Rotational properties of nilpotent groups of diffeomorphisms of surfaces

Javier Ribón
javier@mat.uff.br

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Flows

Theorem (Lima)

Let $X_1, \ldots, X_n$ be vector fields in $\mathbb{S}^2$ such that the flows of X_i and X_j commute for all $1 \leq i, j \leq n$. Then $\mathrm{Sing}(X_1) \cap \ldots \cap \mathrm{Sing}(X_n) \neq \emptyset$.

Close to *Id* diffeomorphisms

Theorem (Bonatti)

Let S be a compact surface of non-vanishing Euler characteristic. Let $f_1, \ldots, f_n$ be pairwise commuting C^1-diffeomorphisms close to Id. Then $\mathrm{Fix}(f_1) \cap \ldots \cap \mathrm{Fix}(f_n) \neq \emptyset$.

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The result holds for nilpotent groups on the sphere (Druck-Fang-Firmo).

Isotopy theory

Idea

Finding models of the group up to isotopy with irreducible elements and then transferring the properties of the model to the initial group.

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Thurston classification

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There exists a common Thurston decomposition for abelian groups.

Theorem (Franks-Handel-Parwani '07)

Let G be an abelian subgroup of $\mathrm{Diff}_{+}^{1}(\mathbb{S}^2)$. Then there exists either a global fixed point or a 2-orbit. Moreover G has a global fixed point if $w(f, g) = 0$ for all $f, g \in G$.

Theorem (Franks-Handel-Parwani '07)

Let G be an abelian subgroup of $\mathrm{Diff}^1_+(\mathbb{S}^2)$. Then there exists either a global fixed point or a 2-orbit. Moreover G has a global fixed point if $w(f,g)=0$ for all $f,g \in G$.

$w : G \times G \to \mathbb{Z}/2\mathbb{Z}$ is a morphism of groups

Theorem (Franks-Handel-Parwani '07)

Let G be an abelian subgroup of $\mathrm{Diff}_{+}^{1}(\mathbb{R}^2)$. Suppose that G has a non-empty compact invariant set. Then there exists a global fixed point.

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Applications

Theorem (R)

Let G be a nilpotent subgroup of $\mathrm{Diff}^1_+(\mathbb{S}^2)$. Suppose that G has an odd finite invariant set. Then there exists either a global fixed point or a 2-orbit.

Consider a fixed point free nilpotent subgroup G of $\mathrm{Diff}_+^1(\mathbb{S}^2)$.

Applications

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Definition

We say that two 2-orbits $\mathcal{O}_1$ and $\mathcal{O}_2$ have the same class if

$$\{f \in G : f_{|\mathcal{O}_1} \equiv Id\} = \{f \in G : f_{|\mathcal{O}_2} \equiv Id\}$$

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If there are 4 classes of 2-orbits then there are 3 independent 2-orbits.

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Abelian case

$\{f \in G : f_{|\mathcal{O}_1} \equiv Id\} \cap \{f \in G : f_{|\mathcal{O}_2} \equiv Id\} = \{f \in G : w(f, g) = 0 \ \forall g \in G\}$

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If there are 4 classes of 2-orbits then there are 3 independent 2-orbits.

Abelian case

There are no 4 different classes of 2-orbits. More precisely there are 3 classes of 2-orbits.

Consider a fixed point free nilpotent G of $\mathrm{Diff}^1_+(\mathbb{S}^2)$.

Nilpotent case

$F =$ union of two 2-orbits or three 2-orbits whose classes are pairwise different.

$G \to [G] \subset \mathrm{Mod}(\mathbb{S}^2, F)$

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Theorem (R)

Let G be a fixed-point-free nilpotent subgroup of $\mathrm{Diff}^1_+(\mathbb{S}^2)$. Then there are either 1 or 3 classes of 2-orbits.

Theorem (Firmo-R)

Let $G = \langle H, \phi \rangle$ be a nilpotent subgroup of $\mathrm{Diff}_0^1(\mathbb{T}^2)$ where H is a normal subgroup of G. Suppose that there exists a ϕ-invariant ergodic measure μ such that the support of μ is contained in $\mathrm{Fix}(H)$ and $\rho_\mu(\phi) = (0, 0)$. Then G has a global fixed point.

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Let G be a nilpotent subgroup of $\mathrm{Homeo}(\mathbb{T}^2)$ (resp. $\mathrm{Homeo}_+(\mathbb{T}^2)$, $\mathrm{Homeo}_0(\mathbb{T}^2)$). Then G''' (resp. G'', G') is irrotational.

Theorem (Firmo-Le Calvez-R)

Let G be a nilpotent subgroup of $\mathrm{Diff}^1(\mathbb{S}^2)$ such that $\mathrm{Fix}(G) = \{\infty\}$. Then:

i) G has a fixed point on S_∞;

ii) if $f \in G$, then ∞ is not isolated in $\mathrm{Fix}(f)$ if $\mathrm{Fix}(f) \neq \{\infty\}$;

iii) for every $f \in G$ and every $z \in \mathrm{Fix}(f) \setminus \{\infty\}$ one has $R_f(z) = \{0\}$;

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The result was already known for abelian groups (Beguin - Le Calvez - Firmo - Miernowski)