Topological models in holomorphic dynamics

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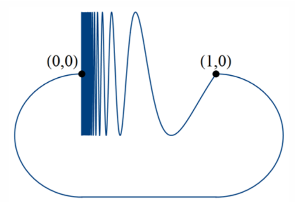
Summary from last time

Describe the topology of some "nice" subsets $K \subset \mathbb{C}$:
Here "nice" means:

- K is compact,
- $\mathbb{C} - K$ is connected,
- K is connected,
- K is locally connected.



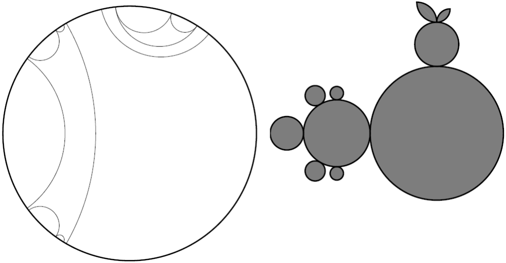
How to do it?

- **Use Uniformization theorem:** $\hat{\mathbb{C}} - K$ is a topological disk $\Rightarrow \exists$ biholom. $\psi : \mathbb{C} - \overline{\mathbb{D}} \to \mathbb{C} - K$,
- **Then:** "locally connected" $\Rightarrow \psi$ extends continuously (not necess. 1 to 1!) to a map $\gamma : S^1 \to \partial K$.

Summary from last time

Definition

We define on the circle S^1 an equivalence relation $\sim_K$ by $t \sim_k t$ iff $\gamma(t) = \gamma(t)$.



Some remarks:

- The equivalence relation has ”no crossing” (unlinked),
- Such an equivalence relation (i.e graph closed + unlinked) can only produce loc. connected spaces.

Summary: application to the case of quadratic dynamics $P_c : z \mapsto z^2 + c$

Main object to study: "Filled Julia set K_c" (for a given c)

$$K_c^+ := \{z \in \mathbb{C} | \text{orbit of } z \text{ is bounded}\}$$

Strong dichotomy: (recall that $\mathcal{M} := \{c \in \mathbb{C} | \text{orbit of } c \text{ is bounded}\}$)

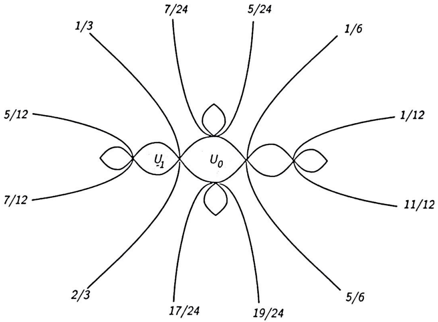
- Case $c \in \mathcal{M}$: then K_c is connected (but not necessarily loc. connected),
- Case $c \notin \mathcal{M}$: then K_c is totally disconnected, a Cantor set.

Added bonus: for K connected, the biholomorphism between $\mathbb{C} - K$ and $\mathbb{C} - \mathbb{D}$ is given by the Böttcher map (conjugacy with $z \mapsto z^2$).
Question: parameters c for which K_c is locally connected?

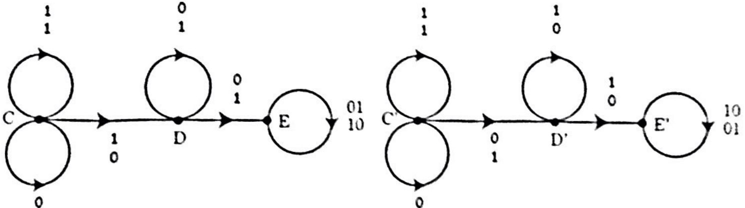
Geometric description of the pinching model for $z \mapsto z^2 - 1$

- **Extension of the conjugacy to the boundary of the disk:** here $\psi := \phi^{-1}$ has a continuous extension $\mathbb{C} - \mathbb{D}(r) \to \mathbb{C} - int(K)$, which induces a continuous map $\gamma : S^1 \to \partial K$ (a semi-conjugacy).
- **Some necessary conditions:**
  - **"rays cannot cross"** $\Rightarrow$ (if $\theta_1 \sim \theta_2$ and $\theta_3 \sim \theta_4$ then the intervals (θ_1, θ_2) and (θ_3, θ_4) are disjoint or nested).
  - **Periodic maps to periodic:** the map γ is a semi-conjugacy...
- **Consequences:** The unique fixed point of the doubling map has to go to a fixed point. The unique 2-cycle $\{1/3, 2/3\}$ has to go to a 2-cycle (impossible, not in K) or a fixed point.
- **Preimages:** the other preimage of $\gamma_{1/3} \cup \gamma_{2/3}$ is $\gamma_{1/6} \cup \gamma_{5/6}$.
- **Further preimages:** $f^{-1}(\gamma_{1/6} \cup \gamma_{5/6}) = \gamma_{1/12} \cup \gamma_{5/12} \cup \gamma_{7/12} \cup \gamma_{11/12}$. The "non-crossing property" will force the correct pairings.

First few pairings for $z \mapsto z^2 - 1$

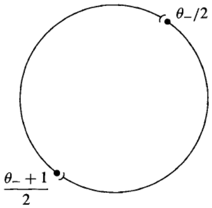


Subshift :$(ex : 1/12 = .00(01)^\infty = .11(10)^\infty)$ do $DE(E^\infty)$



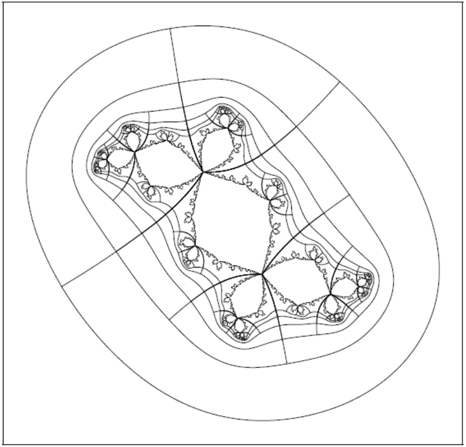
More general recipe

Case of periodic critical point: Pick one "root angle" of the component of the interior of K_c containing c and split the circle as follows:

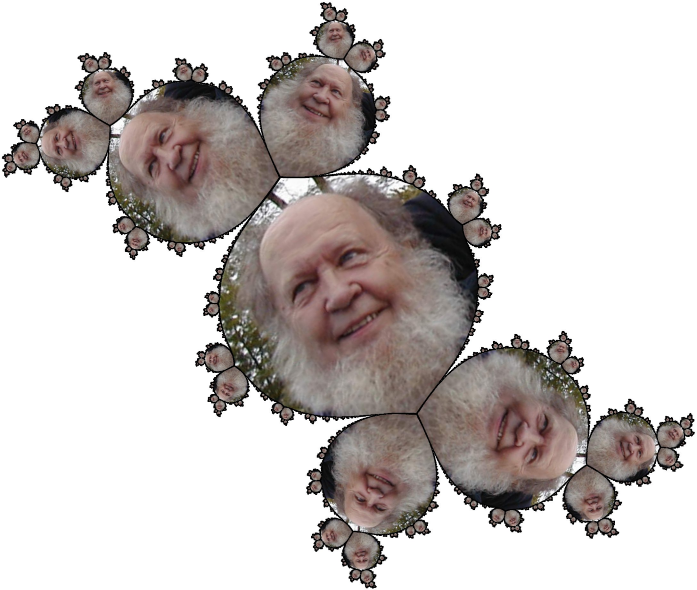


Then for any $t \in S^1$ define the itinerary $I(t)$ under *angle doubling*.
Equivalence relation: $t \sim t'$ if and only if t and t' have the same itineraries.

One more application of the external rays: "Yoccoz puzzles"



Douady's Rabbit (pic by A. Chéritat)



Application to the Mandelbrot set itself

Böttcher coordinate: $\phi_c(z)$ for $P_c : z \mapsto z^2 + c$.

Theorem (Douady-Hubbard)

1. *The function $\phi_c(z)$ is analytic in c and z;*
2. *The function $\theta : c \mapsto \phi_c(c)$ is well-defined in $\mathbb{C} - \mathcal{M}$;*
3. $\boxed{\theta : \widehat{\mathbb{C}} - \mathcal{M} \to \widehat{\mathbb{C}} - \overline{\mathbb{D}} \text{ is a biholomorphism}}$
4. *$\mathcal{M}$ is connected.*

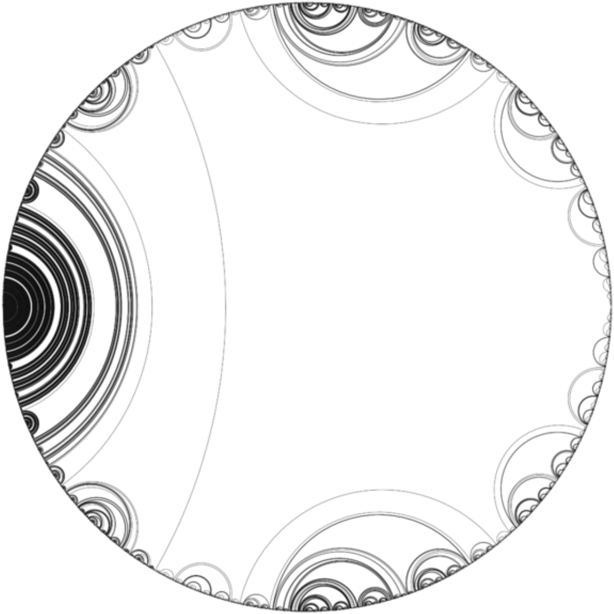
Explicit uniformization of $\widehat{\mathbb{C}} - \mathcal{M}$:

$$\phi_c(c) = c.\prod_{n=0}^{\infty}\left(1 + \frac{c}{P_c^n(c)^2}\right)^{\frac{1}{2^{n+1}}}.$$

Question ("MLC conjecture")

The set $\mathcal{M}$ is locally connected?

Pinched disks IV : what is $\mathcal{M}$ for Thurston?

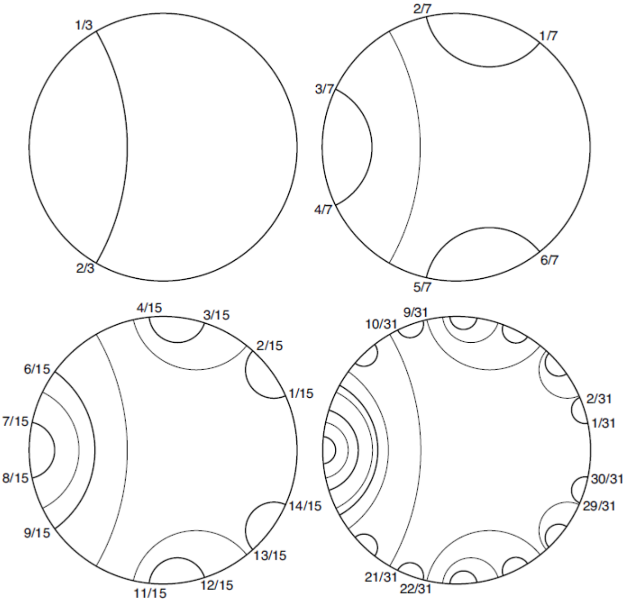


Pinched disks V : description of the lamination associated to $\mathcal{M}$

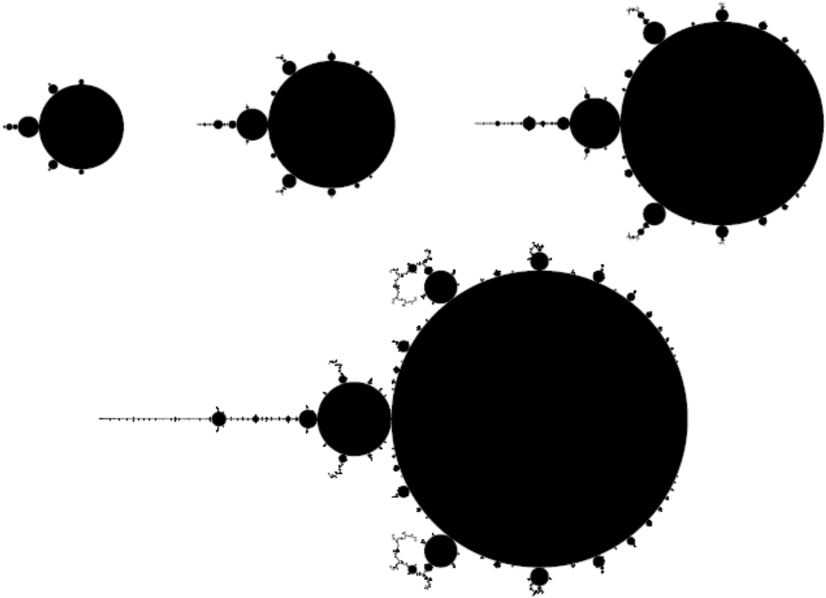
Lavaurs Algorithm:

1. Angles that are 2-periodic under angle doubling map: $\left(\frac{1}{3}, \frac{2}{3}\right)$
2. Angles that are 3-periodic: $\left(\frac{1}{7}, \frac{2}{7}\right), \left(\frac{3}{7}, \frac{4}{7}, \right), \left(\frac{5}{7}, \frac{6}{7}\right)$
3. Angles that are 4-periodic: $\left(\frac{1}{15}, \frac{2}{15}\right), \left(\frac{3}{15}, \frac{4}{15}, \right), \left(\frac{6}{15}, \frac{9}{15}\right), \ldots$
4. $\ldots$

Pinched disks VI : Mandelbrot set



Pinched disks VII : Mandelbrot set (D. Schleicher)



Moving to higher dimensions

1. **Possible dynamical systems:** the list is huge. Here are some examples
   - Polynomials endomorphisms: $\binom{x}{y} \mapsto \binom{x^3+x.y+4}{3x^2y}$
   - Analytic self-maps of $\mathbb{P}^1 \times \mathbb{P}^1$, of $\mathbb{P}^2(\mathbb{C})$, ...
   - Polynomial automorphisms: example $\binom{x}{y} \mapsto \binom{P(x)-ay}{x}$ ("Complex Hénon mappings).
   - etc ...
2. **Polynomial automorphisms of $\mathbb{C}^2$:**

Theorem (Friedland-Milnor)

Let $f \in Aut(\mathbb{C}^2)$. Then either f is conjugated in $Aut(\mathbb{C}^2)$ to a composition of Hénon maps, or it is conjugated to a product of elementary maps of the type $E\binom{x}{y} = \binom{ax+p(y)}{by+c}$ where $ab \neq 0$ and p is a polynomial.

3. **Small perturbations** of quadratic polynomials.

$\mathbb{C}$ and $\mathbb{C}^2$: they look the same

- **Same definitions**: analytic functions (as power series), holomorphic functions (are holomorphic separately, in each variable).
- **Same kind of theorems** example: "bounded holomorphic functions on $\mathbb{C}^2$ must be constant".
- **Exercise:** Prove that the non-escaping set of F intersects any horizontal line $\{x = constant\}$.
- **But nothing is straightforward:** Find an example of a non constant non open analytic map of $\mathbb{C}^2$.

$\mathbb{C}$ and $\mathbb{C}^2$ are different: Fatou-Bieberbach domains

Given $f \in Aut(\mathbb{C}^2)$ such that $\binom{0}{0}$ is an attracting fixed point:

$$f\begin{pmatrix}x\\y\end{pmatrix} = L\begin{pmatrix}x\\y\end{pmatrix} + h\begin{pmatrix}x\\y\end{pmatrix}$$

with

$$L\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}\lambda x\\ \mu y\end{pmatrix} \quad \text{where } 0 < |\mu|^2 < |\lambda| < |\mu| < 1.$$

Theorem

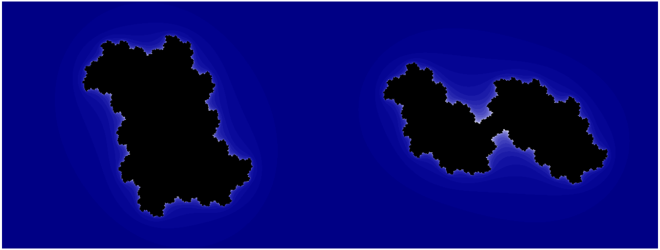
Then the basin U of $\mathbf{0}$ is biholomorphic to $\mathbb{C}^2$.

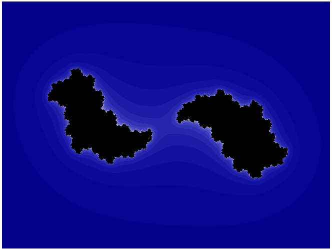
Proof:

- First: prove f is conjugated to L near the origin,
- Then extend this conjugacy by the dynamics: given any $x \in U$, there exists N such that $f^{\circ N}(x)$ is in the domain of definition of ϕ. We can then extend ϕ by the formula

$$\phi(x) = L^{-N}\phi(f^{\circ N})x.$$

Some horizontal slices $y = constant$





$\mathbb{C}$ and $\mathbb{C}^2$: they do not look the same...

Attracting fixed points and linearization in dimension 1: Given

$$f(z) = z(\lambda + O(z)) \quad \text{with } 0 < |\lambda| < 1$$

then near 0 we have a conjugacy

$$\phi \circ f \circ \phi^{-1}(z) = \lambda z.$$

This means that f is *linearizable*: it is conjugate to its linear part.
Attracting fixed points and linearization in dimension 2: Assume

$$f\begin{pmatrix} x \\ y \end{pmatrix} = L\begin{pmatrix} x \\ y \end{pmatrix} + h\begin{pmatrix} x \\ y \end{pmatrix}$$

with

$$L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \lambda x \\ \mu y \end{pmatrix} \quad \text{where } 0 < |\lambda| \leq |\mu| < 1$$

and $\left| h\begin{pmatrix} x \\ y \end{pmatrix} \right| \leq C(|x|^2 + |y|^2|)$ for some C. Then f is not always linearizable at the origin: $f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x/2 \\ y/4 + x^2 \end{pmatrix}$.

Fatou-Bieberbach domains (continued)

Case when $0 < |\mu|^2 < |\lambda| < |\mu| < 1$

Set $\phi_n = L^{-n} \circ f^{\circ n}$, and then study $\phi_{n+1} - \phi_n$:

$$\phi_{n+1} - \phi_n = L^{-(n+1)} \circ h \circ f^{\circ n}.$$

Choose $\epsilon > 0$ so small that $(|\mu| + \epsilon)^2 < |\lambda|$, and $\rho > 0$ so small that there exists C such that

$$\left|\begin{pmatrix} x \\ y \end{pmatrix}\right| < \rho \implies \left|f\begin{pmatrix} x \\ y \end{pmatrix}\right| \leq (|\mu| + \epsilon)\left|\begin{pmatrix} x \\ y \end{pmatrix}\right|$$

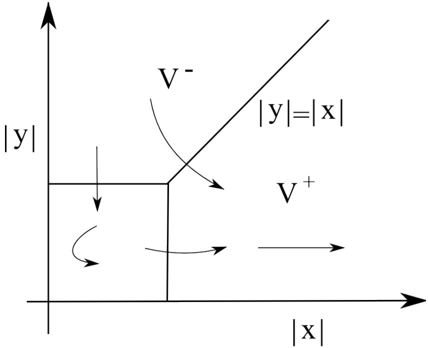
$$\left|h\begin{pmatrix} x \\ y \end{pmatrix}\right| \leq C\left|\begin{pmatrix} x \\ y \end{pmatrix}\right|$$

Then

$$\left|\phi_{n+1}\begin{pmatrix} x \\ y \end{pmatrix} - \phi_n\begin{pmatrix} x \\ y \end{pmatrix}\right| = \left|L^{-(n+1)} \circ h \circ f^{\circ n}\begin{pmatrix} x \\ y \end{pmatrix}\right|$$

$$\leq \frac{1}{|\lambda|^{n+1}} C\Big((|\mu| + \epsilon)^n\Big)^2 = \frac{C}{|\lambda|}\left(\frac{(|\mu| + \epsilon)^2}{|\lambda|}\right)^n.$$

- **Basic properties:** the map $\binom{x}{y} \mapsto \binom{P(x)-ay}{x}$ has inverse $\binom{x}{y} \mapsto \binom{y}{(1/a)(P(y)-x)}$ and constant jacobian equal to a.
- **Crude picture of dynamics:**



Basic invariant sets

- **Filled Julia sets**

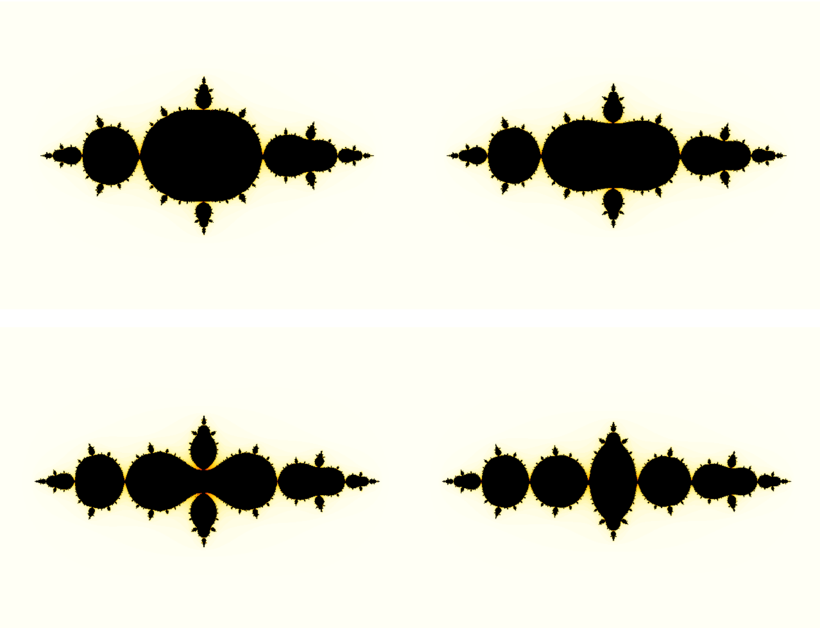
$$K^+ := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{C}^2 \text{ with bounded forward orbit} \right\}$$

- **Escaping set:** $U^+ := \mathbb{C}^2 - K^+$. We have also $U^+ = \bigcup_{n \geq 0} H^{-n}(V^+)$.
- **Julia sets** $J^+ = \partial K^+$, also $J^- = \partial K^-$. We can define $K = K^+ \cap K^-$.
- **Basins of attraction** $W^s(p)$ **of fixed points**.
- **Stable and unstable manifolds of saddle points** p: they are isomorphic to $\mathbb{C}$.
- What is the topology of these sets? Are they connected?
- (partial) answer: $K^\pm$ is always connected.

Theorem (Bedford-Smillie)

- *If p is a sink for f and if B is the basin of attraction of p then $\partial B = J^+$.*
- *If p is a saddle point for f then the stable manifold $W^s(p)$ is dense in J^+.*

Examples of horizontal slices of non-escaping sets



Picture Stable manifold, real slice