Dynamics of Asymptotically Holomorphic Polynomial-like Maps

In memory of Welington de Melo (1946-2016)

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(based on joint work with T. Clark and S. van Strien)

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- **dynamics of asymptotically holomorphic maps** – especially those which are *polynomial-like*;
- **renormalization of one-dimensional dynamical systems** – an area of Dynamics to which Welington de Melo made some fundamental contributions. (Here, we focus on a specific class of one-dimensional systems, namely **unimodal maps**.)

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Here, $\mu_f(z) = \dfrac{\overline{\partial} f(z)}{\partial f(z)}$ is the **complex dilatation** of f at $z \in U$.

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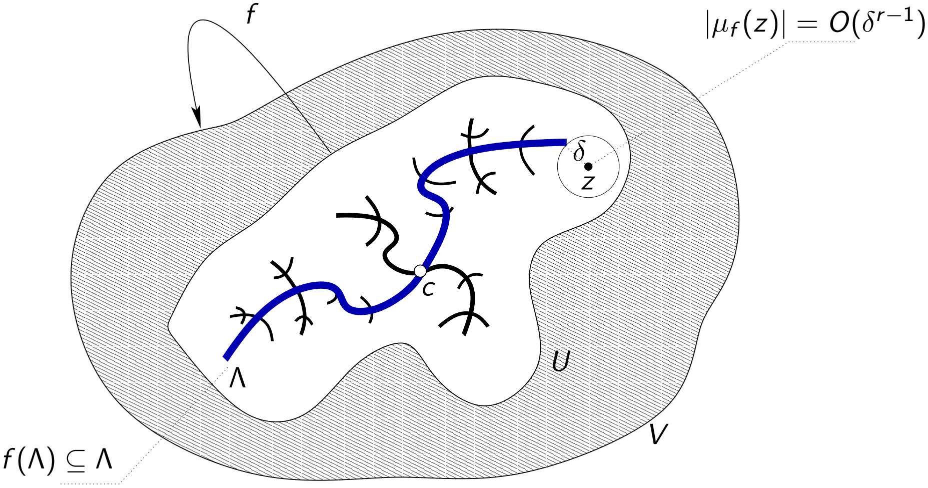

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- Also, in general such a map is **not** uniformly asymptotically conformal (UAC) in the sense of Gardiner and Sullivan.

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- If $R^n f$ is defined for all $n \geq 1$, we say that f is *infinitely renormalizable*.

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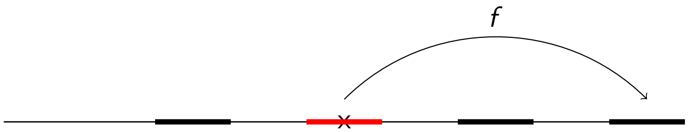

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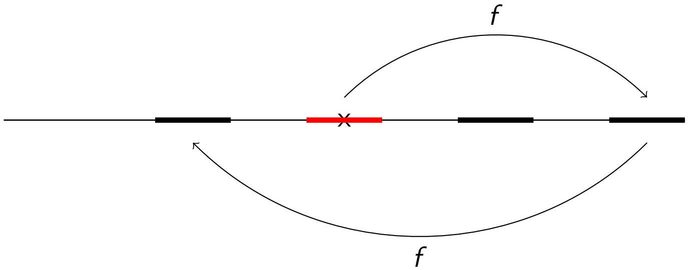

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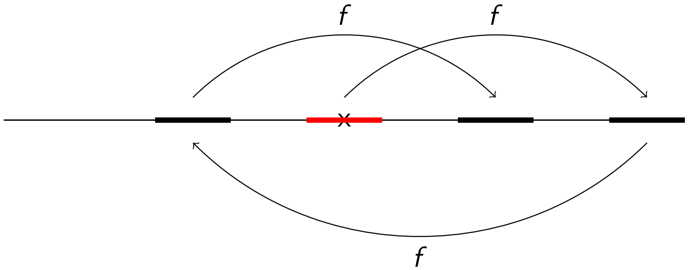

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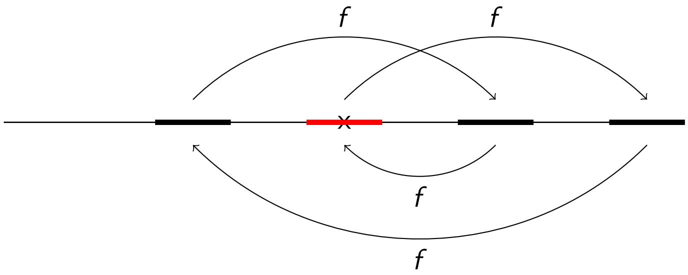

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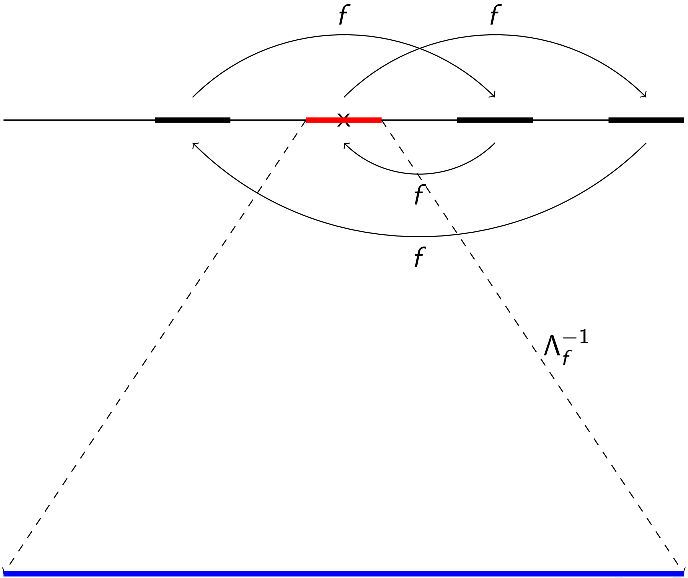

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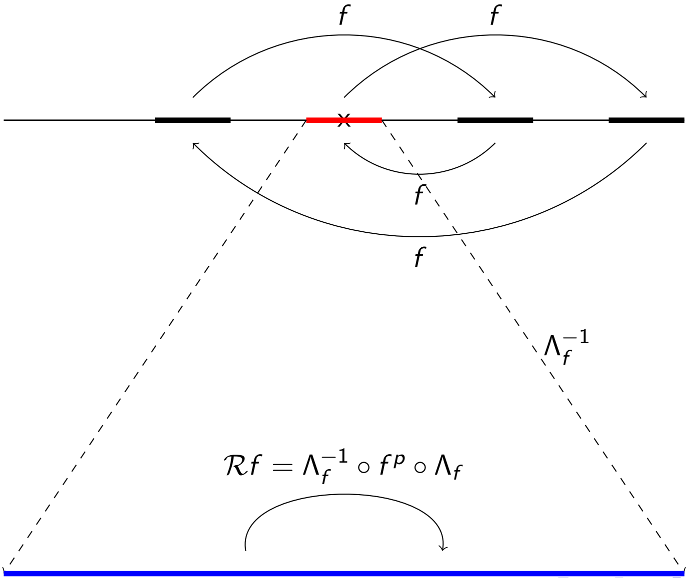

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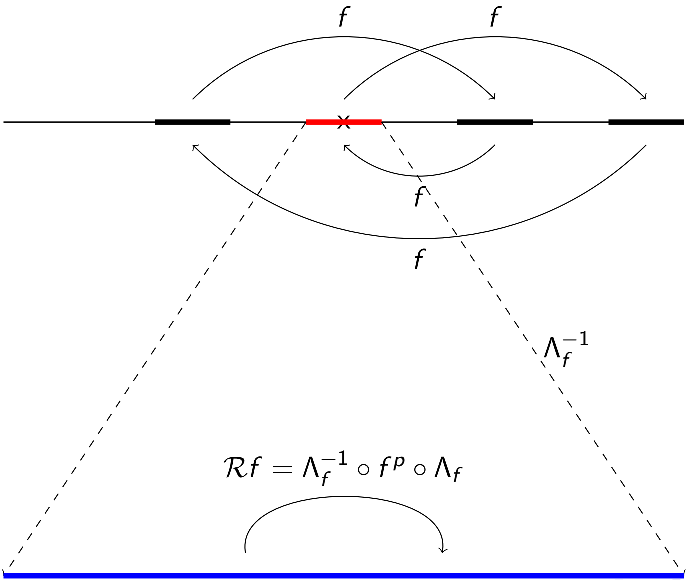

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- The positive integers $a_i = p(R^i f) \geq 2$ are the *renormalization periods* of f, and the q_n's are the *closest return times* of the orbit of the critical point $c = 0$.

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- The *postcritical set of f* is
$$P(f) = \bigcap_{n=0}^{\infty} \bigcup_{i=0}^{q_n - 1} \Delta_{i,n} \ .$$

Sullivan's real bounds

Theorem (Real Bounds)

Let $f : I \to I$ be a C^3 unimodal map as above, and suppose that f is infinitely renormalizable with combinatorial type bounded by $N > 1$. Then there exist constants $K_f > 0$ and $0 < \alpha_f < \beta_f < 1$ such that the following holds for all $n \in \mathbb{N}$.

(i) *If $\Delta \in \mathcal{C}_{n+1}$, $\Delta^* \in \mathcal{C}_n$ and $\Delta \subset \Delta^*$, then $\alpha_f|\Delta^*| \leq |\Delta| \leq \beta_f|\Delta^*|$.*

(ii) *For all $1 \leq i < j \leq q_n - 1$ and each $x \in \Delta_{i,n}$, we have*

$$\frac{1}{K_f}\frac{|\Delta_{j,n}|}{|\Delta_{i,n}|} \;\leq\; |(f^{j-i})'(x)| \;\leq\; K_f\frac{|\Delta_{j,n}|}{|\Delta_{i,n}|}\;.$$

(iii) *We have $\|R^n f\|_{C^2(I)} \;\leq\; K_f$.*

Moreover, there exist positive constants $K = K(N)$, $\alpha = \alpha(N)$, $\beta = \beta(N)$, with $0 < \alpha < \beta < 1$, and $n_0 = n_0(f) \in \mathbb{N}$ such that, for all $n \geq n_0$, the constants K_f, α_f and β_f in (i), (ii) and (iii) above can be replaced by K, α and β, respectively.

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- These bounds become universal at sufficiently deep levels (such bounds are called *beau* by Sullivan).

Complex bounds

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- Sullivan also showed that all limits of renormalization are in fact restrictions of nice complex-analytic maps, namely quadratic-like (or polynomial-like) maps – with good bounds.
- These are the so-called **complex bounds**. Just as the real bounds, the complex bounds are **beau**.

Theorem (Complex bounds)

Let $f : U \to \mathbb{C}$ be an asymptotically holomorphic map of order $r \geq 3$ (with $U \supset I$) and suppose that $f|_I : I \to I$ is an infinitely renormalizable unimodal map with combinatorial type bounded by N. There exist $C = C(N) > 1$ and $n_0 = n_0(f) \in \mathbb{N}$ such that the following statements hold true for all $n \geq n_0$.

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(i) *For each* $0 \leq i \leq q_n - 1$ *there exist Jordan domains* $U_{i,n}, V_{i,n}$*, with piecewise smooth boundaries and symmetric about the real axis, such that* $\Delta_{i,n} \subset U_{i,n} \subset V_{i,n}$*, the* $V_{i,n}$ *are pairwise disjoint, and we have the sequence of surjections*

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This theorem is a straightforward consequence of (a special case of) the complex bounds proved by Clark, van Strien & Trejo in [2].

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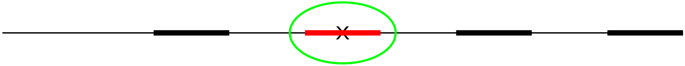

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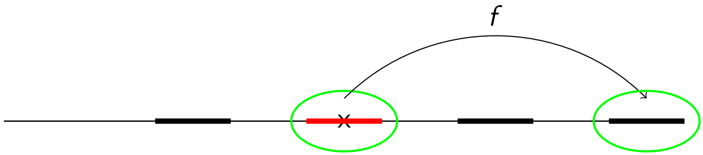

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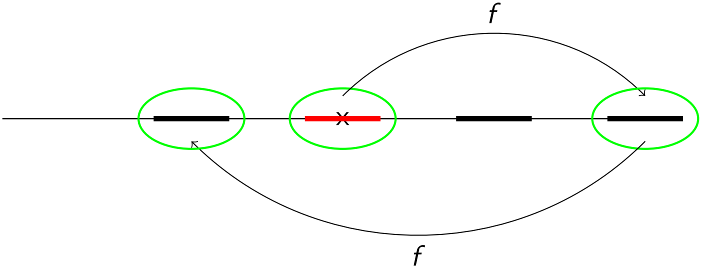

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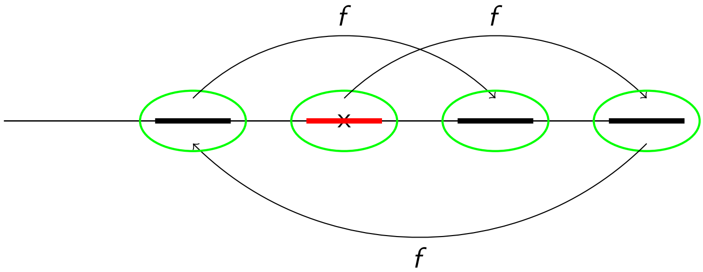

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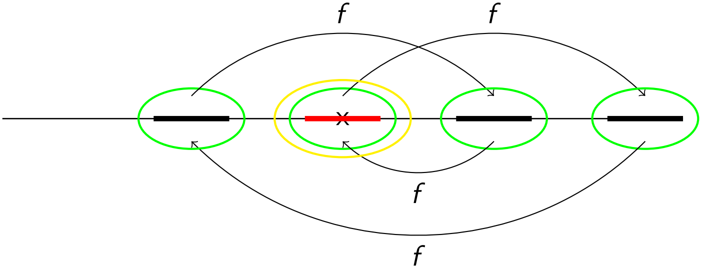

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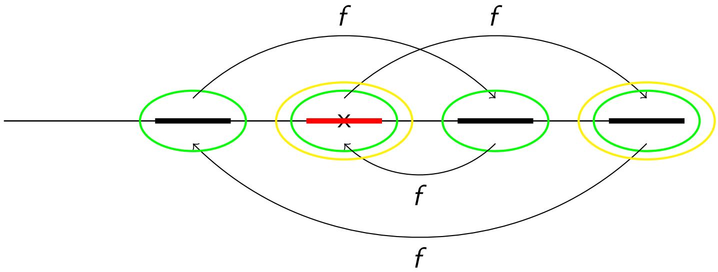

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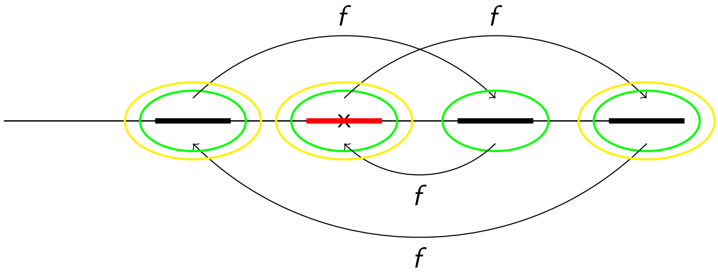

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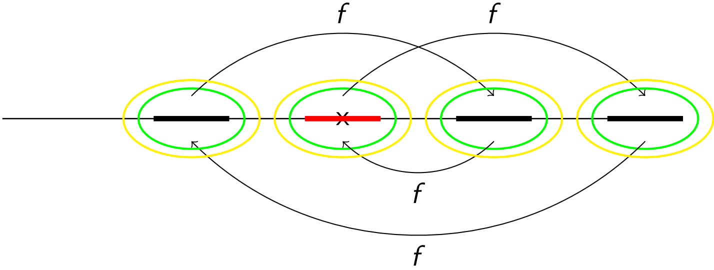

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Let $f : U \to V$ be an infinitely renormalizable AHPL-map of bounded combinatorial type bounded by $N \in \mathbb{N}$, and let $R^n f : U_n \to V_n$, $n \geq 1$, be the sequence of renormalizations of f. There exists a constant $C_f > 0$ such that $\|R^n f\|_{C^2(U_n)} \leq C_f$. Moreover, there exist $C = C(N) > 0$ and $m = m(f) \in \mathbb{N}$ such that $\|R^n f\|_{C^2(U_n)} \leq C$ for all $n \geq m$.

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The proof uses the real as well as the complex bounds. In fact, the complex bounds are essential even to make sure that the renormalizations $R^n f$ appearing above are well-defined AHPL-maps.

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The proof also uses the **chain rule for the second derivative of a composition**.

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Proposition

If $f : U \to V$ *is as in the theorem and* $z \in \mathcal{K}_f$ *is a point whose forward orbit never lands on the real axis, then for all non-zero tangent vectors* $v \in T_z Y$ *we have*

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The idea behind the proof is to use the Stoilow decomposition $f = \phi \circ g$ and show that, as we iterate, **the expansion of the hyperbolic metric by g beats the possible contraction by ϕ at each scale**.

Bounding expansion

Lemma

Let X, Y be hyperbolic Riemann surfaces with $X \subset Y$, and let $g : X \to Y$ be holomorphic univalent and onto. Then for all $x \in X$ and each tangent vector $v \in T_x X$ we have

$$|Dg(x)v|_Y \;\geq\; \Phi(s_{X,Y}(x))^{-1}|v|_Y \;, \tag{2}$$

where $s_{X,Y}(x) = d_Y(x, Y \setminus X)$ and $\Phi(\cdot)$ is the universal function given by

$$\Phi(s) \;=\; \sinh(s)\log\left(\frac{1+e^{-s}}{1-e^{-s}}\right) \;. \tag{3}$$

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- Note that $\Phi(s)$ is a continuous monotone increasing function with $\Phi(0) = 0$ and $\Phi(\infty) = 1$.

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Bounding contraction (cont.)

Proposition

Let $\alpha > 1$ and $\beta > 1$ be given, and suppose $\phi : \mathbb{D} \to \mathbb{D}$ is a C^2 quasiconformal diffeomorphism. If $z \in \mathbb{D}$ is such that

$$\alpha^{-1} \leq \frac{\rho_{\mathbb{D}}(\phi(z))}{\rho_{\mathbb{D}}(z)} \leq \alpha \ ,$$

and

$$\sup_{\zeta \in \Delta_z} |\mu_\phi(\zeta)| \leq c_0(1 - |z|)^\beta \ ,$$

then for each $0 < \theta < 1$ we have

$$J_\phi^h(z) \leq 1 + C_\theta(1 - |z|)^{\beta(1-\theta)} \ ,$$

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In the application, $\beta = r - 1$.

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THANK YOU!

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- They deduced that orbits of renormalization are asymptotic to the *full* renormalization horseshoe.
- Their methods apply to unicritical polynomial-like maps, and yield a unified approach, valid for all (real) combinatorics and all degrees of criticality.

C^r smoothness

- dF, de Melo, Pinto (2006): Established Lanford's conjecture in the space of C^r quadratic unimodal maps. Here r is any real number $\geq 2 + \alpha$, where $\alpha < 1$ is the largest of the Hausdorff dimensions of the post-critical sets of maps in the attractor. The proof combines Lyubich's theorem with Davie's *tour de force*.

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- The authors also went beyond the conjecture, proving that the local stable manifolds of the renormalization operator form a C^0 lamination whose holonomy is $C^{1+\beta}$ for some $\beta > 0$. In particular, every smooth curve which is transversal to such lamination intersects it at a set of constant Hausdorff dimension less than one [3].