1^i,\ldots,k_l^i\}$$

where we have also that $\sum_j k_j^i = l_i$ and $k_j^i \le \dim W_j$.
According proposition 3.11 $P_i(X)$ and $P_i(Y)$ are g-connected,so there is a continuous curve

$$c_i : [0,1] \to Gr(l_i,\mathbb{R}^d)$$

such that $c_i(0) \in P_i(X)$ and $c_i(1) \in P_i(Y)$.
But we need additional properties for these curves to get a continuous curve in F_Θ, that is to say, we must impose:

$$c_{i+1}(t) \subset c_i(t)$$

This means that $c_{i+1}(t)$ is a subspace of $c_i(t)$. With this in mind we construct such a curve
First of all we pick a point in X and another in Y and consider their projection on $Gr(l_i,\mathbb{R}^d)$. The first curve c_s we take as in the construction of the proposition 3.11.connecting the two points. The others curves we will construct iteractivelly.
Supposing we have already the curve c_{i+1} we contruct the next c_i. The curve $c_{i+1}(t)$ we can write according the proof of proposition 3.11 as

$$c_{i+1}(t) = K_1^{i+1}(t) \oplus \cdots \oplus K_l^{i+1}$$

where $K_j^{i+1} \in Gr(k_j^{i+1}, W_j)$.
Now we note that we have for all $j \in \{1,...,l\}$

$$k_j^{i+1} \le k_j^i.$$

To see this consider a $x \in X$ then

$$k_j^{i+1} = \dim \pi_j(P_{i+1}(x)) \text{ and } k_j^i = \dim \pi_j(P_i(x))$$

Since $P_{i+1}(x)$ is a subspace of $P_i(x)$ it follows the observation.
For the construction of the curve c_i we have the terminal points already fixed; $c_i(0)$ has to be the projection of the elected point in X and $c_i(1)$ the projection of the corresponding point in Y. It is then clear that we have :

$$c_{i+1}(0) < c_i(0) \text{ und } c_{i+1}(0) < c_i(0)$$

Since $K_j^{i+1}(t)$ is a k_j^{i+1}-dimensional subvectorspace of W_j , we can complete it to a k_j^i-dimensional subspace of W_j. We are the able to construct a continouous complementation of $K_j^{i+1}(t)$, which we will call $V_j^i(t)$ and such that (see Lemma 3.15 in sec. 3.6):

- $V_j^i(t)$ is a subspace of W_j and is continuous.
- $\dim V_j^i(t) = k_j^i - k_j^{i+1}$ and $V_j^i(t) \cap K_j^{i+1}(t) = \emptyset$.
- $\pi_j(c_i(0)) = \pi_j(c_{i+1}(0)) \oplus V_j^i(0)$ and $\pi_j(c_i(1)) = \pi_j(c_{i+1}(1)) \oplus V_j^i(1)$

Now we define

$$K_j^i(t) = K_j^{i+1}(t) \oplus V_j^i(t) \in Gr(k_j^i, W_j)$$

The curve c_i is made with help of K_j^i:

$$c_i(t) = K_1^i(t) \oplus \cdots \oplus K_l^i(t)$$

These curves have all wanted properties that allows us to define a continuous curve on F_Θ by:

$$c(t) = (c_s(t), \dots c_1(t))$$

We note about this curve that:
(a) $c(0) \in X$ and $c(1) \in Y$.
(b) Each $c_i(t)$ is contained in a minimal set of $Gr(l_i, \mathbb{R}^d)$. These are consequence of the proof of propositon 3.11.
We have still to prove that $c(t)$ is contained in a minimal set.
This follows in the same way we have made in proposition 3.11.
Let $f \in F_\Theta$ such that

$$P_i(f) = x_1^i \oplus \cdots \oplus x_l^i \in Gr(l_i, \mathbb{R}^d). \text{ and } x_j^{i+1} \text{ subspace of } x_j^i$$

We want to prove that f is a almost periodic point, and in this way contained in a minimal set. Note that this is the case for the points $c(t)$.
First we write as usual:

$$g^n(f) = (g^n(x_1^s \oplus \cdots \oplus x_l^s), \dots, g^n(x_1^1 \oplus \cdots \oplus x_l^1))$$

We can decompose g as

$$g = g_1 \oplus \cdots \oplus g_l$$

where$g_i : W_i \to W_i$. then we can write

$$g^n(f) = (g_1^n(x_1^s) \oplus \cdots \oplus g_l^n(x_l^s), \dots, g_1^1(x_1^s) \oplus \cdots \oplus g_l^n(x_l^1))$$

We decompose g_i again as

$$g_j = \rho_j(I_{\alpha_j} \oplus J_{\beta_j} \oplus R_{\theta_1^j} \oplus \cdots \oplus R_{\theta_{n_j}^j})$$

or as already observed in propositin 3.6

$$g_j^n(x_j^i) = (I_{\alpha_j} \oplus {J_{\beta_j}}^n \oplus R_{n\theta_1^j} \oplus \cdots \oplus R_{n\theta_{n_j}^j})(x_j^i)$$

We consider again the mapping:

$$i_f : D \times T^{n_1 + \cdots + n_l} \to F_\Theta$$

$$(\gamma, \theta_1^j, \dots, \theta_{n_j}^j) \mapsto \bigoplus_j (I_{alpha_j} \oplus {J_{\beta_j}}^\gamma \oplus R_{\theta_1^j} \oplus \cdots \oplus R_{\theta_{n_j}^j})(f)$$

$i_f(T)$ is then a compact invariant set on F_Θ and the almost periodic points in $D \times T$ are transformed in almost periodic points on $i_f(T)$. We have proved our assertion and the is also the proposition proved. □

We have as consequence

Corollary 3. *In the flag manifold F_Θ there are at most $C_\Theta(n_1,...,n_l)$ different classes of g-connected minimal sets where*

$$C_\Theta(n_1,...,n_l)=\prod C(l_i,n_1,...,n_l)$$

and $n_i=\dim W_i$.

Proof. This number give the number all possibles multigrade. $\square$

6. Complementary Lemmas

We prove in this section three fundamental Lemmas that we have used in this work.

Lemma 5. *Let $c:[0,1]\to Gr(k,\mathbb{R}^d)$ be a continuous curve. Consider x_0 and $x_1\in Gr(s,\mathbb{R}^d)$, such that $x_0\cap c(0)=\{0\}$ and $x_1\cap c(1)=\{0\}$ as subspaces . Then there is a continuous curve $v:[0,1]\to Gr(s,\mathbb{R}^d)$, such that $c(t)\oplus v(t)$ is continuous with $v(0)=x_0$ and $v(1)=x_1$*

Proof. As $Gr(k,\mathbb{R}^d)$ is a homogeneous space of $SL(d,R)$ we can represent

$$Gr(k,\mathbb{R}^d)=SL(d,R)/G_{c(0)}$$

Let now $\bar{g}\in SL(d,R)$ be an element such that

- $\bar{g}.c(0)=c(1)$
- $\bar{g}.x_0=x_1$

this is possible cause we have: $c(0)\cap x_0=c(1)\cap x_1=\{0\}$ and x_0 and x_1 have the same dimension.
Now we take a lift $g(t)$ of $c(t)$ in $SL(d,R)$ such that $g(1)=\bar{g}$, then we have that

$$c(t)=g(t).c(0)$$

and taking

$$l(t)=g(t).(c(0)\oplus x_0)$$

follows that $l(t)$ is continuous and

$$l(1)=g(1)(c(0)\oplus x_0)=c(1)\oplus x_1$$

Let us consider a $h\in G_{c(0)}$ such that $h.x_0=x_0$ (that is to say: $h.(c(0)\oplus x_0)=c(0)\oplus x_0$), and a curve in $G_{c(0)}$

$$\alpha:[0,1]\to G_{c(0)}$$

such that $\alpha(0)=g(0)^{-1}.h$ and $\alpha(1)=e$. Then the curve

$$u(t)=g(t).\alpha(t)$$

satisfies

(1) $u(t)$ is continuous.
(2) $u(0)=c(0)\oplus x_0$.
(3) $u(1)=c_1\oplus x_1$

Then $v(t)=g(t).\alpha(t).x_0$ is the wanted curve. $\square$

Another importante result is the following

Lemma 6. *Let $f : X \to X$ and $g : Y \to Y$ two homeomorfisms between topological spaces and consider $c : X \to Y$ a continuous conjugation, that is*

$$c \circ f = g \circ c.$$

Then if M is a minimal set in X and $c(M)$ is closed, $c(M)$ is also a minimal set.

Proof. (1) $c(M)$ is an invariante set since we have

$$f(M) = M \implies$$
$$c \circ f(M) = c(M) \implies$$
$$g(c(M)) = c(M)$$

(2) $c(M)$ is closed by assumption.
(3) Suppose that $c(M)$ contains a closed invariant set K. Then $c^{-1}(K)$ is a closed invariant set in X, which implies that $c^{-1}(K) \cap M$ is a closed invariant set contained in M, since it is also not empty it must be M itself, which implies that $K = c(M)$. □

We have also used that each point in a torus T^n is a almost periodic point with respect to translations. Here is a proof of this fact

Lemma 7. *Consider $D = \{-1, 1\}$ and T^n the n-dimensional torus. Moreover we consider for a fixed $\omega \in T^n$ the homeomorphism on $D \times T^n$ given by*

$$t_\omega(\gamma, x) = (-\gamma, x + \omega).$$

Then each point $(\gamma, x) \in D \times T^n$ is a almost periodic point.

Proof. We prove this lemma in two parts, first suppose that $\omega = (\omega_1, \ldots, \omega_n) \in T^n$ is rationally independent, then the point 2ω is also rationally independent and the orbit of a point $(\gamma, x) \in D \times T^n$ we can write as

$$O(\gamma, x) = \{t_\omega^{2n}(\gamma, x) : n \in Z\} \dot\cup \{t_\omega^{2n+1}(\gamma, x) : n \in Z\}$$

We note that since 2ω is rationally independent we have

$$\mathrm{Cl}\{t_\omega^{2n}(\gamma, x) : n \in Z\} = \{\gamma\} \times T^n$$

and

$$\mathrm{Cl}\{t_\omega^{2n+1}(\gamma, x) : n \in Z\} = \{-\gamma\} \times T^n$$

and this is a minimal set.
If ω is not rationally independent then we write $\omega = (\omega_1, \ldots, \omega_n) \in T^n$ and take $x \in T^n$ arbitrary.
We rearrange the vector ω such that the first k components are the maximal rationally independent set in ω. This means:

$$a_1\omega_1 + \cdots + a_k\omega_k = 0, a_i \in Z \Leftrightarrow a_i = 0$$

and

$$\omega_{k+i} = \sum_{j=1}^{k} a_j^i \omega_j$$

We consider then the injection:

$$i_x : D \times T^k \to D \times T^n$$

defined as:

$$i_x(\gamma, (\omega_1, \ldots, \omega_k)) = (\gamma, x + (\omega_1, \ldots, \omega_k, \sum a^i_j \omega_j))$$

for $i = 1, \ldots, n-k$. We note that this map is a conjugation between the homeomorphisms i_ω and $i_{(\omega_1, \cdots, \omega_k)}$ on $D \times T^k$. Since $(\omega_1, \ldots, \omega_k)$ are rationally independent we use the first part of the proof on $D \times T^k$ and then the last lemma to conclude the proof. □

7. Conclusions

We give here a consequence of the above results to the theory of control sets. Let S be a subsemigroup of $SL(d, R)$ with non empty interior acting on a flag manifold. We define a control set as a subset D satisfying
i) int $D \neq \emptyset$,
ii) $D \subset \mathrm{Cl}(Sx)$ for all $x \in D$, and
iii) D is maximal with properties i) and ii).
More on control sets can be found in [T-91]

Proposition 6. *If* $\mathrm{Int}\, S$ *has a semisimple element whose eigenvalues have all the same absolute value then there is only one control set for* S*, namelly the whole flag manifold.*

Proof. In this case we have the whole space as Conley decomposition beeing each point in a minimal set and they all have to be in the same control set. □

In other words, such a semigroup must acts transitivelly on the flag manifold. Another result that comes easily from this theory is that

Proposition 7. *If the semigroup* S *has non empty interior then it has at most finitely many efective* [SMT] *control sets on the flag manifolds*

Proof. It is know (see for instance [SaM86]) that the set of semisimple elements are dense in $SL(d, R)$. Then taking a semisimple element in the interior of S we have that all minimal sets belongs to an efective control set. Now minimal sets having the same multigrade must be in the same control set. So we can have as much control sets as multigrade. This proves the result. □

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