Kuratowski’s Theorem

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ABSTRACT

We present three short proofs of Kuratowski’s theorem on planarity of graphs and discuss applications, extensions, and some related problems.

0. INTRODUCTION

Planar graphs are of great importance in graph theory. They are interesting in their own right and their chromatic, enumerative, hamiltonian, and other properties have been studied in great detail. Furthermore, planar graphs are of some importance in the study of convex polytopes since, by Steinitz’s theorem, the 1-skeletons of the 3-dimensional polytopes are precisely the 3-connected planar graphs. Finally, planar graphs provide an important link between graphs and matroids.

In a classical paper of 1930, Kuratowski [25] characterized the planar graphs.

Kuratowski’s Theorem. A graph is planar if and only if it contains no subdivision of $K_5$ or $K_{3,3}$.

This result was discovered independently by Frink and Smith (see [3, p. 148]) and Pontrjagin (see [6]), and the restriction of Kuratowski’s theorem to cubic graphs was found independently by Menger [29].

The importance of Kuratowski’s theorem is not so much its applications to planar graph theory (in fact, it is used only in relatively few results on planar graphs). It rather lies in the fact that a characterization of planar graphs in terms of essentially a finite number of forbidden subgraphs exists at all. Also, Kuratowski’s theorem has a special ranking among the known planarity criteria not only because of its beauty and simplicity, but also because it implies rather easily the planarity criteria by MacLane [27] and Whitney [60] (as observed by Tutte [47] and rediscovered by O’Neil [30] and Parsons [32]) and other planarity criteria as well ([40]). It should also be noted that, unlike most other planarity criteria, Kuratowski’s theorem provides a useful characterization of the nonplanar graphs. Finally, it seems that almost all known proofs of Kuratowski’s theorem can be turned into polynomially

bounded algorithms for testing planarity of graphs. The asymptotically fastest (but not conceptually simplest) algorithm for this was found by Hopcroft and Tarjan [20]. For the algorithmic aspects of Kuratowski's theorem, the reader is referred to the survey paper of Williamson [59].

Whitney [61] showed how Kuratowski's theorem can be obtained from his planarity criterion [60] by proving the purely combinatorial statement that a graph has a dual graph if and only if it contains no subdivision of $K_5$ or $K_{3,3}$. The first relatively simple proof of Kuratowski's theorem was published in 1954 by Dirac and Schuster [9]. This proof (or a variant of it) is included in several textbooks on graph theory. In Secs. 3, 4, and 5 of this paper we present three other proofs. The first proof is a modification of Burstein's proof [6]. The idea of the proof is implicitly contained in work of Tutte [48]. This proof is more combinatorial in nature, that is the proof of Dirac and Schuster. The proof is rather straightforward, but a complete version of the proof does contain several details of which we leave for the reader.

The second proof which is very short is obtained by combining ideas of Tutte [49] and Kelmans [23]. It is based on a characterization of 3-connected graphs found independently by Barnette and Grünbaum [2] and Tietze [43].

The third proof, due to the author [40], is based on a simple contraction lemma for 3-connected graphs which, in a sense, a "dual" version of the above characterization of 3-connected graphs. This proof is very short, too, and it has the advantage that, without any extra effort, it also establishes the existence of straight-line representations of planar graphs [10, 26, 53] as well as the convex representations of 3-connected planar graphs [45]. A similar and even shorter proof of Kuratowski's theorem was found by Fournier [12, 13] but Case 1 of that proof seems to be incomplete.

In Sec. 6, we discuss problems and results related to Kuratowski's theorem and in Sec. 7, we consider the extensions of the theorem to infinite graphs (only in that section the graphs may be infinite).

1. GEOMETRIC PREQUISITES

In the literature there are often statements on plane graphs which are said to be intuitively clear, but to have complicated formal proofs. We shall now point out that this need not be so. Indeed, the reader will not be assumed to know anything about the Euclidean plane $R^2$ except for the definition of a polygonal arc. Our terminology is essentially that of Harary [17]. However, we shall assume that the edges of a plane graph are polygonal arcs. This definition is more restrictive than the usual definition in which each edge of a plane graph is a Jordan arc, i.e., the image of a continuous 1-1 function $f : [0, 1] \rightarrow R^2$. However, using the uniform continuity of such functions it is easy to re-draw the graph such that the edges are polygonal arcs (see [4, p. 16]). If $G$ is a plane graph and $p$ and $q$ are points in the complement of the point set of $G$, then $p$ and $q$ are said to be $G$-equivalent if they are connected.

class is called a face of $G$. The boundary of a face is defined in the obvious way and clearly any such boundary is a subgraph of $G$. The basic tool for dealing with plane graphs is the following restricted version of the Jordan curve theorem which has a very short proof (see e.g., [51]).

**Theorem 1.1.** If $J$ is a closed polygonal curve, then $J$ has precisely two faces, each of which has $J$ as boundary.

**Proof.** Let $p_0$ be a fixed point on $J$. If $p$ is any point in $R^2 \setminus J$, then it is easy to see that $p$ is equivalent to points arbitrarily close to $p_0$, and it follows that $J$ has at most two faces and that each face has $J$ as a boundary.

It remains to be proved that $J$ has more than one face. For any point $p$ in $R^2 \setminus J$ and any real number $\alpha$ we consider the circle $L$ starting at $p$ and forming the angle $\alpha$ with the $x$-axis. Then $L \cap J$ consists of intervals on $L$. We let $f(p, \alpha)$ be the number of those intervals such that $J$ "enters" the interval from different sides on $L$ (we may call this the number of times $L$ "crosses" $J$). We let $f(p, \alpha) = f(p, \alpha)$ reduced modulo 2. It is easy to see that $f(p, \alpha)$ is a continuous function of $\alpha$ and thus it is a constant function which we call the parity of $p$. Clearly, the parity is the same for all points $p$ on a straight line segment not intersecting $J$ and so the parity is the same for all points in a face of $J$. On the other hand, the parity is not the same for the endpoints on a straight line segment which intersects $J$ in precisely one point and thus $J$ has at least two faces. This completes the proof.

Clearly, one of the faces which is called the exterior of $J$ is unbounded and the other face, the interior of $J$, is bounded.

Using Theorem 1.1, it is easy to prove the following:

**Corollary 1.2.** If $J_1, J_2, J_3$ are polygonal arcs such that any two of these have their endpoints $p$ and $q$ and no other point in common, then $J_1 \cup J_2 \cup J_3$ has precisely three faces with boundaries $J_1 \cup J_2$, $J_1 \cup J_3$, and $J_2 \cup J_3$, respectively.

**Corollary 1.3.** If $G$ is a 2-connected plane multigraph then each face of $G$ is bounded by a cycle of $G$. The number of faces equals $|E(G)| - |V(G)| + 2$.

**Proof (due to W.T. Tutte).** We prove the corollary by induction on the number of edges of $G$. If $G$ has only two edges, Theorem 1.1 applies. So we proceed to the induction step and assume that $G$ is not a cycle. Let $G'$ be a 2-connected proper subgraph of $G$ which is maximal under these restrictions. It is easy to see that $G$ is obtained from $G'$ by adding a path to $G'$ joining two interior vertices such that all intermediate vertices (if any) have degree 2. By
the induction hypothesis, $G'$ satisfies the assertion of Corollary 1.3, and using Corollary 1.2, $G$ does too.

**Lemma 1.4.** If $G$ is a plane graph and $E$ is the edge set of a face boundary of $G$, then $G$ has a plane representation such that $E$ is the boundary of the unbounded face.

The proof is an easy exercise and we leave it for the reader.

These are the only geometric results which are needed for the proofs of Sec. 4 and 5. In the proof of Sec. 3 (and in other contexts as well) the following result is useful.

**Theorem 1.5.** If $J$ and $J'$ are closed polygonal curves of $R^2$, then there is a homeomorphism $\phi$ of $R^2$ into $R^2$ such that $\phi(J) = J'$, and $\phi$ takes polygonal arcs onto polygonal arcs.

By a standard counting argument it follows from the restricted version of Euler's formula given in Corollary 1.3 that a 2-connected plane graph with $n$ vertices has at most $3n-6$ edges and in addition, it has no 3-cycles, it has at most $2n-4$ edges. Thus $K_5$ and $K_{3,3}$ are nonplanar and we have obtained a rigorous proof of the easy part of Kuratowski's theorem:

**Proposition 1.6.** A plane graph contains no subdivision of $K_5$ or $K_{3,3}$.

We shall present three proofs of the nontrivial part of Kuratowski's theorem. Using Lemma 1.4, it is a routine matter to prove that a minimal counterexample to Kuratowski's theorem must be 3-connected (see e.g. [5, p. 152]). So it is sufficient to prove the theorem for 3-connected graphs.

We shall first need a combinatorial lemma.

## 2. OVERLAPPING $C$-COMPONENTS

If $C$ is a cycle of a graph $G$, then a $C$-component of $G$ is either a $K_2$ having its two vertices but not its edge on $C$ or else $C$ consists of a connected component of $G - V(C)$ together with all edges from this component to $C$ and all edges of these edges. The vertices of $V(H) \cap V(C)$ are called the vertices of attachment of $H$. If $H'$ is another $C$-component, the $H$ and $H'$ avoid one another if $H$ has two vertices $x$ and $y$ such that all vertices of attachment of $H$ are on one of the paths of $C$ with ends $x$ and $y$ and all vertices of attachment of $H'$ are on the other path. If $H$ and $H'$ do not avoid one another, they overlap. We say that $H$ and $H'$ are $C$-equivalent if $V(C) \cap V(H') = V(C) \cap V(H)$ and this set has three vertices. Finally, $H$ and $H'$ are skew if $C$ contains four distinct vertices, $x_1, x_2, x_3, x_4$ in that cyclic order such that $x_1$ and $x_3$ are in $H$ and $x_2$ and $x_4$ are in $H'$.

Tutte [50] has investigated $C$-components (under the name bridges) in great detail. We shall use only one of his results. Clearly, two $C$-equivalent bridges and two skew bridges overlap. Conversely, we have

**Lemma 2.1.** If $H$ and $H'$ are overlapping $C$-components, then either they are skew or $C$-equivalent.

**Proof.** Suppose $H$ and $H'$ overlap. If $V(H) \cap V(H') \subseteq V(C)$, then $|V(H') \cap V(C)| = 3$ and $H$ and $H'$ are $C$-equivalent if $|V(H') \cap V(C)| = 3$. Otherwise, they are skew. On the other hand, if $x$ is a vertex of $V(H') \cap V(C)$ not in $H'$, then $V(H') \cap V(C)$ is not contained in the unique segment of $C$ which contains $x$ and has precisely its ends in common with $H'$. But this implies that $H$ and $H'$ are skew.

## 3. A PROOF INVOLVING OVERLAPPING COMPONENTS

Let $G$ be a 2-connected nonplanar graph and suppose Kuratowski's theorem holds for all smaller 2-connected graphs. We shall prove that $G$ has a subdivision of $K_5$ or $K_{3,3}$. Assume therefore this is not the case.

Suppose first that $G$ has a vertex $x$ of degree 2. Then we delete $x$ from $G$ and add the edge between the two neighbors of $G$ if this edge is not already present. By the induction hypothesis, Kuratowski's Theorem holds for the resulting graph and it is easy to see that it holds for $G$ too.

So we assume that each vertex of $G$ has degree at least 3. It is then easy to see that $G$ has a cycle $C$ such that $G$ has a $C$-component which is a $K_5$. If $G$ has no other $C$-components, then $G$ is planar so assume $G$ has at least two $C$-components.

We consider now the overlap graph for $G$ with respect to $C$, i.e., the graph whose vertices are the $C$-components such that two vertices are adjacent if and only if the corresponding $C$-components overlap.

We consider first the case where the overlap graph is bipartite with bipartition $A \cup B$. Say. We represent $A$ by a polygon in the plane and prove that $G$ has a representation such that all $C$-components of $A$ (resp. $B$) are in the interior (respectively exterior) of $C$. Let $H_1, H_2, \ldots, H_p$ be the $C$-components of $A$ and suppose we have already drawn $H_1, H_2, \ldots, H_{k-1}$ (1 $\leq k \leq p$) in the interior of $C$. By hypothesis, $C \cup H_1$ is planar and, by the definition of a $C$-component, $C$ is the boundary of a face in any plane representation of $C \cup H_1$. By Lemma 1.4, we can assume that this face is the unbounded face. Since $H_k$ does not overlap any of $H_1, H_2, \ldots, H_{k-1}$, there is a path $P$ containing all vertices of attachment of $H_k$ such that no intermediate vertex of $P$ is in any $H_i$, $i < k$. Using Corollary 1.3 and Theorem 1.5, we can draw $H_k$ in some bounded face of $C \cup H_1 \cup H_2 \cup \ldots \cup H_{k-1}$. By doing the same (in the exterior of $C$) for the $C$-components of $B$, we obtain a plane representation of $G$, a contradiction.
It remains to prove the purely graph theoretic assertion that any graph $G$ containing a cycle $C$ such that the overlap graph of $G$ with respect to $C$ is nonbipartite has a subdivision of $K_3$ or $K_{3,3}$ as a subgraph. It is sufficient to prove this for any such edge-minimal graph $G$. The overlap graph has an odd cycle with vertices $H_0, H_1, \ldots, H_{2k}$ where the indices are expressed modulo $2k + 1$. We choose $C$ such that $k$ is least possible.

Suppose first $k \geq 2$. The minimality of $k$ implies that $H_1$ overlap $H_{i-1}$ and $H_{i+1}$ and no other $H_i$. In particular, $H_{i-1}$ and $H_{i+1}$ are not C-equivalent for otherwise $H_i$ would also overlap $H_{i+1}$. So any two $C$-components $H_i$ and $H_{i+1}$ are skew by Lemma 2.1. Let $x_j, y_j$ be vertices of $H_{i-1}$ such that $H_i$ intersects each of the segments of $C - \{x_j, y_j\}$ (respectively $C - \{y_j, y_j\}$). Since $H_{i-1}$ and $H_{i+1}$ avoid one another, we can assume that $x_j, y_j, y_j, y_j$ occur on $C$ in that cyclic order (where $x_j = y_j, y_j = x_j$, is possible).

Now let $z_1$ be a vertex of attachment of $H_i$ in the segment of $C$ from $x_j$ to $y_j$ (respectively from $y_j$ to $y_j$) such that $\{x_j, y_j\} \cap \{z_1, z_1\} = \emptyset$ and $P$ be a path from $H_i$ to $z_1$. Since $H_i$ overlaps no $H_{i-1}, j \neq \pm 1$, it follows easily that the union of $P$ and a segment of $C$ is a cycle $C'$ containing all vertices of attachment of each $H_i, i \neq \pm 1, j \neq \pm 1$. Let $H_{i-1}$ be the $C$-component containing $V(C') \setminus V(C)$. Then $H_{i-1}, \ldots, H_{i+1}$ is an odd cycle of the overlap graph of $G$ with respect to $C$. This contradicts the minimality of $k$.

We can therefore assume that $k = 1$, i.e., $H_0, H_1, H_2$ are three pairwise overlapping $C$-components. If they each have just two vertices of attachment, then $G$ contains a subdivision of $K_{3,3}$. So assume $V(H_i) \cap V(C) = \{x_0, x_1, \ldots, x_{m}\}$. For each $i = 1, 2, \ldots, m$, let $e_i$ be an edge of $H_0$ incident with $x_i$. The minimality of $G$ implies that the overlap graph for $G - e_i$ with respect to $C$ is bipartite, so for each $i = 1, 2, \ldots, m, H_0 - e_i$ does not overlap both of $H_i, H_2$. If $m \geq 4$, we easily conclude that $m = 4$ and that the notation can be chosen such that $V(H_1) \cap V(C) = \{x_1, x_2\}$ and $V(H_2) \cap V(C) = \{x_2, x_3\}$. It is easy to find a subdivision of $K_{3,3}$ or $K_3$ in this case. On the other hand, if $m = 3$, then the notation can be chosen such that $H_1$ does not overlap $H_0 - e_i$ for $i = 1, 2$. Since $H_1$ overlaps $H_0$, we conclude that $H_1$ and $H_2$ are C-equivalent. Using the fact that $H_2$ overlaps $H_0$, it is easy to find a subdivision of $K_{3,3}$ in this case (we leave the details for the reader).

This proves the nontrivial part of Kuratowski's theorem for 2-connected graphs.  

In the next two sections we shall prove the theorem for 3-connected graphs only.

4. A PROOF BASED ON A CHARACTERIZATION OF 3-CONNECTED GRAPHS

The proof of Corollary 1.3 shows that any 2-connected graph can be obtained from $K_3$ by successively subdividing an edge through the insertion of a vertex of degree 2 or adding an edge. A well-known characterization of 3-connected graphs was found by Tutte [46]. We shall here use another characterization of 3-connected graphs due to Barnette and Grünbaum [2] and Titov [43]. This result may be formulated as follows:

**Lemma 4.1.** If $G$ is a 3-connected graph with more than four vertices, then $G$ contains an edge $e$ such that $G - e$ is a subdivision of a 3-connected graph.

**Proof.** It is an immediate consequence of Menger's theorem that $G$ contains a subdivision of $K_3$. Now let $H$ be a proper subgraph of $G$ such that $H$ is a subdivision of a 3-connected graph and such that $|E(H)|$ is as large as possible under these conditions. It will suffice to prove that $H$ contains all edges of $G$ except one. We first assume that $H$ is not 3-connected, i.e., $H$ contains a path of length at least two such that they end at vertices $x$ and $y$ such that $x, y > 2$ in $H$. Since $G$ is 3-connected, $G - \{x, y\}$ has a path $P$ from $V(P) \setminus V(H)$ to $V(P) \setminus V(H)$. Since $H \cup P$ is a subdivision of a 3-connected graph, $H \cup P = G$ and hence $P$ has only one edge.

If $H$ is 3-connected, and $V(H) = V(G)$ the lemma is easily verified, so assume that $H$ is 3-connected and that there is a vertex $x \in V(G) \setminus V(H)$. By Menger's theorem, $G$ has three paths $P_1, P_2, P_3$ from $x$ to $V(H)$ such that they have only one in common pair by pair and such that $V(P_i) \cap V(H) = \{x_i\}$ for $i = 1, 2, 3$. If $x_1$ and $x_2$ are adjacent in $H$, we let $e$ denote the edge joining them. But then $H \cup P_1 \cup P_2$ or $(H \cup P_1 \cup P_2) - e$ is a subdivision of a 3-connected graph. This is a contradiction to the maximality of $H$ and the proof is complete. 

We shall now prove that any 3-connected graph $G$ which contains no subdivision of $K_3$ or $K_{3,3}$ has a plane representation. We prove this by induction on $|V(G)| + |E(G)|$. If $|V(G)| = 4$, there is no problem, so assume $|V(G)| \geq 5$. By Lemma 4.1, $G$ contains an edge $e = xy$ such that $G - e$ is a subdivision of a 3-connected graph and by the induction hypothesis, $G - e$ has a plane representation. If $x$ and $y$ are on a cycle which is the boundary of a face of $G - e$, we obtain a plane representation of $G$. So assume this is not the case. Since $G - x$ is 2-connected, it contains a cycle $C$ which is the boundary of the face containing $x$. By the above assumption, $y$ is not on $C$. Let $H_1$ and $H_2$ be the $C$-components of $G - e$ containing $x$ and $y$, respectively. Then $C$ separates $H_1$ and $H_2$, in the sense that one of them, say $H_1$, is in the interior of $C$ and $H_2$ is in the exterior of $C$. By definition of $C$, $H_2$ is the only $C$-component of $G - e$ in the interior of $C$.

The proof is now completed by using arguments similar to those in [49]. We first claim that $H_1$ and $H_2$ overlap. If this were not the case, then $H_1$ would have two vertices $u, v$ of attachment such that $C - u, v, P_j \cup P_k$, where $P_j$ (respectively $P_k$) is a path containing all vertices of attachment of $H_j$ (respectively $H_k$) other than $u$ and $v$. Then it is easy to see that $u$ and $v$ belong to distinct components of $(G - e) - H_1$, $H_2$ (we leave the details for the
reader). Since $G - e$ is a subdivision of a 3-connected graph, we conclude that $H_e$ consist of the path $xwy$. Then $u$ and $v$ are not adjacent in $G - e$ (again because $G - e$ is a subdivision of a 3-connected graph), so $P$, has length at least two. But then $x, y$ and $P_x - \{u, v\}$ belong to three distinct components of $G - \{u, v\}$ and this is not consistent with the assumption that $G - e$ is a subdivision of a 3-connected graph.

So $H_e$ and $H_i$ overlap, and, by Lemma 2.5, either $H_e$ and $H_i$ are skew or equivalent $C$-components.

In the first case it is easy to find a subdivision of $K_{3,3}$ in $G$ and in the second case we consider three paths from $x$ (respectively, $y$) to $C$ in $H_e$, (respectively $H_i$) which have only $x$ (respectively, $y$) in common pair by pair. The union of these six paths together with $C$ and $e$ is a subdivision of $K_{3,3}$.

5. A PROOF BASED ON A CONTRACTION LEMMA FOR 3-CONNECTED GRAPHS

If the graph $G'$ of Lemma 4.1 is planar and $G'$ denotes the 3-connected graph of which $G - e$ is a subdivision, then the dual graph of $G'$ is obtained from the dual graph of $G$ by contracting $e$ and by replacing each 2-cycle (if any) of the resulting multigraph by a single edge. This suggests a dual version of Lemma 4.1 [40].

Lemma 5.1. If $G$ is a 3-connected graph with at least five vertices, then $G$ contains an edge $e$ such that the graph $G/e$ obtained from $G$ by contracting $e$ and replacing each 2-cycle by a single edge is 3-connected.

Proof. Let $e = xy$ be any edge of $G$. If $G/e$ is not 3-connected, then $G$ contains a vertex $z$ such that $G - \{x, y, z\}$ is disconnected. Let $G_1$ be the smallest component of $G - \{x, y, z\}$. Since $G$ is 3-connected, $z$ is joined to $G_1$ by an edge $e_1 = zy$. If $G/e_1$ is not 3-connected, then there is a vertex $v$ such that $G - \{z, u, v\}$ is disconnected, but it is easy to see that the smallest component of this graph is a proper subgraph of $G_1$ (because $G - z$ is 2-connected and hence $G - \{V(G_1) \cup \{z\}\}$ is also 2-connected). Continuing like this, we eventually get the desired edge.

We now use Lemma 5.1 to prove that any 3-connected graph $G$ either contains a subdivision of $K_3$ or $K_{3,3}$, or has a convex representation, i.e., a plane representation such that each face is bounded by a convex polygon. The proof is accomplished by induction on $|V(G)|$.

Clearly, $K_3$ has a convex representation, so we proceed to the induction step. By Lemma 5.1, $G$ has an edge $e$ such that $G/e$ is 3-connected. If $G/e$ contains a subdivision of $K_3$ or $K_{3,3}$, it is easy to see that $G$ does too. So we may assume that $G/e$ has a convex representation. Let $z$ denote the vertex into which $e$ is contracted. Then $(G/e - z)$ is a 2-connected plane graph, and we denote by $C$ the cycle of this graph which is the boundary of the face containing $z$. The neighbors of $x$ in $G$ (except $y$) partition $C$ into segments. If none of these segments contains all neighbors of $y$ (except $x$), then by Lemma 2.5, the $C$-components of $G - e$ containing $x$ and $y$, respectively, are skew or equivalent, and it is easy to see that $G$ contains a subdivision of $K_3$ or $K_{3,3}$.

Assume that one of the above segments contains all neighbors of $y$ (except $x$). We delete from $G/e$ each edge joining $z$ to a vertex which in $G$ is joined to $y$ but not to $x$. We may regard this as a plane representation of $G - y$ and all edges that we delete are contained in the same face of $G - y$. It is now easy to extend $G - y$ to a convex representation of $G$ by placing $y$ sufficiently close to $x$. Figure 1 illustrates the situation where $z$ is on the outer polygon of $G/e$ with neighbors $z_1$ and $z_2$ on that polygon.

This completes the third proof of Kuratowski's theorem.

In [41] it is shown that every 3-connected planar graph has a representation such that each face is bounded by a convex polygon with at most six corners.

6. PROBLEMS AND RESULTS RELATED TO KURATOWSKI'S THEOREM

Using Kuratowski's theorem, one can easily decide if a graph $G$ is planar and has a plane representation such that a given set $A$ of vertices of $G$ are all on the boundary of the unbounded face. This is the case if and only if the graph obtained from $G$ by adding a new vertex and joining it to all vertices of $A$ is planar. If $G$ has such a representation and $A = V(G)$, then $G$ is called outerplanar, and if $A = \{x, y\}$, $x$ and $y$ are said to be close together [49]. Chartrand and Harary [7] characterized the outerplanar graphs as follows:

Theorem 6.1. A graph is outerplanar if and only if it contains no subdivision of $K_4$ or $K_{3,3}$.

\[\text{Figure 1. A detail in the third proof of Kuratowski's theorem.}\]
Two vertices \( x \) and \( y \) in a graph \( G \) are said to be **totally separated** ([49]) if \( G \) contains a cycle \( C \) such that \( x \) and \( y \) belong to overlapping \( C \)-components. Using the results of Sec. 2, it is easy to see that totally separated vertices are far apart, i.e., they are not close together.

Tutte [49] proved that the converse also holds and in [40] it was pointed out that this follows from Kuratowski's theorem.

**Theorem 6.2.** If \( G \) is a planar graph and \( x \) and \( y \) are vertices of \( G \), then either \( x \) and \( y \) are close together or they are totally separated.

Tutte [47] showed how Kuratowski's theorem can be used to obtain a short proof of MacLane's planarity criterion [27] asserting that a graph is planar if and only if it has a 2-basis and Whitney's criterion [60] asserting that a graph is planar if and only if it has a dual graph. The author [40] pointed out that Kuratowski's theorem also implies the planarity criterion of Pournin [11] and that the method of section 5 can be used to derive the result of Tutte [47] (rediscovered by Kelman [23]) that a 3-connected graph is planar if and only if each edge of the graph is contained in precisely two induced nonseparating cycles, i.e., cycles \( C \) with only one \( C \)-component. Such cycles are investigated by Thomassen and Toft [42] and a proof of Kuratowski's theorem based on these investigations is sketched in [42].

Using Whitney's planarity criterion Jaeger [21] obtained a new planarity criterion and Whitney's theorem also plays an important role in the verification of Read and Rosenstiehl's planarity criterion [33]. A planarity criterion depending on Kuratowski's theorem was found by Holton and Little [19].

By Kuratowski's theorem, a cubic graph is planar if and only if it contains no subdivision of \( K_{3,3} \), as proved by Menger [29]. The same holds for any 3-connected graph (except \( K_5 \)), since any 3-connected graph with at least six vertices containing a subdivision of \( K_5 \) also contains a subdivision of \( K_{3,3} \) (see [31, p. 138]). If true, the following conjecture is a planarity criterion involving only \( K_5 \) for a special class of graphs.

**Conjecture 6.5.** Let \( G \) be a 4-connected graph with \( n \) vertices and at least \( 3n-6 \) edges. Then \( G \) is planar if and only if \( G \) contains no subdivision of \( K_5 \).

It can be shown that Conjecture 6.3 implies the conjecture of Dirac [8], that any graph with \( n \) vertices and \( 3n-5 \) or more edges contains a subdivision of \( K_5 \). In [37] it was shown that any graph \( G \) satisfying the assumption of Dirac's conjecture contains a subdivision of the graph \( L \) of Figure 6.1 unless \( G \) is obtained by pasting disjoint copies of \( K_5 \) together along edges. Thus any graph with \( n \) vertices and \( 3n-5 \) or more edges has a subgraph contractible to \( K_5 \) and contains a subdivision of \( K_{3,3} \) if it has the very special structure described above (this also follows from Dirac's results [8]). In [37] it was also shown that any graph with \( n \) vertices and \( 4n-10 \) or more edges contains a subdivision of \( K_5 \).

While Conjecture 6.3, if true, characterizes the 4-connected maximal planar graphs, the 4-connected nonmaximal planar graphs are characterized by Theorem 6.4 found by Jung [22]. We say that a graph \( G \) is 2-linked if, for any four distinct vertices \( x_1, y_1, x_2, y_2 \) of \( G \), \( G \) has two disjoint paths \( P_1, P_2 \) such that \( P_1 \) has \( x_1 \) and \( y_1 \) as ends for \( i = 1, 2 \).

**Theorem 6.4.** A 4-connected graph \( G \) is 2-linked if and only if \( G \) is nonplanar or maximal planar.

While Jung's proof is complicated, a short proof involving Kuratowski's theorem was found by the author [39].

A characterization of all maximal planar graphs in terms of locally Hamiltonian graphs was found by Skupień [35].

A complete list of minimally nonembeddable graphs is not known for other surfaces in general. However, Glover and Huneke [14] showed that such a list is finite for the projective plane and Archdeacon [1] has shown that it consists of 103 graphs.

A Kuratowski-type theorem for higher surfaces has been announced by Volmerhausen [52], but a proof has not appeared. In this connection the following conjecture is of importance (see e.g., [56, p. 61]).

**Conjecture 6.6.** If \( G \) is a graph with \( n \) vertices and \( m \) edges, then \( G \) is planar if and only if \( G \) contains no subdivision of \( K_5 \) or \( K_{3,3} \).

While the proof is complicated, a short proof involving Kuratowski's theorem was found by the author [39].

A characterization of all maximal planar graphs in terms of locally Hamiltonian graphs was found by Skupień [35].

A complete list of minimally nonembeddable graphs is not known for other surfaces in general. However, Glover and Huneke [14] showed that such a list is finite for the projective plane and Archdeacon [1] has shown that it consists of 103 graphs.

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The fundamental character of Kuratowski's theorem is emphasized by the following deep result of Tutte [48] (for the relevant definitions the reader is referred to [48]).

**Theorem 6.7.** A binary matroid is the polygon matroid of a graph if and only if the matroid has no minor which is the bond matroid of $K_5$ or $K_{3,3}$.

Tutte's proof of this theorem is analogous to (though much more difficult than) the proof of Kuratowski's theorem given in Sec. 3. Other connections between planar graphs and matroids are discussed by Weinberg [57] and Welsh [58].

Extensions of a more topological nature of Kuratowski's theorem are described by Kuratowski [26].

### 7. INFINITE PLANAR GRAPHS

Kuratowski's theorem was extended to infinite graphs by Erdős (see [9]) who proved that a countably infinite graph is planar if and only if it contains no subdivision of either of the Kuratowski graphs $K_5$ and $K_{3,3}$. Using this, Wagner [55] characterized planar graphs in general. In this section, a graph is permitted to be infinite.

**Theorem 7.1.** A graph $G$ is planar if and only if

1. $G$ has at most $2^k$ vertices,
2. $G$ has at most countably many vertices of degree at least 3, and
3. $G$ contains no subdivision of $K_5$ or $K_{3,3}$.

If $G$ is a plane graph, then a point $p$ is a vertex accumulation point (abbreviated VAP) of $G$ if, for each positive real number $\varepsilon$, there are infinitely many vertices of $G$ of distance less than $\varepsilon$ from $p$.

If a plane graph $G$ has a cycle $C$ such that infinitely many vertices of $G$ are in the interior of $C$, then clearly $G$ has a VAP. Thus the planar graphs of Figure 3 have no VAP-free representations.

The following result was discovered by Halin [15].

**Theorem 7.2.** A connected locally finite graph has a VAP-free plane representation if and only if it contains no subdivision of $K_5$ or $K_{3,3}$ or any of the graphs of Figure 7.1.

Thus the graphs of Figure 3 are in a sense infinite analogs of the Kuratowski graphs. The analog of Theorem 7.2 for graphs that may have vertices of infinite degree was obtained by Schmidt [34]. The author [38] showed that the graphs of Theorem 7.2 have even VAP-free straight-line representations.

A 2-basis of a graph $G$ is a collection $\mathcal{F}$ of cycles of $G$ such that each edge of $G$ belongs to at most two cycles of $\mathcal{F}$ and each cycle of $G$ has a unique expression as a finite modulo 2 sum of cycles of $\mathcal{F}$. Kuratowski's theorem plays a crucial role in the proof of the following result [40] which is of interest in connection with Theorem 7.2 and which extends MacLane's planarity criterion to infinite graphs.

**Theorem 7.3.** A 2-connected graph $G$ has a 2-basis if and only if $G$ is planar and has a VAP-free representation.

If $G$ and $H$ are graphs, we say that $H$ is a dual graph of $G$ (and $G$ is a predual graph of $H$), if there exists a bijection $\phi: E(G) \to E(H)$ such that a finite edge set $A \subseteq E(G)$ is the edge set of a cycle if and only if $\phi(A)$ is a minimal separating set in $H$. A comprehensive treatment of duality of infinite graphs is given in [41]. We shall here mention two of the most important results related to Kuratowski's theorem (Theorem 7.4 was conjectured in [40]).
Theorem 7.4. A 2-connected multigraph $G$ has a dual graph if and only if $G$ contains no subdivision of $K_5$ or $K_{3,3}$, and contains no two vertices connected by infinitely many pairwise edge-disjoint paths. If $H$ is a planar dual graph of $G$, then $G$ and $H$ can be represented as geometric dual graphs.

If $G$ is an infinite multigraph and we identify two vertices whenever they are joined by infinitely many edge-disjoint paths, the resulting multigraph is called the strong reduction of $G$ (a more precise description is given in [40, 41]).

Theorem 7.5. A multigraph $G$ has a preduall graph if and only if the strong reduction of $G$ contains no subdivision of $K_5$ or $K_{3,3}$.

In [41] it is also shown that the condition in Theorem 7.5 is equivalent to the condition that each block of the strong reduction of $G$ is planar. However, $G$ itself need not be planar.

The following result, proved in [40], is an extension to infinite graphs of a planarity criterion for finite graphs due to Tutte [44].

Theorem 7.6. A graph $G$ contains a subdivision of $K_5$ or $K_{3,3}$ if and only if $G$ contains a cycle such that the overlap graph of $G$ with respect to $C$ is not bipartite.

Halin [16] gave another characterization of the graphs containing no subdivision of $K_5$ or $K_{3,3}$, Roughly speaking, he proved that any such edge-maximal graph can be obtained by pasting a set of countable maximal planar graphs together along edges.

Finally, we note that Kuratowski’s theorem is applied in the result of [39] characterizing all 2-linked graphs.

ACKNOWLEDGMENT

Thanks are due to F. Harary, B. Toft, and K. Villanger for comments on the paper.

References


