Asymptotic Smoothness of the Extensions of Zadeh

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Abstract

The concept of asymptotic smooth transformation was introduced by J. Hale [1] and it is a very important property for a transformation, between complete metric spaces, to have a global attractor. This property has also consequences on the asymptotic stability of attractors. In our work, we study the conditions under which the Zadeh's extension of a continuous map $f : \mathbb{R}^n \to \mathbb{R}^n$ is asymptotically smooth in the complete metric space $\mathcal{F}(\mathbb{R}^n)$ of normal fuzzy sets with the induced Hausdorff metric d_∞ (see Kloeden and Diamond [2]).

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1 Introduction

The question of determining whether a discrete dynamical system on a infinite dimensional vector space has a compact attractor has been studied very intensively in the last ten years. Some new concepts that were irrelevant for the finite dimensional cases shows up to be of decisive importance in infinite dimensional cases. Among these concepts we will center our attention in the asymptotic smoothness of a function.

The concept of asymptotic smoothness was introduced by J. Hale in [1]. Every continuous transformation between finite dimensional vector spaces are asymptotically smooth, this is not the case for infinite dimensional Banach spaces or metric spaces. From the dynamical point of view, an important property of an asymptotic smooth map is that the existence of a compact set that attracts locally points implies that the compact set also attracts locally compact sets. A practical consequence of this result is the use of an iteration procedure to approximate global attractors. In his book, J. Hale also gives some examples of asymptotic smooth transformations in infinite dimensional spaces. Our objective in this paper is to provide another class of such functions using the Zadeh's extensions of continuous transformation in $\mathbb{R}^n$. Extending a map $T : \mathbb{R}^n \to \mathbb{R}^n$ to a map in the fuzzy space $\hat{T} : \mathcal{F}(\mathbb{R}^n) \to \mathcal{F}(\mathbb{R}^n)$ using the extension principle is a standard procedure that is known since the first steps of the theory of fuzzy sets [3]. It is a procedure particularly useful if one knows a deterministic process but has a fuzzy initial condition, for instance see [4]. The properties of the extension principle, in case of any variation of fuzzy dynamical systems, play also a substantial role, due to the so called α-level preservation of the Zadeh's extensions [5, 6, 7, 8]. Properties of the Zadeh's extension connected to the regularity of the function which originated it were studied in the works of Nguyen [9], Barros et al. [10, 11] and Ma et al. [5].

The results we present in this paper are used to analyze the interaction of dynamical properties between a transformation in $\mathbb{R}^n$ and its Zadeh's Extension defined on $\mathcal{F}(\mathbb{R}^n)$[9]. We indicate a procedure to obtain an attractor if it exists.

Attractors of a such discrete dynamical system in the space $\mathcal{F}(\mathbb{R}^n)$ can appear in many problems as an iteration system. Different instances of this applications can be found the works of Cabrelli et al. [12] and Barros et al. [4]:

In [12], Cabrelli et al. introduce the iterated fuzzy sets systems, which orbits are used for approximation of fractals or other images or sets, these sets should be global attractors in $\mathcal{F}(\mathbb{R}^n)$. As a consequence, the transformations $T_n : \mathcal{F}(\mathbb{R}^n) \to \mathcal{F}(\mathbb{R}^n)$ that originate the iterated fuzzy sets systems are assumed to be asymptotically smooth. In this paper the hypothesis that T_n are Zadeh's extensions of Lipschitzian maps in $\mathbb{R}^2$ is sufficient for the existence of a global attractor.

In [4], Barros et al. consider the fuzzification of a classical population growth model. This is obtained considering the Zadeh's extension of the so called Malthus-Verhulst (or logistic) model. This is an appropriated model if one considers environmental fuzziness. The work shows that, besides the known attractors of the deterministic logistic system, there are other fuzzy attractors.

The paper is organized as follows: In the section 2 we recall the main definitions and known results from the theory of dynamical systems and fuzzy set analysis. This material is fully described in the given references. In the section 3 we present a discussion of the properties of compact sets in $\mathcal{F}(\mathbb{R}^n)$ that affect our work. The section 4 contains our main results that give some condition under which the Zadeh's extension of a continuous transformation is asymptotically smooth. In the last section we present some examples.

2 Preliminaries

If X is a metric space and $T : X \to X$ is a continuous transformation then we have a discrete dynamical system. For the basics notations and definitions on dynamical systems we refer to Hale [1]. We say that T is *asymptotically smooth* (see [1]) if, for each nonempty bounded and closed set $B \subset X$ for which $T(B) \subset B$, there is a compact set $J \subset B$ such that J attracts B. We recall that J attracts B if for each neighborhood of J there is a positive n_0 such that $T^n(B)$ is contained in that neighborhood for all $n \geq n_0$ (see Hale [1] page 9). An important property is that if T is an asymptotically smooth map then a set attracts locally points if and only if it attracts locally compact sets. Cooperman [13] and Brumley [14] have given examples where this is not true for general transformations on infinite dimensional Banach Spaces.

The concepts of limit sets of a dynamical systems are classical. Here we will need the following: the ω-*limit* of a subset B of X is given by:

$$\omega(B) = \bigcap_{n>0} \text{cl}(\bigcup_{k\geq n} T^k(B)).$$

It is clear that if B is such that $T(B) \subset B$ then

$$\omega(B) = \bigcap_{n>0} \text{cl}(T^n(B))$$

The next Lemma can be found in Hale [1] (page 11, Cor. 2.2.4)

Lemma 1. *If T is asymptotically smooth and B is a nonempty bounded set such that its positive orbit is bounded, then $\omega(B)$ is nonempty, compact, and invariant and $\omega(B)$ attracts B.*

The problem we addressed here is: What are the conditions for a continuous transformations on $\mathbb{R}^n$ to have its Zadeh's extensions asymptotically smooth? Note that since the Zadeh's extensions of the identity in $\mathbb{R}^n$ is the identity in $\mathcal{F}(\mathbb{R}^n)$, and that not all bounded closed set in $\mathcal{F}(\mathbb{R}^n)$are compact, the identity map doesn't have this property.

Next we recall some definitions on fuzzy metric spaces.

The family of all compact nonempty subsets of $\mathbb{R}^n$ will be denoted as $\mathcal{Q}(\mathbb{R}^n)$. We also set $\mathcal{F}(\mathbb{R}^n)$ for the family of fuzzy sets $u : \mathbb{R}^n \to [0,1]$ whose α-level:

$$[u]^\alpha = \{x \in \mathbb{R}^n : u(x) \geq \alpha\} \quad 0 < \alpha \leq 1 \text{ and } [u]^0 = \text{cl}\{x \in \mathbb{R}^n : u(x) > 0\}$$

are in $\mathcal{Q}(\mathbb{R}^n)$.

It is known that the metric

$$d_\infty(u,v) = \sup_{0 \leq \alpha \leq 1} h([u]^\alpha, [v]^\alpha)$$

where h is the Hausdorff metric in $\mathcal{Q}(\mathbb{R}^n)$, turns the spaces $(\mathcal{F}(\mathbb{R}^n), d_\infty)$ into complete metric spaces [15].

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a mapping, then we define the Zadeh's extension as:

$$\hat{f}(u)(x) = \begin{cases} \sup_{\tau \in f^{-1}(x)} u(\tau) & \text{if} \quad f^{-1}(x) \neq \emptyset \\ 0 & \text{if} \quad f^{-1}(x) = \emptyset \end{cases}$$

for all fuzzy sets u.

The proof of the following results can be found in [10].

Theorem 1. *If $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuous then $\hat{f} : \mathcal{F}(\mathbb{R}^n) \to \mathcal{F}(\mathbb{R}^n)$ is well defined and for all $\alpha \in [0,1]$ we have*

$$[\hat{f}(u)]^\alpha = f([u]^\alpha).$$

We will need also a recent result of Roman-Flores et al. [11]

Theorem 2. *If $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuous then $\hat{f} : \mathcal{F}(\mathbb{R}^n) \to \mathcal{F}(\mathbb{R}^n)$ is continuous with respect to the metric d_∞.*

3 Some facts on compact sets on $\mathcal{F}(\mathbb{R}^n)$

Our strategy is to study the problem using the bare definition of an asymptotically smooth map. So we should be able to determine when a subset of $\mathcal{F}(\mathbb{R}^n)$ is compact or not. In the book of Kloeden and Diamond [2] one finds the characterization of compact sets in the space of fuzzy sets with convex levels which does not fit our purpose since our levels are only compact sets in $\mathbb{R}^n$. The best approach we have found in the literature is the article of Rojas et al. [16]. In that paper it is shown how difficult it is to find a compact set in $\mathcal{F}(\mathbb{R}^n)$. A result that is of great importance here is the following: If K is a compact set in $\mathbb{R}^n$ then $J_K = \{u \in \mathcal{F}(\mathbb{R}^n) : [u]^0 \subset K\}$ is compact if and only if K has diameter zero! Our candidate to be an attractor will be of this type but we have also good properties for this candidate.

Lemma 2. *Let $K \subset \mathbb{R}^n$ be a compact set and $A = \{u \in \mathcal{F}(\mathbb{R}^n) : [u]^0 \subset K\}$. Then A is a bounded closed set of the metric space $(\mathcal{F}(\mathbb{R}^n), d_\infty)$*

Proof: To see that A is bounded note that the distance of A to a point $\hat{0}$ (the characteristic function at 0) is finite. Indeed, denoting h the Hausdorff metric between compact sets

$$d_\infty(\hat{0}, A) = \inf_{u \in A} d_\infty(\hat{0}, u) = \inf_{u \in A} \sup_{\alpha \in [0,1]} h(\{0\}, [u]^\alpha) \leq \sup_{x \in K} d(0, x) = M < \infty$$

Note that this last number M must be less than infinity because K is a compact set in $\mathbb{R}^n$.

Now A is closed. In fact, consider a convergent sequence u_n in A with limit u, the convergence being in the metric d_∞ then we have, in particular, that $h([u_n]^0, [u]^0) \to 0$ and since $[u_n]^0 \subset K$ then $[u]^0 \subset K$ proving that u belongs to A. In other words, A contains all its cluster points and then it is closed. *QED*

4 The asymptotic smoothness problem

The main results will follow from a sequence of lemmas. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous transformation and $\hat{f} : \mathcal{F}(\mathbb{R}^n) \to \mathcal{F}(\mathbb{R}^n)$ its Zadeh's extension which is continuous according Theorem 2. To prove that $\hat{f}$ is asymptotically smooth we have to prove that for each closed bounded set $B \subset \mathcal{F}(\mathbb{R}^n)$ such that $\hat{f}(B) \subset B$ there is a compact set $J \subset B$ that attracts B. We take a closed bounded nonempty B. Note that for each $u \in \hat{f}(B)$ and $v \in B$ such that $\hat{f}(v) = u$ we have $[u]^\alpha = [\hat{f}(v)]^\alpha = f([v]^\alpha)$, using Theorem 1.

Lemma 3. *Let B as above and define $B_\alpha = cl(\bigcup_{u \in B} [u]^\alpha) \subset \mathbb{R}^n$. There exists a compact set $K \subset \mathbb{R}^n$ such that $[u]^0 \subset K$ for all $u \in B$. Hence B_α is bounded for each $\alpha \in [0,1]$*

Proof: Take a point x in $\mathbb{R}^n$ and denote as $\hat{x}$ its characteristic function. Since B is bounded there exists a $r > 0$ such that the ball with center in $\hat{x}$ and radius r contains the entire set B. In other words, for all $u \in B$ $d_\infty(\hat{x}, u) \leq r$. According to a result that can be found in [10] this metric can be written as

$$d_\infty(\hat{x}, u) = \sup_{0 \leq \alpha \leq 1} \inf\{a : [u]^\alpha \subset B_a(x) \text{ and } x \in B_a([u]^\alpha)\}$$

where $B_a(x)$ denote the Euclidean ball centered in x. Then it follows immediately that $[u]^0 \subset B_r(x) \subset \mathrm{cl} B_r(x) = K$. Now since $[u]^\alpha \subset [u]^0$ follows $[u]^\alpha \subset K$ what proves the lemma. *QED*

Lemma 4. *Consider B_α as in Lemma 3. Then B_α are closed bounded and satisfies $f(B_\alpha) \subset B_\alpha$. Therefore there is a compact set $J_\alpha \subset B_\alpha$ that attracts B_α.*

Proof: In fact the only assertion that has to be proved is that $f(B_\alpha) \subset B_\alpha$. The rest follows immediately from definitions, Lemma 2 and the fact that every continuous transformation in $\mathbb{R}^n$ is asymptotically smooth.

Take a x in B_α. By definition x is the limit of a sequence x_n with $x_n \in [u_n]^\alpha$ and $u_n \in B$. Therefore we have

$$f(x_n) \in f([u_n]^\alpha) = [\hat{f}(u_n)]^\alpha \subset B_\alpha.$$

Since f is continuous $f(x_n)$ converges to $f(x) \in B_\alpha$ and this completes this proof. *QED*

Using the Lemma 1 we can define the special compact invariant sets $J_\alpha = \omega(B_\alpha)$. These are the attractors we will consider. Now we can prove

Lemma 5. *For $0 \leq \alpha_1 \leq \alpha_2 \leq 1$ we have $J_{\alpha_2} \subset J_{\alpha_1}$*

Proof: We have that $B_{\alpha_2} \subset B_{\alpha_1}$. Observing that in this case the omega limits can be written as:

$$J_{\alpha_2} = \omega(B_{\alpha_2}) = \bigcap_{n \geq 0} f^n(B_{\alpha_2}) \subset \bigcap_{n \geq 0} f^n(B_{\alpha_1}) = J_{\alpha_1}$$

and then follows the result.

We define $J = \{u \in B : [u]^0 \subset J_0\}$.

Theorem 3. *If $\hat{f} : \mathcal{F}(\mathbb{R}^n) \to \mathcal{F}(\mathbb{R}^n)$ is asymptotically smooth then J is nonempty for all nonempty closed bounded B with $\hat{f}(B) \subset B$.*

Proof: We take a closed bounded B with $\hat{f}(B) \subset B$. Since we are assuming that $\hat{f}$ is asymptotically smooth, we can use Lemma 1 to construct the attractor $K = \omega(B)$ which is also nonempty, compact and invariant. Again take $K_0 = \operatorname{cl} \bigcup_{u \in K} [u]^0$. With respect to K_0 we can assert

a) K_0 is a compact set contained in B_0. And this is clear.

b) K_0 is an invariant set for f (i.e. $f(K_0) = K_0$). In fact, if $x \in K_0$ then we know that x is a limit of a sequence x_n such that $x_n \in [u_n]^0$ and $u_n \in K$ then follows: $f(x) = \lim_{n\to\infty} f(x_n)$. Now

$$f(x_n) \in f([u_n]^0) = [\hat{f}(u_n)]^0 \in K_0$$

this shows that $f(K_0) \subset K_0$.

To prove that $K_0 \subset f(K_0)$, we repeat the process taking $x \in K_0$ and x_n as above. Now since K is invariant for $\hat{f}$ we have that $u_n = \hat{f}(v_n)$ where $v_n \in K$. Then for each $n \geq 0$ we have $x_n = f(y_n)$ where $y_n \in [v_n]^0 \subset K_0$. Choosing a subsequence, if necessary, we take $y = \lim_{n\to\infty} y_n$. By continuity of f follows that $x = f(y)$. Hence K_0 is an invariant set of f.

Now J_0 attracts B_0 and also K_0. This means that for each $\epsilon \geq 0$ there is an n_0 such that for $n \geq n_0$, $f^n(K_0) \subset N(J_0, \epsilon)$ or $K_0 \subset N(J_0, \epsilon)$ using the invariance. Here $N(J_0, \epsilon)$ stands for an ϵ-neighborhood of J_0. This proves that in fact $K_0 \subset J_0$ because ϵ is arbitrary and then $J \neq \emptyset$. *QED*.

In particular if $J = \emptyset$ then $\hat{f}$ isn't asymptotically smooth.

Theorem 4. *Suppose that $\hat{f} : \mathcal{F}(\mathbb{R}^n) \to \mathcal{F}(\mathbb{R}^n)$ is asymptotically smooth, B a nonempty, bounded, closed subset of $\mathcal{F}(\mathbb{R}^n)$such that $\hat{f}(B) \subset B$, $K = \omega(B)$ and $J_\alpha = \omega(B_\alpha)$. Then $K_\alpha = J_\alpha$.*

Proof: The proof that $K_\alpha \subset J_\alpha$ follows as in the above Theorem, only changing the index 0 by α. It remains to prove that $J_\alpha \subset K_\alpha$. We know that K attracts B, therefore given $\epsilon > 0$ there exists n_0 such that for $n \geq n_0$, $\hat{f}^n(B) \subset N(K, \epsilon)$ this implies that for each $\alpha \in [0, 1]$, $f^n(B_\alpha) \subset N^*(K_\alpha, \epsilon)$ where

$$N^*(K_\alpha, \epsilon) = \{x \in \mathbb{R}^n : d(x, K_\alpha) \leq \epsilon\}.$$

It follows from the definition of J_α that

$$J_\alpha \subset \bigcap_{n \geq n_0} f^n(B_\alpha) \subset N^*(K_\alpha, \epsilon).$$

This is true for every $\epsilon > 0$ then follows the result. *QED*.

The set J, according to the above proposition, is the best candidate in which we can find an attractor. It could be the empty set, and in this case the Zadeh's extension isn't asymptotically smooth. We give some examples where we can decide on the smoothness of the transformation and even determine the attractors.

5 Examples

We present two examples.

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous transformation, such that $\{0\}$ is the unique global attractor of f. Then $\hat{f} : \mathcal{F}(\mathbb{R}^n) \to \mathcal{F}(\mathbb{R}^n)$ is asymptotically smooth.

It is clear that $\hat{f} : \mathcal{F}(\mathbb{R}^n) \to \mathcal{F}(\mathbb{R}^n)$ is continuous. We take a bounded closed set $B \in \mathcal{F}(\mathbb{R}^n)$ such that $\hat{f}(B) \subset B$. Now we shall prove that:

(a) $\chi_{\{0\}}$ attracts B, and

(b) $\chi_{\{0\}} \subset B$.

Since the set $\{\chi_{\{0\}}\}$ is compact in $\mathcal{F}(\mathbb{R}^n)$ it follows our result.

To prove (a) note that for each $\alpha \in [0, 1]$ the set $B_\alpha = \mathrm{cl}(\bigcup_{u \in B}[u]^\alpha) \subset \mathbb{R}^n$ is compact, and then $\{0\}$ attracts B_α. This means that for each $\varepsilon > 0$ there is an $n_\alpha \in \mathbb{N}$ such that $f^n(B_\alpha) \subset N(\{0\}, \varepsilon)$ for all $n > n_\alpha$. Here $N(\{0\}, \varepsilon)$ denotes the ε-neighborhood of $\{0\}$. But since $B_\alpha \subset B_0$ we have $f^n(B_\alpha) \subset N(\{0\}, \varepsilon)$ for all $n > n_0$.

For each $u \in B$, it follows that $f^n([u]^\alpha) \subset N(\{0\}, \varepsilon)$ and since f is continuous $[\hat{f}^n(u)]^\alpha \subset N(\{0\}, \varepsilon)$. From this it follows the assertion (a).

For the item (b) we take n_0 such that $d_\infty(\hat{f}^n(B), \chi_{\{0\}}) \leq \varepsilon$ for $n > n_0$. This implies that $\chi_{\{0\}}$ is in a ε-neighborhood of $\hat{f}^n(B)$ and then in a ε-neighborhood of B, because $\hat{f}^n(B) \subset B$. As this last assertion is true for any $\varepsilon > 0$ we must have $\chi_{\{0\}} \in B$. The proof of the result is complete.

As a particular case, we can take $f(x) = Ax$ where A is a linear operator. We can restrict the analysis to the eigenspace associated to the eigenvalues whose absolute values are less than one, and the above result applies.

This next example shows that it is not always true that the Zadeh's extension is asymptotically smooth.

If $f : \mathbb{R}^n \to \mathbb{R}^n$ have a compact set K with infinite points as attractor, then one can easily see that the set

$$B = \{u \in \mathcal{F}(\mathbb{R}^n) : [u]^0 \subset K\}$$

is a bounded closed set for which $\hat{f}(B) = B$. Since B is not compact according to Rojas et al. [16], then $\hat{f}$ is not asymptotically smooth. The meaning of this last example is that if $\hat{f}$ have a global attractor it will be hard to detect it since we can not use the theory for asymptotically smooth transformation.

Theorem 4 can be understood as a way to produce the the fuzzy attractors in a levelwise manner. There is yet the problem to find out the conditions that guarantee the existence of the attractor. This is the project we are involved now. Another study that could be interesting, is to consider the dissipativity and asymptotic smoothness of time continuous dynamical systems in the sense of Hüllermeier [17]. For this, the concept of attractor has to be generalized as in Diamond [18].

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