Complemento: área de uma superfície de revolução

Calcule a área da superfície obtida pela rotação da curva em torno do eixo x:

① $y = \sqrt{1+e^x}$, $0 \leqslant x \leqslant 1$

$$y = \sqrt{1+e^x} \quad \Rightarrow \quad y' = \tfrac{1}{2}(1+e^x)^{-1/2}(e^x) = \frac{e^x}{2\sqrt{1+e^x}} \quad \Rightarrow$$

$$\sqrt{1+(y')^2} = \sqrt{1+\frac{e^{2x}}{4(1+e^x)}} = \sqrt{\frac{4+4e^x+e^{2x}}{4(1+e^x)}} = \sqrt{\frac{(e^x+2)^2}{4(1+e^x)}} = \frac{e^x+2}{2\sqrt{1+e^x}}.$$

$$S = \int_0^1 2\pi y\sqrt{1+(y')^2}\,dx = 2\pi\int_0^1 \sqrt{1+e^x}\,\frac{e^x+2}{2\sqrt{1+e^x}}\,dx = \pi\int_0^1 (e^x+2)\,dx$$

$$= \pi\left[e^x+2x\right]_0^1 = \pi[(e+2)-(1+0)] = \pi(e+1)$$

② $y = \frac{x^3}{6} + \frac{1}{2x}, \ \frac{1}{2} \leq x \leq 1$

$$y = \frac{x^3}{6} + \frac{1}{2x} \ \Rightarrow \ \frac{dy}{dx} = \frac{x^2}{2} - \frac{1}{2x^2} \ \Rightarrow \ \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{\frac{x^4}{4} + \frac{1}{2} + \frac{1}{4x^4}} = \sqrt{\left(\frac{x^2}{2} + \frac{1}{2x^2}\right)^2} = \frac{x^2}{2} + \frac{1}{2x^2} \ \Rightarrow$$

$$S = \int_{1/2}^{1} 2\pi \left(\frac{x^3}{6} + \frac{1}{2x}\right)\left(\frac{x^2}{2} + \frac{1}{2x^2}\right) dx = 2\pi \int_{1/2}^{1} \left(\frac{x^5}{12} + \frac{x}{12} + \frac{x}{4} + \frac{1}{4x^3}\right) dx$$

$$= 2\pi \int_{1/2}^{1} \left(\frac{x^5}{12} + \frac{x}{3} + \frac{x^{-3}}{4}\right) dx = 2\pi \left[\frac{x^6}{72} + \frac{x^2}{6} - \frac{x^{-2}}{8}\right]_{1/2}^{1}$$

$$= 2\pi \left[\left(\tfrac{1}{72} + \tfrac{1}{6} - \tfrac{1}{8}\right) - \left(\tfrac{1}{64 \cdot 72} + \tfrac{1}{24} - \tfrac{1}{2}\right)\right] = 2\pi \left(\tfrac{263}{512}\right) = \tfrac{263}{256}\pi$$

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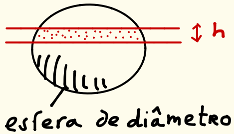
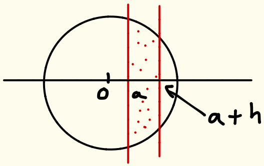


Mostre que a parte da esfera entre 2 planos separados por uma distância h é $\boxed{S = \pi d h}$

Resposta:



Seja $y = \sqrt{R^2 - x^2}$, $y' = \frac{1}{2}\frac{-2x}{\sqrt{R^2-x^2}} \Rightarrow ds = \sqrt{1 + \left(\frac{-x}{\sqrt{R^2-x^2}}\right)^2}\, dx$

$$\Rightarrow S = \int_a^{a+h} 2\pi y \cdot ds = 2\pi \int_a^{a+h} \sqrt{R^2-x^2} \cdot \sqrt{1 + \frac{x^2}{R^2-x^2}}\, dx$$

$$= 2\pi \int_a^{a+h} \sqrt{R^2 - x^2 + x^2}\, dx = 2\pi R \left[x\right]_a^{a+h} = 2\pi R (a+h-a) = 2\pi R h = \pi d h.$$