Resposta: ex. 10: Mostre que $\int_0^{+\infty} \frac{x}{e^x - 1}\,dx = \sum_1^{\infty} \frac{1}{k^2}$.

Resp:

Temos $\int_0^{+\infty} \frac{x}{e^x-1}\,dx = \int_0^{+\infty} \frac{xe^{-x}}{1-e^{-x}}\,dx = \int_0^{+\infty} xe^{-x} \sum_{k=0}^{+\infty} e^{-kx}\,dx$

Conv. dominada $\Rightarrow \int_0^{+\infty} xe^{-x} \sum_{k=0}^{+\infty} e^{-kx}\,dx = \sum_{k=0}^{+\infty} \int_0^{+\infty} xe^{-(k+1)x}\,dx$

$$= \sum_{k=0}^{+\infty} \frac{1}{(k+1)^2} \underbrace{\int_0^{+\infty} ye^{-y}\,dy}_{= 1} \text{ (integração por partes).}$$

$$= \sum_{k=1}^{+\infty} \frac{1}{k^2}.$$

Obs: aqui $\frac{x}{e^x-1}$ é ≥ 0, integrável porque para x suficientemente grande, $\frac{x}{e^x-1} \leq \frac{1}{e^{x/2}}$ integrável.