Lista 2
MAT 2351 / 2013

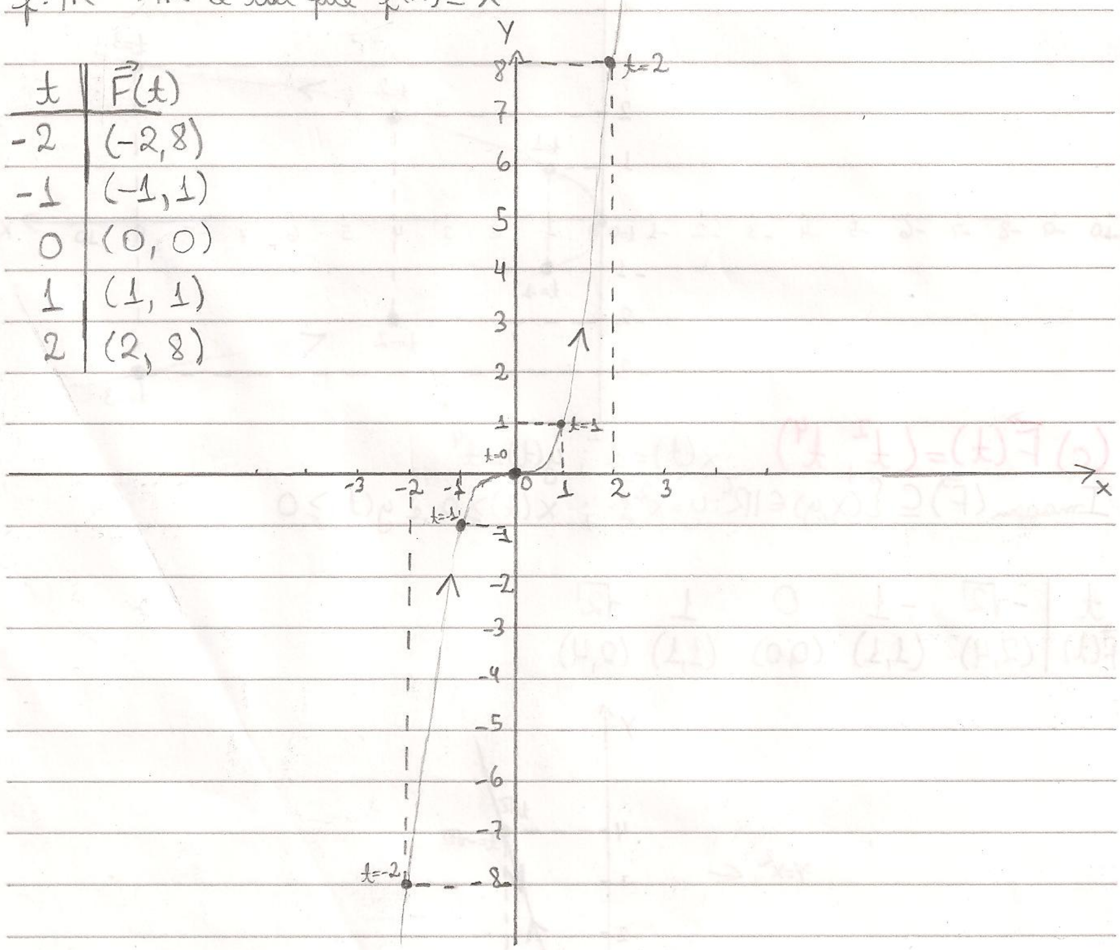
① Desenhe a imagem de $\vec{F}: \mathbb{R} \longrightarrow \mathbb{R}^2$ onde:

(a) $\vec{F}(t) = (t, t^3)$ $\quad x(t) = t;\ y(t) = t^3$

$\text{Imagem}(\vec{F}) \subseteq \{ (x,y) \in \mathbb{R}^2 : y = x^3 \} = \text{Gráfico}(f)$, onde

$f: \mathbb{R} \longrightarrow \mathbb{R}$ é tal que $f(x) = x^3$

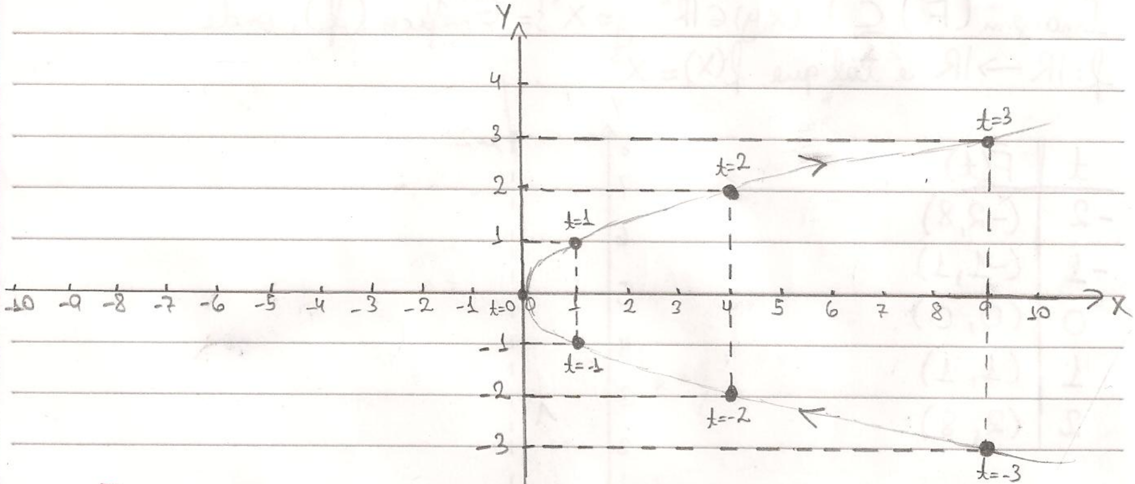
t	$\vec{F}(t)$
-2	$(-2, 8)$
-1	$(-1, 1)$
0	$(0, 0)$
1	$(1, 1)$
2	$(2, 8)$



(b) $\vec{F}(t) = (t^2, t)$ $x(t) = t^2$; $y(t) = t$

$\text{Imagem}(\vec{F}) \subseteq \{(x,y) \in \mathbb{R}^2 : x = y^2\}$

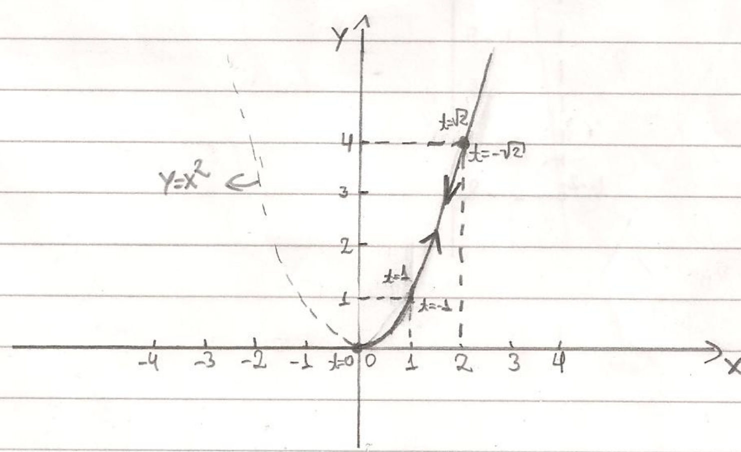
t	-3	-2	-1	0	1	2	3
$\vec{F}(t)$	$(9,-3)$	$(4,-2)$	$(1,-1)$	$(0,0)$	$(1,1)$	$(4,2)$	$(9,3)$



(c) $\vec{F}(t) = (t^2, t^4)$ $x(t) = t^2$; $y(t) = t^4$

$\text{Imagem}(\vec{F}) \subseteq \{(x,y) \in \mathbb{R}^2 : y = x^2\}$; $x(t) \geq 0$ e $y(t) \geq 0$

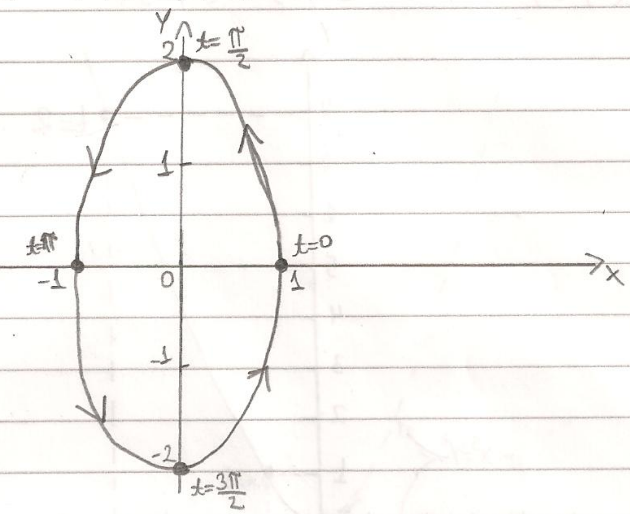
t	$-\sqrt{2}$	-1	0	1	$\sqrt{2}$
$\vec{F}(t)$	$(2,4)$	$(1,1)$	$(0,0)$	$(1,1)$	$(2,4)$



(d) $\vec{F}(t) = (\cos t, 2\sin t)$ $x(t) = \cos t$; $y(t) = 2\sin t$

Imagem $(\vec{F}) = \{(x,y) \in \mathbb{R}^2 : x^2 + \frac{y^2}{4} = 1\}$ = Elipse

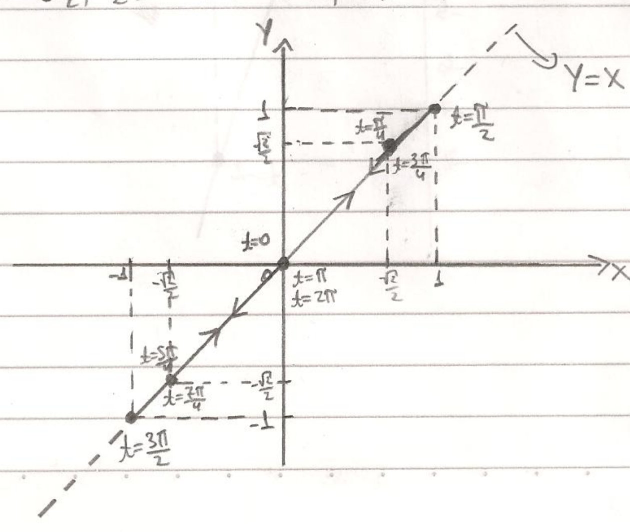
t	0	$\pi/2$	π	$3\pi/2$	2π
$\vec{F}(t)$	$(1,0)$	$(0,2)$	$(-1,0)$	$(0,-2)$	$(1,0)$



(e) $\vec{F}(t) = (\sin t, \sin t)$ $x(t) = y(t) = \sin t$; $x(t) \in [-1,1]$ e $y(t) \in [-1,1], \forall t \in \mathbb{R}$

Imagem $(\vec{F}) \subseteq \{(x,y) \in \mathbb{R}^2 : y = x\}$ = Reta

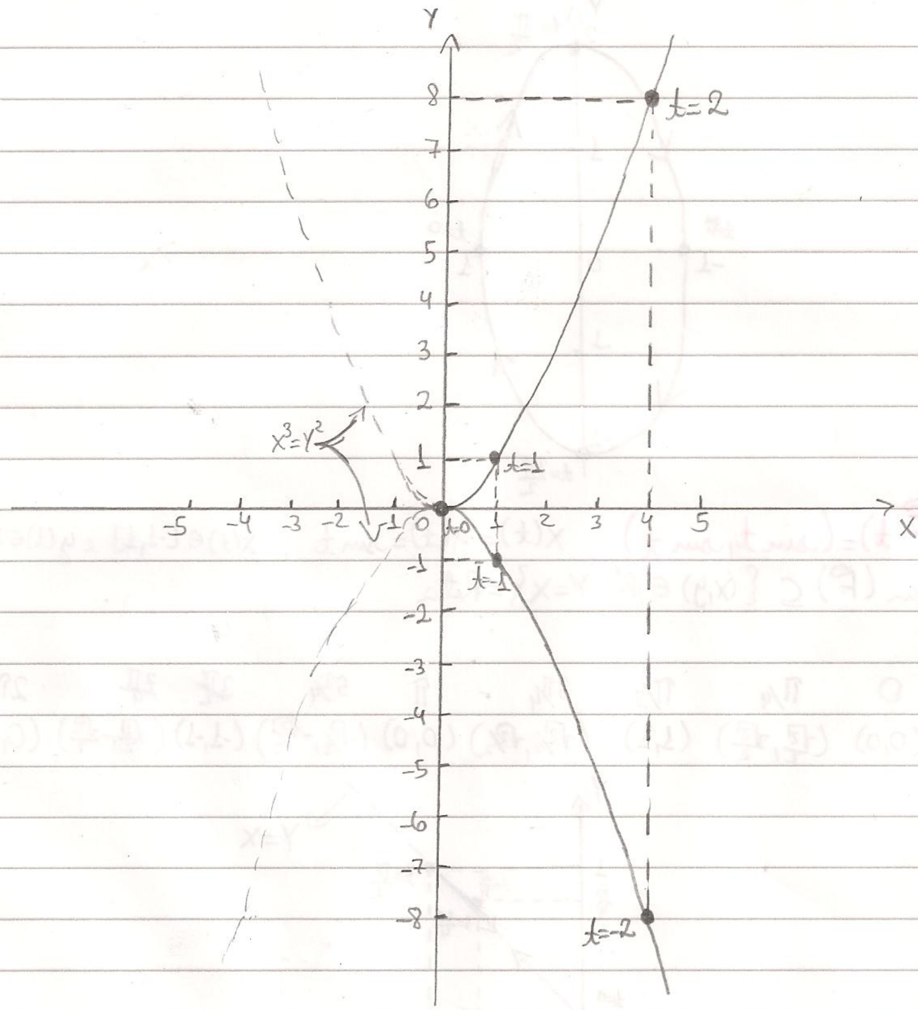
t	0	$\pi/4$	$\pi/2$	$3\pi/4$	π	$5\pi/4$	$3\pi/2$	$7\pi/4$	2π
$\vec{F}(t)$	$(0,0)$	$(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$	$(1,1)$	$(\sqrt{2}/2, \sqrt{2}/2)$	$(0,0)$	$(-\sqrt{2}/2, -\sqrt{2}/2)$	$(-1,-1)$	$(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$	$(0,0)$



(f) $\vec{F}(t) = (t^2, t^3)$ $x(t) = t^2$; $y(t) = t^3$ $x(t) \geq 0$

$\text{Imagem}(\vec{F}) \subseteq \{(x,y) \in \mathbb{R}^2 : x^3 = y^2\} = \{(x,y) \in \mathbb{R}^2 : |y| = |x|^{\frac{3}{2}}\}$

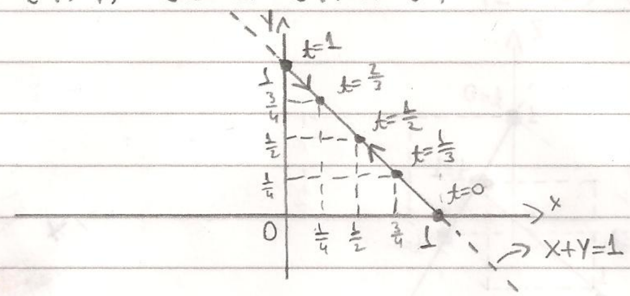
t	-2	-1	0	1	2
$\vec{F}(t)$	$(4,-8)$	$(1,-1)$	$(0,0)$	$(1,1)$	$(4,8)$



(g) $\vec{F}(t) = \left(\cos^2\left(\frac{\pi}{2}t\right), \sin^2\left(\frac{\pi}{2}t\right)\right)$ $x(t) = \cos^2\left(\frac{\pi}{2}t\right)$; $y(t) = \sin^2\left(\frac{\pi}{2}t\right)$

Imagem $(\vec{F}) \subseteq \{(x,y) \in \mathbb{R}^2 : x+y=1\}$ = Reta

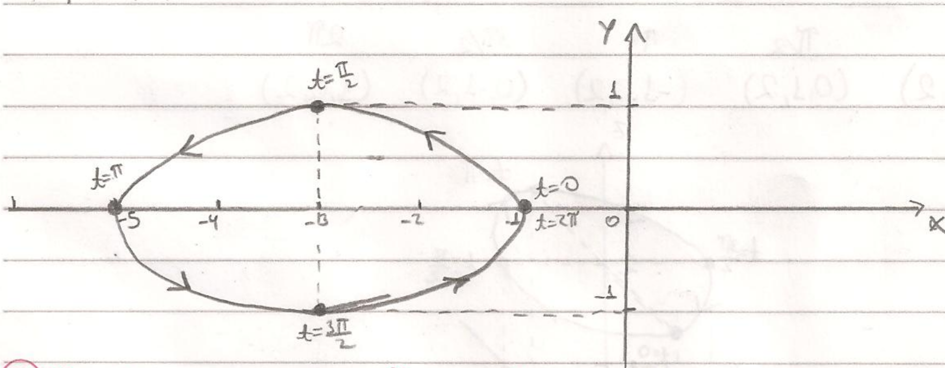
t	0	$\frac{1}{3}$	$1/2$	$2/3$	1
$\vec{F}(t)$	$(1,0)$	$\left(\frac{3}{4},\frac{1}{4}\right)$	$\left(\frac{1}{2},\frac{1}{2}\right)$	$\left(\frac{1}{4},\frac{3}{4}\right)$	$(0,1)$



(h) $\vec{F}(t) = (-3+2\cos t, \sin t)$ $x(t) = -3+2\cos t$; $y(t) = \sin t$

Imagem $(\vec{F}) = \left\{(x,y) \in \mathbb{R}^2 : \frac{(x+3)^2}{4} + y^2 = 1\right\}$ = Elipse

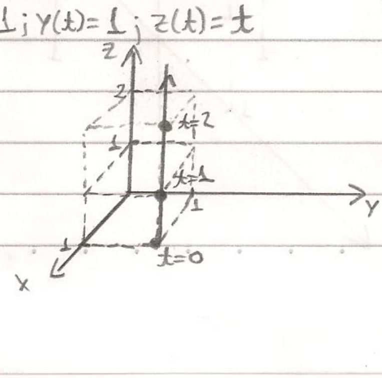
t	0	$\pi/2$	π	$3\pi/2$	2π
$\vec{F}(t)$	$(-1,0)$	$(-3,1)$	$(-5,0)$	$(-3,-1)$	$(-1,0)$



(2) Desenhe a imagem de $\vec{F}: \mathbb{R} \to \mathbb{R}^3$, onde:

(a) $\vec{F}(t) = (1,1,t)$, $t \geq 0$ $x(t) = 1$; $y(t) = 1$; $z(t) = t$

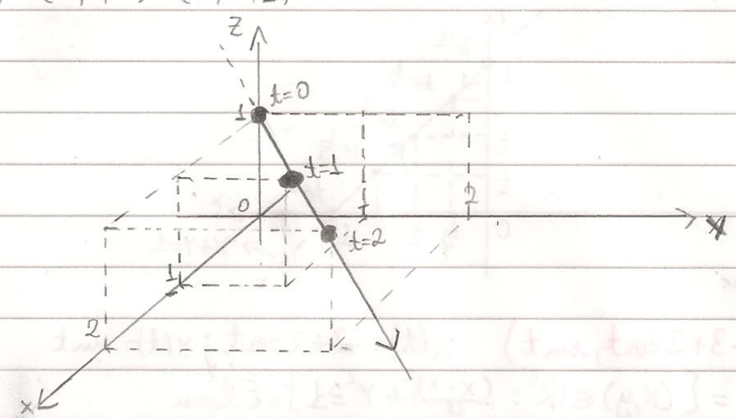
t	$\vec{F}(t)$
0	$(1,1,0)$
1	$(1,1,1)$
2	$(1,1,2)$



(b) $\vec{F}(t) = (t, t, 1)$, $t \geq 0$ $x(t) = t$; $y(t) = t$; $z(t) = 1$

$\text{Imagem}(\vec{F}) \subseteq \underbrace{\{(x,y,z) : x = y\}}_{\text{Plano}} \cap \underbrace{\{(x,y,z) \in \mathbb{R}^3 : z = 1\}}_{\text{Plano}} = \text{Reta}$

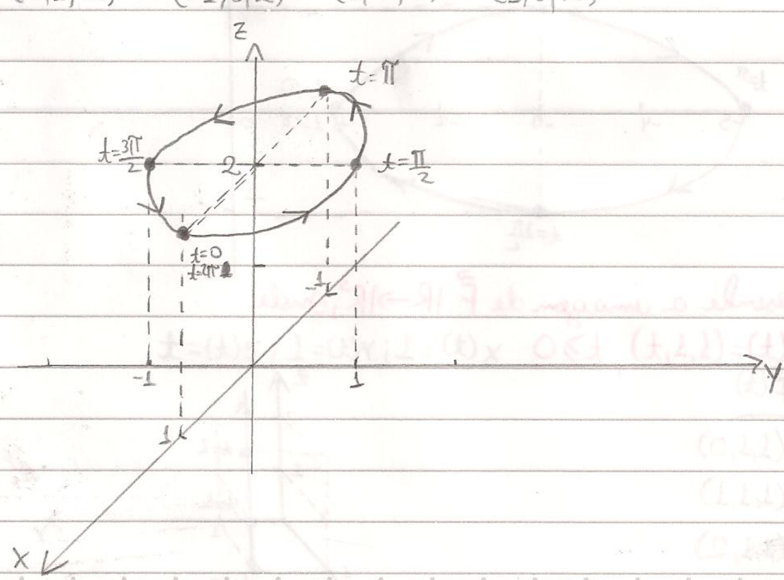
t	0	1	2
$\vec{F}(t)$	(0,0,1)	(1,1,1)	(2,2,1)



(c) $\vec{F}(t) = (\cos t, \sin t, 2)$, $t \in \mathbb{R}$ $x(t) = \cos t$; $y(t) = \sin t$; $z(t) = 2$

$\text{Imagem}(\vec{F}) \subseteq \underbrace{\{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 = 1\}}_{\text{Cilindro}} \cap \underbrace{\{(x,y,z) \in \mathbb{R}^3 : z = 2\}}_{\text{Plano}} = \text{Círculo}$

t	0	$\pi/2$	π	$3\pi/2$	2π
$\vec{F}(t)$	(1,0,2)	(0,1,2)	(-1,0,2)	(0,-1,2)	(1,0,2)

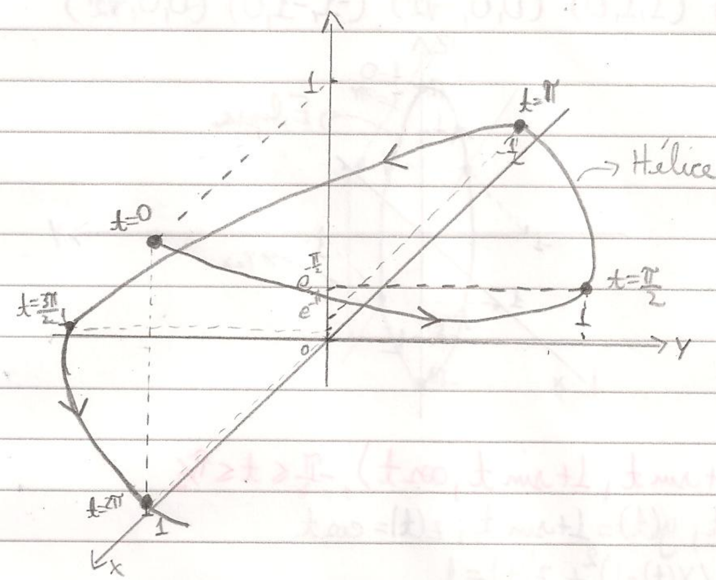


(d) $\vec{F}(t) = (\cos t, \sin t, e^{-t}),\ t \geq 0$

$x(t) = \cos t;\ y(t) = \sin t;\ z(t) = e^{-t}$

$\text{Imagem}(\vec{F}) \subseteq \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2=1\}$ = Cilindro e $z(t)$ é decrescente

t	0	$\pi/2$	π	$3\pi/2$	2π
$\vec{F}(t)$	$(1,0,1)$	$(0,1,e^{-\frac{\pi}{2}})$	$(-1,0,e^{-\pi})$	$(0,-1,e^{-\frac{3\pi}{2}})$	$(1,0,e^{-2\pi})$

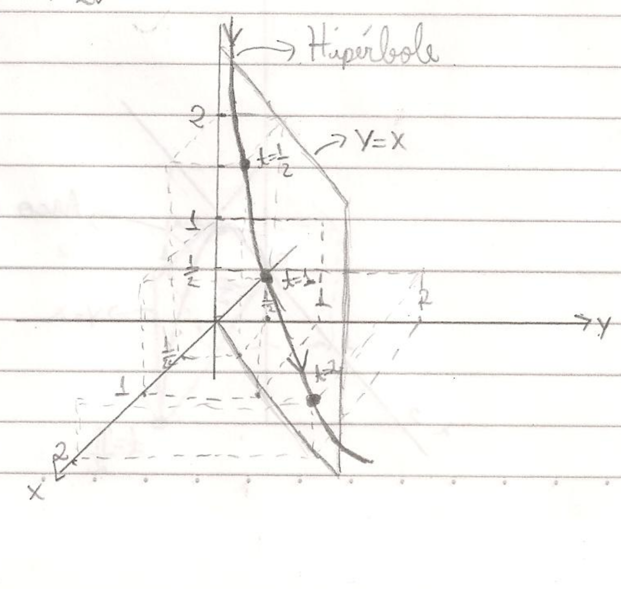


(e) $\vec{F}(t) = (t, t, 1/t),\ t > 0$

$x(t) = t;\ y(t) = t;\ z(t) = 1/t$

$x(t) = y(t)$ [Plano] e $z(t) = \frac{1}{x(t)}$

t	$1/2$	1	2
$\vec{F}(t)$	$(\frac{1}{2},\frac{1}{2},2)$	$(1,1,1)$	$(2,2,\frac{1}{2})$

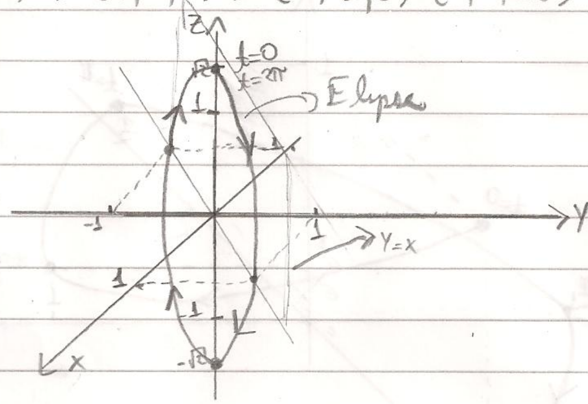


(f) $\vec{F}(t) = (\sin t, \sin t, \sqrt{2}\cos t)$, $0 \leq t \leq 2\pi$

$x(t) = \sin t$; $y(t) = \sin t$; $z(t) = \sqrt{2}\cos t$

$x(t) = y(t)$ e $x^2(t) + \frac{z^2(t)}{2} = 1$ (Elipse)

t	0	$\pi/2$	π	$3\pi/2$	2π
$\vec{F}(t)$	$(0,0,\sqrt{2})$	$(1,1,0)$	$(0,0,-\sqrt{2})$	$(-1,-1,0)$	$(0,0,\sqrt{2})$

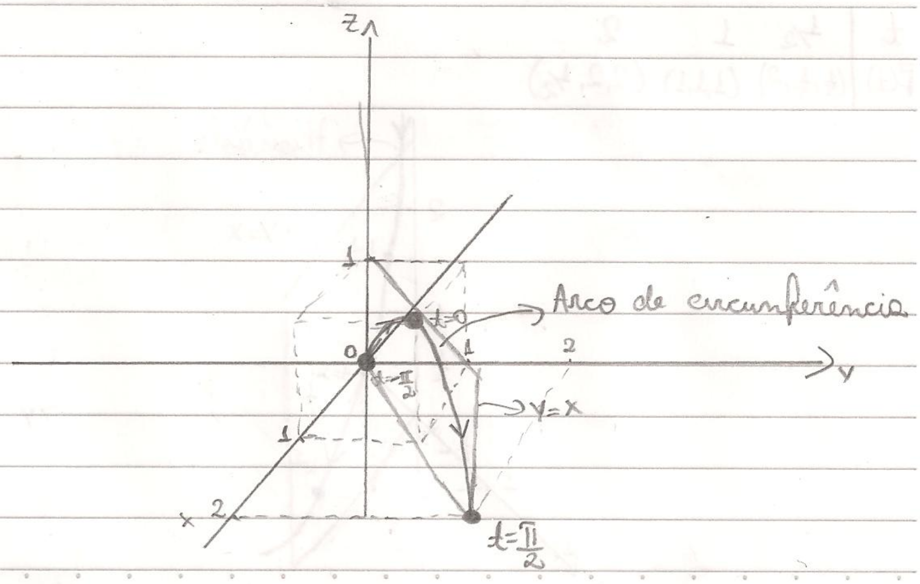


(g) $\vec{F}(t) = (1+\sin t, 1+\sin t, \cos t)$, $-\frac{\pi}{2} \leq t \leq \pi/2$

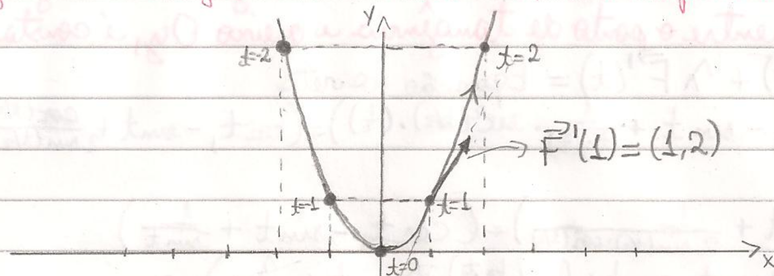
$x(t) = 1+\sin t$; $y(t) = 1+\sin t$; $z(t) = \cos t$

$x(t) = y(t)$ e $(x(t)-1)^2 + z^2(t) = 1$

t	$-\pi/2$	0	$\pi/2$
$\vec{F}(t)$	$(0,0,0)$	$(1,1,1)$	$(2,2,0)$



③ Determine o ponto de interseção do eixo Ox com a reta tangente à trajetória de $\vec{F}(t) = (t, t^2)$ no ponto $\vec{F}(1)$.



$\vec{F}'(t) = (1, 2t) \Rightarrow \vec{F}'(1) = (1, 2) \Rightarrow r: \vec{F}(1) + \lambda \vec{F}'(1) \Rightarrow$

$r(\lambda) = (1,1) + \lambda(1,2) = (1+\lambda, \underline{1+2\lambda})$

$r(\bar{\lambda}) \in Ox \Rightarrow 1 + 2\bar{\lambda} = 0 \Rightarrow \bar{\lambda} = -1/2 \Rightarrow r(\bar{\lambda}) = r(-1/2) = \boxed{(1/2, 0)}$

④ Determine a equação da reta tangente à trajetória de $\vec{F}(t) = (2t^2+1, t-1, 3t^3)$ no ponto $\vec{F}(t_0)$ em que a imagem de $\vec{F}$ fura xz.

$\vec{F}(t_0) \in \text{Plano}_{xz} \Rightarrow t_0 - 1 = 0 \Rightarrow t_0 = 1 \Rightarrow r: \vec{F}(1) + \lambda \vec{F}'(1)$

$\vec{F}(1) = (3, 0, 3)$ e $\vec{F}'(t) = (4t, 1, 9t^2) \Rightarrow \vec{F}'(1) = (4, 1, 9) \Rightarrow$

$\boxed{r(\lambda) = (3,0,3) + \lambda(4,1,9) = (3+4\lambda, \lambda, 3+9\lambda)}$

⑤ Mostre que as retas tangentes à trajetória de $\vec{F}(t) = (3t, 3t^2, 2t^3)$ fazem um ângulo constante com a reta $y=0, z=x$.

O vetor diretor da reta é $(1,0,1) = \vec{v}$.

O ângulo pedido é dado por: $\cos\theta(t) = \dfrac{\vec{F}'(t) \cdot (1,0,1)}{\|\vec{F}'(t)\| \cdot \|(1,0,1)\|} =$

$= \dfrac{(3, 6t, 6t^2) \cdot (1,0,1)}{\sqrt{3^2 + (6t)^2 + (6t^2)^2} \cdot \sqrt{1^2 + 0^2 + 1^2}} = \dfrac{3 + 6t^2}{\sqrt{9 + 36t^2 + 36t^4} \cdot \sqrt{2}} = \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}}{2}$

$\therefore \theta(t) = \dfrac{\pi}{4}, \forall t \in \mathbb{R}$

⑥ Sendo $\vec{F}(t) = \left(\sin t, \cos t + \ln\left(\operatorname{tg}\frac{t}{2}\right)\right)$, $t \in \,]0, \pi[$, mostre que o comprimento do segmento da reta tangente à trajetória de $\vec{F}$, compreendido entre o ponto de tangência e o eixo Oy, é constante igual a 1.

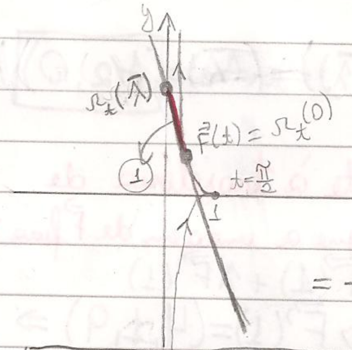
$r_t(\lambda) = \vec{F}(t) + \lambda \vec{F}'(t)$ = Equação da reta

$\vec{F}'(t) = \left(\cos t, -\sin t + \frac{1}{\operatorname{tg}\left(\frac{t}{2}\right)} \cdot \sec^2(t/2) \cdot \left(\frac{1}{2}\right)\right) = \left(\cos t, -\sin t + \frac{\cos(t/2)}{2\sin(t/2) \cdot \cos^2(t/2)}\right) =$

$= \left(\cos t, -\sin t + \frac{1}{2\sin(t/2) \cdot \cos(t/2)}\right) = \left(\cos t, -\sin t + \frac{1}{\sin t}\right)$

$r_t(\lambda) = \left(\sin t + \lambda \cos t, \cos t + \ln\left(\operatorname{tg}\frac{t}{2}\right) - \lambda \sin t + \frac{\lambda}{\sin t}\right) \Rightarrow$

$r_t(\bar{\lambda}) \in Oy \Leftrightarrow \sin t + \bar{\lambda} \cos t = 0 \Rightarrow \boxed{\bar{\lambda} = -\operatorname{tg} t}$



$d(r_t(0), r_t(\bar{\lambda})) =$

$= d\left(\left(\sin t, \cos t + \ln\left(\operatorname{tg}\frac{t}{2}\right)\right), \left(0, \cos t + \ln\left(\operatorname{tg}\frac{t}{2}\right) - \bar{\lambda}\left(\sin t - \frac{1}{\sin t}\right)\right)\right) =$

$= \sqrt{(\sin t - 0)^2 + \left[\bar{\lambda}\left(\sin t - \frac{1}{\sin t}\right)\right]^2} = \sqrt{\sin^2 t + \left[(-\operatorname{tg} t)\left(\frac{\sin^2 t - 1}{\sin t}\right)\right]^2} =$

$= \sqrt{\sin^2 t + \left[\left(-\frac{\sin t}{\cos t}\right)\left(\frac{-\cos^2 t}{\sin t}\right)\right]^2} = \sqrt{\sin^2 t + \cos^2 t} = \sqrt{1} = 1$ ∎

⑦ Verifique que $\vec{F}(t) = \left(t, \frac{1+t}{t}, \frac{1-t^2}{t}\right)$, $t > 0$, tem sua trajetória contida em um plano.

Se $\vec{F}(t)$ pertencer a um plano Π para todo $t > 0$, então existe um vetor $\vec{n}$ normal a todas as direções desse plano, isto é, $\exists\, \vec{n} \neq (0,0,0)$ tal que $\vec{n} \perp (\vec{F}(t) - \vec{F}(t_0))$, para todo $t > 0$. Tomando $t_0 = 1$, temos que $\vec{F}(t_0) = (1, 2, 0)$. Tomando $t_1 = 2$ e $t_2 = 3$, temos que $v_1 = \vec{F}(t_1) - \vec{F}(t_0) = \left(1, -\frac{1}{2}, -\frac{3}{2}\right)$ e $v_2 = \vec{F}(t_2) - \vec{F}(t_0) = \left(2, -\frac{2}{3}, -\frac{8}{3}\right)$ são tais que $v_1 \wedge v_2 = \left(\frac{1}{3}, -\frac{1}{3}, \frac{1}{3}\right)$ é perpendicular a v_1 e v_2. Tome $\vec{n} = (1, -1, 1)$ e temos que $\vec{n} \cdot (\vec{F}(t) - \vec{F}(t_0)) = (1, -1, 1) \cdot \left(t - 1, \frac{1}{t} - 1, \frac{1}{t} - t\right) =$
$= t - 1 - \frac{1}{t} + 1 + \frac{1}{t} - t = 0$, $\forall t > 0$. $\therefore$ $\vec{F}(t) \in \Pi = \{\vec{v} \in \mathbb{R}^3 : (\vec{v} - (1,2,0)) \perp \vec{n}\} =$

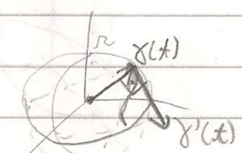
$= \{(x, y, z) \in \mathbb{R}^3 : x - y + z + 1 = 0\}$

⑧ A imagem de uma curva $\gamma: I \subset \mathbb{R} \to \mathbb{R}^3$ está contida na esfera de raio r: $S^2 = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 = r^2\}$. Verifique que $\gamma(t) \cdot \gamma'(t) = 0$ para todo $t \in I$. Interprete geometricamente.

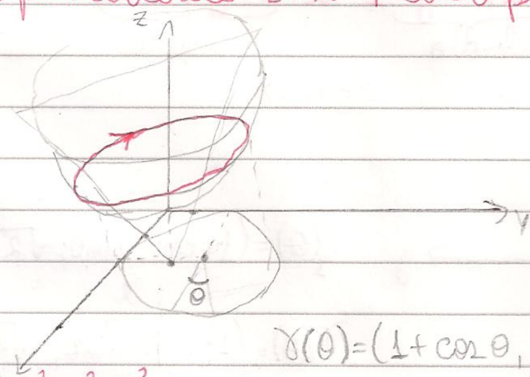
Sejam $\gamma(t) = (x(t), y(t), z(t))$ e $f: I \to \mathbb{R}$ definida por $f(t) = x^2(t) + y^2(t) + z^2(t)$.

Então, temos que $f(t) = r^2$, $\forall t \in I$, isto é, f é uma função constante.
Logo, $f'(t) = 0$, $\forall t \in I$ (f é diferenciável, pois γ é diferenciável).
Mas $f'(t) = 2x(t) \cdot x'(t) + 2y(t)y'(t) + 2 \cdot z(t) \cdot z'(t)$ e
$\gamma(t) \cdot \gamma'(t) = (x(t), y(t), z(t)) \cdot (x'(t), y'(t), z'(t)) = x(t) \cdot x'(t) + y(t) \cdot y'(t) + z(t) \cdot z'(t) =$
$= \dfrac{f'(t)}{2} = \dfrac{0}{2} = 0$, $\forall t \in I$. ∎

Interpretação: $\gamma(t)$ e $\gamma'(t)$ são ortogonais:



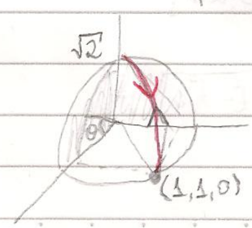
⑨ Determine uma curva γ cuja imagem é a intersecção:
(a) do parabolóide $z = x^2 + y^2$ com o plano $z = 2x + 2y - 1$



$x^2 + y^2 = 2x + 2y - 1 \Rightarrow$
$\underbrace{(x-1)^2 + (y-1)^2 = 1}_{\text{Eq. cilindro de raio 1 com centro em } (1,1,t)}$

$\gamma(\theta) = (1 + \cos\theta, 1 + \sin\theta, 2\cos\theta + 2\sin\theta + 1)$, $\theta \in [0, 2\pi]$

(b) de $\underbrace{x^2 + y^2 + z^2 = 2}_{\text{esfera}}$, $x \geq 0$, $y \geq 0$ e $z \geq 0$, com o plano $y = x$

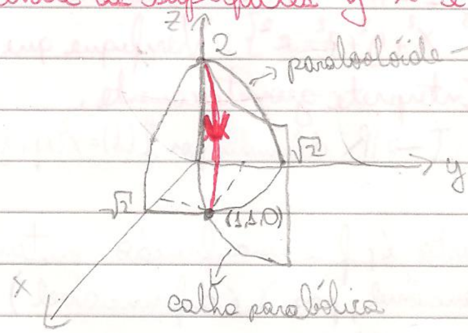


$\overbrace{2x^2 + z^2 = 1}^{\text{elipse no plano } xz} \Rightarrow \gamma(\theta) = \left(\dfrac{\cos\theta}{\sqrt{2}}, \dfrac{\cos\theta}{\sqrt{2}}, \sin\theta\right)$, $\theta \in [0, 2\pi]$

ou

$\gamma(t) = (t, t, \sqrt{2 - 2t^2})$, $t \in [0, 1]$

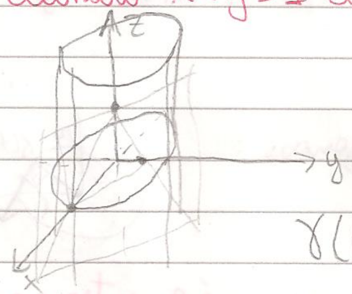
(c) entre as superfícies $y=x^2$ e $z=2-x^2-y^2$, $x\geq 0$, $y\geq 0$ e $z\geq 0$



$z=2-x^2-x^4$

$\gamma(t)=(t,t^2,2-t^2-t^4)$, $t\in[0,1]$

(d) do cilindro $x^2+y^2=1$ com o plano $x+2y+z=1$



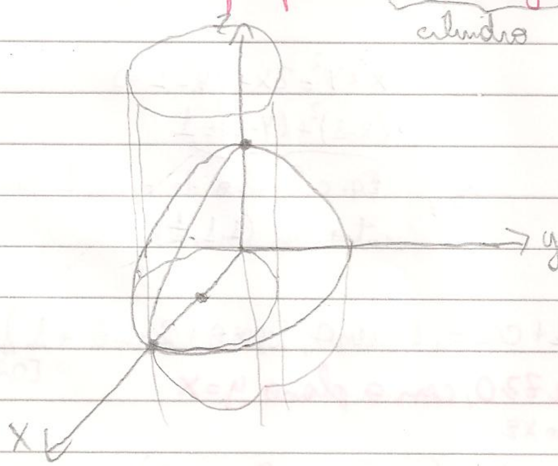
$z=1-x-2y$

$x=\cos\theta$ e $y=\sin\theta \Rightarrow z=1-\cos\theta-2\sin\theta$

$\gamma(\theta)=(\cos\theta,\sin\theta,1-\cos\theta-2\sin\theta)$, $\theta\in[0,2\pi]$

(e) entre as superfícies $\underbrace{(x-1)^2+y^2=1}_{\text{cilindro}}$ e $\underbrace{x^2+y^2+z^2=4}_{\text{esfera}}$

$z=4-2x$



$\gamma_1(\theta)=(1+\cos\theta,\sin\theta,\sqrt{2-2\cos\theta})$

e

$\gamma_2(\theta)=(1+\cos\theta,\sin\theta,-\sqrt{2-2\cos\theta})$

$t\in[0,2\pi]$

ou

$\gamma_1(t)=\left(2-\frac{t^2}{2},\sqrt{t^2-\frac{t^4}{4}},t\right)$ e $\gamma_2(t)=\left(2-\frac{t^2}{2},-\sqrt{t^2-t^4/4},t\right)$

$t\in[-2,2]$

10) Calcule o comprimento da curva dada:

(a) $\gamma(t) = (t\cos t,\ t\sin t),\ 0 \leq t \leq 2\pi$

$\gamma'(t) = (1\cdot\cos t + t\cdot(-\sin t),\ 1\cdot\sin t + t\cdot\cos t) = (\cos t - t\sin t,\ \sin t + t\cos t)$

$$C = \int_0^{2\pi} \|\gamma'(t)\|\, dt = \int_0^{2\pi} \sqrt{(\cos t - t\sin t)^2 + (\sin t + t\cos t)^2}\, dt = \int_0^{2\pi} (\cos^2 t + t^2\sin^2 t -$$

$$- \cancel{2t\sin t\cos t} + \sin^2 t + t^2\cos^2 t + \cancel{2t\sin t\cos t})^{\frac{1}{2}} dt = \int_0^{2\pi} \sqrt{1+t^2}\, dt =$$

$\boxed{\begin{array}{l} t = \operatorname{tg}\theta \\ dt = \sec^2\theta\, d\theta \end{array}}$

$$= \int_0^{\operatorname{arctg}(2\pi)} \sec^3\theta\, d\theta = \frac{1}{2}\Big[\sec\theta\cdot\operatorname{tg}\theta + \ln|\sec\theta + \operatorname{tg}\theta|\Big]_0^{\operatorname{arctg} 2\pi} =$$

$$= \frac{1}{2}\Big[\sqrt{1+4\pi^2}\cdot 2\pi + \ln|\sqrt{1+4\pi^2} + 2\pi| - 0\Big] =$$

$$= \pi\sqrt{1+4\pi^2} + \frac{\ln(\sqrt{1+4\pi^2} + 2\pi)}{2}$$

(b) $\gamma(t) = (2t-1,\ t+1),\ 1 \leq t \leq 2$

$$\gamma'(t) = (2, 1) \Rightarrow C = \int_1^2 \|\gamma'(t)\|\, dt = \int_1^2 \sqrt{2^2+1^2}\, dt = \int_1^2 \sqrt{5}\, dt =$$

$$= \Big[\sqrt{5}\,t\Big]_1^2 = \sqrt{5}$$

(c) $\gamma(t) = (\cos t,\ \sin t,\ e^{-t}),\ 0 \leq t \leq \pi$

$$\gamma'(t) = (-\sin t,\ \cos t,\ -e^{-t}) \Rightarrow C = \int_0^{\pi} \|\gamma'(t)\|\, dt = \int_0^{\pi} \sqrt{(-\sin t)^2 + \cos^2 t + (-e^{-t})^2}\, dt =$$

$$= \int_0^{\pi} \sqrt{\sin^2 t + \cos^2 t + e^{-2t}}\, dt = \int_0^{\pi} \sqrt{1 + e^{-2t}}\, dt =$$

(d) $\gamma(t) = (e^{-t}\cos t,\ e^{-t}\sin t,\ e^{-t}),\ 0 \leq t \leq 1$

$\gamma'(t) = (-e^{-t}\cos t - e^{-t}\sin t,\ -e^{-t}\sin t + e^{-t}\cos t,\ -e^{-t})$

$$C = \int_0^1 \|\gamma'(t)\|\, dt = \int_0^1 \sqrt{(-e^{-t}\cos t - e^{-t}\sin t)^2 + (e^{-t}\cos t - e^{-t}\sin t)^2 + (e^{-t})^2}\, dt = \int_0^1 e^{-t}(\cos^2 t + \sin^2 t +$$

$$+ 2\cancel{\sin t}\cos t + \cos^2 t + \sin^2 t - 2\cancel{\sin t}\cos t + 1)^{\frac{1}{2}} dt = \int_0^1 3e^{-t}\, dt = \Big[-3e^{-t}\Big]_0^1 = \Big[-3e^{-1} + 3\cdot e^{-0}\Big] =$$

$$= 3\cdot(1 - e^{-1})$$