Topological models in holomorphic dynamics

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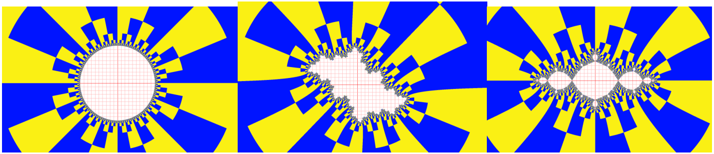
Workshop on Dynamics, Numeration and Tilings 2013

Overview of the mini course

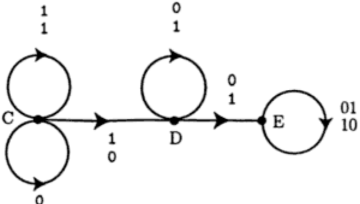
- **Iterating $z \mapsto z^2 + c$ in $\mathbb{C}$:** basic invariant set, the "filled Julia set" :

$$K_c^+ := \{z \in \mathbb{C} | \text{orbit of } z \text{ is bounded}\}$$

1. **At infinity $z^2 + c$ is "like" z^2:**

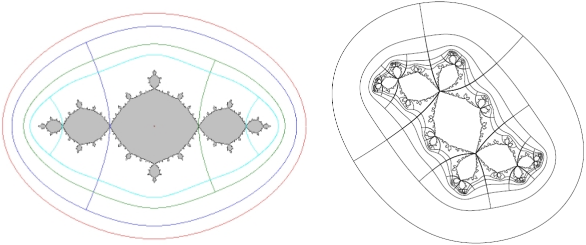


this will produce identifications of the following type:

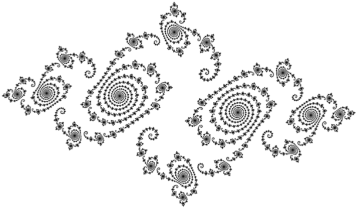


Overview

- **Iterating $z \mapsto z^2 + c$:**
  1. **Dynamical partitions:** "Yoccoz puzzles"



  2. **Moving Cantor sets:** opposite situation, for every c large, K_c is Cantor set, moving with s.



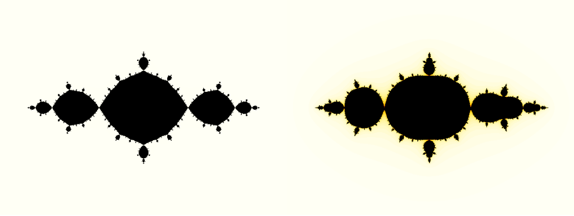
Now each Cantor set $\simeq \{0,1\}^{\mathbb{N}}$, so loops in parameter space produce automorphisms of $\Sigma_2 := \{0,1\}^{\mathbb{N}}$.

Overview

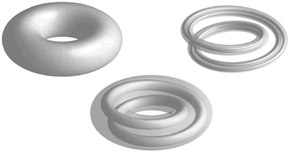
- **Higher dimensional version:** to simplify we study polynomial maps: so-called Hénon maps

$$H : \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{C}^2 \mapsto \begin{pmatrix} x^2 + c - ay \\ x \end{pmatrix}$$

- **Standard thing to do:** plot ”non-escaping sets”

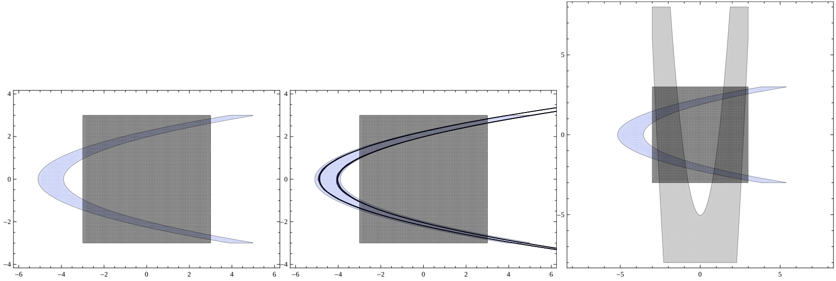


- **Use dynamics at ∞:** to produce invariant partitions



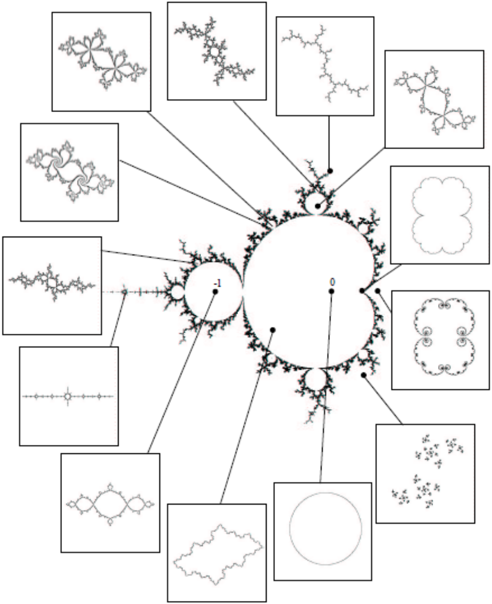
Overview

- **Higher dimensional version:** what is the analogue of the "shift locus"?

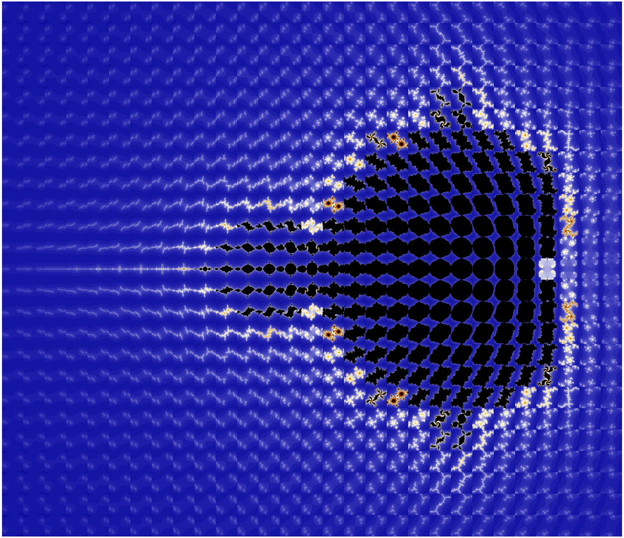


- **Horseshoe locus:** dynamics is conjugated to the full 2-shift.
- **Moving around in the Horseshoe locus:** will produce automorphisms of the full 2-shift.

Roadmap to quadratic dynamics

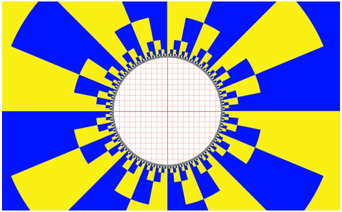


Roadmap to quadratic dynamics 2: Pic by Kawahira

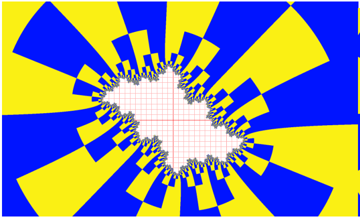
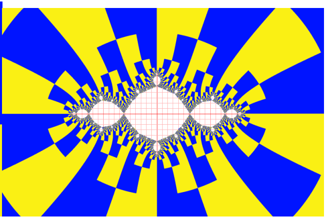


The simplest dynamics

1. **The squaring map:** $z \mapsto z^2$.

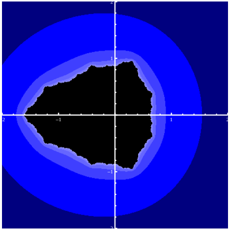


2. **Key fact:** as simple as it is, this case will somehow generate all the others.

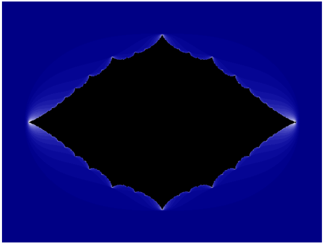



Dynamics and fixed points: super-attracting fixed points

1. **Super-attracting fixed points:** $f(z_0) = z_0$ and $f'(z_0) = 0$.
2. **Example 1:** $z_0 = -1$ for $N(z) = \frac{z^2+1}{2z}$.
3. **Example 2:** $z_0 = 0$ for $f(z) = z^2 + z^3$.
4. **Example 3:** $z_0 = 0$ for $g(z) = \frac{z^2}{1+0.24*z^2}$



Example 2: basin of attraction of 0



Example 3: basin of attraction of 0

Super-attracting fixed points: Böttcher's coordinate

1. **The case of a quadratic poynomial $P_c : z \mapsto z^2 + c$:**

Theorem

There exists a biholomorphism ϕ in a neighborhood U of $z_0 = \infty$ such that the following diagram commutes:

$$\begin{array}{ccc} U & \xrightarrow{\phi} & V \\ \downarrow{P_c} & & \downarrow{z^2} \\ U & \xrightarrow{\phi} & V \end{array}$$

2. **Demonstration**: (Idea: define $'' \sqrt[2^n]{P_c^{\circ n}(z)} ''$)

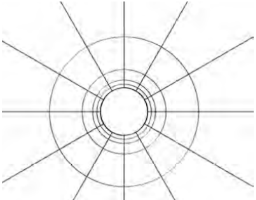
1. **Demonstration**: (Idea: define " $\sqrt[2^n]{P_c^{\circ n}(z)}$") More precisely: at infinity $\frac{P(z)}{z^2} \sim 1 + \epsilon(z)$, so we can define $\psi(z) := \log \frac{P(z)}{z^2}$ and thus write

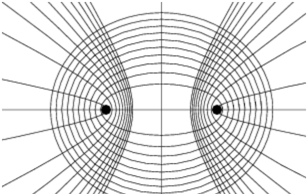
$$P(z) = z^2 . \exp \psi(z).$$

Then, by induction, we get:
$P^n(z) = z^{2^n} \exp(2^{n-1}\psi(z) + \ldots + \psi(P^{n-1}(z)))$. Finally the 2^n-th root we wanted can be taken as

$$z . \exp \left(\frac{1}{2}\psi(z) + \ldots \frac{1}{2^n}\psi(P^{n-1}(z)) \right).$$

2. **Explicit example:** the so-called Joukowsky transform $z \mapsto z + 1/z$ is one such map for $z \mapsto z^2 - 2$.





Consequences of the existence of the Böttcher coordinate I

Theorem (Douady-Hubbard)

Let f be a polynomial map of degree $d \geq 2$. If the filled Julia set contains all the finite critical points of f, then the two sets K and $J = \partial K$ are connected and the complement of K is isomorphic to $\mathbb{C} - \overline{\mathbb{D}}$ under an isomorphism

$$\hat{\phi} : \mathbb{C} - K \to \mathbb{C} - \overline{\mathbb{D}},$$

such that

$$\hat{\phi} \circ f(z) = \phi(z)^d.$$

When at least one critical point of f belongs to $\mathbb{C} - K$, then K and J have an uncountable number of connected components.

Connected basin of infinity U: critical points do not escape

- **Exhaustion of** U: U is the increasing union of $U_n := f^{-n}(\widehat{\mathbb{C}} - \mathbb{D}_R)$.
- **Claim:** each restriction $f : U_{n+1} \to U_n$ is a branched double cover, with the ∞ being the only branch point.
- **Claim**: all the U_n are topological disks (so is their union) (Hint: given $f : S \to T$ we get $\chi(S) = 2.\chi(T) - \sum_{br.pts}(deg_a f - 1)$, remembering that $\chi(S) = 2 - 2g - n$.
- **Plane topology:** G open connected set is simply connected iff $\widehat{\mathbb{C}} - G$ is connected.

Pinched disks description of compact sets in $\mathbb{C}$

We consider sets $K \subset \mathbb{C}$ such that:

- K is compact,
- $\mathbb{C} - K$ is connected (i.e K is "full"),
- K is connected,
- K is locally connected.

Lemma

Let $K \subset \mathbb{C}$ be compact, connected and full. Then one can find in a unique way a radius $r \geq 0$ and a biholomorphism $\phi : \mathbb{C} - K \to \mathbb{C} - \overline{\mathbb{D}}(r)$ such that $\phi(z)/z \to 1$ as $z \to \infty$.

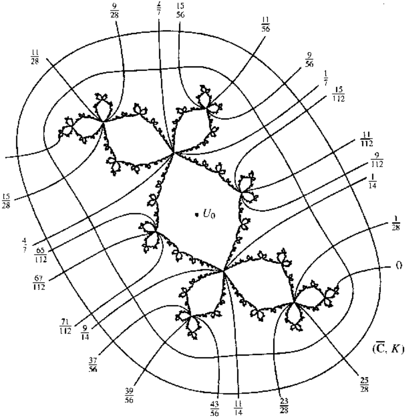
Lemma

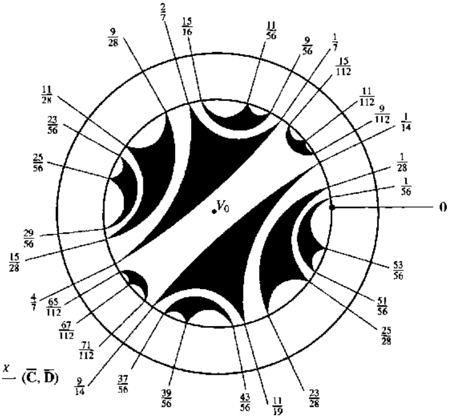
Let K be as above and locally connected. Then $\psi := \phi^{-1}$ has a continuous extension $\mathbb{C} - \mathbb{D}(r) \to \mathbb{C} - int(K)$, which induces a continuous map $\gamma : S^1 \to \partial K$.

Equivalence relation defining the pinched disks models (Thurston)

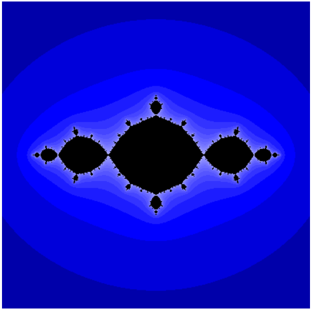
Definition

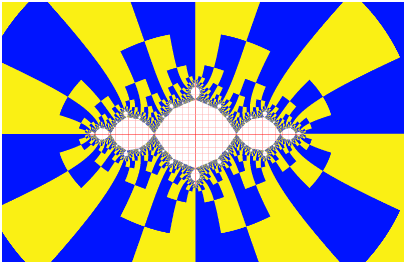
We define on the circle S^1 an equivalence relation $\sim_K$ by $t \sim_k t$ iff $\gamma(t) = \gamma(t)$.

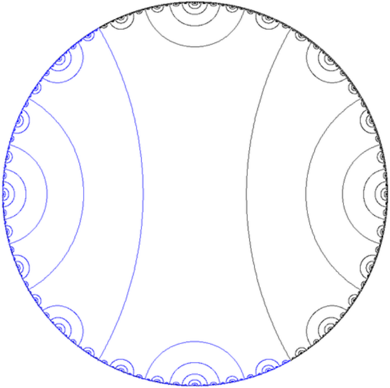




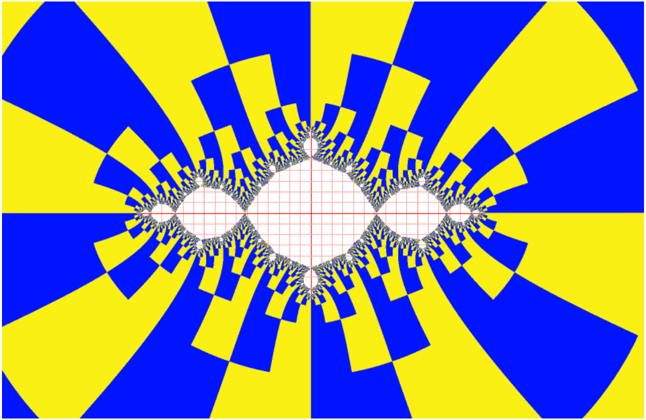
More pinched disks: $z \mapsto z^2 - 1$







More about: $z \mapsto z^2 - 1$



- **Distinguished types of points:**
  - **"Tips"**: they correspond to separations between blue and yellow.
  - **"Cut points"**: they correspond to words of the type 010101010101...

Geometric description of the pinching model for $z \mapsto z^2 - 1$

- **Extension of the conjugacy to the boundary of the disk:** here $\psi := \phi^{-1}$ has a continuous extension $\mathbb{C} - \mathbb{D}(r) \to \mathbb{C} - int(K)$, which induces a continuous map $\gamma : S^1 \to \partial K$ (a semi-conjugacy).
- **Some necessary conditions:**
  - **"rays cannot cross"** $\Rightarrow$ (if $\theta_1 \sim \theta_2$ and $\theta_3 \sim \theta_4$ then the intervals (θ_1, θ_2) and (θ_3, θ_4) are disjoint or nested).
  - **Periodic maps to periodic:** the map γ is a semi-conjugacy...
- **Consequences:** The unique fixed point of the doubling map has to go to a fixed point. The unique 2-cycle $\{1/3, 2/3\}$ has to go to a 2-cycle (impossible, not in K) or a fixed point.
- **Preimages:** the other preimage of $\gamma_{1/3} \cup \gamma_{2/3}$ is $\gamma_{1/6} \cup \gamma_{5/6}$.
- **Further preimages:** $f^{-1}(\gamma_{1/6} \cup \gamma_{5/6}) = \gamma_{1/12} \cup \gamma_{5/12} \cup \gamma_{7/12} \cup \gamma_{11/12}$. The "non-crossing property" will force the correct pairings.