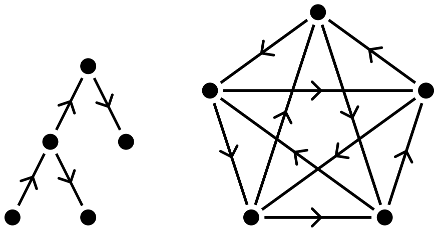
Unavoidable trees in tournaments

Richard Mycroft   Tássio Naia

20 April 2016

Tournaments & Oriented Trees

Oriented tree T on n vertices, tournament G



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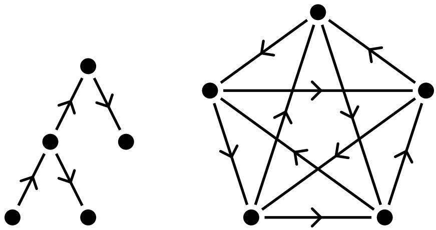

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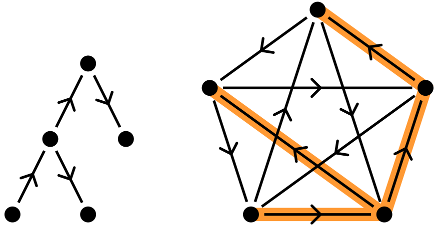

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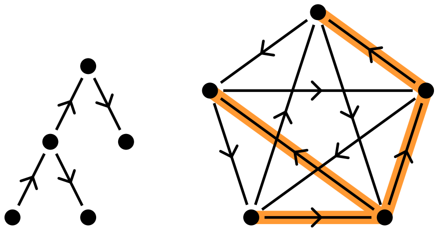

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Definition (unavoidable trees)

A (oriented) tree T with $|V(T)| = n$ is unavoidable if every tournament on n vertices contains a copy of T.

Unavoidable trees — examples

Directed paths (Rédei 1934) $\bullet \to \bullet \to \bullet \cdots \bullet \to \bullet$

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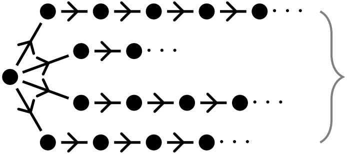
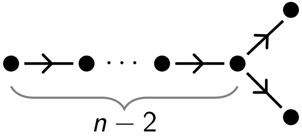
Some claws (Saks & Sós 84; Lu '93; Lu, Wang & Wong '98)



$\leq \left(\frac{3}{8} + \frac{1}{200}\right) n$ branches

Examples — non-unavoidable trees



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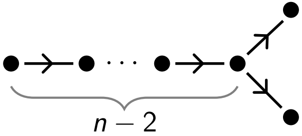

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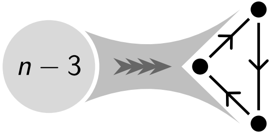

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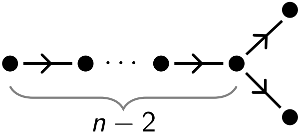

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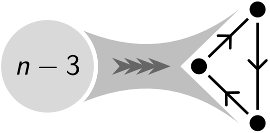

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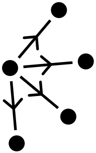
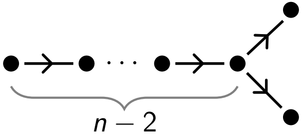


5 vertices

Examples — non-unavoidable trees



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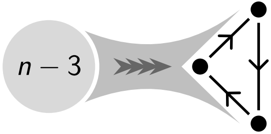

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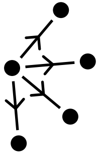

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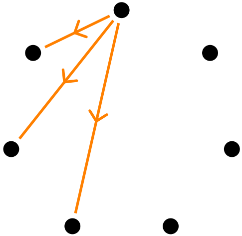

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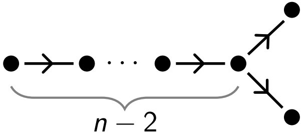

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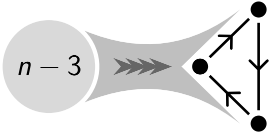

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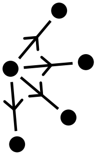

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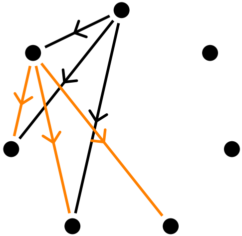

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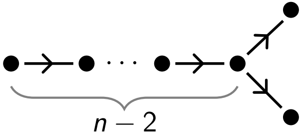

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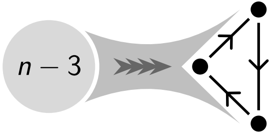

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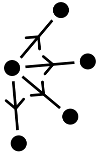

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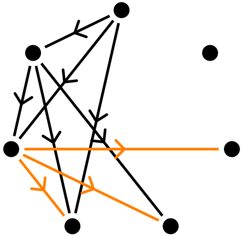

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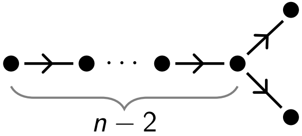

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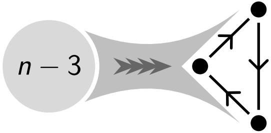

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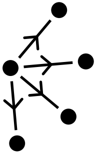

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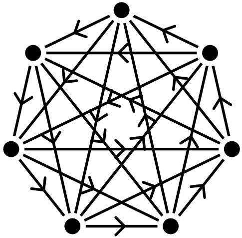

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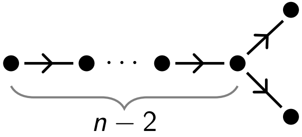

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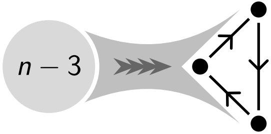

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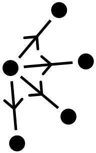

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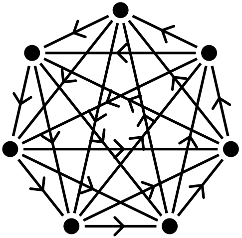

3-regular: $2 \cdot 5 - 3$ vertices

Conjecture and proofs

Sumner's conjecture (1971)

Every oriented tree on n vertices is contained in every tournament on $2n-2$ vertices.

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publ.	**who**	**tournament size**
1982	Chung	$n^{1+o(n)}$
1983	Wormald	$n\log_2(2n/e)$
1991	Häggkvist & Thomason	$12n$ and also $(4+o(n))n$
2002	Havet	$38n/5-6$
2000	Havet & Thomassé	$(7n-5)/2$
2004	El Sahili	$3n-3$
2011	Kühn, Mycroft & Osthus	$2n-2$ for large n

Embedding bounded-degree trees

Theorem (Kühn, Mycroft & Osthus, 2011)

For all $\alpha, \Delta > 0$ there exists n_0 such that if $n > n_0$, each tournament on $(1+\alpha)n$ vertices contains any tree T on n vertices with $\Delta(T) \leq \Delta$.

When can we do better?

Question (Alon)

Which trees are unavoidable?

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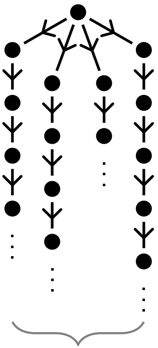
Paths, some claws



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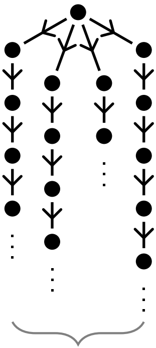
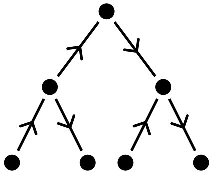
Paths, some claws  , this tree: 

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A family of examples – alternating trees

Alternating trees are rooted trees $\mathbb{B}_\ell$

$\mathbb{B}_1$: $\bullet$

$r(\mathbb{B}_1)$

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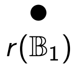

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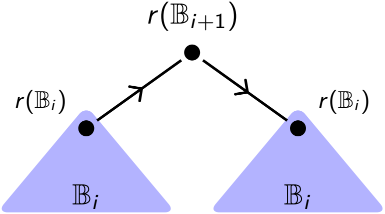

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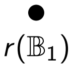

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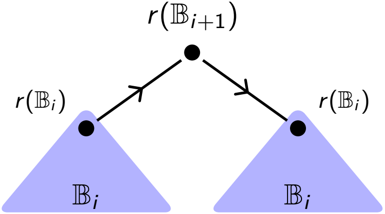

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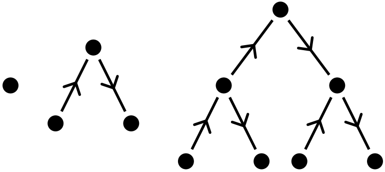

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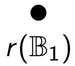

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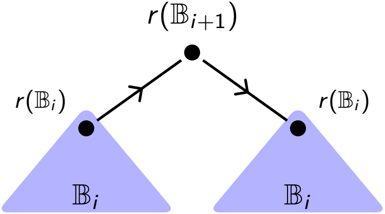

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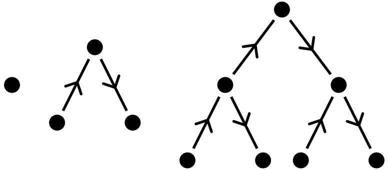

Theorem (Mycroft, N. 2016$^+$)

For ℓ large enough, $\mathbb{B}_\ell$ is unavoidable.

More examples – balanced q-ary trees

q-ary tree are rooted trees $\mathbb{B}_\ell^q$ $\quad q \in \mathbb{N}$

$\mathbb{B}_1^q$:

$r(\mathbb{B}_1^q)$

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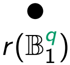

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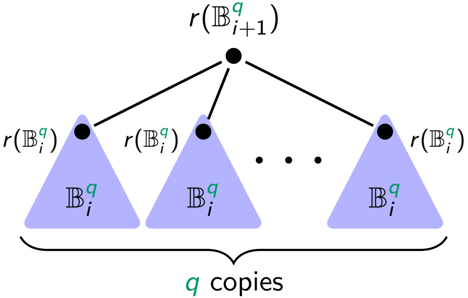

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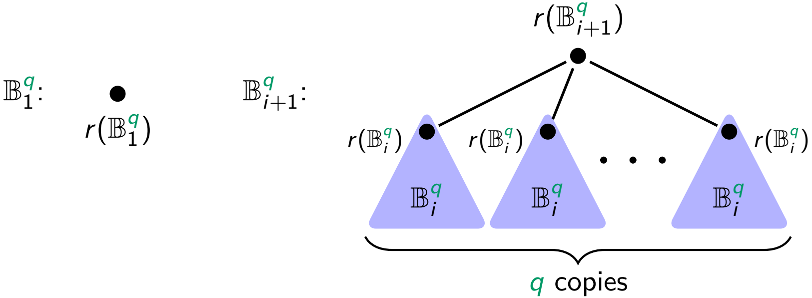

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For each $q \in \mathbb{N}$, if ℓ large enough then almost all orientations of $\mathbb{B}_\ell^q$ are unavoidable.

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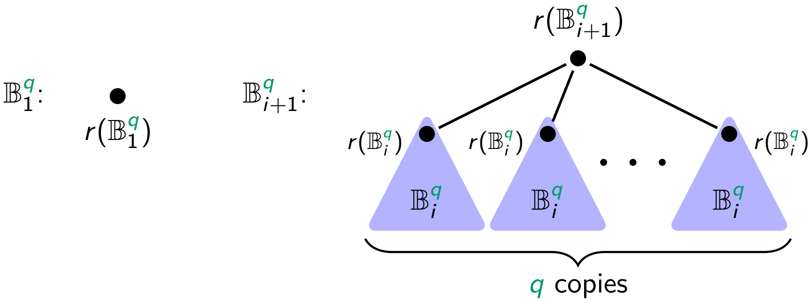

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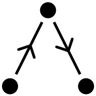
The method works a much wider class of trees.

Some definitions and a property of $\mathbb{B}_\ell$

$\mathbb{B}_2$ is a cherry:



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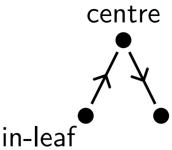

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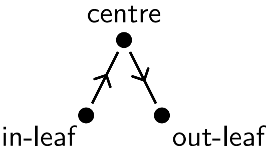

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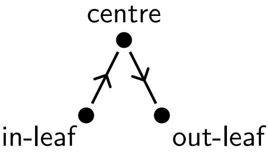

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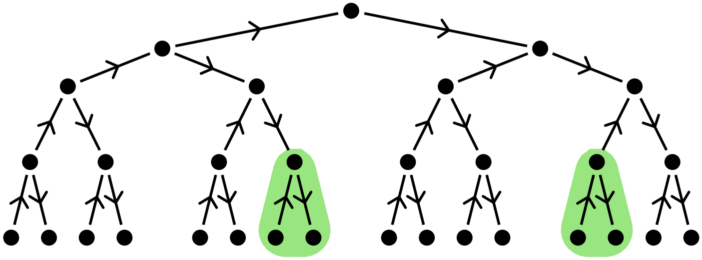

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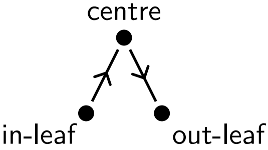

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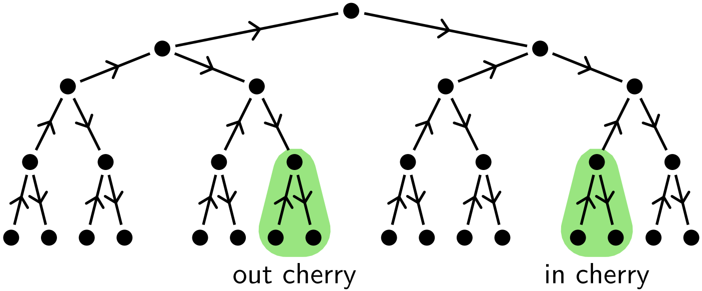

Characterization of large tournaments

Theorem (Kühn, Mycroft, Osthus 2011)

Large tournaments contain either a large strong cut or a large robust expander of linear minimum semidegree.

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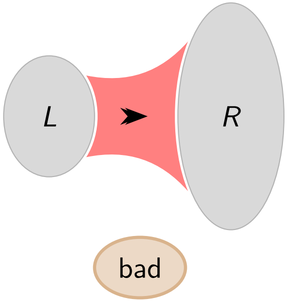

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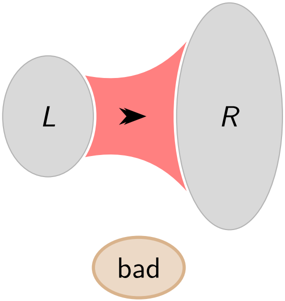

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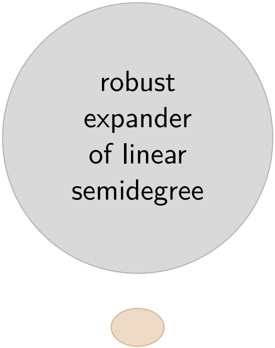

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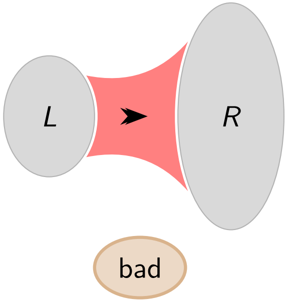
Theorem (Kühn, Osthus, Treglown 2010)

A large robust expander of linear minimum semidegree contains a regular cycle of cluster tournaments.



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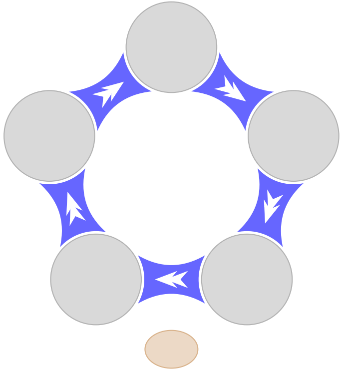
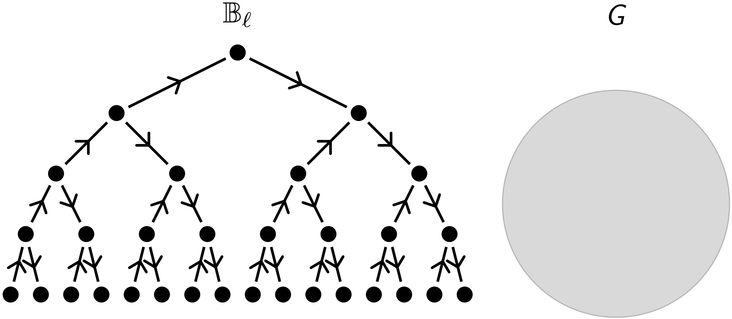


Embedding $\mathbb{B}_\ell$ to G (general scheme)



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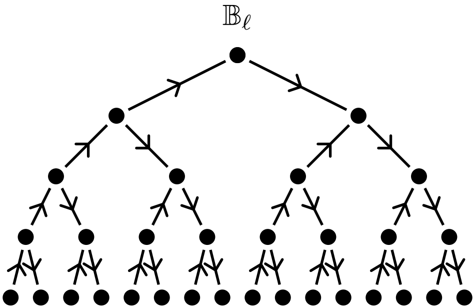

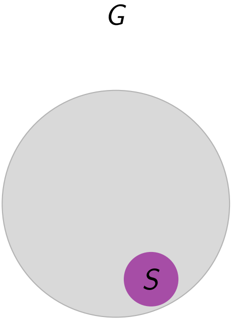

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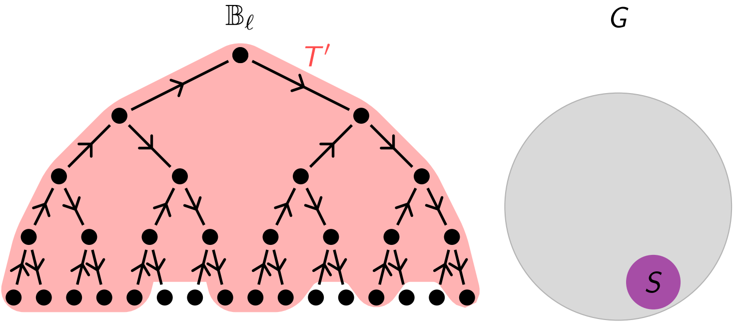

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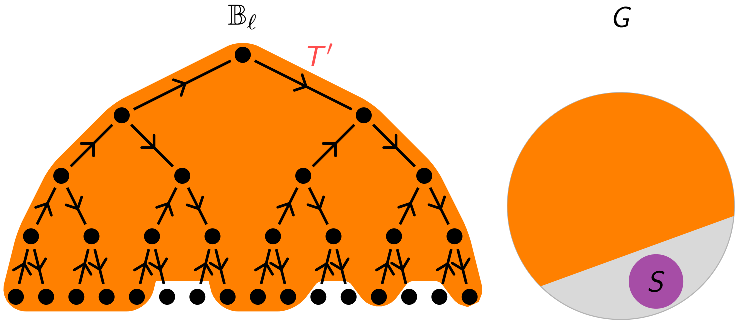

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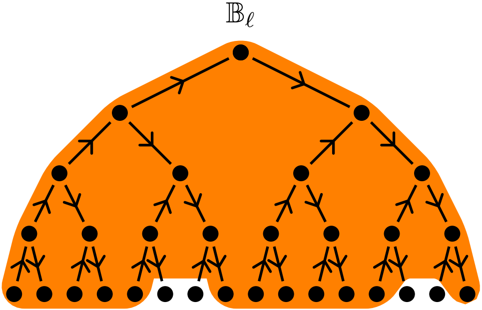

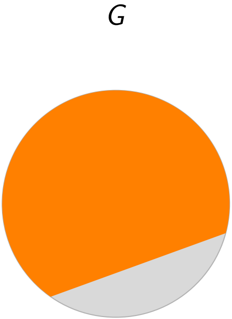

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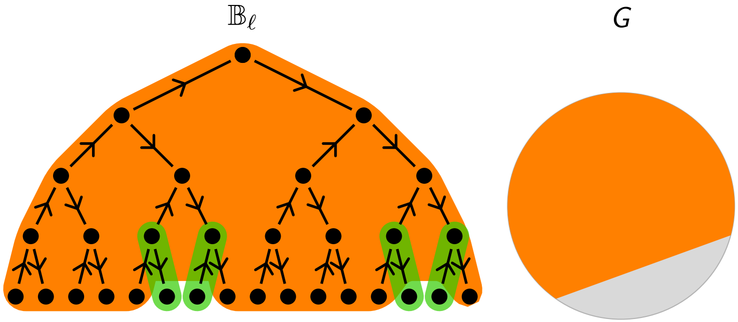

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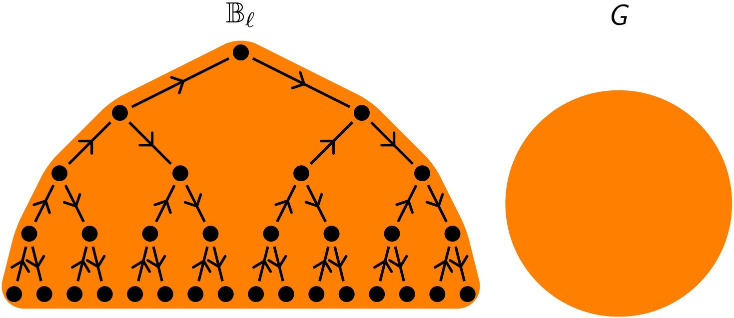

Beyond binary trees

Theorem (R. Mycroft, N., 2016^+)

For all $q > 0$ there exists n_0 such that if $n > n_0$ almost all orientations of every "roughly balanced" q-ary tree on n vertices are unavoidable.

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Work in progress

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Questions

How about unbounded degree? (hopefully soon!)
How about the binary arborescence?

Quick Reference