NOTES ON MECHANICS AND RIGID BODIES

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1. Quick review of affine spaces

By an *affine space* (over some field of scalars $\mathbb{K}$) we mean a nonempty set $\mathcal{P}$ endowed with an action

$$\mathcal{V} \times \mathcal{P} \ni (v, p) \longmapsto v + p \in \mathcal{P}$$

of the additive group of a vector space $\mathcal{V}$ (over the field of scalars $\mathbb{K}$) which is transitive and has no fixed points. For $p \in \mathcal{P}$ and $v \in \mathcal{V}$, we take $p + v$ to mean the same as $v + p$. Given $p, q \in \mathcal{P}$, we denote by $p - q$ the unique $v \in \mathcal{V}$ such that $p = q + v$. The *dimension* of $\mathcal{P}$ is, by definition, the dimension of $\mathcal{V}$. We call $\mathcal{V}$ the vector space *parallel* to $\mathcal{P}$. For each $v \in \mathcal{V}$, the map $\mathcal{P} \ni p \mapsto p + v \in \mathcal{P}$ is called a *translation* of $\mathcal{P}$.

Given affine spaces $\mathcal{P}_1$ and $\mathcal{P}_2$ parallel, respectively, to vector spaces $\mathcal{V}_1$ and $\mathcal{V}_2$, then a map $\Omega : \mathcal{P}_1 \to \mathcal{P}_2$ is called *affine* if there exists a (automatically unique) linear map $\Omega_0 : \mathcal{V}_1 \to \mathcal{V}_2$ such that

$$\Omega(p + v) = \Omega(p) + \Omega_0(v),$$

for all $p \in \mathcal{P}_1$, $v \in \mathcal{V}_1$. We call Ω_0 the *underlying linear map* of Ω. An *affine isomorphism* is a bijective affine map. Its inverse is automatically affine.

Every vector space $\mathcal{V}$ can be regarded as an affine space parallel to itself by considering the action $\mathcal{V} \times \mathcal{V} \to \mathcal{V}$ given by the addition of $\mathcal{V}$. An affine map between vector spaces (regarded as affine spaces) is the same as the composition of a linear map with a translation. For an arbitrary affine space $\mathcal{P}$, parallel to a vector space $\mathcal{V}$, we have for each point $O \in \mathcal{P}$ an affine isomorphism $\mathcal{P} \to \mathcal{V}$ given by

$$\mathcal{P} \ni p \longmapsto p - O \in \mathcal{V}. \tag{1.1}$$

This is the unique affine map from $\mathcal{P}$ to $\mathcal{V}$ that sends O to the origin of $\mathcal{V}$ and whose underlying linear map is the identity of $\mathcal{V}$.

For the following subsections, we consider a fixed affine space $\mathcal{P}$ parallel to a vector space $\mathcal{V}$.

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1.1. **Subspaces and quotients.** Given a vector subspace $\mathcal{W}$ of $\mathcal{V}$, we consider the action of $\mathcal{W}$ on $\mathcal{P}$ given by the restriction of the action of $\mathcal{V}$. The orbits

$$p+\mathcal{W}=\big\{p+w : w\in\mathcal{W}\big\}$$

of this action are called the *affine subspaces* of $\mathcal{P}$ parallel to $\mathcal{W}$. Obviously, $p+\mathcal{W}$ is an affine space parallel to $\mathcal{W}$ and the inclusion map of $p+\mathcal{W}$ in $\mathcal{P}$ is an affine map whose underlying linear map is the inclusion map of $\mathcal{W}$ in $\mathcal{V}$. The orbit space

$$\mathcal{P}/\mathcal{W}=\big\{p+\mathcal{W} : p\in\mathcal{P}\big\}$$

has a natural structure of an affine space parallel to the quotient $\mathcal{V}/\mathcal{W}$, with action given by

$$\mathcal{V}/\mathcal{W}\times\mathcal{P}/\mathcal{W}\ni(v+\mathcal{W})+(p+\mathcal{W})\longmapsto(v+p)+\mathcal{W}\in\mathcal{P}/\mathcal{W}.$$

The quotient map $\mathcal{P}\to\mathcal{P}/\mathcal{W}$ is an affine map whose underlying linear map is the quotient map $\mathcal{V}\to\mathcal{V}/\mathcal{W}$.

Given an arbitrary nonempty subset S of $\mathcal{P}$, then intersection of all affine subspaces of $\mathcal{P}$ containing S is an affine subspace of $\mathcal{P}$ and it is obviously the smallest affine subspace of $\mathcal{P}$ containing S. We call it the affine subspace *spanned* by S.

1.2. **Linear combinations and geometric dependence.** Let $(p_\lambda)_{\lambda\in\Lambda}$ be a family of points of $\mathcal{P}$ and let $(a_\lambda)_{\lambda\in\Lambda}$ be a family of scalars. Assume that $(a_\lambda)_{\lambda\in\Lambda}$ is *almost null*, i.e., the set $\big\{\lambda\in\Lambda : a_\lambda\ne0\big\}$ is finite. If $\sum_{\lambda\in\Lambda}a_\lambda=1$, we define the linear combination $\sum_{\lambda\in\Lambda}a_\lambda p_\lambda$ to be the point of $\mathcal{P}$ given by

$$\sum_{\lambda\in\Lambda}a_\lambda p_\lambda=O+\sum_{\lambda\in\Lambda}a_\lambda(p_\lambda-O)\in\mathcal{P},\tag{1.2}$$

where $O\in\mathcal{P}$ is arbitrarily chosen. The righthand side of (1.2) is easily seen to be independent of the choice of $O\in\mathcal{P}$. If $\sum_{\lambda\in\Lambda}a_\lambda=0$, we define the linear combination $\sum_{\lambda\in\Lambda}a_\lambda p_\lambda$ to be the element of the vector space $\mathcal{V}$ given by

$$\sum_{\lambda\in\Lambda}a_\lambda p_\lambda=\sum_{\lambda\in\Lambda}a_\lambda(p_\lambda-O)\in\mathcal{V},\tag{1.3}$$

where $O\in\mathcal{P}$ is arbitrarily chosen. Again, the righthand side of (1.3) is independent of the choice of $O\in\mathcal{P}$. If Λ is nonempty, the set

$$\left\{\sum_{\lambda\in\Lambda}a_\lambda p_\lambda : (a_\lambda)_{\lambda\in\Lambda}\text{ almost null family of scalars with }\sum_{\lambda\in\Lambda}a_\lambda=1\right\}\tag{1.4}$$

is an affine subspace of $\mathcal{P}$ parallel to the vector subspace of $\mathcal{V}$ given by:

$$\left\{\sum_{\lambda\in\Lambda}a_\lambda p_\lambda : (a_\lambda)_{\lambda\in\Lambda}\text{ almost null family of scalars with }\sum_{\lambda\in\Lambda}a_\lambda=0\right\}.\tag{1.5}$$

Moreover, (1.4) is precisely the affine subspace spanned by $\{p_\lambda : \lambda \in \Lambda\}$. Note that (1.5) coincides with the vector subspace of $\mathcal{V}$ spanned by

$$\{p_\lambda - p_\mu : \lambda, \mu \in \Lambda\}.$$

The family $(p_\lambda)_{\lambda\in\Lambda}$ is said to be *geometrically independent* if for every almost null family of scalars $(a_\lambda)_{\lambda\in\Lambda}$ with $\sum_{\lambda\in\Lambda} a_\lambda = 0$, if $\sum_{\lambda\in\Lambda} a_\lambda p_\lambda = 0$, then $a_\lambda = 0$, for all $\lambda \in \Lambda$. If the family $(p_\lambda)_{\lambda\in\Lambda}$ is geometrically independent, then every point of the affine subspace spanned by $\{p_\lambda : \lambda \in \Lambda\}$ can be written in a unique way as a linear combination $\sum_{\lambda\in\Lambda} a_\lambda p_\lambda$, with $(a_\lambda)_{\lambda\in\Lambda}$ an almost null family of scalars with $\sum_{\lambda\in\Lambda} a_\lambda = 1$. If the family $(p_\lambda)_{\lambda\in\Lambda}$ is not geometrically independent, it is called *geometrically dependent*. Given an arbitrary index $\lambda_0 \in \Lambda$, we have that the family $(p_\lambda)_{\lambda\in\Lambda}$ is geometrically independent if and only if the family $(p_\lambda - p_{\lambda_0})_{\lambda\in\Lambda\setminus\{\lambda_0\}}$ is linearly independent.

1.3. **Products and spaces of maps.** If $(\mathcal{P}_\lambda)_{\lambda\in\Lambda}$ is a family of affine spaces, with $\mathcal{V}_\lambda$ the parallel vector space to $\mathcal{P}_\lambda$, then the product $\prod_{\lambda\in\Lambda} \mathcal{P}_\lambda$ has a natural structure of affine space parallel to the product vector space $\prod_{\lambda\in\Lambda} \mathcal{V}_\lambda$; the action is given by:

$$\prod_{\lambda\in\Lambda} \mathcal{V}_\lambda \times \prod_{\lambda\in\Lambda} \mathcal{P}_\lambda \ni \big((v_\lambda)_{\lambda\in\Lambda}, (p_\lambda)_{\lambda\in\Lambda}\big) \longmapsto (v_\lambda + p_\lambda)_{\lambda\in\Lambda} \in \prod_{\lambda\in\Lambda} \mathcal{P}_\lambda.$$

In particular, for an arbitrary set Λ, the set $\mathcal{P}^\Lambda$ of all maps form Λ to $\mathcal{P}$ has a natural structure of affine space parallel to $\mathcal{V}^\Lambda$. If $\mathcal{P}_1$ and $\mathcal{P}_2$ are affine spaces parallel, respectively, to vector spaces $\mathcal{V}_1$ and $\mathcal{V}_2$, then the set $\mathrm{Aff}(\mathcal{P}_1, \mathcal{P}_2)$ of all affine maps from $\mathcal{P}_1$ to $\mathcal{P}_2$ is an affine subspace of $\mathcal{P}_2^{\mathcal{P}_1}$ and its parallel vector space is $\mathrm{Aff}(\mathcal{P}_1, \mathcal{V}_2)$.

1.4. **Manifold structure.** If $\mathcal{P}$ is real and of finite dimension n, then $\mathcal{P}$ has a natural structure of a differentiable manifold of dimension n given by the maximal atlas containing the affine isomorphisms from $\mathcal{P}$ to $\mathbb{R}^n$. For each $p \in \mathcal{P}$, we identify the tangent space $T_p\mathcal{P}$ with the parallel vector space $\mathcal{V}$ using the differential at p of the map (1.1). This identification is independent of the choice of the point $O \in \mathcal{P}$.

2. Spacetime and units

For simplicity, we will consider units of mass, length and times to be fixed, so that masses, lengths and intervals of time will be identified with real numbers[1]. Let n be a fixed nonnegative integer. By a *Galilean spacetime with n space dimensions* we mean a real affine space $\mathcal{E}$ with $n+1$ dimensions parallel to a vector space $\mathcal{E}_0$ endowed with:

- a nonzero linear functional $\mathfrak{t} : \mathcal{E}_0 \to \mathbb{R}$ (the *time functional*);

[1]Otherwise, the appropriate mathematical formalism would be to consider separate abstract one-dimensional vector spaces for masses, lengths and times, which would make the presentation a bit cumbersome. We observe that our simplified presentation also entails a choice of time-orientation.

- a (positive definite) inner product $\langle \cdot, \cdot \rangle$ on the kernel of $\mathfrak{t}$.

Points of $\mathcal{E}$ are *events* and, for any pair of events $e_1, e_2 \in \mathcal{E}$, the number $\mathfrak{t}(e_2 - e_1)$ is the *elapsed time* between e_1 and e_2. We write

$$\mathcal{V} = \mathrm{Ker}(\mathfrak{t});$$

this is the vector space consisting of ordinary physical vectors connecting simultaneous events. Since $\mathcal{V}$ is endowed with an inner product, we obtain a notion of distance between simultaneous events. For physics, of course, the interesting case is $n = 3$, but we will be working with arbitrary n and make occasional remarks about the case $n = 3$.

The quotient $\mathcal{T} = \mathcal{E}/\mathcal{V}$ is a one-dimensional affine space parallel to $\mathcal{E}_0/\mathcal{V}$ and we call it the space of *instants of time*. We identify $\mathcal{E}_0/\mathcal{V}$ with $\mathbb{R}$ through $\mathfrak{t}$, so that $\mathcal{T}$ becomes an affine space parallel to $\mathbb{R}$. The quotient map $\mathcal{E} \to \mathcal{T}$ is a fibration and the smooth sections $q : \mathcal{T} \to \mathcal{E}$ of this fibration are *particle trajectories*. Since the tangent space $T_t\mathcal{T}$ of $\mathcal{T}$ at each point $t \in \mathcal{T}$ is identified with $\mathbb{R}$, we can consider the vector field $\frac{\mathrm{d}}{\mathrm{d}t}$ on $\mathcal{T}$ that is constant and equal to $1 \in \mathbb{R}$. We can use this vector field to define *time derivatives* of particle trajectories, obtaining a *velocity* $\frac{\mathrm{d}q}{\mathrm{d}t}(t)$ and an *acceleration* $\frac{\mathrm{d}^2q}{\mathrm{d}t^2}(t)$, for each instant $t \in \mathcal{T}$. Note that (absolute!) velocities are elements of the affine space $\mathfrak{t}^{-1}(1)$ (which is parallel to $\mathcal{V}$) and accelerations are elements of the vector space $\mathcal{V}$, while differences of velocities (relative velocities) are elements of $\mathcal{V}$.

Physical space has no (absolute) meaning in a Galilean spacetime: we have one different physical space for each instant of time (the various physical spaces are the fibers of $\mathcal{E} \to \mathcal{T}$). In order to obtain one single physical space, we make use of an *inertial frame*, i.e., a one-dimensional vector subspace $\mathcal{Z}$ of $\mathcal{E}_0$ not contained in $\mathcal{V}$. The quotient $\mathcal{P} = \mathcal{E}/\mathcal{Z}$ is then an n-dimensional affine space parallel to $\mathcal{E}_0/\mathcal{Z}$ and we call it *physical space for the given inertial frame* $\mathcal{Z}$. Since $\mathcal{E}_0 = \mathcal{V} \oplus \mathcal{Z}$, we can naturally identify the quotient $\mathcal{E}_0/\mathcal{Z}$ with $\mathcal{V}$, so that $\mathcal{P}$ becomes an affine space parallel to $\mathcal{V}$. We have an affine isomorphism

$$\mathcal{E} \ni e \longmapsto \big(e + \mathcal{V}, e + \mathcal{Z}\big) \in \mathcal{T} \times \mathcal{P}$$

which allows us to identify particle trajectories with smooth maps from $\mathcal{T}$ to $\mathcal{P}$. Velocities and accelerations are then both identified with elements of $\mathcal{V}$. More generally, one could use a *noninertial frame* to obtain a single physical space: that is, one considers an affine space $\mathcal{P}$ parallel to $\mathcal{V}$ and an isomorphism between $\mathcal{E} \to \mathcal{T}$ and $\mathcal{T} \times \mathcal{P} \to \mathcal{T}$ regarded as fiber bundles whose fibers are metric affine spaces. More explicitly, one considers a smooth map

$$\varphi : \mathcal{E} \longrightarrow \mathcal{P}$$

whose restriction to every fiber of $\mathcal{E} \to \mathcal{T}$ is an affine bijection whose underlying linear map $\mathcal{V} \to \mathcal{V}$ is a (linear) isometry. We then obtain a diffeomorphism

$$\mathcal{E} \ni e \longmapsto \big(e + \mathcal{V}, \varphi(e)\big) \in \mathcal{T} \times \mathcal{P}$$

which again allows us to identify particle trajectories with smooth maps $\mathcal{T} \to \mathcal{P}$. When dealing with a concrete physics problem using noninertial frames one has to add *fictional forces* to the dynamics, i.e., forces that appear because of the choice of frame and are not due to some real physical interaction.

3. Basic definitions

Throughout the rest of the text we consider a fixed n-dimensional real affine space $\mathcal{P}$ with corresponding vector space $\mathcal{V}$, where n is a nonnegative integer. The vector space $\mathcal{V}$ is endowed with an inner product $\langle \cdot, \cdot \rangle$. We think of $\mathcal{P}$ as physical space (see Section 2). We also consider a fixed finite set Λ which we think of as the *set of all particles*.

Definition 3.1. We consider the following objects:

- a *particle configuration*, i.e., a family $(q_\lambda)_{\lambda\in\Lambda}$ of elements of $\mathcal{P}$;
- a *family of velocities*, i.e., a family $(\dot q_\lambda)_{\lambda\in\Lambda}$ of elements of $\mathcal{V}$;
- a *family of forces*, i.e., a family $(F_\lambda)_{\lambda\in\Lambda}$ of elements of $\mathcal{V}$;
- a *family of masses*, i.e., a family $(m_\lambda)_{\lambda\in\Lambda}$ os positive real numbers.

With these objects we make the following definitions.

- the *total mass* is the positive real number given by:
$$M = \sum_{\lambda\in\Lambda} m_\lambda > 0;$$
- the *center of mass* is the point given by:
$$C = \sum_{\lambda\in\Lambda} \frac{m_\lambda}{M}\, q_\lambda \in \mathcal{P};$$
- the *velocity of the center of mass* is the vector given by:
$$\dot C = \sum_{\lambda\in\Lambda} \frac{m_\lambda}{M}\, \dot q_\lambda \in \mathcal{V};$$
- the *total (linear) momentum* with respect to a *reference velocity* $\dot O \in \mathcal{V}$ is the vector given by:
$$p_{\dot O} = \sum_{\lambda\in\Lambda} m_\lambda(\dot q_\lambda - \dot O) = M(\dot C - \dot O) \in \mathcal{V};$$
- the *total kinetic energy* with respect to a reference velocity $\dot O \in \mathcal{V}$ is the nonnegative real number given by:
$$T_{\dot O} = \frac{1}{2}\sum_{\lambda\in\Lambda} m_\lambda \|\dot q_\lambda - \dot O\|^2 \geq 0;$$
- the *total power* with respect to a reference velocity $\dot O \in \mathcal{V}$ is the real number given by:
$$P_{\dot O} = \sum_{\lambda\in\Lambda} \langle F_\lambda, \dot q_\lambda - \dot O\rangle \in \mathbb{R};$$

- the *total angular momentum* with respect to a *reference point* $O \in \mathcal{P}$ and a reference velocity $\dot{O} \in \mathcal{V}$ is the element of the exterior product $\mathcal{V} \wedge \mathcal{V}$ given by:
$$L_{O,\dot{O}} = \sum_{\lambda\in\Lambda} m_\lambda (q_\lambda - O) \wedge (\dot{q}_\lambda - \dot{O}) \in \mathcal{V} \wedge \mathcal{V};$$
- the *total force* is the vector given by:
$$F = \sum_{\lambda\in\Lambda} F_\lambda \in \mathcal{V};$$
- the *total torque* with respect to a reference point $O \in \mathcal{P}$ is the element of the exterior product $\mathcal{V} \wedge \mathcal{V}$ given by:
$$\tau_O = \sum_{\lambda\in\Lambda} (q_\lambda - O) \wedge F_\lambda \in \mathcal{V} \wedge \mathcal{V}.$$

Remark 3.2. If $n = 3$ and $\mathcal{V}$ is oriented, we can identify the exterior product $\mathcal{V} \wedge \mathcal{V}$ with $\mathcal{V}$ itself by identifying the exterior product $v \wedge w$ of two vectors $v, w \in \mathcal{V}$ with their vector product. With this identification, angular momentum and torque can be regarded as elements of $\mathcal{V}$.

We state some readily checkable useful formulas that relate quantities defined with respect to different points of reference and velocities of reference.

Theorem 3.3. *Consider objects like in Definition 3.1. Let $O_1, O_2 \in \mathcal{P}$ be points of reference and $\dot{O}_1, \dot{O}_2 \in \mathcal{V}$ be velocities of reference. The following formulas hold:*
$$p_{\dot{O}_2} = p_{\dot{O}_1} + M(\dot{O}_1 - \dot{O}_2);$$
$$T_{\dot{O}_2} = T_{\dot{O}_1} + M\langle \dot{C} - \tfrac{1}{2}(\dot{O}_1 + \dot{O}_2), \dot{O}_1 - \dot{O}_2\rangle;$$
$$P_{\dot{O}_2} = P_{\dot{O}_1} + \langle F, \dot{O}_1 - \dot{O}_2\rangle;$$
$$\begin{aligned} L_{O_2,\dot{O}_2} = L_{O_1,\dot{O}_1} &+ M(O_1 - O_2) \wedge (\dot{C} - \dot{O}_1) + M(C - O_1) \wedge (\dot{O}_1 - \dot{O}_2) \\ &+ M(O_1 - O_2) \wedge (\dot{O}_1 - \dot{O}_2); \end{aligned}$$
$$\tau_{O_2} = \tau_{O_1} + (O_1 - O_2) \wedge F.$$

Proof. Straightforward computation. □

Corollary 3.4. *The total torque is independent of reference point if the total force is zero.* □

Corollary 3.5. *The total angular momentum is independent of reference velocity if the reference point is the center of mass, i.e.:*
$$L_{C,\dot{O}_1} = L_{C,\dot{O}_2},$$
for all $\dot{O}_1, \dot{O}_2 \in \mathcal{V}$. □

Corollary 3.6. *For an arbitrary reference velocity $\dot{O} \in \mathcal{V}$, we have:*
$$T_{\dot{O}} = T_{\dot{C}} + \frac{1}{2} M \|\dot{C} - \dot{O}\|^2.$$ □

4. Inner forces

Definition 4.1. Given a particle configuration $(q_\lambda)_{\lambda\in\Lambda}$, we call the subspace of $\mathcal{V}$ spanned by
$$\big\{q_\lambda - q_\mu : \lambda, \mu \in \Lambda\big\}$$
the *parallel subspace* of $(q_\lambda)_{\lambda\in\Lambda}$.

Note that, if Λ is nonempty, this is precisely the parallel subspace to the affine subspace spanned by $\big\{q_\lambda : \lambda \in \Lambda\big\}$.

Definition 4.2. A *system of internal forces* for the particle configuration $(q_\lambda)_{\lambda\in\Lambda}$ is a family $(F_{\lambda\mu})_{\lambda,\mu\in\Lambda}$ of elements of $\mathcal{V}$ such that $F_{\lambda\mu}$ is a scalar multiple of $q_\lambda - q_\mu$, for all $\lambda, \mu \in \Lambda$, and such that the *action and reaction law*
$$F_{\lambda\mu} = -F_{\mu\lambda}$$
is satisfied, for all $\lambda, \mu \in \Lambda$. We think of $F_{\lambda\mu}$ as the force exerted on the particle λ by the particle μ. The family of forces $(F_\lambda)_{\lambda\in\Lambda}$ defined by
$$F_\lambda = \sum_{\mu\in\Lambda} F_{\lambda\mu}, \quad \lambda \in \Lambda$$
is said to be *induced* by the system of internal forces $(F_{\lambda\mu})_{\lambda,\mu\in\Lambda}$.

The main goal of this section is to prove the following result.

Proposition 4.3. *Let $(q_\lambda)_{\lambda\in\Lambda}$ be a particle configuration and $(F_\lambda)_{\lambda\in\Lambda}$ be a family of forces. Denote by $\mathcal{W}$ the parallel subspace of $(q_\lambda)_{\lambda\in\Lambda}$. The following conditions are equivalent:*

(a) *$(F_\lambda)_{\lambda\in\Lambda}$ is induced by some system of internal forces;*
(b) *$F_\lambda \in \mathcal{W}$, for all $\lambda \in \Lambda$, and both the total force and the total torque of $(F_\lambda)_{\lambda\in\Lambda}$ are zero.*

We will need some preliminary results. In what follows, we consider the vector space $\mathcal{V}^\Lambda$ endowed with the inner product:
$$\langle (v_\lambda)_{\lambda\in\Lambda}, (w_\lambda)_{\lambda\in\Lambda}\rangle = \sum_{\lambda\in\Lambda} \langle v_\lambda, w_\lambda\rangle, \quad (v_\lambda)_{\lambda\in\Lambda}, (w_\lambda)_{\lambda\in\Lambda} \in \mathcal{V}^\Lambda. \tag{4.1}$$

Lemma 4.4. *Let $(q_\lambda)_{\lambda\in\Lambda}$ be a particle configuration. The set of families of forces $(F_\lambda)_{\lambda\in\Lambda}$ whose corresponding total force and total torque are zero is a subspace of $\mathcal{V}^\Lambda$. The orthogonal complement of such subspace is the space of all families of the form*
$$\big(\Omega(q_\lambda)\big)_{\lambda\in\Lambda},$$
with Ω varying over the set of affine maps $\Omega : \mathcal{P} \to \mathcal{V}$ whose underlying linear map $\Omega_0 : \mathcal{V} \to \mathcal{V}$ is anti-symmetric.

Proof. For a given reference point $O \in \mathcal{P}$, the map
$$\mathcal{V}^\Lambda \ni (F_\lambda)_{\lambda\in\Lambda} \longmapsto \Big(\sum_{\lambda\in\Lambda} F_\lambda, \sum_{\lambda\in\Lambda} (q_\lambda - O) \wedge F_\lambda\Big) \in \mathcal{V} \times (\mathcal{V} \wedge \mathcal{V}) \tag{4.2}$$

is linear and its kernel is precisely the set of families of forces whose corresponding total force and total torque are zero. The annihilator of the kernel of (4.2) is equal to the image of the adjoint of (4.2). We will make the following identifications: the dual space of a finite product of vector spaces will be identified with the product of the dual spaces, the dual space $\mathcal{V}^*$ of $\mathcal{V}$ will be identified with $\mathcal{V}$ itself using the inner product, the dual space of $\mathcal{V} \wedge \mathcal{V}$ will be identified in the canonical way with the space of anti-symmetric bilinear forms in $\mathcal{V}$ and an anti-symmetric linear operator $\Omega_0 : \mathcal{V} \to \mathcal{V}$ will be identified with the anti-symmetric bilinear form $\langle \Omega_0 \cdot, \cdot \rangle$. Using such identifications, the adjoint of (4.2) is easily computed and given by

$$\mathcal{V} \times \mathrm{Lin_a}(\mathcal{V}) \ni (v, \Omega_0) \longmapsto \big(\Omega_0(q_\lambda - O) + v\big)_{\lambda \in \Lambda} \in \mathcal{V}^\Lambda, \tag{4.3}$$

where $\mathrm{Lin_a}(\mathcal{V})$ denotes the space of anti-symmetric linear endomorphisms of $\mathcal{V}$. Since the identification of the dual space of $\mathcal{V}^\Lambda$ with $\mathcal{V}^\Lambda$ coincides with the identification given by the inner product of $\mathcal{V}^\Lambda$, we have that the image of (4.3) is equal to the orthogonal complement of the kernel of (4.2). The conclusion follows. □

Lemma 4.5. *Let $(q_\lambda)_{\lambda \in \Lambda}$ be a particle configuration. The set of families of forces $(F_\lambda)_{\lambda \in \Lambda}$ that are induced by some system of internal forces is a subspace of $\mathcal{V}^\Lambda$. Its orthogonal complement is given by:*

$$\big\{(F_\lambda)_{\lambda \in \Lambda} \in \mathcal{V}^\Lambda : \langle F_\lambda - F_\mu, q_\lambda - q_\mu \rangle = 0, \text{ for all } \lambda, \mu \in \Lambda\big\}.$$

Proof. The systems of internal forces for the particle configuration $(q_\lambda)_{\lambda \in \Lambda}$ are of the form

$$\big(a_{\lambda\mu}(q_\lambda - q_\mu)\big)_{\lambda, \mu \in \Lambda},$$

with $(a_{\lambda\mu})_{\lambda,\mu \in \Lambda}$ a real symmetric matrix index by $\Lambda \times \Lambda$. It follows that the set of families of forces induced by some system of internal forces is the image of the linear map

$$\phi : M^{\mathrm{s}}_\Lambda(\mathbb{R}) \ni (a_{\lambda\mu})_{\lambda,\mu \in \Lambda} \longmapsto \Big(\sum_{\mu \in \Lambda} a_{\lambda\mu}(q_\lambda - q_\mu)\Big)_{\lambda \in \Lambda} \in \mathcal{V}^\Lambda,$$

where we have denoted by $M^{\mathrm{s}}_\Lambda(\mathbb{R})$ the space of real symmetric matrices indexed by $\Lambda \times \Lambda$. For $\rho, \theta \in \Lambda$, denote by $A_{\rho\theta} \in M^{\mathrm{s}}_\Lambda(\mathbb{R})$ the symmetric matrix given by:

$$(A_{\rho\theta})_{\lambda\mu} = \begin{cases} 1, & \text{if } (\lambda, \mu) = (\rho, \theta) \text{ or } (\lambda, \mu) = (\theta, \rho), \\ 0, & \text{otherwise.} \end{cases}$$

Since $\phi(A_{\rho\rho}) = 0$, for all $\rho \in \Lambda$, it follows that the image of ϕ is spanned by

$$\big\{\phi(A_{\rho\theta}) : \rho, \theta \in \Lambda,\ \rho \neq \theta\big\}.$$

Moreover, for $\rho, \theta \in \Lambda$, $\rho \neq \theta$, and $(F_\lambda)_{\lambda \in \Lambda} \in \mathcal{V}^\Lambda$, we have:

$$\langle (F_\lambda)_{\lambda \in \Lambda}, \phi(A_{\rho\theta}) \rangle = \langle F_\rho - F_\theta, q_\rho - q_\theta \rangle.$$

The conclusion follows. □

Lemma 4.6. *Let $(q_\lambda)_{\lambda\in\Lambda}$ be a particle configuration and denote by $\mathcal{W}$ its parallel subspace. For a family of forces $(F_\lambda)_{\lambda\in\Lambda}$, the following conditions are equivalent:*

(a) $\langle F_\lambda - F_\mu, q_\lambda - q_\mu\rangle = 0$, *for all* $\lambda, \mu \in \Lambda$*;*
(b) *there exists an affine map* $\Omega : \mathcal{P} \to \mathcal{V}$*, with underlying linear map* $\Omega_0 : \mathcal{V} \to \mathcal{V}$ *anti-symmetric, such that* $F_\lambda - \Omega(q_\lambda) \in \mathcal{W}^\perp$*, for all* $\lambda \in \Lambda$*.*

Proof. Assume (b). Given $\lambda, \mu \in \Lambda$, we have that $F_\lambda - F_\mu$ and $\Omega(q_\lambda) - \Omega(q_\mu)$ differ by an element of $\mathcal{W}^\perp$ and that $q_\lambda - q_\mu \in \mathcal{W}$. Therefore

$$\langle F_\lambda - F_\mu, q_\lambda - q_\mu\rangle = \langle \Omega(q_\lambda) - \Omega(q_\mu), q_\lambda - q_\mu\rangle = \langle \Omega_0(q_\lambda - q_\mu), q_\lambda - q_\mu\rangle = 0,$$

proving (a). Now assume (a) and let us prove (b). Obviously we can assume that Λ is nonempty. Fix $\lambda \in \Lambda$ and write

$$e_\theta = q_\theta - q_\lambda,$$

for all $\theta \in \Lambda$. Since

$$q_\theta - q_\rho = e_\theta - e_\rho,$$

for all $\theta, \rho \in \Lambda$, we have that $\mathcal{W}$ is spanned by $\big\{e_\theta : \theta \in \Lambda\big\}$. Let Θ be a subset of Λ such that $(e_\theta)_{\theta\in\Theta}$ is a basis of $\mathcal{W}$. Let $\Omega_0 : \mathcal{W} \to \mathcal{V}$ be the unique linear map such that

$$\Omega_0(e_\theta) = F_\theta - F_\lambda,$$

for all $\theta \in \Theta$. We have:

$$\langle \Omega_0(e_\theta), e_\theta\rangle = \langle F_\theta - F_\lambda, q_\theta - q_\lambda\rangle = 0, \tag{4.4}$$

for all $\theta \in \Theta$. Moreover:

$$\langle \Omega_0(e_\theta) - \Omega_0(e_\rho), e_\theta - e_\rho\rangle = \langle F_\theta - F_\rho, q_\theta - q_\rho\rangle = 0, \tag{4.5}$$

for all $\theta, \rho \in \Theta$. From (4.4) and (4.5) we obtain:

$$\langle \Omega_0(e_\theta), e_\rho\rangle + \langle \Omega_0(e_\rho), e_\theta\rangle = 0,$$

for all $\theta, \rho \in \Theta$, from which it follows that:

$$\langle \Omega_0(w), w'\rangle + \langle \Omega_0(w'), w\rangle = 0, \tag{4.6}$$

for all $w, w' \in \mathcal{W}$. Equality (4.6) implies[2] that Ω_0 admits an anti-symmetric linear extension to $\mathcal{V}$, which we also denote by Ω_0. Let $\Omega : \mathcal{P} \to \mathcal{V}$ be an affine map whose underlying linear map is Ω_0 and such that $\Omega(q_\lambda) = F_\lambda$. We have:

$$\Omega(q_\theta) = \Omega(q_\lambda + e_\theta) = \Omega(q_\lambda) + \Omega_0(e_\theta) = F_\lambda + (F_\theta - F_\lambda) = F_\theta,$$

for all $\theta \in \Theta$. Set

$$F'_\theta = F_\theta - \Omega(q_\theta),$$

[2]One way to prove this is to use matrices. Pick an orthonormal basis of $\mathcal{V}$ whose $k = \dim(\mathcal{W})$ first vectors form a basis of $\mathcal{W}$. The matrix representation of Ω_0 is then an $n \times k$ matrix whose top $k \times k$ block is anti-symmetric. Such matrix can then be completed to an anti-symmetric $n \times n$ matrix.

for all $\theta \in \Lambda$, so that:

$$F'_\theta = 0,$$

for all $\theta \in \Theta$ and also for $\theta = \lambda$. We will conclude the proof by showing that F'_θ is in $\mathcal{W}^\perp$, for all $\theta \in \Lambda$. Clearly

$$\langle F'_\theta - F'_\rho, q_\theta - q_\rho \rangle = \langle F_\theta - F_\rho, q_\theta - q_\rho \rangle - \langle \Omega_0(q_\theta - q_\rho), q_\theta - q_\rho \rangle = 0,$$

for all $\rho, \theta \in \Lambda$. In particular, setting $\rho = \lambda$, we obtain

$$\langle F'_\theta, q_\theta - q_\lambda \rangle = \langle F'_\theta, e_\theta \rangle = 0,$$

for all $\theta \in \Lambda$. Now, for $\theta \in \Lambda$ and $\rho \in \Theta$, we have

$$0 = \langle F'_\theta - F'_\rho, q_\theta - q_\rho \rangle = \langle F'_\theta, e_\theta - e_\rho \rangle = -\langle F'_\theta, e_\rho \rangle,$$

which shows that F'_θ is in $\mathcal{W}^\perp$ and concludes the proof. □

Proof of Proposition 4.3. Denote by $\mathcal{I}$ the subspace of $\mathcal{V}^\Lambda$ consisting of families of forces that are induced by some system of internal forces and by $\mathcal{Z}$ the subspace of $\mathcal{V}^\Lambda$ consisting of families of forces whose total force and total torque are zero. We have to prove that:

$$\mathcal{I} = \mathcal{Z} \cap (\mathcal{W}^\Lambda).$$

Since $\mathcal{V}^\Lambda$ is finite-dimensional, it is sufficient to prove that:

$$\mathcal{I}^\perp = \big(\mathcal{Z} \cap (\mathcal{W}^\Lambda)\big)^\perp.$$

Noticing that

$$\big(\mathcal{Z} \cap (\mathcal{W}^\Lambda)\big)^\perp = \mathcal{Z}^\perp + (\mathcal{W}^\Lambda)^\perp = \mathcal{Z}^\perp + (\mathcal{W}^\perp)^\Lambda,$$

the conclusion is easily obtained from Lemmas 4.4, 4.5 and 4.6. □

5. Particles in motion

By a *particle configuration in motion* we mean a family $(q_\lambda)_{\lambda\in\Lambda}$ of smooth maps $q_\lambda : \mathcal{T} \to \mathcal{P}$, where $\mathcal{T}$ is an affine space parallel to $\mathbb{R}$ (the space of instants of time, see Section 2). Obviously, we can identify a particle configuration in motion with a smooth map $q : \mathcal{T} \to \mathcal{P}^\Lambda$ by setting

$$q(t) = \big(q_\lambda(t)\big)_{\lambda\in\Lambda},$$

for all $t \in \mathcal{T}$. We then obtain, for each $t \in \mathcal{T}$, a particle velocity

$$\dot{q}_\lambda(t) = \frac{\mathrm{d}q_\lambda}{\mathrm{d}t}(t) \in \mathcal{V}$$

and thus a family of velocities $\big(\dot{q}_\lambda(t)\big)_{\lambda\in\Lambda}$. Given a family of masses $(m_\lambda)_{\lambda\in\Lambda}$, we define the *family of resultant forces* $\big(F_\lambda(t)\big)_{\lambda\in\Lambda}$ by

$$F_\lambda(t) = m_\lambda \frac{\mathrm{d}^2 q_\lambda}{\mathrm{d}t^2}(t) \in \mathcal{V}, \tag{5.1}$$

for each $t \in \mathcal{T}$; the family of resultant forces yields then a total resultant force $F(t) \in \mathcal{V}$, for each $t \in \mathcal{T}$. The curves $F_\lambda : \mathcal{T} \to \mathcal{V}$, $\lambda \in \Lambda$, and $F : \mathcal{T} \to \mathcal{V}$ are obviously smooth. Newton's law says that $F_\lambda(t)$ is actually the sum of all the forces acting on the particle λ, in case we are using an inertial frame (appropriate fictitious forces should be included, in the case of a noninertial frame). For each $t \in \mathcal{T}$, the particle configuration $\big(q_\lambda(t)\big)_{\lambda\in\Lambda}$ (with masses $(m_\lambda)_{\lambda\in\Lambda}$) has a center of mass $C(t) \in \mathcal{P}$ and the family of velocities $\big(\dot{q}_\lambda(t)\big)_{\lambda\in\Lambda}$ defines a velocity of center of mass $\dot{C}(t) \in \mathcal{V}$. We thus obtain smooth curves $C : \mathcal{T} \to \mathcal{P}$ and $\dot{C} : \mathcal{T} \to \mathcal{V}$ and the equality

$$\dot{C}(t) = \frac{\mathrm{d}C}{\mathrm{d}t}(t)$$

holds, for all $t \in \mathcal{T}$.

Consider now a *moving point*, i.e., a smooth curve $O : \mathcal{T} \to \mathcal{P}$. For each $t \in \mathcal{T}$ we then have a reference point $O(t) \in \mathcal{P}$ and a reference velocity

$$\dot{O}(t) = \frac{\mathrm{d}O}{\mathrm{d}t}(t),$$

which can be used to define total linear momentum, total kinetic energy, total power, total angular momentum and total torque, which yield smooth curves:

$$\begin{gathered}
\mathcal{T} \ni t \longmapsto p_{\dot{O}}(t) = p_{\dot{O}(t)} \in \mathcal{V}, \quad \mathcal{T} \ni t \longmapsto T_{\dot{O}}(t) = T_{\dot{O}(t)} \in \mathbb{R}, \\
\mathcal{T} \ni t \longmapsto P_{\dot{O}}(t) = P_{\dot{O}(t)} \in \mathbb{R}, \quad \mathcal{T} \ni t \longmapsto L_{O,\dot{O}}(t) = L_{O(t),\dot{O}(t)} \in \mathcal{V} \wedge \mathcal{V}, \\
\mathcal{T} \ni t \longmapsto \tau_O(t) = \tau_{O(t)} \in \mathcal{V} \wedge \mathcal{V}.
\end{gathered}$$

Theorem 5.1. *Given a particle configuration in motion $(q_\lambda)_{\lambda\in\Lambda}$, a family of masses $(m_\lambda)_{\lambda\in\Lambda}$, a moving point O and defining all the corresponding objects as above, then the following formulas hold:*

$$\begin{gathered}
M\frac{\mathrm{d}^2C}{\mathrm{d}t^2}(t) = F(t), \\
\frac{\mathrm{d}p_{\dot{O}}}{\mathrm{d}t}(t) = F(t) - M\frac{\mathrm{d}^2O}{\mathrm{d}t^2}(t), \\
\frac{\mathrm{d}T_{\dot{O}}}{\mathrm{d}t}(t) = P_{\dot{O}}(t) - \Big\langle \frac{\mathrm{d}^2O}{\mathrm{d}t^2}(t), p_{\dot{O}}(t)\Big\rangle, \\
\frac{\mathrm{d}L_{O,\dot{O}}}{\mathrm{d}t}(t) = \tau_O(t) + \big(O(t) - C(t)\big) \wedge \Big(M\frac{\mathrm{d}^2O}{\mathrm{d}t^2}(t)\Big),
\end{gathered} \tag{5.2}$$

for all $t \in \mathcal{T}$.

Proof. Straightforward computation. □

Corollary 5.2. *Under the same conditions of Theorem 5.1, if O has constant velocity, then:*

$$\frac{\mathrm{d}p_{\dot{O}}}{\mathrm{d}t}(t) = F(t),$$

$$\frac{\mathrm{d}T_{\dot{O}}}{\mathrm{d}t}(t) = P_{\dot{O}}(t),$$

$$\frac{\mathrm{d}L_{O,\dot{O}}}{\mathrm{d}t}(t) = \tau_O(t),$$

for all $t \in \mathcal{T}$. □

Corollary 5.3. *Under the same conditions of Theorem 5.1, we have:*

$$\frac{\mathrm{d}T_{\dot{C}}}{\mathrm{d}t}(t) = P_{\dot{C}}(t),$$

$$\frac{\mathrm{d}L_{C,\dot{C}}}{\mathrm{d}t}(t) = \tau_C(t), \tag{5.3}$$

for all $t \in \mathcal{T}$.

Proof. For the first equality, note that $p_{\dot{C}} = 0$. □

5.1. **Mechanics with constraints.** Let $\mathcal{Q}$ be an embedded submanifold of the affine space $\mathcal{P}^\Lambda$. We consider a configuration in motion $q : \mathcal{T} \to \mathcal{P}^\Lambda$ that is forced to satisfy the constraint given by $\mathcal{Q}$, i.e., such that the image of q is in $\mathcal{Q}$. The resultant force (5.1) is decomposed as

$$F_\lambda(t) = F_\lambda^{\mathrm{ext}}(t) + F_\lambda^{\mathrm{cons}}(t), \quad t \in \mathcal{T}, \tag{5.4}$$

with $F_\lambda^{\mathrm{cons}}(t)$ the *constraint reaction* (whatever is forcing the configuration to remain inside the constraint) and $F_\lambda^{\mathrm{ext}}(t)$ the *external forces* (forces not related to the constraint). We combine all the forces $F_\lambda(t)$ into a single vector

$$\hat{F}(t) = \big(F_\lambda(t)\big)_{\lambda\in\Lambda} \in \mathcal{V}^\Lambda$$

and, similarly, we define $\hat{F}^{\mathrm{ext}}$ and $\hat{F}^{\mathrm{cons}}$. The family of masses $(m_\lambda)_{\lambda\in\Lambda}$ are combined to form a linear operator $\hat{m} : \mathcal{V}^\Lambda \to \mathcal{V}^\Lambda$ defined by

$$\hat{m}\big((\dot{q}_\lambda)_{\lambda\in\Lambda}\big) = (m_\lambda \dot{q}_\lambda)_{\lambda\in\Lambda}, \quad (\dot{q}_\lambda)_{\lambda\in\Lambda} \in \mathcal{V}^\Lambda.$$

Endowing $\mathcal{V}^\Lambda$ with the inner product $\langle\cdot,\cdot\rangle$ defined in (4.1), then $\hat{m}$ becomes a positive symmetric operator and

$$\langle\cdot,\cdot\rangle_m = \langle \hat{m}\,\cdot,\cdot\rangle$$

defines a new inner product on $\mathcal{V}^\Lambda$. This is the unique inner product on $\mathcal{V}^\Lambda$ such that the total kinetic energy is given by

$$T_{\dot{O}} = \frac{1}{2}\big\langle (\dot{q}_\lambda - \dot{O})_{\lambda\in\Lambda}, (\dot{q}_\lambda - \dot{O})_{\lambda\in\Lambda}\big\rangle_m,$$

for every family of velocities $(\dot q_\lambda)_{\lambda\in\Lambda} \in \mathcal{V}^\Lambda$ and some reference velocity $\dot O$. Using the operator $\hat m$, equation (5.1) can be written as:

$$\hat m\Big(\frac{\mathrm{d}^2 q}{\mathrm{d}t^2}(t)\Big) = \hat F(t). \tag{5.5}$$

We now make the assumption that we have an *ideal constraint*, namely, that the constraint reaction $\hat F^{\mathrm{cons}}(t)$ belongs to the orthogonal complement of the tangent space $T_{q(t)}\mathcal{Q}$ with respect to the inner product $\langle\cdot,\cdot\rangle$. With this assumption, the constraint reaction does no work, i.e., it makes no contribution to the energy balance. From equations (5.5) and (5.4), we obtain the following equality of linear functionals on $T_{q(t)}\mathcal{Q}$:

$$\Big\langle \hat m\Big(\frac{\mathrm{d}^2 q}{\mathrm{d}t^2}(t)\Big),\cdot\Big\rangle\Big|_{T_{q(t)}\mathcal{Q}} = \langle \hat F^{\mathrm{ext}}(t),\cdot\rangle|_{T_{q(t)}\mathcal{Q}} \in (T_{q(t)}\mathcal{Q})^*, \quad t\in\mathcal{T}.$$

We can rewrite this equality using the inner product $\langle\cdot,\cdot\rangle_m$:

$$\Big[\Big\langle \frac{\mathrm{d}^2 q}{\mathrm{d}t^2}(t),\cdot\Big\rangle_m\Big]\Big|_{T_{q(t)}\mathcal{Q}} = \langle \hat F^{\mathrm{ext}}(t),\cdot\rangle|_{T_{q(t)}\mathcal{Q}} \in (T_{q(t)}\mathcal{Q})^*, \quad t\in\mathcal{T}.$$

Let g denote the Riemannian metric on $\mathcal{Q}$ given by the restriction of $\langle\cdot,\cdot\rangle_m$ and consider $\mathcal{Q}$ endowed with the corresponding Levi-Civita connection. If $\frac{\mathrm{D}}{\mathrm{d}t}$ denotes the covariant derivative in the direction of the vector field $\frac{\mathrm{d}}{\mathrm{d}t}$, then $\frac{\mathrm{D}}{\mathrm{d}t}\frac{\mathrm{d}q}{\mathrm{d}t}(t)$ is simply the $\langle\cdot,\cdot\rangle_m$-orthogonal projection of $\frac{\mathrm{d}^2q}{\mathrm{d}t^2}(t)$ onto $T_{q(t)}\mathcal{Q}$; thus:

$$g\Big(\frac{\mathrm{D}}{\mathrm{d}t}\frac{\mathrm{d}q}{\mathrm{d}t}(t),\cdot\Big) = \langle \hat F^{\mathrm{ext}}(t),\cdot\rangle|_{T_{q(t)}\mathcal{Q}} \in (T_{q(t)}\mathcal{Q})^*, \quad t\in\mathcal{T}. \tag{5.6}$$

Typically, the external forces will be specified as smooth functions of time, position and velocity, so that (5.6) will become a second-order differential equation for curves on the manifold $\mathcal{Q}$; such equation has a unique solution once an initial position and velocity are specified. If q satisfies (5.6), then (5.5) and (5.4) will be satisfied for some constraint reaction $\hat F^{\mathrm{cons}}(t)$ that is $\langle\cdot,\cdot\rangle$-orthogonal to $T_{q(t)}\mathcal{Q}$. An explicit formula for $\hat F^{\mathrm{cons}}(t)$ can be written in terms of $q(t)$, $\frac{\mathrm{d}q}{\mathrm{d}t}(t)$ and $\hat F^{\mathrm{ext}}(t)$. We obtain this formula below.

For each $q\in\mathcal{Q}$, denote by $(T_q\mathcal{Q})^\perp$ the orthogonal complement of $T_q\mathcal{Q}$ in $\mathcal{V}^\Lambda$ with respect to the inner product $\langle\cdot,\cdot\rangle$. The orthogonal complement of $T_q\mathcal{Q}$ with respect to $\langle\cdot,\cdot\rangle_m$ is then equal to $\hat m^{-1}[(T_q\mathcal{Q})^\perp]$; it follows that

$$\mathcal{V}^\Lambda = T_q\mathcal{Q}\oplus \hat m^{-1}[(T_q\mathcal{Q})^\perp]$$

and thus

$$\mathcal{V}^\Lambda = \hat m[T_q\mathcal{Q}]\oplus (T_q\mathcal{Q})^\perp.$$

Denote by $\alpha_q : T_q\mathcal{Q}\times T_q\mathcal{Q}\to \hat m^{-1}[(T_q\mathcal{Q})^\perp]$ the second fundamental form of $\mathcal{Q}$ in $\mathcal{P}^\Lambda$, where $\mathcal{P}^\Lambda$ is endowed with the Riemannian metric that is constant and equal to $\langle\cdot,\cdot\rangle_m$. We then have

$$\frac{\mathrm{d}^2q}{\mathrm{d}t^2}(t) = \frac{\mathrm{D}}{\mathrm{d}t}\frac{\mathrm{d}q}{\mathrm{d}t}(t) + \alpha_{q(t)}\Big(\frac{\mathrm{d}q}{\mathrm{d}t}(t),\frac{\mathrm{d}q}{\mathrm{d}t}(t)\Big), \quad t\in\mathcal{T},$$

with $\frac{\mathrm{D}}{\mathrm{d}t}\frac{\mathrm{d}q}{\mathrm{d}t}(t)$ in $T_{q(t)}\mathcal{Q}$ and $\alpha_{q(t)}\big(\frac{\mathrm{d}q}{\mathrm{d}t}(t), \frac{\mathrm{d}q}{\mathrm{d}t}(t)\big)$ in $\hat{m}^{-1}[(T_{q(t)}\mathcal{Q})^{\perp}]$. Thus:

$$\hat{m}\Big(\frac{\mathrm{d}^2 q}{\mathrm{d}t^2}(t)\Big) = \hat{m}\Big(\frac{\mathrm{D}}{\mathrm{d}t}\frac{\mathrm{d}q}{\mathrm{d}t}(t)\Big) + \hat{m}\Big(\alpha_{q(t)}\Big(\frac{\mathrm{d}q}{\mathrm{d}t}(t), \frac{\mathrm{d}q}{\mathrm{d}t}(t)\Big)\Big), \quad t \in \mathcal{T}, \tag{5.7}$$

with $\hat{m}\big(\frac{\mathrm{D}}{\mathrm{d}t}\frac{\mathrm{d}q}{\mathrm{d}t}(t)\big)$ in $\hat{m}[T_{q(t)}\mathcal{Q}]$ and $\hat{m}\big(\alpha_{q(t)}\big(\frac{\mathrm{d}q}{\mathrm{d}t}(t), \frac{\mathrm{d}q}{\mathrm{d}t}(t)\big)\big)$ in $(T_{q(t)}\mathcal{Q})^{\perp}$. Denote by $\pi : \mathcal{V}^{\Lambda} \to (T_{q(t)}\mathcal{Q})^{\perp}$ the projection determined by the direct sum decomposition

$$\mathcal{V}^{\Lambda} = \hat{m}[T_{q(t)}\mathcal{Q}] \oplus (T_{q(t)}\mathcal{Q})^{\perp}.$$

From (5.7) and (5.5), we obtain:

$$\pi\big(\hat{F}(t)\big) = \hat{m}\Big(\alpha_{q(t)}\Big(\frac{\mathrm{d}q}{\mathrm{d}t}(t), \frac{\mathrm{d}q}{\mathrm{d}t}(t)\Big)\Big), \quad t \in \mathcal{T}.$$

Using (5.4) and the fact that $\hat{F}^{\mathrm{cons}}(t)$ is in $(T_{q(t)}\mathcal{Q})^{\perp}$, we finally conclude that:

$$\hat{F}^{\mathrm{cons}}(t) = \hat{m}\Big(\alpha_{q(t)}\Big(\frac{\mathrm{d}q}{\mathrm{d}t}(t), \frac{\mathrm{d}q}{\mathrm{d}t}(t)\Big)\Big) - \pi\big(\hat{F}^{\mathrm{ext}}(t)\big), \quad t \in \mathcal{T}.$$

6. Some exterior algebra

Consider the canonical embedding

$$\bigwedge_k \mathcal{V} \longrightarrow \bigotimes_k \mathcal{V}$$

of the k-th exterior power of $\mathcal{V}$ into the k-th tensor power of $\mathcal{V}$ defined by

$$v_1 \wedge v_2 \wedge \cdots \wedge v_k \longmapsto \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma)(v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(k)}),$$

for all $v_1, \ldots, v_k \in \mathcal{V}$, where S_k denotes the group of permutations of $\{1, 2, \ldots, k\}$ and $\mathrm{sgn}(\sigma)$ denotes the sign of a permutation σ. The inner product of $\mathcal{V}$ induces a *contraction map*

$$\Big(\bigotimes_{k+1} \mathcal{V}\Big) \times \mathcal{V} \longrightarrow \bigotimes_k \mathcal{V} \tag{6.1}$$

defined by

$$(v_1 \otimes v_2 \otimes \cdots \otimes v_{k+1}) \lrcorner\, w = \langle v_1, w\rangle(v_2 \otimes \cdots \otimes v_{k+1}),$$

for all $v_1, \ldots, v_{k+1}, w \in \mathcal{V}$. Through the canonical embedding of the exterior product into the tensor product, the contraction map (6.1) restricts to a contraction map

$$\Big(\bigwedge_{k+1} \mathcal{V}\Big) \times \mathcal{V} \longrightarrow \bigwedge_k \mathcal{V} \tag{6.2}$$

which is given by

$$(v_1 \wedge v_2 \wedge \cdots \wedge v_{k+1}) \lrcorner\, w = \sum_{i=1}^{k+1} (-1)^{i+1} \langle v_i, w\rangle (v_1 \wedge v_2 \wedge \cdots \wedge \widehat{v_i} \wedge \cdots \wedge v_{k+1}),$$

for all $v_1, \ldots, v_{k+1}, w \in \mathcal{V}$, where the hat indicates that the term has been omitted. We endow the exterior product $\bigwedge_k \mathcal{V}$ with the inner product defined by

$$\langle v_1 \wedge v_2 \wedge \cdots \wedge v_k, w_1 \wedge w_2 \wedge \cdots \wedge w_k \rangle = \det\big(\langle v_i, w_j \rangle\big)_{i,j=1,\ldots,k},$$

for all $v_1, \ldots, v_k, w_1, \ldots, w_k \in \mathcal{V}$. It is easily checked that the following formula holds:

$$\langle \omega \lrcorner z, \rho \rangle = \langle \omega, z \wedge \rho \rangle,$$

for all $\omega \in \bigwedge_{k+1} \mathcal{V}$, $\rho \in \bigwedge_k \mathcal{V}$ and $z \in \mathcal{V}$; in other words, for all $z \in \mathcal{V}$, the linear map

$$z \wedge \cdot : \bigwedge_k \mathcal{V} \longrightarrow \bigwedge_{k+1} \mathcal{V}$$

is the adjoint of the linear map:

$$\cdot \lrcorner z : \bigwedge_{k+1} \mathcal{V} \longrightarrow \bigwedge_k \mathcal{V}.$$

We are only concerned with the contraction map (6.2) for $k = 1$, which is a map

$$(\mathcal{V} \wedge \mathcal{V}) \times \mathcal{V} \longrightarrow \mathcal{V} \tag{6.3}$$

given by

$$(v \wedge w) \lrcorner z = \langle v, z \rangle w - \langle w, z \rangle v,$$

for all $v, w, z \in \mathcal{V}$. It is easily seen that the map

$$\mathcal{V} \wedge \mathcal{V} \ni \omega \longmapsto \omega \lrcorner \cdot \in \mathrm{Lin}_{\mathrm{a}}(\mathcal{V}) \tag{6.4}$$

is an isomorphism onto the space $\mathrm{Lin}_{\mathrm{a}}(\mathcal{V})$ of anti-symmetric linear endomorphisms of $\mathcal{V}$. The space $\mathrm{Lin}_{\mathrm{a}}(\mathcal{V})$ has the structure of a Lie algebra, with Lie bracket defined by

$$[\Omega_0, \Omega_0'] = \Omega_0 \circ \Omega_0' - \Omega_0' \circ \Omega_0, \quad \Omega_0, \Omega_0' \in \mathrm{Lin}_{\mathrm{a}}(\mathcal{V}).$$

We can then endow $\mathcal{V} \wedge \mathcal{V}$ with a Lie bracket by requiring (6.4) to be a Lie algebra isomorphism. More explicitly, we set:

$$[\omega, \omega'] \lrcorner \cdot = [\omega \lrcorner \cdot, \omega' \lrcorner \cdot], \quad \omega, \omega' \in \mathcal{V} \wedge \mathcal{V}. \tag{6.5}$$

Remark 6.1. As observed in Remark 3.2, if $n = 3$ and $\mathcal{V}$ is oriented, we can identify $\mathcal{V} \wedge \mathcal{V}$ with $\mathcal{V}$ using the vector product. It follows from the formula for the iterated vector product

$$(v \wedge w) \wedge z = \langle v, z \rangle w - \langle w, z \rangle v, \quad v, w, z \in \mathcal{V},$$

that, under the identification of $\mathcal{V} \wedge \mathcal{V}$ with $\mathcal{V}$, the contraction map (6.3) is identified with the vector product itself. Moreover, since the vector product of $\mathcal{V}$ turns $\mathcal{V}$ into a Lie algebra, we know that the adjoint representation

$$\mathcal{V} \ni \omega \longmapsto \omega \wedge \cdot \in \mathrm{Lin}_{\mathrm{a}}(\mathcal{V})$$

is a Lie algebra homomorphism (which turns out to be an isomorphism in this case). Thus, under the identification of $\mathcal{V} \wedge \mathcal{V}$ with $\mathcal{V}$, the Lie bracket (6.5) of $\mathcal{V} \wedge \mathcal{V}$ is also identified with the vector product.

7. Rigid motion

Recall that the set $\mathrm{Aff}(\mathcal{P}, \mathcal{P})$ of all affine maps from $\mathcal{P}$ to $\mathcal{P}$ is an affine space parallel to the vector space $\mathrm{Aff}(\mathcal{P}, \mathcal{V})$ (see Subsection 1.3). Given an arbitrary point $O \in \mathcal{P}$, we obtain an affine isomorphism

$$\mathrm{Lin}(\mathcal{V}, \mathcal{V}) \times \mathcal{V} \longrightarrow \mathrm{Aff}(\mathcal{P}, \mathcal{P}) \tag{7.1}$$

given by $(\Omega_0, v) \mapsto \Omega$, with $\Omega(p) = O + \Omega_0(p - O) + v$, for all $\Omega_0 \in \mathrm{Lin}(\mathcal{V}, \mathcal{V})$, $v \in \mathcal{V}$ and $p \in \mathcal{P}$, where $\mathrm{Lin}(\mathcal{V}, \mathcal{V})$ denote the space of all linear transformations from $\mathcal{V}$ to $\mathcal{V}$. The subset $\mathrm{GAff}(\mathcal{P})$ of $\mathrm{Aff}(\mathcal{P}, \mathcal{P})$ consisting of affine isomorphisms is a group (under composition) and an open subset of $\mathrm{Aff}(\mathcal{P}, \mathcal{P})$; hence it is a Lie group and its Lie algebra is identified, as a vector space, with $\mathrm{Aff}(\mathcal{P}, \mathcal{V})$. The affine isomorphism (7.1) restricts to a smooth diffeomorphism

$$\mathrm{GL}(\mathcal{V}) \times \mathcal{V} \longrightarrow \mathrm{GAff}(\mathcal{P}), \tag{7.2}$$

where $\mathrm{GL}(\mathcal{V})$ denotes the Lie group of linear isomorphisms of $\mathcal{V}$. This diffeomorphism becomes a Lie group isomorphism if $\mathrm{GL}(\mathcal{V}) \times \mathcal{V}$ is endowed with the semidirect product structure defined by

$$(\Omega_0, v) \cdot (\Theta_0, w) = \big(\Omega_0 \circ \Theta_0, \Omega_0(w) + v\big), \quad \Omega_0, \Theta_0 \in \mathrm{GL}(\mathcal{V}),\ v, w \in \mathcal{V}.$$

Let $\mathrm{Iso}(\mathcal{P})$ denote the subgroup of $\mathrm{GAff}(\mathcal{P})$ consisting of *affine isometries*, i.e., affine maps whose underlying linear map is a linear isometry of $\mathcal{V}$. We have that $\mathrm{Iso}(\mathcal{P})$ is a closed Lie subgroup of $\mathrm{GAff}(\mathcal{P})$ and (7.2) restricts to a diffeomorphism

$$\mathrm{O}(\mathcal{V}) \times \mathcal{V} \longrightarrow \mathrm{Iso}(\mathcal{P}), \tag{7.3}$$

where $\mathrm{O}(\mathcal{V})$ denotes the orthogonal group of $\mathcal{V}$, i.e., the closed Lie subgroup of $\mathrm{GL}(\mathcal{V})$ consisting of linear isometries. The Lie algebra of $\mathrm{O}(\mathcal{V})$ is the space $\mathrm{Lin}_{\mathrm{a}}(\mathcal{V})$ of anti-symmetric linear endomorphisms of $\mathcal{V}$ and the Lie algebra of $\mathrm{Iso}(\mathcal{P})$ is the subspace of $\mathrm{Aff}(\mathcal{P}, \mathcal{V})$ consisting of affine maps whose underlying linear map is anti-symmetric.

The connected component of the identity of the Lie group $\mathrm{Iso}(\mathcal{P})$ is the group $\mathrm{Iso}_0(\mathcal{P})$ consisting of affine maps whose underlying linear map is in the group $\mathrm{SO}(\mathcal{V})$ of orientation-preserving linear isometries of $\mathcal{V}$ (which is the connected component of the identity of the Lie group $\mathrm{O}(\mathcal{V})$). Obviously, the diffeomorphism (7.3) restricts to a diffeomorphism

$$\mathrm{SO}(\mathcal{V}) \times \mathcal{V} \longrightarrow \mathrm{Iso}_0(\mathcal{P}).$$

We consider the smooth action of $\mathrm{Iso}(\mathcal{P})$ on $\mathcal{P}^\Lambda$ given by

$$g \cdot (q_\lambda)_{\lambda \in \Lambda} = \big(g(q_\lambda)\big)_{\lambda \in \Lambda}, \quad g \in \mathrm{Iso}(\mathcal{P}),\ (q_\lambda)_{\lambda \in \Lambda} \in \mathcal{P}^\Lambda.$$

Proposition 7.1. *The orbits of the action of* $\mathrm{Iso}(\mathcal{P})$ *on* $\mathcal{P}^\Lambda$ *are closed and thus*[3] *are embedded submanifolds of* $\mathcal{P}^\Lambda$*.*

Proof. Obviously we can assume that Λ is nonempty. Let $(q_\lambda)_{\lambda\in\Lambda} \in \mathcal{P}^\Lambda$ be given and let us show that the orbit of $(q_\lambda)_{\lambda\in\Lambda}$ under $\mathrm{Iso}(\mathcal{P})$ is closed. Fix $\lambda_0 \in \Lambda$ and let $\mathcal{W}$ be the subspace of $\mathcal{V}$ spanned by

$$\big\{q_\lambda - q_{\lambda_0} : \lambda \in \Lambda \setminus \{\lambda_0\}\big\}.$$

Let Θ be a subset of $\Lambda \setminus \{\lambda_0\}$ such that $(q_\lambda - q_{\lambda_0})_{\lambda\in\Theta}$ is a basis of $\mathcal{W}$. For each $(q'_\lambda)_{\lambda\in\Lambda} \in \mathcal{P}^\Lambda$, there exists a unique linear map $\Omega_0 : \mathcal{W} \to \mathcal{V}$ such that

$$\Omega_0(q_\lambda - q_{\lambda_0}) = q'_\lambda - q'_{\lambda_0}, \tag{7.4}$$

for all $\lambda \in \Theta$. The affine map $\Omega : q_{\lambda_0} + \mathcal{W} \to \mathcal{P}$ defined by

$$\Omega(q) = q'_{\lambda_0} + \Omega_0(q - q_{\lambda_0}), \quad q \in q_{\lambda_0} + \mathcal{W},$$

is the unique affine map with $\Omega(q_\lambda) = q'_\lambda$, for all $\lambda \in \Theta \cup \{\lambda_0\}$. We have that $(q'_\lambda)_{\lambda\in\Lambda}$ is in the orbit of $(q_\lambda)_{\lambda\in\Lambda}$ under $\mathrm{Iso}(\mathcal{P})$ if and only if Ω_0 is a linear isometric embedding and (7.4) holds for all $\lambda \in \Lambda$. The conclusion is obtained by observing that the set of linear isometric embeddings from $\mathcal{W}$ to $\mathcal{V}$ is a closed subset of the space of all linear maps from $\mathcal{W}$ to $\mathcal{V}$ and that the mapping $(q'_\lambda)_{\lambda\in\Lambda} \mapsto \Omega_0$ is continuous. □

Definition 7.2. By a *configuration in rigid motion* we mean a particle configuration in motion whose image is contained in an orbit of the action of $\mathrm{Iso}(\mathcal{P})$ on $\mathcal{P}^\Lambda$.

Obviously, the image of a configuration in motion is connected and thus the image of a configuration in rigid motion is contained in a connected component of an orbit of $\mathrm{Iso}(\mathcal{P})$. Since the connected components[4] of the orbits of $\mathrm{Iso}(\mathcal{P})$ are the orbits of $\mathrm{Iso}_0(\mathcal{P})$, we can replace $\mathrm{Iso}(\mathcal{P})$ with $\mathrm{Iso}_0(\mathcal{P})$ in Definition 7.2.

Proposition 7.3. *If* $\mathcal{Q}$ *is an orbit of the action of* $\mathrm{Iso}(\mathcal{P})$ *(or of* $\mathrm{Iso}_0(\mathcal{P})$*) on* $\mathcal{P}^\Lambda$ *then the tangent space of* $\mathcal{Q}$ *at a point* $(q_\lambda)_{\lambda\in\Lambda}$ *is the space of all families* $\big(\Omega(q_\lambda)\big)_{\lambda\in\Lambda}$ *with* $\Omega : \mathcal{P} \to \mathcal{V}$ *running over all affine maps whose underlying linear map is anti-symmetric.*

Proof. The tangent space of an orbit $\mathcal{Q}$ at $(q_\lambda)_{\lambda\in\Lambda}$ is the image of the differential at the identity of the map

$$\mathrm{Iso}(\mathcal{P}) \ni g \longmapsto g \cdot (q_\lambda)_{\lambda\in\Lambda} \in \mathcal{Q}.$$

Such differential is given by

$$T_{\mathrm{Id}}\,\mathrm{Iso}(\mathcal{P}) \ni \Omega \longmapsto \big(\Omega(q_\lambda)\big)_{\lambda\in\Lambda} \in T_{(q_\lambda)_{\lambda\in\Lambda}}\mathcal{Q}.$$ □

[3] An orbit of a smooth action of a Lie group is an embedded submanifold if and only if it is locally closed, i.e., it is the intersection of a closed subset and an open subset.

[4] If a Lie group G acts continuously and transitively on a manifold M and if G_0 is the connected component of the identity of G, then the orbits of the action of G_0 on M are open, closed and connected. Hence, the orbits of G_0 are the connected components of M.

Corollary 7.4. *If $\mathcal{Q}$ is an orbit of the action of* $\mathrm{Iso}(\mathcal{P})$ *(or of* $\mathrm{Iso}_0(\mathcal{P})$*) on* $\mathcal{P}^\Lambda$*, then the admissible reaction forces of the ideal constraint $\mathcal{Q}$ at a particle configuration $(q_\lambda)_{\lambda\in\Lambda} \in \mathcal{Q}$ (i.e., the orthogonal complement of $T_{(q_\lambda)_{\lambda\in\Lambda}}\mathcal{Q}$ in $\mathcal{V}^\Lambda$) are precisely the families of forces whose total force and total torque are zero.*

Proof. Follows from Lemma 4.4. □

Definition 7.5. Let $(q_\lambda)_{\lambda\in\Lambda}$ be a particle configuration. We say that a family of velocities $(\dot{q}_\lambda)_{\lambda\in\Lambda}$ is *admissible for rigid motion* (for the given particle configuration) if $(\dot{q}_\lambda)_{\lambda\in\Lambda}$ belongs to the tangent space at $(q_\lambda)_{\lambda\in\Lambda}$ of the orbit of $\mathrm{Iso}(\mathcal{P})$ (or of $\mathrm{Iso}_0(\mathcal{P})$) passing through $(q_\lambda)_{\lambda\in\Lambda}$. By Proposition 7.3, this is equivalent to the existence of an affine map $\Omega : \mathcal{P} \to \mathcal{V}$, with anti-symmetric underlying linear map, such that

$$\dot{q}_\lambda = \Omega(q_\lambda),$$

for all $\lambda \in \Lambda$. If $\omega \in \mathcal{V}\wedge\mathcal{V}$ is such that $\omega \lrcorner \cdot$ is the underlying linear map of Ω, then we call ω an *angular velocity* for $(\dot{q}_\lambda)_{\lambda\in\Lambda}$.

Since two anti-symmetric linear maps that agree on a codimension one subspace are equal, it follows that if the parallel subspace to the particle configuration $(q_\lambda)_{\lambda\in\Lambda}$ has codimension at most one, then there exists exactly one angular velocity for a family of velocities $(\dot{q}_\lambda)_{\lambda\in\Lambda}$, so we call it *the* angular velocity of $(\dot{q}_\lambda)_{\lambda\in\Lambda}$.

8. Tensor of inertia

Definition 8.1. Given a particle configuration $(q_\lambda)_{\lambda\in\Lambda}$, a family of masses $(m_\lambda)_{\lambda\in\Lambda}$ and a reference point $O \in \mathcal{P}$, we define the corresponding *tensor of inertia* as the linear operator

$$I_O : \mathcal{V}\wedge\mathcal{V} \longrightarrow \mathcal{V}\wedge\mathcal{V}$$

defined by

$$I_O(\omega) = \sum_{\lambda\in\Lambda} m_\lambda (q_\lambda - O) \wedge \big(\omega \lrcorner (q_\lambda - O)\big),$$

for all $\omega \in \mathcal{V}\wedge\mathcal{V}$.

Note that, since $z \wedge \cdot$ is the adjoint of $\cdot \lrcorner z$, it follows that

$$\langle I_O(\omega_1), \omega_2\rangle = \sum_{\lambda\in\Lambda} m_\lambda \langle \omega_1 \lrcorner (q_\lambda - O), \omega_2 \lrcorner (q_\lambda - O)\rangle, \tag{8.1}$$

for all $\omega_1, \omega_2 \in \mathcal{V}\wedge\mathcal{V}$. In particular, I_O is a symmetric positive semi-definite operator whose kernel consists of those $\omega \in \mathcal{V}\wedge\mathcal{V}$ such that $\omega \lrcorner \cdot$ annihilate the subspace spanned by $\big\{q_\lambda - O : \lambda \in \Lambda\big\}$.

Remark 8.2. Assume that $n = 3$ and $\mathcal{V}$ is oriented. Identifying $\mathcal{V}\wedge\mathcal{V}$ with $\mathcal{V}$ through the vector product (Remark 3.2), we obtain:

$$\langle I_O(\omega), \omega\rangle = \sum_{\lambda\in\Lambda} m_\lambda \|\omega \wedge (q_\lambda - O)\|^2,$$

for all $\omega \in \mathcal{V}$. Note that, if $\|\omega\| = 1$, then $\|\omega \wedge (q_\lambda - O)\|$ is equal to the distance from q_λ to the line $O + \mathbb{R}\omega$.

Proposition 8.3. *Let $(q_\lambda)_{\lambda\in\Lambda}$ be a particle configuration, $(m_\lambda)_{\lambda\in\Lambda}$ be a family of masses, $(F_\lambda)_{\lambda\in\Lambda}$ be a family of forces and $\Omega : \mathcal{P} \to \mathcal{V}$ be an affine map whose underlying linear map $\Omega_0 : \mathcal{V} \to \mathcal{V}$ is anti-symmetric. Let $\omega \in \mathcal{V} \wedge \mathcal{V}$ be such that $\Omega_0 = \omega \lrcorner \cdot$. Consider the family of velocities*

$$\dot{q}_\lambda = \Omega(q_\lambda), \quad \lambda \in \Lambda.$$

Pick $O \in \mathcal{P}$ and set $\dot{O} = \Omega(O)$. The following formulas hold:

$$T_{\dot{O}} = \frac{1}{2}\langle I_O(\omega), \omega\rangle; \tag{8.2}$$

$$P_{\dot{O}} = \langle \tau_O, \omega\rangle; \tag{8.3}$$

$$L_{O,\dot{O}} = I_O(\omega). \tag{8.4}$$

Proof. Note that

$$\dot{q}_\lambda - \dot{O} = \Omega_0(q_\lambda - O) = \omega \lrcorner (q_\lambda - O), \quad \lambda \in \Lambda.$$

Formula (8.4) then follows directly from the definitions of I_O and $L_{O,\dot{O}}$ and formula (8.2) follows directly from (8.1) and the definition of $T_{\dot{O}}$. Finally, formula (8.3) follows from the definitions of τ_O and $P_{\dot{O}}$ and the fact that $(q_\lambda - O) \wedge \cdot$ is the adjoint of $\cdot \lrcorner (q_\lambda - O)$. □

Corollary 8.4. *Under the conditions in the statement of Proposition 8.3, we have:*

$$T_{\dot{C}} = \frac{1}{2}\langle I_C(\omega), \omega\rangle;$$

$$P_{\dot{C}} = \langle \tau_C, \omega\rangle;$$

$$L_{C,\dot{C}} = I_C(\omega).$$

Proof. Simply note that $\Omega(C) = \dot{C}$. □

Proposition 8.5. *Given a particle configuration $(q_\lambda)_{\lambda\in\Lambda}$, a family of masses $(m_\lambda)_{\lambda\in\Lambda}$ and two reference points $O_1 \in \mathcal{P}$, $O_2 \in \mathcal{P}$, we have:*

$$\begin{aligned} I_{O_2}(\omega) = I_{O_1}(\omega) &+ M(O_1 - O_2) \wedge \big(\omega \lrcorner (C - O_1)\big) \\ &+ M(C - O_1) \wedge \big(\omega \lrcorner (O_1 - O_2)\big) + M(O_1 - O_2) \wedge \big(\omega \lrcorner (O_1 - O_2)\big), \end{aligned}$$

for all $\omega \in \mathcal{V} \wedge \mathcal{V}$.

Proof. Straightforward computation. □

9. Derivative in a moving frame

Let $\mathcal{V}$ be a finite dimensional real vector space, G be a Lie group and

$$\rho : G \longrightarrow \mathrm{GL}(\mathcal{V})$$

be a smooth representation of G on $\mathcal{V}$. Let M be a differentiable manifold and X be a vector field on M. For any smooth map $f : M \to \mathcal{V}$ we denote, as usual, by $X(f)$ the map from M to $\mathcal{V}$ given by

$$X(f)(m) = \mathrm{d}f(m) \cdot X(m), \quad m \in M.$$

Fix a smooth map $g : M \to G$. We define a map $\nabla^g_X f : M \to \mathcal{V}$ by setting

$$(\nabla^g_X f)(m) = \rho\big(g(m)\big) \cdot \big(X(\tilde{f})(m)\big), \quad m \in M,$$

where $\tilde{f} : M \to \mathcal{V}$ is the smooth map defined by

$$\tilde{f}(m) = \rho\big(g(m)\big)^{-1} \cdot f(m), \quad m \in M.$$

In other words, ∇^g is the connection on the trivial vector bundle $M \times \mathcal{V}$ obtained by taking the push-forward of the trivial connection by the vector bundle isomorphism given by

$$M \times \mathcal{V} \ni (m, v) \longmapsto \big(m, \rho\big(g(m)\big) \cdot v\big) \in M \times \mathcal{V}.$$

If M is (an open subset of) an affine vector space $\mathcal{T}$ parallel to $\mathbb{R}$ then, as before, we denote by $\frac{\mathrm{d}}{\mathrm{d}t}$ the (derivative with respect to) the canonical vector field of M that has value 1 at every point. We then denote by $\frac{\mathrm{D}^g}{\mathrm{d}t}$ the covariant derivative with respect to that same vector field, i.e.:

$$\frac{\mathrm{D}^g}{\mathrm{d}t} = \nabla^g_{\frac{\mathrm{d}}{\mathrm{d}t}}.$$

Let us now write a more explicit formula for $\nabla^g_X f$. Denote by $\mathfrak{g}$ the Lie algebra of G and by $\bar{\rho}$ the differential of ρ at the identity $1 \in G$:

$$\bar{\rho} = \mathrm{d}\rho(1) : \mathfrak{g} \longrightarrow \mathrm{Lin}(\mathcal{V}, \mathcal{V}),$$

which is a representation of the Lie algebra $\mathfrak{g}$ in $\mathcal{V}$, i.e., a Lie algebra homomorphism of $\mathfrak{g}$ to $\mathrm{Lin}(\mathcal{V}, \mathcal{V})$. Using that ρ is a homomorphism we obtain:

$$\mathrm{d}\rho(h) \cdot (Zh) = \bar{\rho}(Z) \circ \rho(h), \quad h \in G,\ Z \in \mathfrak{g},$$

where $Zh \in T_h G$ denotes the image of Z by the differential at 1 of the map given by right translation by h. Let $\Omega_X : M \to \mathfrak{g}$ be the smooth map such that

$$\mathrm{d}g(m) \cdot X(m) = \Omega_X(m) g(m), \quad m \in M.$$

Then:

$$\begin{aligned} X(\tilde{f}) &= -\big[\rho\big(g(m)\big)^{-1} \circ \big[\big(\mathrm{d}\rho\big(g(m)\big) \circ \mathrm{d}g(m)\big) \cdot X(m)\big] \circ \rho\big(g(m)\big)^{-1}\big] \cdot f(m) \\ &\quad + \rho\big(g(m)\big)^{-1} \cdot \big(X(f)(m)\big) \\ &= -\big[\rho\big(g(m)\big)^{-1} \circ \big[\bar{\rho}\big(\Omega_X(m)\big) \circ \rho\big(g(m)\big)\big] \circ \rho\big(g(m)\big)^{-1}\big] \cdot f(m) \\ &\quad + \rho\big(g(m)\big)^{-1} \cdot \big(X(f)(m)\big) \\ &= -\big[\rho\big(g(m)\big)^{-1} \circ \bar{\rho}\big(\Omega_X(m)\big)\big] \cdot f(m) + \rho\big(g(m)\big)^{-1} \cdot \big(X(f)(m)\big), \end{aligned}$$

for all $m \in M$. Hence:

$$(\nabla^g_X f)(m) = X(f)(m) - \bar{\rho}\big(\Omega_X(m)\big) \cdot f(m), \quad m \in M. \tag{9.1}$$

Definition 9.1. Let $\mathcal{V}$, $\mathcal{W}$ be real finite dimensional vector spaces and let $\rho^{\mathcal{V}} : G \to \mathrm{GL}(\mathcal{V})$, $\rho^{\mathcal{W}} : G \to \mathrm{GL}(\mathcal{W})$ be smooth representations of the Lie group G. A map $\phi : \mathrm{dom}(\phi) \subset \mathcal{V} \to \mathcal{W}$ is said to be *intertwining* if $\mathrm{dom}(\phi)$ is invariant by $\rho^{\mathcal{V}}$ and

$$\phi\big(\rho^{\mathcal{V}}(g) \cdot v\big) = \rho^{\mathcal{W}}(g) \cdot \phi(v), \tag{9.2}$$

for all $g \in G$ and $v \in \mathrm{dom}(\phi)$.

Lemma 9.2. *Let $\mathcal{V}$, $\mathcal{W}$ be real finite dimensional vector spaces,*

$$\rho^{\mathcal{V}} : G \longrightarrow \mathrm{GL}(\mathcal{V}), \quad \rho^{\mathcal{W}} : G \longrightarrow \mathrm{GL}(\mathcal{W})$$

be smooth representations of the Lie group G and $\phi : \mathcal{V} \to \mathcal{W}$ be an intertwining linear map. Consider the Lie algebra representations

$$\bar{\rho}^{\mathcal{V}} : \mathfrak{g} \longrightarrow \mathrm{Lin}(\mathcal{V}, \mathcal{V}), \quad \bar{\rho}^{\mathcal{W}} : \mathfrak{g} \longrightarrow \mathrm{Lin}(\mathcal{W}, \mathcal{W})$$

given by the differential at $1 \in G$ of $\rho^{\mathcal{V}}$ and $\rho^{\mathcal{W}}$, respectively. We have that ϕ is also intertwining *for $\bar{\rho}^{\mathcal{V}}$ and $\bar{\rho}^{\mathcal{W}}$ in the sense that*

$$\phi \circ \bar{\rho}^{\mathcal{V}}(Z) = \bar{\rho}^{\mathcal{W}}(Z) \circ \phi,$$

for all $Z \in \mathfrak{g}$.

Proof. Simply differentiate the equality

$$\phi \circ \rho^{\mathcal{V}}(g) = \rho^{\mathcal{W}}(g) \circ \phi, \quad g \in G$$

with respect to g, at $g = 1$, in the direction of $Z \in T_1 G = \mathfrak{g}$. □

Proposition 9.3. *Let $\mathcal{V}$, $\mathcal{W}$ be real finite dimensional vector spaces and let $\rho^{\mathcal{V}} : G \to \mathrm{GL}(\mathcal{V})$, $\rho^{\mathcal{W}} : G \to \mathrm{GL}(\mathcal{W})$ be smooth representations of the Lie group G. Let $\phi : \mathrm{dom}(\phi) \subset \mathcal{V} \to \mathcal{W}$ be a smooth intertwining map defined in an open subset* $\mathrm{dom}(\phi)$ *of $\mathcal{V}$. If $f : M \to \mathrm{dom}(\phi)$ is a smooth map, then*

$$\nabla^g_X(\phi \circ f)(m) = \mathrm{d}\phi\big(f(m)\big) \cdot (\nabla^g_X f)(m), \quad m \in M.$$

Proof. If $u = \phi \circ f$, then

$$\tilde{u}(m) = \rho^{\mathcal{W}}\big(g(m)^{-1}\big) \cdot u(m) = \phi\big(\tilde{f}(m)\big), \quad m \in M,$$

so that:

$$\begin{aligned}(\nabla^g_X u)(m) &= \rho^{\mathcal{W}}\big(g(m)\big) \cdot \big(X(\tilde{u})(m)\big)\\ &= \rho^{\mathcal{W}}\big(g(m)\big) \cdot \mathrm{d}\phi\big(\tilde{f}(m)\big) \cdot \big(X(\tilde{f})(m)\big), \quad m \in M.\end{aligned} \tag{9.3}$$

Differentiating (9.2) with respect to v, we get

$$\mathrm{d}\phi\big(\rho^{\mathcal{V}}(g) \cdot v\big) \circ \rho^{\mathcal{V}}(g) = \rho^{\mathcal{W}}(g) \circ \mathrm{d}\phi(v),$$

for all $v \in \mathrm{dom}(\phi)$. It follows that

$$\begin{aligned}\mathrm{d}\phi\big(\tilde{f}(m)\big) &= \mathrm{d}\phi\big[\rho^{\mathcal{V}}\big(g(m)^{-1}\big) \cdot f(m)\big]\\ &= \rho^{\mathcal{W}}\big(g(m)^{-1}\big) \circ \mathrm{d}\phi\big(f(m)\big) \circ \rho^{\mathcal{V}}\big(g(m)\big), \quad m \in M.\end{aligned} \tag{9.4}$$

The conclusion follows by substituting (9.4) into (9.3). □

Corollary 9.4. *Let $\mathcal{V}$, $\mathcal{W}$ be real finite dimensional vector spaces and let $\rho^{\mathcal{V}} : G \to \mathrm{GL}(\mathcal{V})$, $\rho^{\mathcal{W}} : G \to \mathrm{GL}(\mathcal{W})$ be smooth representations of the Lie group G. Let $\phi : \mathcal{V} \to \mathcal{W}$ be a linear intertwining map. If $f : M \to \mathcal{V}$ is a smooth map, then*

$$\nabla^g_X(\phi \circ f)(m) = \phi\big((\nabla^g_X f)(m)\big), \quad m \in M. \qquad \square$$

Corollary 9.5. *Let $\mathcal{V}$, $\mathcal{W}$ be real finite dimensional vector spaces and let $\rho^{\mathcal{V}} : G \to \mathrm{GL}(\mathcal{V})$, $\rho^{\mathcal{W}} : G \to \mathrm{GL}(\mathcal{W})$ be smooth representations of the Lie group G. Let $\mathcal{V} \times \mathcal{W}$ be endowed with the product representation*

$$\rho^{\mathcal{V}\times\mathcal{W}}(g) \cdot (v, w) = \big(\rho^{\mathcal{V}}(g) \cdot v, \rho^{\mathcal{W}}(g) \cdot w\big), \quad g \in G,\ v \in \mathcal{V},\ w \in \mathcal{W}.$$

If $f = (f_1, f_2) : M \to \mathcal{V} \times \mathcal{W}$ is a smooth map, then

$$(\nabla^g_X f)(m) = \big((\nabla^g_X f_1)(m), (\nabla^g_X f_2)(m)\big), \quad m \in M.$$

Proof. Apply Corollary 9.4 by taking ϕ equal to the projection maps of $\mathcal{V} \times \mathcal{W}$. $\square$

Corollary 9.6. *Let $\mathcal{V}$, $\mathcal{W}$ and $\mathcal{Z}$ be real finite dimensional vector spaces and let $\rho^{\mathcal{V}} : G \to \mathrm{GL}(\mathcal{V})$, $\rho^{\mathcal{W}} : G \to \mathrm{GL}(\mathcal{W})$ and $\rho^{\mathcal{Z}} : G \to \mathrm{GL}(\mathcal{Z})$ be smooth representations of the Lie group G. Let*

$$\mathcal{V} \times \mathcal{W} \ni (v, w) \longmapsto v \star w \in \mathcal{Z}$$

be an intertwining bilinear map, where $\mathcal{V} \times \mathcal{W}$ is endowed with the product representation of $\rho^{\mathcal{V}}$ and $\rho^{\mathcal{W}}$. Given smooth maps

$$f_1 : M \longrightarrow \mathcal{V} \quad \text{and} \quad f_2 : M \longrightarrow \mathcal{W},$$

set

$$(f_1 \star f_2)(m) = f_1(m) \star f_2(m), \quad m \in M.$$

We have:

$$\big(\nabla^g_X(f_1 \star f_2)\big)(m) = (\nabla^g_X f_1)(m) \star f_2(m) + f_1(m) \star (\nabla^g_X f_2)(m), \quad m \in M. \qquad \square$$

9.1. **Functors of representations.** Consider the category $\underline{\mathfrak{Vec}}$ whose objects are real finite-dimensional vector spaces and the morphisms are linear isomorphisms. We say that a functor $\mathfrak{F} : \underline{\mathfrak{Vec}} \to \underline{\mathfrak{Vec}}$ is *smooth* if the map

$$\mathrm{GL}(\mathcal{V}) \ni \Omega \longrightarrow \mathfrak{F}(\Omega) \in \mathrm{GL}\big(\mathfrak{F}(\mathcal{V})\big) \tag{9.5}$$

is smooth, for any real finite-dimensional vector space $\mathcal{V}$. Examples of smooth functors are

- $\mathfrak{F}(\mathcal{V}) = \mathcal{V}^*$, $\mathfrak{F}(\Omega) = \Omega^{*-1}$,
- $\mathfrak{F}(\mathcal{V}) = \bigotimes_k \mathcal{V}$, $\mathfrak{F}(\Omega) = \bigotimes_k \Omega$,
- $\mathfrak{F}(\mathcal{V}) = \bigwedge_k \mathcal{V}$, $\mathfrak{F}(\Omega) = \bigwedge_k \Omega$,
- $\mathfrak{F}(\mathcal{V}) = \mathrm{Lin}(\mathcal{V}, \mathcal{V})$, $\mathfrak{F}(\Omega)(\Theta) = \Omega \circ \Theta \circ \Omega^{-1}$.

Since (9.5) is a smooth group homomorphism, it follows that a smooth representation $\rho : G \to \mathrm{GL}(\mathcal{V})$ yields a smooth representation of G on $\mathfrak{F}(\mathcal{V})$ by considering the composition of ρ with (9.5). We will denote this representation of G on $\mathfrak{F}(\mathcal{V})$ by ρ as well.

Denote by $\underline{\mathfrak{Vec}'}$ the category whose objects are real finite-dimensional vector spaces and morphisms are smooth maps. Given smooth functors

$$\mathfrak{F}_1 : \underline{\mathfrak{Vec}} \longrightarrow \underline{\mathfrak{Vec}}, \quad \mathfrak{F}_2 : \underline{\mathfrak{Vec}} \longrightarrow \underline{\mathfrak{Vec}}$$

we can regard them as functors

$$\mathfrak{F}_1 : \underline{\mathfrak{Vec}} \longrightarrow \underline{\mathfrak{Vec}'}, \quad \mathfrak{F}_2 : \underline{\mathfrak{Vec}} \longrightarrow \underline{\mathfrak{Vec}'}$$

simply by composing them with the inclusion of $\underline{\mathfrak{Vec}}$ in $\underline{\mathfrak{Vec}'}$. Let ϕ be a natural transformation from $\mathfrak{F}_1$ to $\mathfrak{F}_2$ regarded as functors taking values in $\underline{\mathfrak{Vec}'}$. This means that, for each real finite-dimensional vector space $\mathcal{V}$, we have a smooth map

$$\phi_{\mathcal{V}} : \mathfrak{F}_1(\mathcal{V}) \longrightarrow \mathfrak{F}_2(\mathcal{V})$$

and that, given real finite-dimensional vectors spaces $\mathcal{V}$, $\mathcal{W}$ and a linear isomorphism $\Omega : \mathcal{V} \to \mathcal{W}$, the diagram

$$\begin{tikzcd}
\mathfrak{F}_1(\mathcal{V}) \arrow[r, "\phi_{\mathcal{V}}"] \arrow[d, "\mathfrak{F}_1(\Omega)"'] & \mathfrak{F}_2(\mathcal{V}) \arrow[d, "\mathfrak{F}_2(\Omega)"] \\
\mathfrak{F}_1(\mathcal{W}) \arrow[r, "\phi_{\mathcal{W}}"'] & \mathfrak{F}_2(\mathcal{W})
\end{tikzcd}$$

commutes. Clearly, given a smooth representation $\rho : G \to \mathrm{GL}(\mathcal{V})$, then $\phi_{\mathcal{V}} : \mathfrak{F}_1(\mathcal{V}) \to \mathfrak{F}_2(\mathcal{V})$ is a smooth intertwining map.

10. Fundamental equations for rigid motion

Let $q : \mathcal{T} \to \mathcal{Q}$ be a configuration in rigid motion, with $\mathcal{Q}$ an orbit of $\mathrm{Iso}(\mathcal{P})$, and let $(q^0_\lambda)_{\lambda\in\Lambda}$ be an arbitrary point of $\mathcal{Q}$. Since the map

$$\mathrm{Iso}(\mathcal{P}) \ni g \longmapsto g \cdot (q^0_\lambda)_{\lambda\in\Lambda} \in \mathcal{Q} \tag{10.1}$$

is a smooth fibration, the smooth map $q : \mathcal{T} \to \mathcal{Q}$ admits a lifting

$$g : \mathcal{T} \longrightarrow \mathrm{Iso}(\mathcal{P})$$

through (10.1), so that

$$q_\lambda(t) = g(t)(q^0_\lambda), \quad t \in \mathcal{T},\ \lambda \in \Lambda.$$

We can write

$$\frac{\mathrm{d}g}{\mathrm{d}t}(t) = \Omega(t) \circ g(t), \quad t \in \mathcal{T},$$

with Ω a smooth curve in the Lie algebra of $\mathrm{Iso}(\mathcal{P})$. For each $t \in \mathcal{T}$, let $g_0(t)$ be the underlying linear map of $g(t)$, $\Omega_0(t)$ be the underlying linear map of $\Omega(t)$ and $\omega(t) \in \mathcal{V} \wedge \mathcal{V}$ be such that $\Omega_0(t) = \omega(t) \lrcorner\, \cdot$, so that g_0 is a smooth curve in $\mathrm{O}(\mathcal{V})$, Ω_0 is a smooth curve in $\mathrm{Lin_a}(\mathcal{V})$ and ω is a smooth curve in $\mathcal{V}\wedge\mathcal{V}$. We have

$$\frac{\mathrm{d}g_0}{\mathrm{d}t}(t) = \Omega_0(t)\circ g_0(t), \quad t\in\mathcal{T}. \tag{10.2}$$

Setting

$$\dot{q}_\lambda(t) = \frac{\mathrm{d}q_\lambda}{\mathrm{d}t}(t), \quad t\in\mathcal{T},$$

then

$$\dot{q}_\lambda(t) = \Omega(t)\big(q_\lambda(t)\big), \quad t\in\mathcal{T},$$

so that $\omega(t)$ is an angular velocity for the family of velocities $\big(\dot{q}_\lambda(t)\big)_{\lambda\in\Lambda}$. Pick a moving point $O:\mathcal{T}\to\mathcal{P}$ that *follows the same pattern of motion as the particle configuration in motion*, i.e., such that

$$O(t) = g(t)\big(O^0\big), \quad t\in\mathcal{T},$$

for some point $O^0\in\mathcal{P}$. For instance, the center of mass[5] has this property. Setting

$$\dot{O}(t) = \frac{\mathrm{d}O}{\mathrm{d}t}(t), \quad t\in\mathcal{T},$$

we have

$$\dot{O}(t) = \Omega(t)\big(O(t)\big), \quad t\in\mathcal{T}.$$

We then have:

$$\dot{q}_\lambda(t) = \dot{O}(t) + \omega(t) \lrcorner \big(q_\lambda(t) - O(t)\big), \quad t\in\mathcal{T},\ \lambda\in\Lambda.$$

If $I_{O(t)}$ denotes the tensor of inertia of the particle configuration $\big(q_\lambda(t)\big)_{\lambda\in\Lambda}$ with respect to $O(t)$, then Proposition 8.3 yields

$$T_{\dot{O}(t)} = \frac{1}{2}\big\langle I_{O(t)}\big(\omega(t)\big),\omega(t)\big\rangle,$$

$$P_{\dot{O}(t)} = \langle\tau_{O(t)},\omega(t)\rangle,$$

$$L_{O(t),\dot{O}(t)} = I_{O(t)}\big(\omega(t)\big), \tag{10.3}$$

where we are using the same notation of Section 5.

Consider the Lie group $G=\mathrm{Iso}(\mathcal{P})$ and the canonical representation[6]

$$\rho: G\longrightarrow \mathrm{GL}(\mathcal{V})$$

which carries each element of G to its underlying linear map. We also denote by ρ any representation obtained from ρ using a smooth functor (Subsection 9.1) and by $\bar{\rho}$ the corresponding representation of the Lie algebra $\mathfrak{g}$ of G obtained by differentiating ρ at the identity. We will denote by $\frac{\mathrm{D}^g}{\mathrm{d}t}$ the

[5]You could also use the center of mass with respect to a different (nonphysical) family of masses for the given particle configuration. More generally, pick a map $f:\mathcal{P}^\Lambda\to\mathcal{P}$ that is *intertwining* in the sense that $f\circ g^\Lambda = g\circ f$ for all $g\in\mathrm{Iso}(\mathcal{P})$, where $g^\Lambda:\mathcal{P}^\Lambda\to\mathcal{P}^\Lambda$ is the map given by the action of g on $\mathcal{P}^\Lambda$. Then define $O(t)$ by applying f to the particle configuration $\big(q_\lambda(t)\big)_{\lambda\in\Lambda}$, for all $t\in\mathcal{T}$.

[6]In what follows, we could, alternatively, take $G=\mathrm{O}(\mathcal{V})$, ρ the inclusion of G in $\mathrm{GL}(\mathcal{V})$ and consider the connection defined from the smooth map $g_0:\mathcal{T}\to\mathrm{O}(\mathcal{V})$.

covariant derivative with respect to the vector field $\frac{\mathrm{d}}{\mathrm{d}t}$ using the connection defined from the smooth map $g : \mathcal{T} \to G$ as in Section 9. We set:

$$\Omega_0^g(t) = \rho\big(g(t)\big)^{-1} \cdot \Omega_0(t), \quad I_{O(t)}^g = \rho\big(g(t)\big)^{-1} \cdot I_{O(t)}, \qquad t \in \mathcal{T}.$$

We note that

$$\Omega_0^g(t) = g_0(t)^{-1} \circ \Omega_0(t) \circ g_0(t),$$

so that:

$$\frac{\mathrm{d}g_0}{\mathrm{d}t}(t) = g_0(t) \circ \Omega_0^g(t), \quad t \in \mathcal{T}. \tag{10.4}$$

Moreover, $I_{O(t)}^g$ is the tensor of inertia of the particle configuration $(q_\lambda^0)_{\lambda \in \Lambda}$ with respect to the reference point O^0, so that the map

$$\mathcal{T} \ni t \longmapsto I_{O(t)}^g \in \mathrm{Lin}(\mathcal{V} \wedge \mathcal{V})$$

is constant. Thus

$$\frac{\mathrm{D}^g I_O}{\mathrm{d}t}(t) = 0, \tag{10.5}$$

where $I_O : \mathcal{T} \to \mathrm{Lin}(\mathcal{V} \wedge \mathcal{V})$ is defined by $I_O(t) = I_{O(t)}$.

The representation ρ of $G = \mathrm{Iso}(\mathcal{P})$ in $\mathrm{Lin_a}(\mathcal{V})$ is simply the composition of the homomorphism $\mathrm{Iso}(\mathcal{P}) \to \mathrm{O}(\mathcal{V})$ that carries an affine map to its underlying linear map with the adjoint representation of $\mathrm{O}(\mathcal{V})$ in its Lie algebra. Thus, the representation $\bar{\rho}$ of the Lie algebra $\mathfrak{g}$ in $\mathrm{Lin_a}(\mathcal{V})$ is simply the composition of the homomorphism $\mathfrak{g} \to \mathrm{Lin_a}(\mathcal{V})$ that carries an affine map to its underlying linear map with the adjoint representation of the Lie algebra $\mathrm{Lin_a}(\mathcal{V})$ in itself. More explicitly:

$$\bar{\rho}(\Theta) \cdot \Phi_0 = [\Theta_0, \Phi_0],$$

for all $\Theta \in \mathfrak{g}$, $\Phi_0 \in \mathrm{Lin_a}(\mathcal{V})$, where Θ_0 denotes the underlying linear map of Θ. Since the linear isomorphism (6.4) is intertwining for the representations ρ of G, it is also intertwining for the representations $\bar{\rho}$ of $\mathfrak{g}$ (Lemma 9.2). Thus, the representation $\bar{\rho}$ of $\mathfrak{g}$ in $\mathcal{V} \wedge \mathcal{V}$ is given by

$$\bar{\rho}(\Theta) \cdot \phi = [\theta, \phi],$$

for all $\Theta \in \mathfrak{g}$, $\phi \in \mathcal{V} \wedge \mathcal{V}$, where $\theta \in \mathcal{V} \wedge \mathcal{V}$ is such that $\theta \lrcorner \cdot$ is the underlying linear map of Θ and the Lie bracket of $\mathcal{V} \wedge \mathcal{V}$ is defined in (6.5). Using (9.1) we obtain:

$$\frac{\mathrm{D}^g \omega}{\mathrm{d}t}(t) = \frac{\mathrm{d}\omega}{\mathrm{d}t}(t) - \bar{\rho}\big(\Omega(t)\big) \cdot \omega(t) = \frac{\mathrm{d}\omega}{\mathrm{d}t}(t) - [\omega(t), \omega(t)] = \frac{\mathrm{d}\omega}{\mathrm{d}t}(t). \tag{10.6}$$

Since the bilinear pairing

$$\mathrm{Lin}(\mathcal{V} \wedge \mathcal{V}, \mathcal{V} \wedge \mathcal{V}) \times (\mathcal{V} \wedge \mathcal{V}) \longrightarrow \mathcal{V} \wedge \mathcal{V}$$

given by evaluation is intertwining, we have (Corollary 9.6):

$$\frac{\mathrm{D}^g I_O(\omega)}{\mathrm{d}t}(t) = \frac{\mathrm{D}^g I_O}{\mathrm{d}t}(t)\big(\omega(t)\big) + I_{O(t)}\Big(\frac{\mathrm{D}^g \omega}{\mathrm{d}t}(t)\Big),$$

where $I_O(\omega) : \mathcal{T} \to \mathcal{V} \wedge \mathcal{V}$ is defined by $\big(I_O(\omega)\big)(t) = I_{O(t)}\big(\omega(t)\big)$. Using (10.5) and (10.6) we obtain

$$\frac{\mathrm{D}^g I_O(\omega)}{\mathrm{d}t}(t) = I_{O(t)}\Big(\frac{\mathrm{d}\omega}{\mathrm{d}t}(t)\Big).$$

Thus, using again (9.1):

$$\begin{aligned}\frac{\mathrm{d}I_O(\omega)}{\mathrm{d}t}(t) &= \frac{\mathrm{D}^g I_O(\omega)}{\mathrm{d}t}(t) + \bar{\rho}\big(\Omega(t)\big) \cdot \big(I_{O(t)}\big(\omega(t)\big)\big) \\ &= \frac{\mathrm{D}^g I_O(\omega)}{\mathrm{d}t}(t) + \big[\omega(t), I_{O(t)}\big(\omega(t)\big)\big] \\ &= I_{O(t)}\Big(\frac{\mathrm{d}\omega}{\mathrm{d}t}(t)\Big) + \big[\omega(t), I_{O(t)}\big(\omega(t)\big)\big].\end{aligned}$$

Using now (10.3), we get:

$$\frac{\mathrm{d}L_{O,\dot{O}}}{\mathrm{d}t} = I_{O(t)}\Big(\frac{\mathrm{d}\omega}{\mathrm{d}t}(t)\Big) + \big[\omega(t), I_{O(t)}\big(\omega(t)\big)\big].$$

We now restrict ourselves to the case where O is the center of mass. Equation (5.3) then yields:

$$\tau_{C(t)} = I_{C(t)}\Big(\frac{\mathrm{d}\omega}{\mathrm{d}t}(t)\Big) + \big[\omega(t), I_{C(t)}\big(\omega(t)\big)\big], \tag{10.7}$$

where $\tau_{C(t)}$ denotes the total torque with respect to the center of mass. Since the torque of the reaction forces is zero, $\tau_{C(t)}$ is also equal to the total external torque with respect to the center of mass. Setting

$$\tau^g_{C(t)} = \rho\big(g(t)\big)^{-1} \cdot \tau_{C(t)}, \quad \omega^g(t) = \rho\big(g(t)\big)^{-1} \cdot \omega(t), \qquad t \in \mathcal{T},$$

we obtain from (10.7) and (10.6) that:

$$\tau^g_{C(t)} = I^g_{C(t)}\Big(\frac{\mathrm{d}\omega^g}{\mathrm{d}t}(t)\Big) + \big[\omega^g(t), I^g_{C(t)}\big(\omega^g(t)\big)\big]. \tag{10.8}$$

Equation (10.7) is the fundamental equation for rigid motion and equation (10.8) is the fundamental equation for rigid motion written in the reference frame that rotates with the object. The advantage of (10.8) is that $I^g_{C(t)}$ is independent of t, since it is just the tensor of inertia of the particle configuration $(q^0_\lambda)_{\lambda\in\Lambda}$ with respect to its center of mass. Equation (10.7) must be coupled with (10.2), keeping in mind that $\Omega_0(t) = \omega(t) \lrcorner \cdot$. Similarly, equation (10.8) must be coupled with (10.4), keeping in mind that $\Omega^g_0(t) = \omega^g(t) \lrcorner \cdot$. Moreover, equation (5.2) should be used to determine the motion of the center of mass, keeping in mind that the total force $F(t)$ equals the total external force, since reaction forces have zero total force.

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