AULA DO DIA 23/04/2018

Logo, considerando o ponto $\left(\frac{1}{2}, \frac{\pi}{2}\right) \in A$, temos

$$Df\left(\frac{1}{2}, \pi/2\right) = \begin{bmatrix} \cos \pi/2 & -\frac{1}{2} \operatorname{sen} \frac{\pi}{2} \\ \operatorname{sen} \pi/2 & \frac{1}{2} \cos \frac{\pi}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1/2 \\ 1 & 0 \end{bmatrix}$$

Essa matriz é a derivada da função f no ponto $(1/2, \pi/2)$. Além disso,

$$f(1/2, \pi/2) = \left(\frac{1}{2} \cos \frac{\pi}{2}, \frac{1}{2} \operatorname{sen} \frac{\pi}{2}\right) = (0, 1/2).$$

Do <u>exercício 2 da L2 complementar</u> temos que

$$Df^{-1}(0,1/2) = \left[Df(1/2, \pi/2)\right]^{-1}$$

Ou seja,

$$\left[\begin{array}{cc|cc} 0 & -1/2 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right] \underset{\substack{L_1 := L_2 \\ L_2 := L_1}}{\xrightarrow{\sim}} \left[\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & -1/2 & 1 & 0 \end{array}\right]$$

$$\underset{L_2 := (-2)L_2}{\longrightarrow} \left[\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 1 & -2 & 0 \end{array}\right].$$

$$\left[\begin{array}{cc} 0 & 1 \\ -2 & 0 \end{array}\right] = \left[Df(1/2, \pi/2)\right]^{-1}$$

Portanto,

$$Df^{-1}(0,1/2) = \left[Df(1/2, \pi/2)\right]^{-1} = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}$$

FUNÇÕES DEFINIDAS IMPLICITAMENTE

DEFINIÇÃO 17: A função $y = g(x)$ é definida implicitamente pela equação $f(x,y) = 0$ se, para todo $x \in \text{Dom}(g)$,

$$f(x, g(x)) = 0$$

Exemplo : $y = x$ é definida implicitamente pela equação $f(x,y) = 0$, onde $f(x,y) = y - x$.

(OBS: $\frac{\partial f}{\partial x} := D_1 f$ e $\frac{\partial f}{\partial y} := D_2 f$)

Considere que a função $y = g(x)$ esteja definida implicitamente pela equação $f(x,y) = 0$. Suponha que f e g sejam funções diferenciáveis. Então:

$$f(x, g(x)) = 0 \quad , \quad \forall \, x \in \mathrm{Dom}(g)$$

$$\Rightarrow \quad \frac{d}{dx} f(x, g(x)) = 0 \quad , \quad \forall \, x \in \mathrm{Dom}(g)$$

$$\Rightarrow \quad \frac{\partial f}{\partial x}(x, g(x)) \cdot \overbrace{\frac{dx}{dx}}^{=1} + \frac{\partial f}{\partial y}(x, g(x)) \cdot \frac{dg}{dx} = 0$$

$$\Rightarrow \quad \frac{\partial f}{\partial y}(x, g(x)) \cdot \frac{dg}{dx} = -\frac{\partial f}{\partial x}(x, g(x))$$

$$\Rightarrow \quad \frac{dg}{dx} = \frac{-\frac{\partial f}{\partial x}(x, g(x))}{\frac{\partial f}{\partial y}(x, g(x))}$$, desde que

$$\frac{\partial f}{\partial y}(x, g(x)) \neq 0$$

DEFINIÇÃO 18: A função $x = h(y)$ é definida implicitamente pela equação $f(x,y) = 0$ se, para todo $y \in Dom(h)$,

$$f(h(y), y) = 0.$$

De forma análoga, se f e h são funções diferenciáveis e $\frac{\partial f}{\partial x}(h(y), y) \neq 0$, então:

$$\frac{dx}{dy} = \frac{-\frac{\partial f}{\partial y}(h(y), y)}{\frac{\partial f}{\partial x}(h(y), y)}.$$

Exemplo: A função diferenciável $y = y(x)$ está definida implicitamente pela equação

$$y^3 + xy + x^3 = 3.$$

Calcule $\frac{dy}{dx}$.

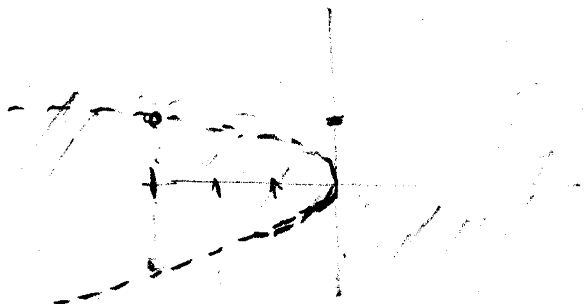
SOLUÇÃO:

$$y^3 + xy + x^3 = 3 \iff \underbrace{y^3 + xy + x^3 - 3}_{f(x,y) :=} = 0$$

Como f e y(x) são funções diferenciáveis temos que:

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}(x,y)}{\frac{\partial f}{\partial y}(x,y)} = \frac{-(y+3x^2)}{3y^2+x}$$

desde que $3y^2+x \neq 0$, ou seja, que $x \neq -3y^2$.



(*) Verificar a conta usando apenas técnicas do Cálculo I

Suponha que a função diferenciável $z = g(x,y)$ seja dada implicitamente pela equação $f(x,y,z) = 0$, onde f é diferenciável num aberto de $\mathbb{R}^3$. Assim,

$$f(x,y,g(x,y)) = 0 \quad, \forall (x,y) \in Dom(g) \Rightarrow$$

$$\frac{\partial}{\partial x} f(x,y,g(x,y)) = 0 \quad, \forall (x,y) \in Dom(g)$$

$$\frac{\partial f}{\partial x}(x,y,z)\cdot\underbrace{\frac{\partial (x)}{\partial x}}_{1}+\frac{\partial f}{\partial y}(x,y,z)\cdot\underbrace{\frac{\partial y}{\partial x}}_{0}+\frac{\partial f}{\partial z}(x,y,z)\cdot\frac{\partial z}{\partial x}=0$$

$$\frac{\partial f}{\partial x}(x,y,z)+\frac{\partial f}{\partial z}(x,y,z)\cdot\frac{\partial z}{\partial x}=0 \implies$$

$$\frac{\partial z}{\partial x}=\frac{-\frac{\partial f}{\partial x}(x,y,z)}{\frac{\partial f}{\partial z}(x,y,z)}$$

desde que $\frac{\partial f}{\partial z}(x,y,z)\neq 0$.

Da mesma forma verifica-se que:

$$\frac{\partial z}{\partial y}=\frac{-\frac{\partial f}{\partial y}(x,y,z)}{\frac{\partial f}{\partial z}(x,y,z)}$$ desde que

$$\frac{\partial f}{\partial z}(x,y,z)\neq 0.$$

Exemplo: A função diferenciável $z=z(x,y)$ é dada implicitamente pela equação:

$$xyz+x^3+y^3+z^3=5$$

Calcule $\frac{\partial z}{\partial x}(x,y,z)$:

SOLUÇÃO:

$$\underbrace{xyz + x^3 + y^3 + z^3 - 5}_{f(x,y,z) :=} = 0$$

Logo,

$$\frac{\partial z}{\partial x} = \frac{-\frac{\partial f}{\partial x}(x,y,z)}{\frac{\partial f}{\partial z}(x,y,z)} = \frac{-(yz + 3x^2)}{xy + 3z^2}$$

desde que $xy + 3z^2 \neq 0$,

AULA DO DIA 23/04/2018

Suponha que as funções diferenciáveis $y = y(x)$ e $z = z(x)$ sejam definidas no intervalo aberto I e que sejam dadas implicitamente pelo sistema:

$$\begin{cases} F(x,y,z) = 0 \\ G(x,y,z) = 0 \end{cases}$$

Onde F, G são funções diferenciáveis num aberto do $\mathbb{R}^3$.

Logo, $\forall\, x \in I$

$$F(x, y(x), z(x)) = 0 \quad \text{e} \quad G(x, y(x), z(x)) = 0$$

Ou seja, a imagem da curva $\gamma(x) = (x, y(x), z(x))$ está contida na intersecção das superfícies de nível 0 da F e G.

$$\begin{cases} \dfrac{d}{dx}\left(F(x, y(x), z(x))\right) = \dfrac{d}{dx}(0) = 0 \\[2ex] \dfrac{d}{dx}\left(G(x, y(x), z(x))\right) = \dfrac{d}{dx}(0) = 0 \end{cases}$$

$$\Downarrow$$

$$\begin{cases} \dfrac{\partial F}{\partial x} \cdot \overset{=1}{\dfrac{dx}{dx}} + \dfrac{\partial F}{\partial y} \cdot \dfrac{dy}{dx} + \dfrac{\partial F}{\partial z} \cdot \dfrac{dz}{dx} = 0 \\[2ex] \dfrac{\partial G}{\partial x} \cdot \underset{=1}{\dfrac{dx}{dx}} + \dfrac{\partial G}{\partial y} \cdot \dfrac{dy}{dx} + \dfrac{\partial G}{\partial z} \cdot \dfrac{dz}{dx} = 0 \end{cases}$$

$$\Downarrow$$

$$
\begin{cases}
\dfrac{\partial F}{\partial y}\cdot\dfrac{dy}{dx}+\dfrac{\partial F}{\partial z}\cdot\dfrac{dz}{dx}=-\dfrac{\partial F}{\partial x}\\[2ex]
\dfrac{\partial G}{\partial y}\cdot\dfrac{dy}{dx}+\dfrac{\partial G}{\partial z}\cdot\dfrac{dz}{dx}=-\dfrac{\partial G}{\partial x}
\end{cases}
$$

$$\Downarrow$$

$$
\begin{bmatrix}
\dfrac{\partial F}{\partial y} & \dfrac{\partial F}{\partial z}\\[2ex]
\dfrac{\partial G}{\partial y} & \dfrac{\partial G}{\partial z}
\end{bmatrix}
\begin{bmatrix}
\dfrac{dy}{dx}\\[2ex]
\dfrac{dz}{dx}
\end{bmatrix}
=
\begin{bmatrix}
-\dfrac{\partial F}{\partial x}\\[2ex]
-\dfrac{\partial G}{\partial x}
\end{bmatrix}
$$

$\Downarrow$ (Regra de Cramer)

$$
\frac{dy}{dx}=-\frac{\begin{vmatrix}\dfrac{\partial F}{\partial x} & \dfrac{\partial F}{\partial z}\\[2ex] \dfrac{\partial G}{\partial x} & \dfrac{\partial G}{\partial z}\end{vmatrix}}{\begin{vmatrix}\dfrac{\partial F}{\partial y} & \dfrac{\partial F}{\partial z}\\[2ex] \dfrac{\partial G}{\partial y} & \dfrac{\partial G}{\partial z}\end{vmatrix}}
\quad \text{e} \quad
\frac{dz}{dx}=-\frac{\begin{vmatrix}\dfrac{\partial F}{\partial y} & \dfrac{\partial F}{\partial x}\\[2ex] \dfrac{\partial G}{\partial y} & \dfrac{\partial G}{\partial x}\end{vmatrix}}{\begin{vmatrix}\dfrac{\partial F}{\partial y} & \dfrac{\partial F}{\partial z}\\[2ex] \dfrac{\partial G}{\partial y} & \dfrac{\partial G}{\partial z}\end{vmatrix}}
$$

desde que, $\forall\, x \in I$, $\begin{vmatrix}\dfrac{\partial F}{\partial y} & \dfrac{\partial F}{\partial z}\\[2ex] \dfrac{\partial G}{\partial y} & \dfrac{\partial G}{\partial z}\end{vmatrix} \neq 0$ em $\gamma(x)$.

NOTAÇÃO:

$$\frac{\partial(F,G)}{\partial(y,z)} := \begin{vmatrix} \frac{\partial F}{\partial y} & \frac{\partial F}{\partial z} \\ \frac{\partial G}{\partial y} & \frac{\partial G}{\partial z} \end{vmatrix}$$

$$\frac{\partial(F,G)}{\partial(x,z)} := \begin{vmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial z} \\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial z} \end{vmatrix}$$

$$\frac{\partial(F,G)}{\partial(y,x)} := \begin{vmatrix} \frac{\partial F}{\partial y} & \frac{\partial F}{\partial x} \\ \frac{\partial G}{\partial y} & \frac{\partial G}{\partial x} \end{vmatrix}$$

Exemplo:

$y = y(x)$ e $z = z(x)$ funções diferenciáveis em $\mathbb{R}$, dadas implicitamente pelo sistema

$$\begin{cases} 2x + y - z = 3 \\ x + y + z = 1 \end{cases}$$

Vamos calcular $\frac{dy}{dx}$ e $\frac{dz}{dx}$:

1ª maneira:

$$\begin{cases} \dfrac{d}{dx}(2x+y-z) = \dfrac{d}{dx}(3) = 0 \\ \dfrac{d}{dx}(x+y+z) = \dfrac{d}{dx}(1) = 0 \end{cases}$$

$$\Downarrow$$

$$\begin{cases} 2 + \dfrac{dy}{dx} - \dfrac{dz}{dx} = 0 \\ 1 + \dfrac{dy}{dx} + \dfrac{dz}{dx} = 0 \end{cases}$$

$$\Downarrow$$

$$\begin{cases} \dfrac{dy}{dx} - \dfrac{dz}{dx} = -2 \\ \dfrac{dy}{dx} + \dfrac{dz}{dx} = -1 \end{cases} \quad +$$

$$2 \cdot \frac{dy}{dx} = -3$$

$$\frac{dy}{dx} = -\frac{3}{2} \,//$$

$$-\frac{3}{2} + \frac{dz}{dx} = -1 \implies \frac{dz}{dx} = -1 + \frac{3}{2} = \frac{1}{2} \,//$$

2ª maneira:

Defina $\begin{cases} F(x,y,z) := 2x + y - z - 3 \\ G(x,y,z) := x + y + z - 1 \end{cases}$.

$\frac{\partial F}{\partial x}(x,y,z) = 2$ $\quad \frac{\partial F}{\partial y}(x,y,z) = 1$ $\quad \frac{\partial F}{\partial z}(x,y,z) = -1$

$\frac{\partial G}{\partial x}(x,y,z) = 1$ $\quad \frac{\partial G}{\partial y}(x,y,z) = 1$ $\quad \frac{\partial G}{\partial z}(x,y,z) = 1$

Como $\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}, \frac{\partial G}{\partial x}, \frac{\partial G}{\partial y}, \frac{\partial G}{\partial z}$ são funções constantes, temos que são contínuas no $\mathbb{R}^3$. Portanto F, G são funções de classe C^1 em $\mathbb{R}^3$ e logo diferenciáveis em $\mathbb{R}^3$. Assim

$$\frac{dy}{dx} = - \frac{\begin{vmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial z} \\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial z} \end{vmatrix}}{\begin{vmatrix} \frac{\partial F}{\partial y} & \frac{\partial F}{\partial z} \\ \frac{\partial G}{\partial y} & \frac{\partial G}{\partial z} \end{vmatrix}} = - \frac{\begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix}}$$

$$= -\frac{3}{2} \quad //$$

$$\frac{dz}{dx} = - \frac{\begin{vmatrix} \frac{\partial F}{\partial y} & \frac{\partial F}{\partial x} \\ \frac{\partial G}{\partial y} & \frac{\partial G}{\partial x} \end{vmatrix}}{\begin{vmatrix} \frac{\partial F}{\partial y} & \frac{\partial F}{\partial z} \\ \frac{\partial G}{\partial y} & \frac{\partial G}{\partial z} \end{vmatrix}} = - \frac{\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix}}{2}$$

$$= \frac{1}{2} \,//.$$