C

Instituto Federal de Educação, Ciência e Tecnologia de São Paulo.

1a. Avaliação de Matemática - T210.

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N^o Pront. : ______________________

Assinatura : ______________________

Questão	Nota
1	
2	
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Total	

Q1. (2,0) Pede-se:

a) Se $a = \cos\left(\frac{\pi}{5}\right)$ e $b = \sin\left(\frac{\pi}{5}\right)$, mostre que $\left(\cos\left(\frac{\pi}{5}\right) + i\sin\left(\frac{\pi}{5}\right)\right)^{54} = -a + bi$.

b) Se $z_1, z_2 \in \mathbb{C}$ são tais que $|z_1| = |z_2| = 1$ e $|z_1 - z_2| = \sqrt{2}$, mostre que $z_1^2 + z_2^2 = 0$.

SOLUÇÃO :

a) Note que $\frac{\pi}{5} = \frac{180^\circ}{5} = 36^\circ$. Chame de $z = \cos\left(\frac{\pi}{5}\right) + i\,\text{sen}\left(\frac{\pi}{5}\right)$.

Logo $|z| = \sqrt{\cos^2\left(\frac{\pi}{5}\right) + \text{sen}^2\left(\frac{\pi}{5}\right)} = \sqrt{1} = 1$. Com isso,

$$z^{54} = \left(\cos\left(\frac{\pi}{5}\right) + i\,\text{sen}\left(\frac{\pi}{5}\right)\right)^{54} = 1^{54}\left(\cos\left(54 \cdot \frac{\pi}{5}\right), \text{sen}\left(54 \cdot \frac{\pi}{5}\right)\right)$$

$$= \left(\cos(54 \cdot 36^\circ), \text{sen}(54 \cdot 36^\circ)\right)$$

$$= \left(\cos(1944^\circ), \text{sen}(1944^\circ)\right)$$

$$= \left(\cos(144^\circ), \text{sen}(144^\circ)\right)$$

$$= \left(\cos(4 \cdot 36^\circ), \text{sen}(4 \cdot 36^\circ)\right)$$

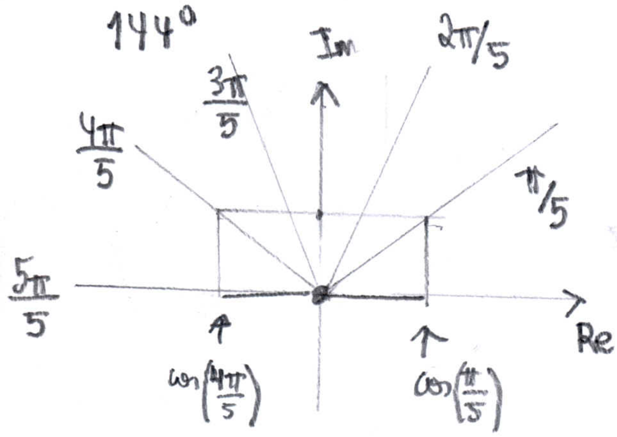
$$= \left(\cos\left(\frac{4\pi}{5}\right), \text{sen}\left(\frac{4\pi}{5}\right)\right)$$

$$= \left(-\cos\left(\frac{\pi}{5}\right), \text{sen}\left(\pi/5\right)\right)$$

$$= -\cos\left(\frac{\pi}{5}\right) + i\,\text{sen}\left(\frac{\pi}{5}\right)$$

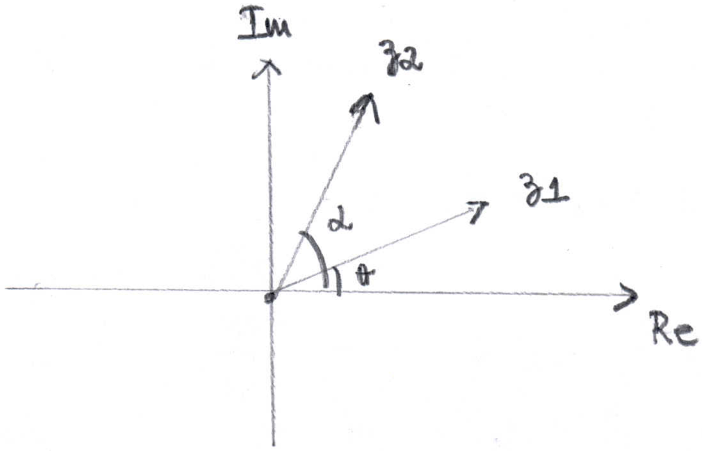
$$= -a + ib$$

$$\begin{array}{r|l} 1944^\circ & 360^\circ \\ \hline -1800^\circ & 5 \\ \hline 144^\circ & \end{array}$$



CONTINUAÇÃO DA Q1:

b) Representa z_1 e z_2 na forma trigonométrica:



$$z_1 = |z_1|(\cos\theta, \text{sen}\,\theta) = 1\cdot(\cos\theta, \text{sen}\,\theta)$$

$$z_2 = |z_2|(\cos\alpha, \text{sen}\,\alpha) = 1\cdot(\cos\alpha, \text{sen}\,\alpha)$$

Note que:

$$z_1 - z_2 = (\cos\theta - \cos\alpha, \text{sen}\,\theta - \text{sen}\,\alpha)$$

$$\sqrt{2} = |z_1 - z_2| = \sqrt{(\cos\theta - \cos\alpha)^2 + (\text{sen}\,\theta - \text{sen}\,\alpha)^2} \Longrightarrow$$

$$(\cos\theta - \cos\alpha)^2 + (\text{sen}\,\theta - \text{sen}\,\alpha)^2 = 2$$

$$\cos^2\theta - 2\cos\theta\cos\alpha + \cos^2\alpha + \text{sen}^2\theta - 2\,\text{sen}\,\theta\,\text{sen}\,\alpha + \text{sen}^2\alpha = 2$$

Como $\begin{cases} |z_1| = 1, \text{ então } \cos^2\theta + \text{sen}^2\theta = 1; \\ \text{e} \\ |z_2| = 1, \text{ então } \cos^2\alpha + \text{sen}^2\alpha = 1 \end{cases}$

Logo,

$$\cancel{1} - 2\cos\theta\cos\alpha + \cancel{1} - 2\,\text{sen}\,\theta\,\text{sen}\,\alpha = \cancel{2} \Longrightarrow$$

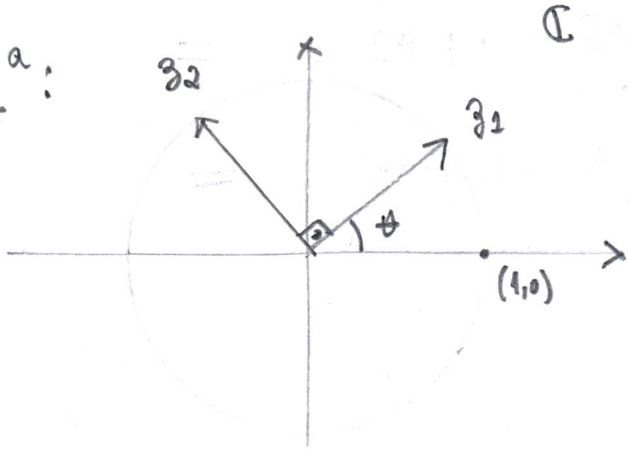
$$-2\left(\cos\theta\cos\alpha + \text{sen}\,\theta\,\text{sen}\,\alpha\right) = 0 \Longrightarrow$$

$$\underbrace{\cos\theta\cos\alpha + \text{sen}\,\theta\,\text{sen}\,\alpha}_{} = 0 \Longrightarrow$$

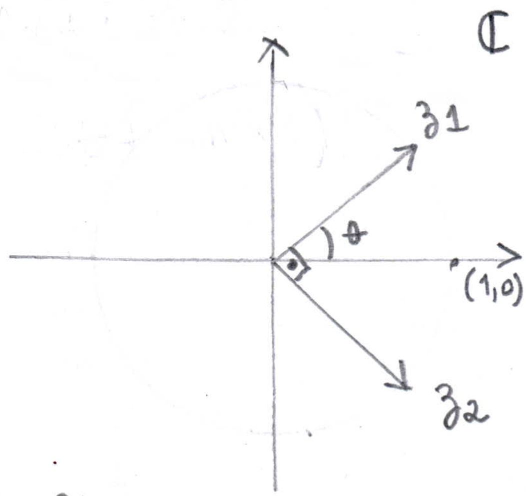
$$\cos(\theta - \alpha) = 0$$

Logo o ângulo entre z_1 e z_2 é $\frac{\pi}{2}$. Por isso, temos 2 possibilidades:

1ª:



2ª:



Ou seja, $\theta - \alpha = \frac{\pi}{2} \Longrightarrow \alpha = \theta - \frac{\pi}{2}$, com isso:

$$z_1^2 + z_2^2 = (\cos\theta + i\,\text{sen}\,\theta)^2 + (\cos\alpha + i\,\text{sen}\,\alpha)^2$$

$$= (\cos\theta + i\,\text{sen}\,\theta)^2 + \left(\cos\left(\theta - \frac{\pi}{2}\right) + i\,\text{sen}\left(\theta - \frac{\pi}{2}\right)\right)^2$$

Agora, lembre que:

$$\cos\left(\theta - \frac{\pi}{2}\right) = \cos(\theta)\cdot\overbrace{\cos\left(\frac{\pi}{2}\right)}^{=0} + \text{sen}(\theta)\cdot\overbrace{\text{sen}\left(\frac{\pi}{2}\right)}^{=1} = \text{sen}(\theta)$$

$$\text{sen}\left(\theta - \frac{\pi}{2}\right) = \text{sen}(\theta)\cdot\underbrace{\cos\left(\frac{\pi}{2}\right)}_{=0} - \underbrace{\text{sen}\left(\frac{\pi}{2}\right)}_{=1}\cos(\theta) = -\cos(\theta)$$

CONTINUAÇÃO Q1-b:

Logo:

$$z_1^2 + z_2^2 = (\cos(\theta) + i\,\text{sen}(\theta))^2 + (\text{sen}(\theta) - i\cos(\theta))^2$$

mas,

$$(\text{sen}(\theta) - i\cos(\theta))^2 = (\text{sen}\,\theta, -\cos\theta)^2$$

$$= (\text{sen}\,\theta, -\cos\theta) \cdot (\text{sen}\,\theta, -\cos\theta)$$

$$= (\text{sen}^2\theta - \cos^2\theta, -2\,\text{sen}\,\theta\cos\theta)$$

$$= (-(\cos^2\theta - \text{sen}^2\theta), -2\,\text{sen}\,\theta\cos\theta)$$

$$= (-\cos 2\theta, -2\,\text{sen}\,2\theta)$$

$$\overset{(*)}{=} -\cos 2\theta - i\,\text{sen}\,2\theta$$

Portanto:

$$z_1^2 + z_2^2 = (\cos(\theta) + i\,\text{sen}(\theta))^2 + (\text{sen}(\theta) - i\cos(\theta))^2$$

$$\overset{(*)}{=} \cancel{\cos(2\theta)} + i\,\cancel{\text{sen}(2\theta)} + (-\cancel{\cos(2\theta)} - i\,\cancel{\text{sen}(2\theta)})$$

$$= 0 \;!!!$$

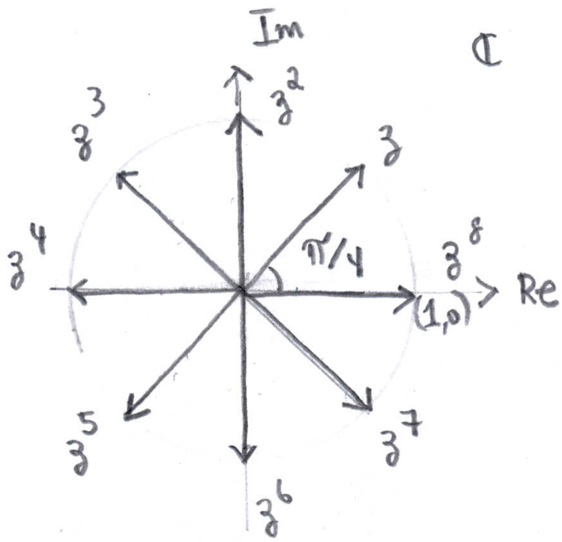
Q2. (3,0)Pede-se:

a) Se $z = \dfrac{1+i}{\sqrt{2}}$, quanto vale $\left|\sum_{n=1}^{47} z^n\right|$?

b) Seja $f : \mathbb{R} \to \mathbb{C}$ uma função definida por $f(x) = 2\cos(x) + i2\sin(x)$. Mostre que para todo $x_1, x_2 \in \mathbb{R}$

$$2\, f(x_1 + x_2) = f(x_1)f(x_2).$$

c) É verdade que $(1-i)^{16} = 256$? Justifique.



SOLUÇÃO :

$$z = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i$$

o argumento de z é $\dfrac{\pi}{4}$. Assim,

$$z = \left(\cos\frac{\pi}{4}, \operatorname{sen}\frac{\pi}{4}\right)$$, Por isso,

$$z^2 = \left(\cos 2.\frac{\pi}{4}, \operatorname{sen} 2.\frac{\pi}{4}\right) = \left(\cos\frac{\pi}{2}, \operatorname{sen}\frac{\pi}{2}\right) = (0,1) = i$$

$$z^3 = \left(\cos 3\frac{\pi}{4}, \operatorname{sen}\frac{3\pi}{4}\right) = \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$$

$$z^4 = (\cos\pi, \operatorname{sen}\pi) = (-1,0) \qquad z^5 = \left(\cos\frac{5\pi}{4}, \operatorname{sen}\frac{5\pi}{4}\right) = \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$$

$$z^6 = \left(\cos\frac{6\pi}{4}, \operatorname{sen}\frac{6\pi}{4}\right) = \left(\cos\frac{3\pi}{2}, \operatorname{sen}\frac{3\pi}{2}\right) = (0,-1) = -i$$

$$z^7 = \left(\cos\frac{7\pi}{4}, \operatorname{sen}\frac{7\pi}{4}\right) = \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$$

$$z^8 = \left(\cos\frac{8\pi}{4}, \operatorname{sen}\frac{8\pi}{4}\right) = (\cos 2\pi, \operatorname{sen} 2\pi) = (1,0).$$

CONTINUAÇÃO Q2

a) Veja que apartir de z^8 as potências se repetem.

Além disso,

$$0 = \sum_{n=1}^{8} z^n = \sum_{n=9}^{16} z^n = \sum_{n=17}^{24} z^n = \sum_{n=25}^{32} z^n = \sum_{n=33}^{40} z^n$$

e também,

$$\sum_{n=1}^{47} z^n = \underbrace{\sum_{n=1}^{8} z^n + \sum_{n=9}^{16} z^n + \sum_{n=17}^{24} z^n + \sum_{n=25}^{32} z^n + \sum_{n=33}^{40} z^n}_{=0} + \sum_{n=41}^{47} z^n$$

$$= \sum_{n=41}^{47} z^n = \sum_{n=1}^{7} z^n = z^4$$

Logo, $\left| \sum_{n=1}^{47} z^n \right| = |z^4| = |(-1,0)| = \sqrt{(-1)^2 + 0^2} = 1$

b) Sejam $x_1, x_2 \in \mathbb{R}$, assim,

$$f(x_1) = 2\cos(x_1) + i\,2\,sen(x_1) = 2(\cos(x_1), sen(x_1))$$

$$f(x_2) = 2\cos(x_2) + i\,2\,sen(x_2) = 2(\cos(x_2), sen(x_2$$

$$f(x_1 + x_2) = 2\cos(x_1 + x_2) + i\,2\,sen(x_1 + x_2) =$$

$$2(\cos(x_1 + x_2), sen(x_1 + x_2))$$

Logo:

$$2\,f(x_1+x_2) = 2\left(2\left(\cos(x_1+x_2),\ \text{sen}(x_1+x_2)\right)\right)$$

$$= 2\left(2\left(\cos x_1 \cos x_2 - \text{sen}\, x_1\, \text{sen}\, x_2,\ \text{sen}\, x_1 \cos x_2 + \text{sen}\, x_2 \cos x_1\right)\right)$$

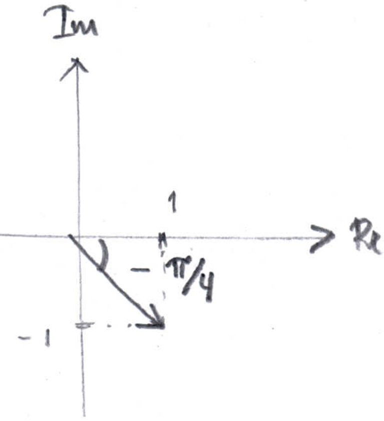
$$= 2\left(2\left(\cos x_1,\ \text{sen}\, x_1\right)\cdot\left(\cos x_2,\ \text{sen}\, x_2\right)\right)$$

$$= 2\left(\cos x_1,\ \text{sen}\, x_1\right)\cdot 2\left(\cos x_2,\ \text{sen}\, x_2\right)$$

$$= f(x_1)\cdot f(x_2).$$

c) $1 - i = (1,-1)$

$$= \sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right),\ \text{sen}\left(-\frac{\pi}{4}\right)\right)$$



Logo:

$$(1-i)^{16} = (\sqrt{2})^{16}\left(\cos\left(16\cdot\left(-\frac{\pi}{4}\right)\right),\ \text{sen}\left(16\left(-\frac{\pi}{4}\right)\right)\right)$$

$$= 2^{8}\left(\cos(-4\pi),\ \text{sen}(-4\pi)\right)$$

$$= 256\,(1,0) = 256$$

Logo a afirmação é verdadeira!

Q3. (1,5) Dada a expressão $A = \dfrac{1 - ix}{2x - i}$, em que $x \in \mathbb{R}$ e $i = (0, 1)$, encontre os valores de x que tornam A um número:

a) Imaginário puro.

b) Real.

Em ambos os casos determinar A.

SOLUÇÃO:

$$A = \frac{1 - ix}{2x - i} = \frac{(1, -x) \cdot (2x, 1)}{(2x, -1) \cdot (2x, 1)} = \frac{(2x + x,\ 1 - 2x^2)}{(4x^2 + 1,\ 0)}$$

$$= \left(\frac{3x}{4x^2+1},\ \frac{1-2x^2}{4x^2+1} \right)$$

a) Para que A seja imaginário puro, devemos ter $\dfrac{3x}{4x^2+1} = 0$, logo $3x = 0 \Rightarrow x = 0$.

Assim, $A = \dfrac{1 - i \cdot 0}{2 \cdot 0 - i} = \dfrac{1}{-i} \cdot \dfrac{i}{i} = \dfrac{i}{-(-1)} = i$

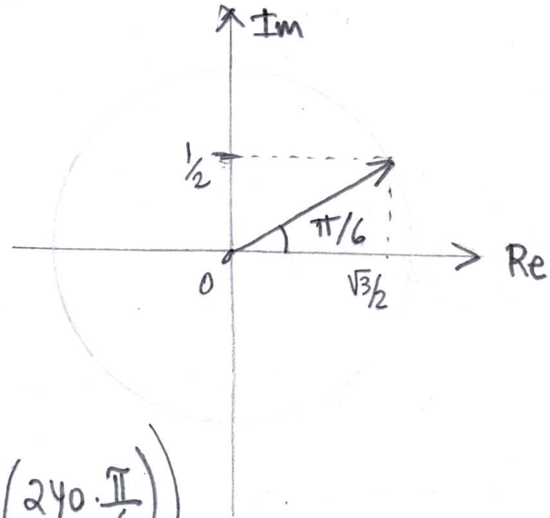
b) Para que A seja um número real, devemos ter $\dfrac{1-2x^2}{4x^2+1} = 0 \Rightarrow 1 - 2x^2 = 0 \Rightarrow x = \pm \dfrac{1}{\sqrt{2}} = \pm \dfrac{\sqrt{2}}{2}$

Logo, $A = \dfrac{2 - \sqrt{2}\, i}{2\sqrt{2} - 2i}$ se $x = \dfrac{\sqrt{2}}{2}$ ou $A = -\dfrac{2 + \sqrt{2}\, i}{2\sqrt{2} + 2i}$ se $x = -\dfrac{\sqrt{2}}{2}$

Q4. (2,0) Encontrar as representações cartesiana, algébrica e trigonométrica, das potências abaixo:

a) $\left(\dfrac{\sqrt{3}}{2}+\dfrac{1}{2}i\right)^{240}$.

b) $\left(\dfrac{\sqrt{2}}{1+i}\right)^{90}$



SOLUÇÃO :

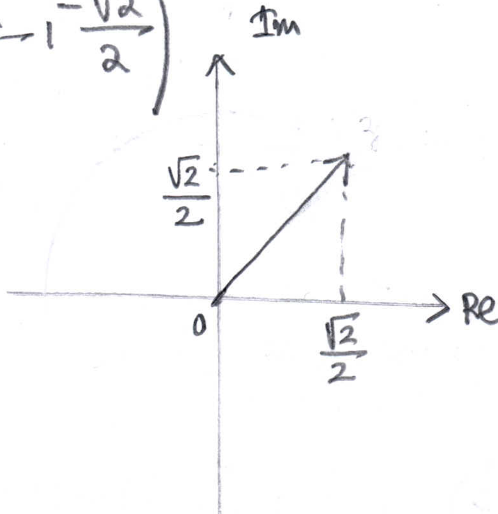
a) $\left(\dfrac{\sqrt{3}}{2}+\dfrac{1}{2}i\right)^{240} = 1^{240}\left(\cos\left(240\cdot\dfrac{\pi}{6}\right), \operatorname{sen}\left(240\cdot\dfrac{\pi}{6}\right)\right)$

$= (\cos(40\pi), \operatorname{sen}(40\pi))$ (rep. trigonométrica)

$= (1, 0) = 1$ ← rep. algébrica

↑ rep. cartesiana

b) $\dfrac{\sqrt{2}}{1+i} = \dfrac{(\sqrt{2},0)\cdot(1,-1)}{(1,1)\cdot(1,-1)} = \dfrac{(\sqrt{2},-\sqrt{2})}{(2,0)} = \left(\dfrac{\sqrt{2}}{2}, -\dfrac{\sqrt{2}}{2}\right)$



Logo, $\left(\dfrac{\sqrt{2}}{1+i}\right)^{90} = \left(\dfrac{\sqrt{2}}{2}, -\dfrac{\sqrt{2}}{2}\right)^{90}$

$= \left(\cos\dfrac{90\pi}{4}, \operatorname{sen}\dfrac{90\pi}{4}\right)$

$= \left(\cos\dfrac{\pi}{2}, \operatorname{sen}\dfrac{\pi}{2}\right)$ ← rep. trigonométrica.

$= (0, 1) = i$ ← rep. algébrica.

↑ rep. cartesiana

Q5. (1,5) Pede-se:

a) Encontre os números complexos de módulo $\sqrt{2}$ que se encontram sobre a reta $y = 2x - 1$.

b) Chamando de z e w os complexos do ítem a), calcule zw e $\overline{z}w$.

SOLUÇÃO:

Seja $z = (x, y) \in \mathbb{C}$. Se z se encontra sobre a reta $y = 2x - 1$, então $z = (x, y) = (x, 2x-1)$. Como $|z| = \sqrt{2}$, temos:

$$\sqrt{x^2 + (2x-1)^2} = \sqrt{2} \Rightarrow$$

$$x^2 + (2x-1)^2 = 2 \Rightarrow$$

$$x^2 + 4x^2 - 4x + 1 = 2 \Rightarrow$$

$$5x^2 - 4x - 1 = 0$$

$$x = \frac{4 \pm \sqrt{36}}{10} = \frac{4 \pm 6}{10} \begin{cases} x_1 = 1 \\ x_2 = \frac{-2}{10} = -\frac{1}{5} \end{cases}$$

Logo $z = (1, 1)$ e $w = \left(-\frac{1}{5}, -\frac{7}{5}\right)$.

b) $zw = (1,1) \cdot \left(-\frac{1}{5}, -\frac{7}{5}\right) = \left(\frac{6}{5}, -\frac{8}{5}\right)$

$$\overline{z} \cdot w = (1, -1)\left(-\frac{1}{5}, -\frac{7}{5}\right) = \left(-\frac{8}{5}, -\frac{6}{5}\right)$$

♠♡♢♣∞ **Questão Extra**(1,0)

Sejam $z_1, z_2 \in \mathbb{C}$ com $z_1 \neq 0$. Nosso objetivo é mostrar que se $\dfrac{5z_2}{7z_1}$ é um imaginário puro, então $\left|\dfrac{2z_1+3z_2}{2z_1-3z_2}\right| = 1$. Para isso:

a) Multiplique $\dfrac{5z_2}{7z_1}$ por $\dfrac{\overline{z_1}}{\overline{z_1}}$ e conclua, usando a hipótese, que $x_1x_2 + y_1y_2 = 0$.

b) Usando o ítem a), conclua que $z_1\overline{z_2} + \overline{z_1}z_2 = 0$.

c) Verifique, usando o ítem b), que $\left|\dfrac{2z_1+3z_2}{2z_1-3z_2}\right|^2 = 1$. (Dica: lembre que $|w|^2 = w\overline{w}$).

d) O que você conclui do ítem c)?

SOLUÇÃO:

Chame $z_2 = (x_2, y_2)$ e $z_1 = (x_1, y_1)$. Logo:

$$\frac{5z_2}{7z_1}\cdot\frac{\overline{z_1}}{\overline{z_1}} = \frac{5}{7}\,\frac{(x_2,y_2)\cdot(x_1,-y_1)}{|z_1|^2} = \frac{5}{7}\left(\frac{x_2x_1+y_2y_1}{x_1^2+y_1^2}, \frac{-x_2y_1+y_2x}{x_1^2+y_1^2}\right.$$

Como $\dfrac{5z_2}{7z_1}$ é imaginário puro, temos $\dfrac{x_2x_1+y_2y_1}{x_1^2+y_1^2} = 0$,

logo $x_2x_1 + y_2y_1 = 0$.

b) $z_1\overline{z_2} + \overline{z_1}z_2 = (x_1,y_1)(x_2,-y_2) + (x_1,-y_1)(x_2,y_2)$

$= (x_1x_2+y_1y_2, -\cancel{x_1y_2}+\cancel{y_1x_2}) + (x_1x_2+y_1y_2, \cancel{x_1y_2}-\cancel{x_2y_1})$

$= (2\underbrace{(x_1x_2+y_1y_2)}_{=0\ (\text{letra a})}, 0) = (0,0) = 0.$

c) $$\left|\frac{2z_1+3z_2}{2z_1-3z_2}\right|^2 = \frac{|2z_1+3z_2|^2}{|2z_1-3z_2|^2} = \frac{(2z_1+3z_2)\overline{(2z_1+3z_2)}}{(2z_1-3z_2)\overline{(2z_1-3z_2)}}$$

$$= \frac{(2z_1+3z_2)(2\overline{z_1}+3\overline{z_2})}{(2z_1-3z_2)(2\overline{z_1}-3\overline{z_2})} = \frac{4z_1\overline{z_1}+6z_1\overline{z_2}+6z_2\overline{z_1}+9z_2\overline{z_2}}{4z_1\overline{z_1}-6z_1\overline{z_2}-6z_2\overline{z_1}+9z_2\overline{z_2}}$$

$$= \frac{4|z_1|^2+6\overbrace{(z_1\overline{z_2}+z_2\overline{z_1})}^{=0 \text{ (letra b)}}+9|z_2|^2}{4|z_1|^2-6\underbrace{(z_1\overline{z_2}+z_2\overline{z_1})}_{=0 \text{ (letra b)}}+9|z_2|^2} = \frac{4|z_1|^2+9|z_2|^2}{4|z_1|^2+9|z_2|^2} = 1$$

d) Como $\left|\frac{2z_1+3z_2}{2z_1-3z_2}\right|^2 = 1$, então

$\left|\frac{2z_1+3z_2}{2z_1-3z_2}\right| = 1$. E o resultado do exercício está verificado!