Spanning trees of dense digraphs

Tássio Naia

ICGT 2018

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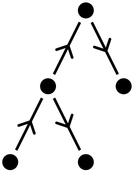
Joint work with Richard Mycroft

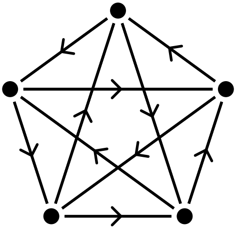
CIÊNCIA SEM FRONTEIRAS

ICGT 2018

UNIVERSITY OF BIRMINGHAM

Is there a copy of T in G?

tree T 

tournament G 

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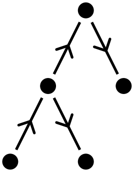

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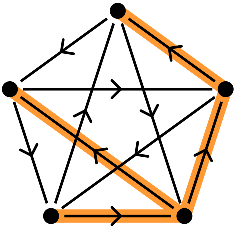

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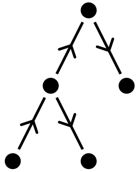

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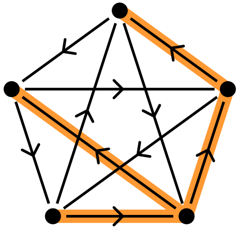
tournament G



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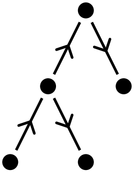
tree T 

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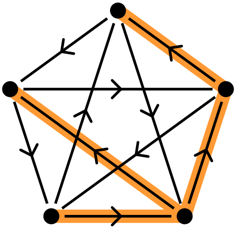
tournament G 

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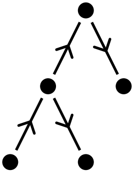

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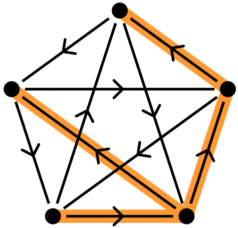

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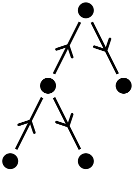

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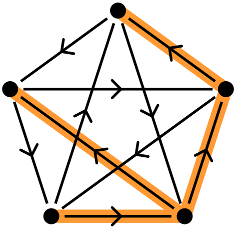

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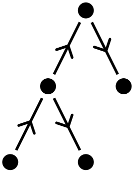

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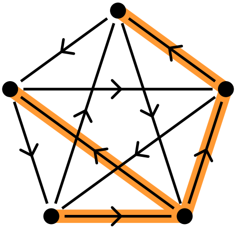

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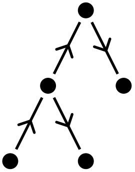

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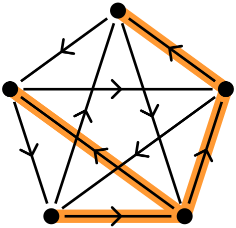

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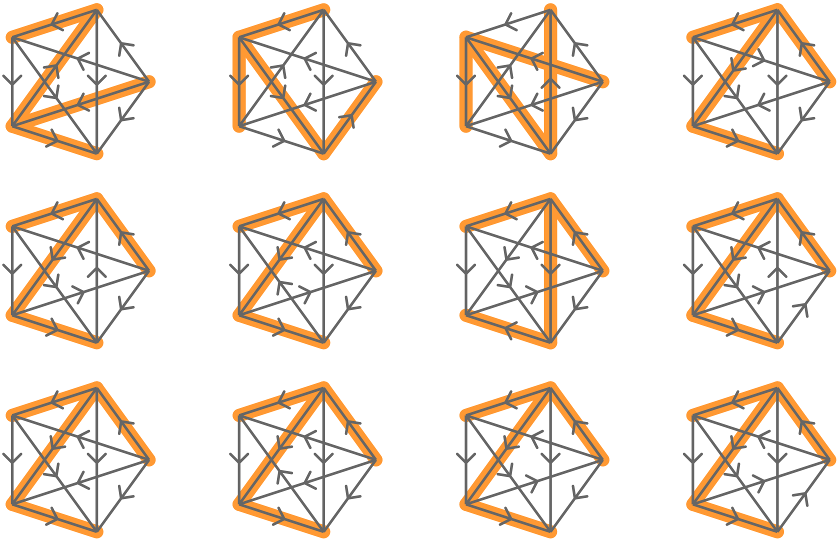
Proof by picture



Unavoidable trees — examples

Directed paths (Rédei '34) $\bullet \rightarrow \bullet \rightarrow \bullet \cdots \bullet \rightarrow \bullet$

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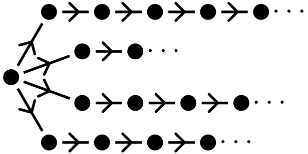
Some claws (Saks & Sós '84; Lu '93; Lu, Wang & Wong '98)



$\leq \left(\frac{3}{8} + \frac{1}{200}\right) n$ branches

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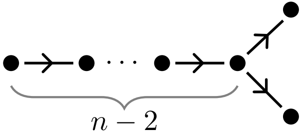

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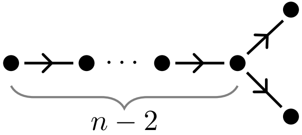

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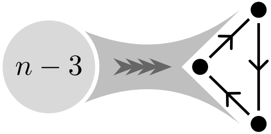

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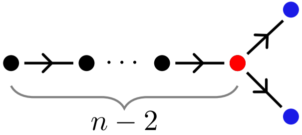

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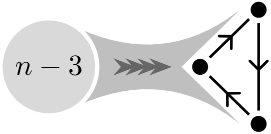

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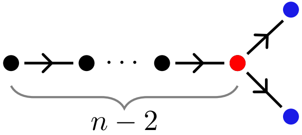

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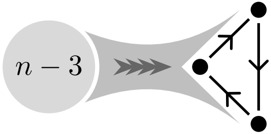

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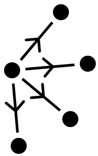
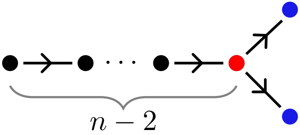


5 vertices

Avoidable trees — examples



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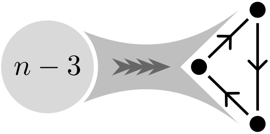

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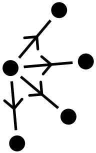

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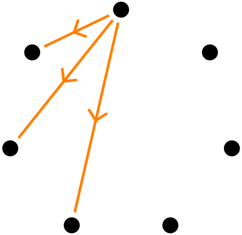

3-regular

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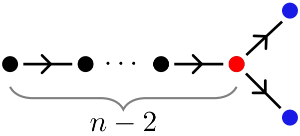

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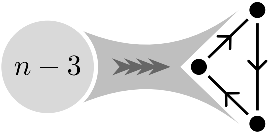

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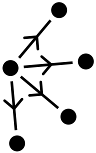

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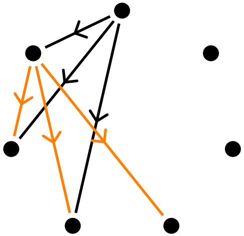

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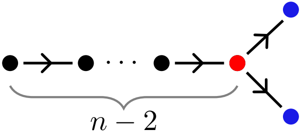

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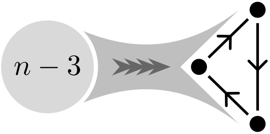

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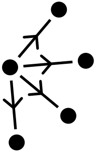

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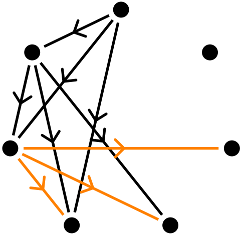

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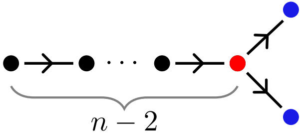

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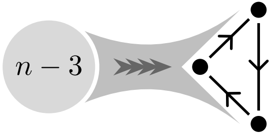

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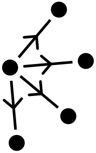

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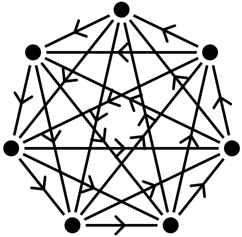

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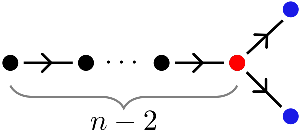

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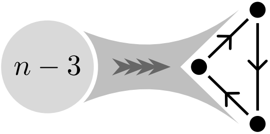

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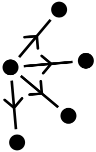

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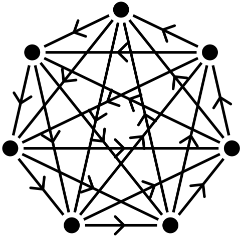

3-regular: $2 \cdot 5 - 3$ vertices

Avoidable trees, a longstanding conjecture

Conjecture (Sumner, 1971)

? $r(T) \leq 2n-2$ for all T. $v(T) = n$

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? T has ℓ_T leaves $\implies$ $r(T) \leq n + \ell_T - 1$. $v(T) = n$

Typical behaviour (1) — an approximate result

Kühn, Mycroft & Osthus, 2011

For all $\varepsilon, \Delta > 0$ and large n

$$\Delta(T) \leq \Delta \quad \Longrightarrow \quad r(T) \leq (1+\varepsilon)n. \qquad v(T) = n$$

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1^{st} moment + Cayley by Moon Lemma

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Mycroft, N., 2018

Havet and Thomassé's conjecture holds for almost every T.

Typical behaviour (2) — a probabilistic result

$\mathcal{T}_n := \{\, n\text{-vertex trees} \,\}$ oriented, labelled

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Bender, Wormald, 1988

There exists

$\mathcal{T} \subseteq \mathcal{T}_n$ with $|\mathcal{T}| = \big(1 - o(1)\big)|\mathcal{T}_n|$ and
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$$T \in \mathcal{T},\ \ G \in \mathcal{G} \quad \Longrightarrow \quad T \subseteq G.$$

Typical behaviour (2) — a probabilistic result

$\mathcal{T}_n := \{\, n\text{-vertex trees} \,\}$ oriented, labelled
$\mathcal{G}_n := \{\, n\text{-vertex tournaments} \,\}$ labelled

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Mycroft, N., 2018

Indeed!

Almost all trees are unavoidable!

Mycroft, N., 2018

There exists $\mathcal{T}'_n \subseteq \mathcal{T}_n$ of size $(1 - o(1))|\mathcal{T}_n|$ s.t.
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A sufficient condition which is typical!

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Ingredients:

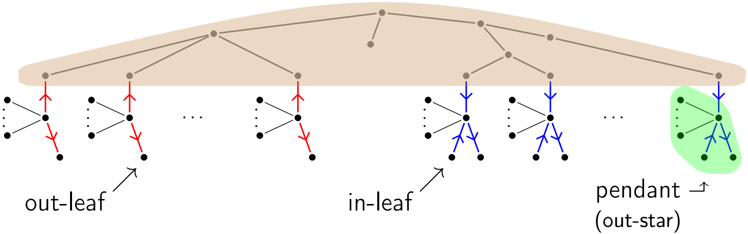
Mycroft, N., 2018

$\Delta(T) \leq \text{polylog}(n) \implies (1+\varepsilon)n$

Kühn, Mycroft & Osthus, 2011

Structure of large tournaments
$+$ randomized embedding algorithm

Nice trees



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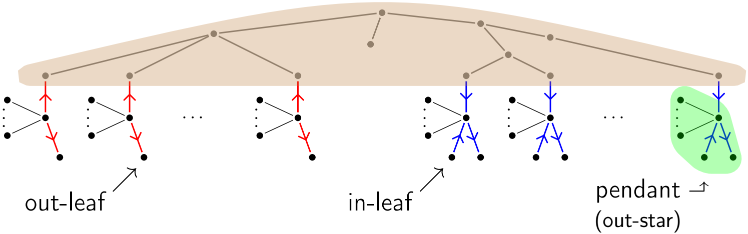

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$\forall \alpha > 0$, oriented tree T on n vertices is α-nice if

- αn pendant in-stars with an outleaf
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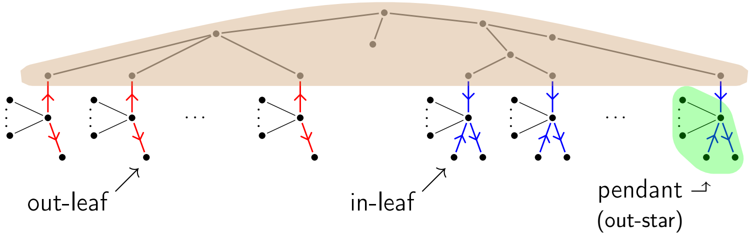

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Mycroft, N., 2018

For all $\alpha, C > 0$ and large n

T is $\boxed{\alpha\text{-nice}}$ and $\boxed{\Delta(T) \le (\log n)^C}$ $\implies$ T is unavoidable.

Nice trees are nice, and ALSO typical

Mycroft, N., 2018

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Moon, 1970

Almost all n-vertex trees have max. degree $(1 \pm o(1)) \left(\frac{\log n}{\log \log n}\right)$.

Basic lemma

Almost all n-vertex trees are $\frac{1}{250}$-nice.

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$\implies$ Almost all trees are unavoidable.

Beyond tournaments — Dirac-type results

Komlós, Sárközy, Szemerédi, 1995

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tree T, graph G s.t. $n = v(T) = v(G)$

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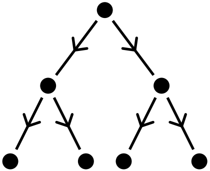
Work in progress:

1. Let $\Delta(T)$ grow with n
2. More general spanning subdigraphs (i.e.: allow some cycles)

Open questions

Q How about other oriented trees?

e.g., few leaves, or outbranchings



Q Characterise unavoidable trees

Q Orientations of n-chromatic graphs (Burr's conjecture)

Conjecture (Burr, 1980)

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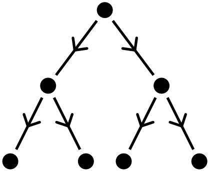

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Thanks! Other questions? Please ask!

Quick Reference

Paths and claws

Avoidable trees

Sumner

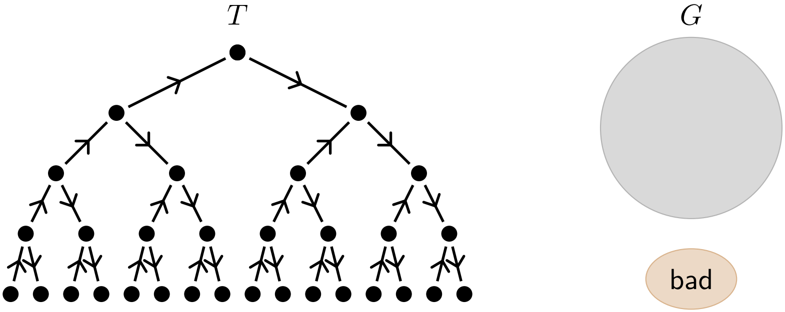
Bounded degree result

“Unavoidable” conjecture

Result summary

Nice trees are nice — our sufficient condition

Embedding T to G (general scheme)



Embedding T to G (general scheme)

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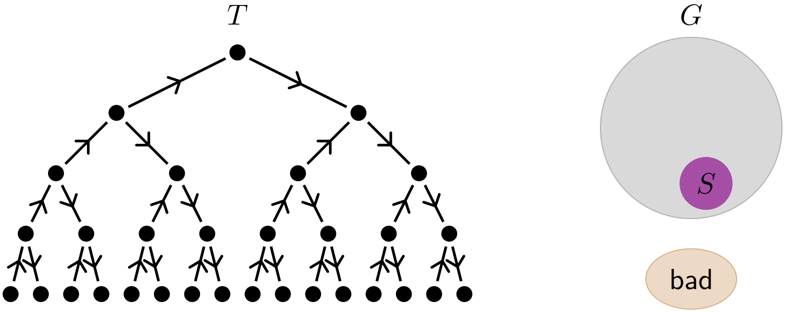

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- reserve a small set $S \subseteq G$
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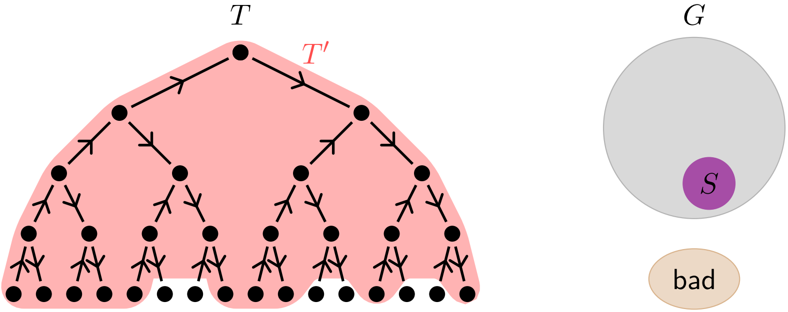

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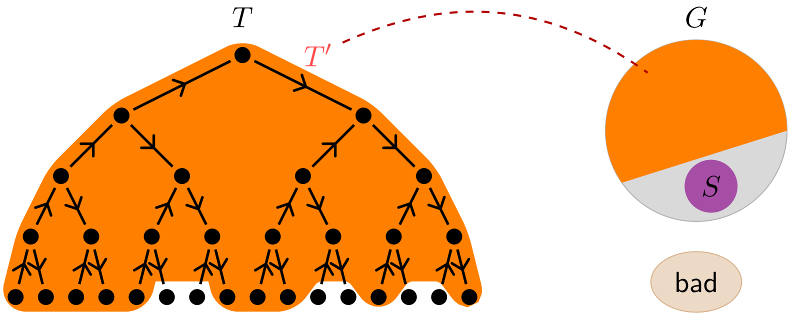

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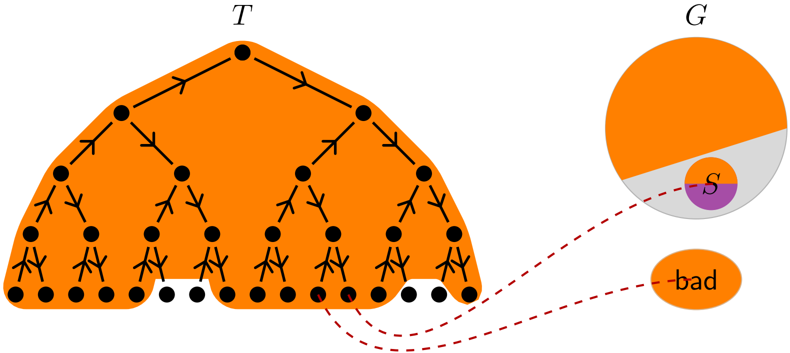

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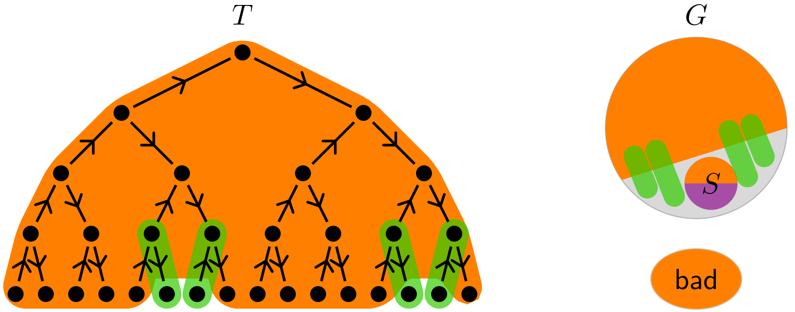

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